Robotics and Autonomous Systems 173 (2024) 104620

Contents lists available at ScienceDirect

Robotics and Autonomous Systems

journal homepage: www.elsevier.com/locate/robot

# Defense against smart invaders with swarms of sweeping agents

Roee M. Francos *, Alfred M. Bruckstein

*Faculty of Computer Science, Technion- Israel Institute of Technology, Haifa, 320003, Israel*

ARTICLE INFO

*Keywords:*
Multiple mobile robot systems
Motion and path planning for multi agent systems
Aerial robots
Teamwork analysis
Robot surveillance and security

ABSTRACT

The goal of this research is to devise guaranteed defense policies that allow to protect a given region from the entrance of smart mobile invaders by detecting them using a team of defending agents equipped with identical line sensors. By designing cooperative defense strategies that ensure all invaders are detected, conditions on the defenders' speed are derived. Successful accomplishment of the defense task implies invaders with a known limit on their speed cannot slip past the defenders and enter the guarded region undetected. The desired outcome of the defense protocols is to defend the area and additionally to expand it as much as possible. Expansion becomes possible if the defenders' speed exceeds a critical speed that is necessary to only defend the initial region. We present results on the total search time, critical speeds and maximal expansion possible for two types of novel pincer-movement defense processes, circular and spiral, for any even number of defenders. The proposed spiral process allows to detect invaders at nearly the lowest theoretically optimal speed, and if this speed is exceeded, it also allows to expand the protected region almost to the maximal area.

## 1. Introduction

The objective of this paper is to develop efficient guaranteed defense search strategies in which a swarm of $n$ defending agents must guarantee the detection of an unknown number of smart invaders from entering a region which the defenders guard. An initially given circular region of radius $R_0$ is assumed not to contain mobile invaders at the beginning of the sweep protocol, and is referred to as the initial protected region. The invaders may attempt to move into the protected region from any point outside of the initial protected region and try to enter the protected region (the region where invaders are not located) at a maximal speed of $V_T$, known to the defenders, without being detected by the defending agents. Thus, intruders may slightly enter the protected region, permitting that they are guaranteed to be detected during the current sweep around the region by the defenders. All defenders move at a speed $V_s > V_T$ and detect the invaders with linear sensors of length $2r$. Once a defender's sensor touches a particular location, potential invaders that might have been present there are detected and therefore are "eliminated".

Each guaranteed defense strategy requires a minimal speed that depends on the trajectory of the sweeping defenders and imposes a lower bound on the speed of the defenders. This critical speed is derived to ensure the satisfaction of the guarding task. Increasing the speed above the lower bound enables the defending agents to not only complete the guarding task but also to expand the guarded region as well.

Performing an efficient defense protocol requires that the footprint of the defenders' sensors minimally overlaps the protected region, thus allowing them to detect invaders further away from the protected region and stop their advance. Defenders moving at speeds higher than the critical speed can expand the protected region by performing sweeps around the initial area they need to guard. This extended protected region can grow up to a circular area with a maximum radius, determined by their additional speed, sensing abilities, and the chosen sweeping strategy. This research paper develops two guaranteed defense search protocols for a swarm consisting of an even number of defenders that sweep the region. There are two goals for each developed defense strategy, defending the initial protected region in the defense task and performing the maximal expansion task in which the defenders execute their defense strategy until the protected region reaches its maximal defendable area, by employing novel pincer-movement search strategies. The proposed defense protocols are based on pairs of defenders that move towards each other thus entrapping invaders and halting their advance into the protected region.

### 1.1. Overview of related research

Multi-agent search problems have been an active area of research for almost a century, where early works by [1] focused on designing algorithms for detecting ships and submarines from surveillance aircraft in the English channel during the second world war. Multi-agent search

* Corresponding author.
*E-mail address:* roee.francos@cs.technion.ac.il (R.M. Francos).

https://doi.org/10.1016/j.robot.2024.104620
Received 1 July 2022; Received in revised form 14 December 2023; Accepted 1 January 2024
Available online 5 January 2024

tasks involve searching for static or mobile targets and can take place in environments that range from being fully or partially known to being completely unknown [2–4].

In case the targets being searched are static, searching the entire area in which the targets are located will surely result in their detection. Therefore, in such scenarios, the goal in designing an optimal searching algorithm is to find a traversal path for the searching agents that locates all targets in minimal time. In case the targets are mobile, detection is not always guaranteed since the targets' movements might prevent the searchers from detecting them. This situation can occur even in closed and confined environments, in which the targets cannot exit the borders of the area being searched. In this paper we address the detection of a more challenging type of target, a smart mobile target that may perform evasive maneuvers by detecting and responding to the movements of the defending team in order to avoid being intercepted by the defenders that wish to prevent it from entering the protected region. Smart targets, which in the context of this paper are referred to as invaders, are assumed to have full knowledge of the defenders' strategy and to use that knowledge, to the best of their ability, in order to devise a counter strategy that allows them to enter the protected region without being caught.

In this work we are interested in developing guaranteed defense strategies against smart invaders, implying that regardless of the infiltration plan the invaders choose and their resulting trajectories, they will all be detected by the defending team. In [5], Vincent et al. investigate guaranteed detection of smart targets in a channel environment using a team of detecting sweeping agents and [6] provides optimal strategies to the same problem.

In [7], McGee et al. study how to defend a given planar circular region against the entrance of smart intruders. The intruders do not have any maneuverability restrictions besides an upper limit on their speed. The defenders are equipped with sensors that detect intruders that are inside a disk shaped region around them. The considered search pattern is composed of spiral and linear sections.

Somewhat related problems are pursuit-evasion games, in which the pursuers' goal is to detect and catch the evaders and the evaders goal is to avoid being detected and caught by the pursuing team. There are several variants of pursuit-evasion games which include different combinations of single and multiple evaders and pursuers settings. Pursuit-evasion games were also applied to address defending a region from the entrance of intruders. Such works are [8–10] by Shishika et al. which investigate perimeter defense games and emphasize the cooperation between pursuers to improve the defense tactic. In [8], members of the defending team of agents cooperate and form defender pairs by moving in a "pincer movement" to prevent intruders from entering a convex region in the plane. Cooperation between the defender sub-teams, allows to extend the winning regions of the defender team compared to performing uncooperative defense strategies.

In [11], pursuit–evasion problems involving multiple pursuers and multiple evaders (MPME) are studied. Pursuers and evaders are all assumed to be identical, and pursuers follow either a constant bearing or a pure pursuit strategy. The problem is simplified by adopting a dynamic divide and conquer approach, where at every time instant each evader is assigned to a set of pursuers based on the instantaneous positions of all the players. The original MPME problem is decomposed to a sequence of simpler multiple pursuers single evader (MPSE) problems by testing whether a pursuer is relevant or redundant against each evader, by using Apollonius circles. Then, only the relevant pursuers participate in the MPSE pursuit of each evader. Recent surveys on pursuit evasion problems are [12–14].

In [12], a taxonomy of search problems is presented. The paper highlights algorithms and results arising from different assumptions on searchers, evaders and environments and discusses potential field applications for these approaches. The authors focus on a number of pursuit-evasion games that are directly connected to robotics and not on differential games which are the focus of the other cited surveys. The paper concentrates on adversarial pursuit-evasion games on graphs and in polygonal environments where the objective is to maximize the worst-case performance on the search or capture time and on probabilistic search scenarios where the objective is the optimization of the expected value of the search objective, such as the maximal probability of detection or minimal capture time. In [13], a survey on pursuit problems with 1 pursuer versus 2 evaders or 2 pursuers versus 1 evader are formulated as a dynamic game and solved with general methods of zero-sum differential games. In [14], the authors present a recent survey on pursuit-evasion differential games and classify the papers according to the numbers of participating players: single-pursuer single-evader (SPSE), MPSE, one- pursuer multiple-evaders (SPME) and MPME.

In [15], a two-player differential game in which a pursuer aims to capture an evader before it escapes a circular region is investigated. The state space, comprised of pursuer and evader locations, is divided into evader and defender winning regions. In each region the players try to execute their optimal strategies. The players' strategy depends on the state of the system (if it is in the capture or escape regions), and the proposed approach guarantees that if the state of the system is in the winning region of one of the players, and that player executes its prescribed optimal move, then they are guaranteed to win. The players move at a constant speed and the pursuer is faster than the evader. The players' controls are the instantaneous heading angles. The game is a two-termination set differential game, i.e., the game ends when either player wins. In [16], the problem of a border defense differential game where $M$ pursuers cooperate in order to optimally catch $N$ evaders before they reach the border of the region and escape is investigated. The members of the pursuer team exchange information between the team members and decide on the discrete assignment of pursers to evaders in an on-line manner. Furthermore, the game is a perfect information differential game where both pursuers and evaders have access to all state variables, which are the locations of all players, as well all their dynamics and velocities. The pursuers in this setting are assumed to have greater speeds than the evaders. The game takes place in a simple half-plane environment, and ends when all evaders are either caught or reach the border and escape.

Devising multi-robot perimeter patrol policies for adversarial settings in which an opponent has complete knowledge of the robots' patrol strategy are developed in [17,18]. Possessing information about the robots' patrol policy enables the smart opponent to attempt to enter the perimeter undetected at the location with the highest probability of success. In order to prevent the opponent to utilize its knowledge on the strategy of the robot team, randomness is introduced into the robots' perimeter patrol algorithm, thus preventing the opponent from having full knowledge of the chosen patrol strategy and consequently increasing the chances to detect it.

The problems considered in the papers [19,20], research related problems of boundary patrolling by a team of defending agents. In [19] the objective of the patrolling team is to minimize an idleness measure, defined as the maximal time interval for which a boundary point remains unvisited. All patrolling agents are equipped with a pointwise sensor that detects invaders once they coincide with their location. The defending agents have distinct maximal speeds at which they move, and their aim is to protect the region from an intruder that attempts to enter the region from an unknown location on the boundary. The assumption is that the intruder needs a certain amount of time to penetrate the boundary. Hence, if the longest time interval at which any given point on the boundary remains unvisited by a patrolling agent is less than the intruders' penetration time, all intruders are guaranteed to be detected. The patrolling agents traverse the boundary of the region, and the authors propose cyclic strategies in which all defending agents move in the same direction around the boundary to detect invaders as well as partition-based strategies in which the environment is partitioned into different sections, each patrolled by a different defending agent. The paper provides theoretical bounds and results when 2,3 and 4 agents perform the patrolling task and discuss the optimality of the solutions given the model's assumptions.

In [20], the authors investigate the problem of patrolling the boundary of a given disk shaped region using robots equipped with sensors with a given visibility range measured on the boundary. As in our setting, all invaders that are within the robots' sensors range are detected. The robots have distinct visibility ranges and may also have distinct speeds. The goal is again to minimize the unseen time intervals at all points on the boundary of the region by proposing same direction and different direction patrolling algorithms. In the described setting of different visibilities and same speeds the authors show that their protocol achieves optimal idle times. In case agents have different speeds and different visibility ranges an optimal algorithm is provided for 2 patrolling agents.

The problems investigated in the papers [21,22] are closely related to the defense problem we investigate in this paper. In [21], a dynamic boundary defense problem against radially advancing intruders is investigated. The paper considers a single defending vehicle that is charged with preventing a set of intruders that are stochastically generated at a given rate according to a Poisson process on the boundary of a circular region from entering an inner circular perimeter guarded by the defending vehicle. The goal of the defending vehicle is to maximize the capture fraction of the incoming intruders by choosing the order of intruders it detects by moving and coinciding with their location. The paper develops several vehicle routing policies for low and high target arrival rates and compares these approaches to an optimal upper bound on the capture fraction of the intruders that is independent of the particular defense strategy implemented by the defending vehicle.

In [22], a perimeter defense problem where a defending vehicle must guard a given one dimensional line segment from the entrance of intruders is investigated. The paper proposes several online algorithms that enable interception of intruders and provides theoretical guarantees on the performance of these algorithms in several different parameter regimes in which the speed of the intruders and the perimeter of the region to be guarded varies. The proposed online algorithms provide results that allow provable guaranteed capturing of all, half or a fourth of the intruders compared to an optimal offline algorithm based on increasingly difficult parameter regimes and discusses the parameter regimes at which it is more beneficial for the defender to implement a certain protocol.

### *1.2. Contributions*

In this paper, we provide several theoretical and experimental contributions to multi-agent search and coordinated motion planning literature. In the considered scenario, a defending team of agents has to protect an initial region from the entrance of an unknown number of smart invaders, that have superior sensing and planning capabilities compared to the defender team. An analysis on the defenders' trajectories and critical speeds that enable the successful completion of the defense task is provided. Additionally, if possible, an additional goal for the defender team is to optimally expand the region which they guard to the maximal allowable size that still enables the defender team to detect any number of smart invaders that may attempt to enter the protected region. Hence, the total search times and the maximal attainable protected area are also reported. Extensive theoretical and numerical analysis is performed for both the defense and maximal expansion tasks.

- We propose two types of novel guarding and expansion strategies for any even number of defenders:
  - Circular defense pincer sweep strategy
  - Spiral defense pincer sweep strategy
- Based on geometric and dynamic constraints we establish the necessary critical speed for each defense protocol to be successful and derive analytical expressions for the search times and radius of maximal expansion for the two types of search patterns.
- We show that the spiral defense pincer sweep strategy enables defenders that sweep with speeds that are only slightly above the theoretical lower bound to ensure all invaders attempting to enter the protected region are detected.
- We compare between the circular and spiral defense pincer sweep expansion strategies and highlight the advantages of the spiral strategy in both enabling the expansion of the region to a larger size and the search time required to reach it.
- We provide a quantitative comparison between the developed pincer-based defense protocols and the corresponding same-direction sweep protocols that are regarded as the state-of-the-art in defense against smart invaders. We prove that the corresponding pincer-based protocols yield lower critical speeds, shorter time to increase the protected region to its maximal size as well as the ability to expand the protected region to a larger area compared to same-direction protocols.
- We demonstrate through empirical simulations conducted with MATLAB and NetLogo the theoretical results and present the evolution of the guarding and expansion strategies graphically.

### *1.3. Paper organization*

This article is organized as follows. Section 2 addresses the motivation for employing pincer-based defense strategies, introduces essential concepts and considerations used in pincer-based detection, and compares our developed protocols with related works. Section 3 presents an optimal bound on the speed of defenders employing the guarding task. This lower bound is independent of the actual implemented defense protocol. The obtained lower bound is used as a one of the comparison metrics for evaluation the performance of different guard/defense protocols. Section 4 provides an analysis of the defense and maximal expansion problems when the defender team performs the circular defense pincer sweep process. Section 5 presents an analysis of the defense and maximal expansion problems when the defender team performs the spiral defense pincer sweep process. Section 6 provides a comparison between the circular and spiral defense pincer defense strategies and highlights the advantages of using the proposed spiral defense pincer protocol. Section 7 compares prevalent approaches for defense against smart invaders which are considered the state-of-the-art in defense against smart invaders and compares these approaches to the pincer-based defense protocols developed in this work, proving the superiority of pincer-based approaches. The last section draws conclusions from the performed analysis and provides some interesting future research directions.

## 2. Pincer-based defense

This research focuses on developing a guaranteed defense protocol of an initial region from the entrance of an unknown number of smart invaders. The region is protected by employing a multi-agent team of identical cooperating defenders that sweep around the protected region and detect invaders that attempt to enter it. The defenders possess a linear sensor of length $2r$ with which they detect invaders that intersect their field-of-view. The only information the defenders have is that invaders may be located at any point outside of an initial circular region of radius $R_0$, referred to as the initial protected region at the beginning of the defense process.

The proposed defense strategies are deterministic and pre-planned, and therefore they can be accomplished by using simple agents-like defenders. All defenders move with a speed of $V_s$ which is measured at the center of a defender's sensor. The invaders move at a maximal speed of $V_T$, and do not have any maneuverability restrictions.

The time it takes the defender team to expand the protected region to the maximal defendable size obviously depends on the applied defense protocol. Two types of defense strategies are investigated, circular and spiral. When defenders perform the maximal expansion task, their goal is to iteratively increase the radius of the protected region after each sweep, up to the maximal defendable size of the region. At the beginning of the circular defense pincer sweep process only half the length of the defenders' sensors is outside of the protected region, i.e. a footprint of length $r$, while the other half is inside the region in order to catch invaders that may move inside the region while the search progresses. At the beginning of the spiral defense pincer sweep process the entire length of the defenders' sensors is outside the protected region, i.e. a footprint of length $2r$.

The basic idea in performing pincer-based defense is requiring defenders to search for invaders in opposite directions instead of equally distributing the defenders around the region and letting them search in the same direction. Defenders performing pincer movements address the worst-case scenario of invader entrance to the protected region from the "most dangerous points", points situated at the edges of their sensors nearest to the protected region's boundary. Invaders located at these points have the maximal amount of time to advance towards the protected region during defender movement and thus if invaders that attempt to infiltrate the region from these points are detected, invaders trying to enter from other points are detected as well. For an extensive discussion about the comparison between pincer-based sweeps and same-direction sweeps in search tasks against smart evaders see [23,24].

Successfully completing the defense and maximal expansion tasks with the lowest possible critical speed is one of the performance criteria for an efficient defense strategy. Pincer-based defense procedures result in lower critical speeds compared to their same-direction counterparts, and hence are chosen in the developed defense protocols. The discussed pincer-based strategies can be performed with any even number of defenders. The basic idea of pincer-based defense is to decompose the defender team into pairs that are placed back-to-back at the start of the protocol. Each defender in a pair moves in an opposite direction, counter-clockwise or clockwise. When two defenders meet at a location after the completion of a sweep, implying that their sensors are again back-to-back, they switch their movement direction. The direction switches are performed every time a defender meets the defender scanning the adjacent angular section to its section. Based on the numbers of defenders performing the defense task, the protected region is portioned into equal angular sectors, where each sector is searched by a different defender. The discussed defense protocols can be applied to both 2 dimensional defense tasks on the plane or in 3 dimensional defense tasks where the defenders and invaders are drone-like agents that fly over the protected region. In the 3 dimensional defense tasks, defenders fly above invaders and detect their locations while implementing the same planar defense tasks as in the described 2 dimensional protocols.

The first considered defense protocol is the circular defense sweep protocol. The circular defense protocol is a simple method that enables defenders with basic motion capabilities to carry out their defense and achieve maximal expansion. However, due to its simplicity, it may not be the most optimal strategy. Hence, we propose the spiral defense pincer sweep process that provides an improved protocol that uses spiral scans, drawing inspiration from [7]. The spiraling-in trajectories of the defenders allow them to track the "wavefront" of the expanding protected region, thus detecting invaders at the furthest possible locations from the invader region. At last, we compare and discuss the obtained results of the two defense strategies. The evaluation metrics for the defense strategies include the minimal defender speed required for successful defense of the initial protected region, the time to expand the protected region to the maximal defendable area as well as the maximal feasible protected region's radius resulting from the defense protocol. All these quantities are expressed as a function of the search parameters $R_0$, $r$, $V_T$ and the number of defenders, $n$.

Illustrative simulations demonstrating the evolution of the defense processes were generated using NetLogo software [25] and are presented in Figs. 2 and 3. Green areas show locations that were searched and hence do not contain invaders and red areas indicate locations where potential invaders may be present. Blue areas represent locations that belong to the initial protected region that does not contain invaders. Fig. 2 shows the cleaning progress during the expansion of the protected region when 6 defenders perform the circular defense pincer sweep process.Fig. 3 shows the evolution of the defense process during the expansion of the protected region when 4 defenders perform the spiral defense pincer sweep process. It is worth emphasizing that recently searched areas are shrinking due to the advancement of invaders into the cleared regions. This is depicted in Figs. 2 and 3 in the initial sweep by the decrease of blue and green areas that turn red, and in subsequent sweeps when green areas become red. A location in the environment turns green once a defender's sensor touches that particular location. Once the defender leaves that location, invaders can attempt to enter it again, thus after some time that depends on the expansion speed of the intruders, a green location can turn to be red again.

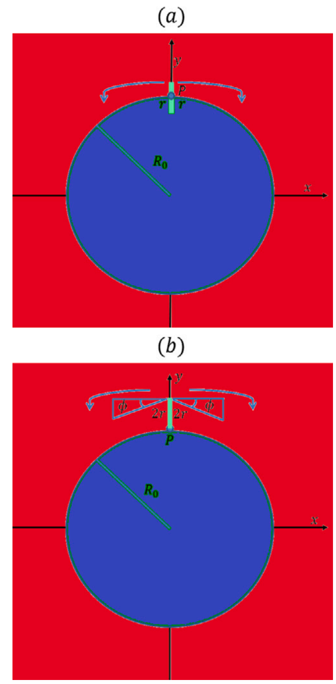


**Fig. 1.** (a) - Initial placement of 2 defenders performing the circular defense pincer sweep process. (b) - Initial placement of 2 defenders performing the spiral defense pincer sweep process. Defenders' sensors are shown in green and red areas indicate locations where potential invaders may be present. Blue areas represent locations that belong to the initial protected region that does not contain invaders. The angle $\phi$ is the angle between the tip of a defender's sensor and the normal to of the edge of the protected region. $\phi$ is an angle that depends on the ratio between the defender and invader speeds. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Note that in the considered problems, the search is continued until the expansion of the protected region reaches the maximal attainable radius, and afterwards the defenders continuously patrol around this radius.

### 2.1. Comparison to related research

In our previous work [26], the confinement of an unknown number of smart evaders that are originally located somewhere inside a given circular region to their original domain is investigated. By deploying a line formation of searching agents or alternatively a single agent with an equivalent sized linear sensor that sweep inside and around the region, guaranteed detection protocols are developed. In case the speed of the agents in the line formation exceeds a critical speed, they may decrease the evader region by performing a search protocol consisting

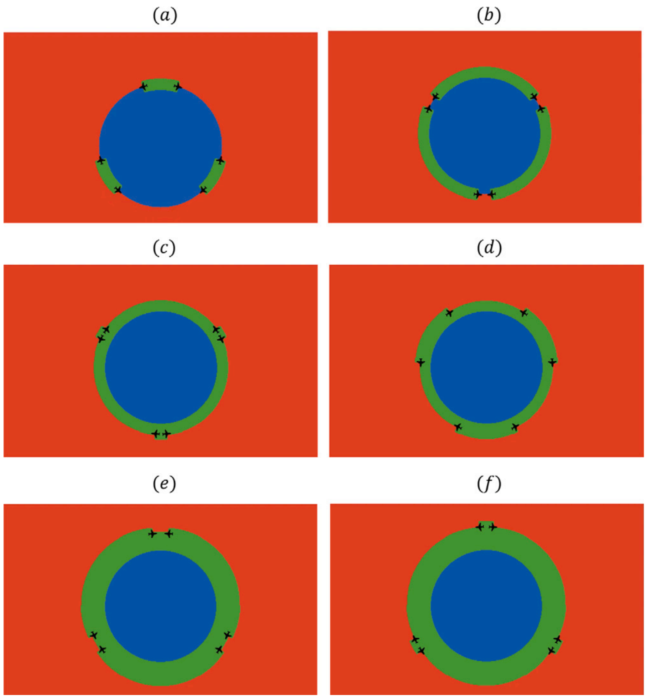


**Fig. 2.** Swept areas and protected region status for different times in a scenario where 6 defenders perform the circular defense pincer sweep process. (a) - Beginning of first sweep. (b) - Towards the completion of the first sweep. (c) - Beginning of the second sweep. (d) - Midway of the second sweep. (e) - End of the fourth sweep. (f) - Beginning of fifth sweep. Green areas show locations that were searched and hence do not contain invaders and red areas indicate locations where potential invaders may be present. Blue areas represent locations that belong to the initial protected region that does not contain invaders. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

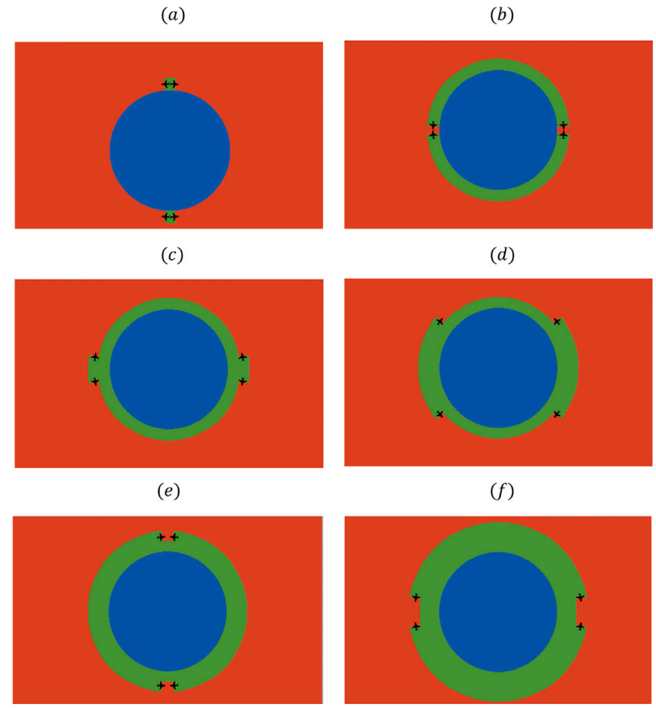


**Fig. 3.** Swept areas and protected region status for different times in a scenario where 4 defenders perform the spiral defense pincer sweep process. (a) - Beginning of first sweep. (b) - End of the first sweep. (c) - Beginning of the second sweep. (d) - Midway of the second sweep. (e) - End of the second sweep. (f) - Towards the end of the third sweep. Green areas show locations that were searched and hence do not contain invaders and red areas indicate locations where potential invaders may be present. Blue areas represent locations that belong to the initial protected region that does not contain invaders. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of alternating circular sweeps around the region that are followed by inward advancement steps toward the center of the evader region (the region in which evaders are located). A proof in the paper shows that since the evaders are smart, a search pattern that uses circular sweeps cannot completely clean the evader region, Therefore, after the evader region is reduced by the circular sweeping protocol to a region that is bounded in a circle with a radius equal to half the formation's sensing range, the search pattern must be changed in order to perform a set of end-game maneuvers that guarantee the detection of all smart evaders in the region.

In [23], we consider teams of agents that perform pincer sweep search strategies with linear sensors, in order to detect all smart evaders that try to escape from a given region. The paper presents two multi-agent pincer sweep search strategies, circular and spiral, that can be applied with any even number of sweeping agents. The results obtained from the paper show that performing the circular pincer sweep process, where pairs of defenders sweep towards each other allow for lower critical speeds and shorter sweep times until the entire evader region is searched and cleared from evaders compared to a circular sweep process in which the defenders are equally distributed around the evader region and all sweep in the same direction. A circular sweep process in which the searchers all rotate in the same direction is the extension of [26] to a scenario where the defenders are distributed equally around the region and perform the circular sweep protocol described in [26]. Therefore, defenders that rotate in the same direction have to perform an end-game scenario similar to the one described in [26] in order to completely clear the region from evaders, a set of maneuvers that is unnecessary when using the circular pincer sweep protocol described in [23].

The second type of sweep pattern that is developed in [23] is the spiral pincer sweep process that allows to complete the search of the evader region in a significantly shorter time and at lower critical speeds compared to the circular pincer sweep process. The critical speed of the spiral pincer sweep process approaches the theoretical lower bound on the critical speed.

This work aims to solve the dual problem to the problem investigated in [23,26], which is to protect a given initial region that does not contain invaders from entering the region undetected, and if possible to expand the protected region to the maximal defendable size possible. The first task this work is concerned with, the guarding task is analogous to the confinement task in [23,26], in the sense that it aims to keep the protected region's radius constant after the defenders complete a full sweep around the region. The maximal expansion task, in which after each full sweep around the protected region, its radius increases is the dual problem to the constriction of the evader region in [23,26].

In contrast to the algorithms described in [23,26] the defense process does not terminate when the defenders expand the protected region to its maximal size and they must continuously sweep around the region to keep the intruders that are outside of the protected region from entering it. Additionally, this work presents for each developed pincer sweep defense process the maximal radius that the protected region can be extended into and presents an analysis on the trade-off between approaching the maximal protected region's radius and the sweep time it takes to reach it. Alternatively to the barrier placement problem in [7], our approach emphasizes the usage of cooperation between the defenders by using pincer sweeps, calculates the maximal defendable region's size and presents analytical solutions to all aspects of the defense and maximal expansion problems against smart invaders.

Related to our work, [19] interestingly investigates movements of equally spaced defenders that move in the same direction and also an approach that allows defenders to sweep in opposite directions along a fixed circular boundary. Under the assumptions of the model and the constraint that all agents must patrol in the same direction, the authors prove optimality results for a small number of agents that perform the boundary patrolling. The developed protocol uses a subset of agents with sufficiently high speeds. The chosen agents all move with the maximal speed of the slowest agent. The authors next investigate the case at which defending agents can move in both directions and conjecture that when using 3 or more patrolling agents, in order to obtain optimal results, patrolling strategies must utilize opposite direction sweeps in various settings.

Despite related objectives, there are fundamental differences between the goal of the work [19] and ours. In our work, the defenders are equipped with line sensors versus the pointwise sensors of defending agents in [19]. While the pointwise sensors may be thought of as a particular case of line sensors that are located at the defender's center, thereby reducing the visibility range of the defenders, the analysis of protocols with line sensors is radically different. The entire analysis and results are based on having a non-zero visibility range due to line sensing, and hence are not applicable in case of point sensing. Furthermore, the analysis of the most dangerous points invaders can enter the protected region from, the critical speed calculation and the maximal expansion protocols are also unique to our work. Finally, we show that in the adversarial intruder setting we investigate, it is always better to perform pincer-based defense protocols compared to performing same-direction defense protocols.

An additional difference is that while [19] seeks to protect stationary points on the boundary of the region, in our setting we assume the invaders team continuously moves and makes optimal maneuvering decisions against any possible defense protocol that attempts to detect its members. Furthermore, in our setting we provide analytical and provable solutions to any even number of defenders and can precisely determine how close an algorithm is to an optimal solution by using the critical speed metric and by comparing the maximal radius the protected region can be increased into related to the theoretical maximal circle that can be protected using defenders with a given speed limit. At last, in our work we show that it is always more beneficial to use an increasing number of defenders.

Similarly to our setting, the authors explore the patrolling of a circular boundary. While the sensors used in [20] resemble the line sensors used in our work, they consider intruders that appear at random boundary locations and must stay there for a certain amount of time to penetrate the boundary as opposed to our model in which intruders are in constant motion with a bounded speed and can maneuver optimally to avoid detection. This assumption avoids the inherently two-dimensional nature of the problem that we solve. In case all robots have the same speed and the same visibility range the algorithms described are somewhat similar to the same-dimensional algorithms we investigate, however a major distinction in the analysis arises since in our protocols the invaders are may briefly enter the protected region before being detected at a later stage of a sweep cycle. This necessitates an analysis of the possible spread of invaders into the protected region from all points outside of the boundary, which is quite different from the setting of the problem investigated in [20]. The possibility of expanding the guarded boundary and the ensuing analysis is not considered in the work of [20] as well.

Related to our work [21] develops a policy independent upper bound on the capture fraction that is similar to the policy independent lower bound on the critical speed we develop. Additionally, the paper considers that intruders move at lower speeds compared to the defender which is similar to our assumption. In contrast to our setting, that guarantees intruder detection for worst-case scenarios, the approach in [21] is an online approach that focuses on algorithms that incorporate replanning of the vehicle's trajectory as new information regarding the intruders' locations is obtained.

Additionally, the paper considers that a discrete number of invaders are generated stochastically according to a given rate while in our work we assume that intruders are continuously generated at every time instance at every location that is not part of the protected region at that instance. Furthermore, our work considers a multi-agent setting where algorithms are developed for any even number of defenders and not only for a single defending agent.

Another distinction is that the sensor used in [21] is a pointwise sensor, differing from the line sensors we use in our work. Finally, [21] uses a probabilistic approach aiming to maximize the capture fraction of intruders, which is a fundamentally different setting from our approach that seeks to develop deterministic defense protocols whose performance is guaranteed against worst-case actions utilized by the defending team's adversarial opponents, and to expand the protected perimeter if possible.

Related to our work [22] provides theoretical lower and upper bounds on the performance of the proposed defense protocols and characterizes the quality or competitiveness of the algorithms based on the performance for worst case inputs. This approach is similar to the analysis we perform in our setting by characterizing the "most dangerous points" that intruders can enter the protected region from and developing defense strategies that ensure that if intruders attempting to enter the region from such points are detected, all other intruders will be detected as well. An additional similarity to our work is that the performance of the defense protocols in [22] do not depend on the number or placement of intruders.

Albeit the similarities there are many distinctions from [22]. Amongst them is that in our setting the environment to be guarded is 2 dimensional, does not contain a fixed perimeter, the defenders are equipped with line sensors and not pointwise sensors, that information on the locations of intruders arrives online and it not assumed that intruders arrive all the time from all locations on the boundaries of the region, and that we investigate a multi-agent setting in which we particularly focus on the distribution of the defending agents in contrast to the single vehicle case investigate in [22].

Importantly, in contrast to [22], we are only interested in defense protocols that allow guaranteed detection of all intruders under the considered parameter regimes and not only a fraction of them. Since we can calculate a theoretical lower bound that is independent of the implemented defense protocol for one of our performance metrics, the critical speed, for which attaining a lower value is better, we are able to compare all the provably guaranteed defense protocols we develop in order to analyze their quality. The spiral pincer defense protocol allows us to approach this theoretical lower bound for any even number of defenders. Furthermore, we prove in our paper that the spiral pincer defense protocol is better across all performance metrics (critical speed, time until maximal expansion and maximal protected region's radius), parameter regimes and number of defenders. Hence, if the orchestrator of the defense protocol has availability to robots that can implement this protocol they should always choose to use it over other protocols, differing from the selection of different defense protocols for different parameter regimes as described in [22].

The research conducted in [8–10] also investigates protecting a region from the entrance of invaders and uses pincer movements between pairs of defenders as well. However, the goal of the defender team in these works is to intercept the largest possible number of invaders contrary to our approach which develops a defense protocol that ensures detection of all invaders, regardless of their numbers, and consequently sets necessary requirements on the defender team in order to achieve its goal. Furthermore, in [8–10], the invaders' locations are constantly known to the defenders hence this information assists the defender team in planning and coordinating its movements and its allocation of defender pairs. Conversely, our approach does not assume any knowledge on the invaders' locations or their number.

## 3. A universal bound on cleaning rate

In this section we present an optimal bound on the cleaning rate of a defender with a linear shaped sensor. This bound is independent of the particular defense pattern employed. For each of the proposed defense methods we then compare the resulting cleaning rate to the optimal derived bound in order to compare between different defense protocols. We denote the defender's speed as $V_s$, the sensor length as $2r$, the protected region's initial radius as $R_0$ and the maximal speed of an invading agent as $V_T$. The maximal cleaning rate occurs when the footprint of the sensor outside the protected region is maximal. For a line shaped sensor of length $2r$ this happens when the entire length of the sensor is fully outside the protected region and it moves perpendicular to its orientation. The rate of sweeping when this happens has to be higher than the minimal decrease rate of the protected region (given its total area) otherwise no sweeping process can ensure detection of all invaders. We analyze the defense process when the defender swarm comprises $n$ identical agents. The smallest defender's speed satisfying this requirement is defined as the critical speed and denoted by $V_{LB}$, we have:

**Theorem 1.** *No sweeping process is able to successfully complete the defense task if its speed, $V_s$, is less than,*

$$\mathrm{V}_{LB} = \frac{\pi R_0 V_T}{nr} \tag{1}$$

For proof see [26]. The desired outcome is that after the first sweep the protected region is within a circle with a greater radius than the initial protected region's radius.

## 4. Circular defense pincer sweep process

At first, we study a scenario in which a multi-agent team comprising of $n$ agents, referred to as defenders, perform the defense task. The number of defenders, $n$, is even, and all defenders are identical and are equipped with a linear sensor with a length of $2r$. The initial defenders' configuration at the start of the defense protocol is such that each defender has half of its linear sensor outside of the protected region, i.e. a length of $r$. Fig. 4 shows an illustration of the dynamic evolution of searched regions and regions containing invaders throughout the defense task when 4 defending agents perform the circular defense pincer protocol.

### 4.1. The defense task and critical speed analysis

Using pincer movements between adjacent pairs of defenders leverages the symmetry between the trajectories of nearby defenders in order to impede the entrance of invaders to the protected region from the most dangerous points invaders can enter from (by using similar arguments to the proof provided in [26]. Hence, the defenders' critical speed is computed based on the time required for a defender to scan the sector allocated for it, i.e. an angular section of $\frac{2\pi}{n}$. In case the defenders' speeds exceed the critical speed required for successfully implementing the defense/guarding task, the defenders can advance outwards together from the center of the protected region after completing a sweep. A full sweep or iteration refers to a defender's scan of the sector of the protected region it guards spanning an angle of $\frac{2\pi}{n}$. Hence, the scanned angle each defender guards is a function of the participating defenders in the defense task. When defenders perform the defense task, they change their scanning direction after the completion of a full sweep.

**Theorem 2.** *The circular critical speed equals twice the optimal minimal critical speed,*

$$V_c = 2V_{LB} \tag{2}$$

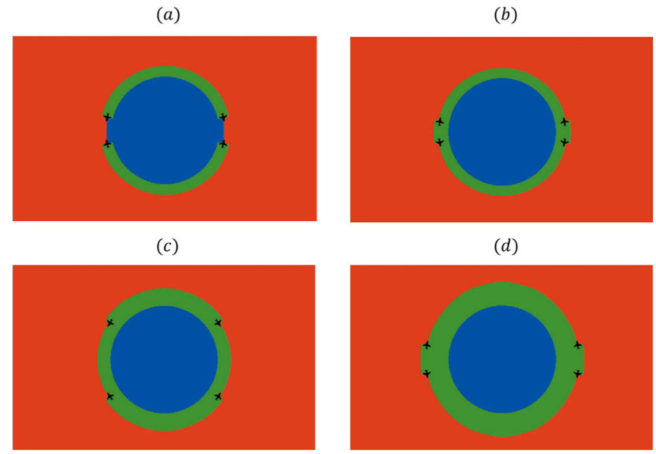


**Fig. 4.** Swept areas and protected region status for different times in a scenario where 4 defenders perform the circular defense pincer sweep process. (a) - Beginning of first sweep. (b) - Towards the end of the first sweep. (c) - Beginning of second sweep. (d) - Midway of the third sweep. Green areas show locations that were searched and hence do not contain invaders and red areas indicate locations where potential invaders may be present. Blue areas represent locations that belong to the initial protected region that does not contain invaders. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Proof.** Every defender performing the circular defense pincer sweep process has a sensor length of $r$ inside the protected region, to ensure no invader enters the protected region without being detected by the defenders. Hence, in order to catch all invaders, the spread from any potential location inside the invader region (the region outside of the protected region where invaders might be located), has to be restricted to a radius less than $r$ from its origin point at the start of the defense protocol. Therefore, during an angular traversal of $\frac{2\pi}{n}$ around a protected region with a radius of $R_0$, this requirement implies that,

$$\frac{2\pi R_0}{nV_s} \leq \frac{r}{V_T} \tag{3}$$

Hence, in order to detect all invaders, the defenders' speed has to exceed,

$$V_s \geq \frac{2\pi R_0 V_T}{nr} \tag{4}$$

The critical speed for the circular defense pincer sweep process is obtained when (4) is satisfied with equality.

$$V_c = \frac{2\pi R_0 V_T}{nr} \quad \square \tag{5}$$

The obtained result matches the observation that the circular critical speed for defending a region against the entrance of smart invaders should equal the critical speed required for confining smart evaders (developed in [23]) inside a region having the same size.

### 4.2. The maximal expansion task

#### 4.2.1. Number of sweeps analysis

**Lemma 4.1.** *The maximal radius that $n$ circularly sweeping defenders, with a linear sensor of length $2r$, a given speed $V_s$ and a maximal invader speed of $V_T$ can safely protect against the entrance of invaders is,*

$$\bar{R}_{N_c} = \frac{nV_s r}{2\pi V_T} \tag{6}$$

**Proof.** Given $n$ defenders having a fixed speed of $V_s$ that employ the circular defense pincer protocol, while denoting by $R_{N_c}$ the maximal boundable radius, replacing it with $R_0$ in (4) and rearranging terms yields,

$$R_{N_c} \leq \frac{nV_s r}{2\pi V_T} \tag{7}$$

The maximal boundable radius is attained when (7) is satisfied with equality. □

**Theorem 3.** *For a defender team with $n$ defenders, for which $n$ is even, performing the circular defense pincer sweep process, the number of sweeps required for the defender team to expand the protected region to be bounded by a circle with a radius that is $\varepsilon$ close to the maximal boundable radius $\bar{R}_{N_c}$ is,*

$$N_n = \left\lceil \frac{\ln\left(\frac{-2\pi V_T \varepsilon}{2\pi R_0 V_T - nV_s r}\right)}{\ln\left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)} \right\rceil \tag{8}$$

**Proof.** Denote by $\Delta V > 0$ the excess speed of the defender above the critical speed. Hence, the defender's speed is $V_s = V_c + \Delta V$. The time required for a defender to circularly sweep around its allocated section is,

$$T_{circular\,i} = \frac{2\pi R_i}{n(V_c + \Delta V)} \tag{9}$$

Since $V_s = V_c + \Delta V$, $T_{circular\,i}$ may also be expressed as,

$$T_{circular\,i} = \frac{2\pi R_i}{nV_s} \tag{10}$$

Depending on the number of participating defenders and the iteration number, the distance a defender can advance outwards from the center of the protected region after completing an iteration is,

$$\delta_i(\Delta V) = r - V_T T_{circular\,i}\ , \ \ 0 \leq \delta_i(\Delta V) \leq r \tag{11}$$

In the expression $\delta_i(\Delta V)$, $\Delta V$ is the excess speed of a defender with respect to the critical speed. Denote by $i$ the number of full sweeps defenders perform around the protected region, in which the first sweep occurs when $i = 0$. After completing a full sweep, the defenders move outwards from the center of the protected region with the inner tips of their sensors pointing to the center of the protected region. At times in which defenders move outwards, they do so with a speed of $V_s$. This motion continues up to the location in which the defenders begin their next sweep once half of their sensor is inside the expanding wavefront of the invading region. During the outward advancements no invaders are detected, while the protected region continues to shrink.

The time it takes defenders to move outwards up to the point where half of their sensors are outside of the protected region depends on the relative speed between the defenders outward motion and the invader region's inwards expansion and is given by (13). As the defenders gradually head outward from the center of the protected region, the protected region continuous to shrink. Hence, in order for no invader to enter the region, defenders must advance outwards to a lesser extent than $\delta_i(\Delta V)$. This quantity is denoted by $\delta_{i_{eff}}(\Delta V)$, and depends on the ratio between the speed in which defenders move outwards from the center of the protected region and the sum of the defender and invader region spread speeds. $\delta_{i_{eff}}(\Delta V)$ is the actual distance defenders are allowed to progress outwards after each sweep so that they meet the wavefront of the expanding invader region at the point where half of their sensors are over the invader region. Therefore, the distance a defender may advance outwards after completing a sweep around the protected region is,

$$\delta_{i_{eff}}(\Delta V) = \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{12}$$

The outward advancement time depends on the iteration number. It is denoted by $T_{out_i}$ and is expressed as,

$$T_{out_i} = \frac{\delta_{i_{eff}}(\Delta V)}{V_s} = \frac{rV_s - 2\pi R_i V_T}{nV_s\left(V_s + V_T\right)} \tag{13}$$

The index $i$ in $T_{out_i}$ denotes the iteration number in which the advancement takes place. After the defenders complete their sweep, the protected region is bounded by a circle with a larger radius compared to the previous sweep. Thus, the new radius of the circle bounding the protected region is given by,

$$R_{i+1} = R_i + \delta_{i_{eff}}(\Delta V) = R_i + \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{14}$$

Replacing the value of $\delta_i(\Delta V)$ from (11) into (14) results in,

$$R_{i+1} = R_i + \delta_{i_{eff}}(\Delta V) = R_i + \frac{rV_s}{V_s + V_T} - \frac{2\pi R_i V_T}{n\left(V_s + V_T\right)} \tag{15}$$

Rearranging terms yields,

$$R_{i+1} = R_i\left(1 - \frac{2\pi V_T}{n\left(V_s + V_T\right)}\right) + \frac{rV_s}{V_s + V_T} \tag{16}$$

Denote the coefficients $c_1$ and $c_2$ by,

$$c_1 = \frac{rV_s}{V_s + V_T},\, c_2 = 1 - \frac{2\pi V_T}{n\left(V_s + V_T\right)} \tag{17}$$

Hence, (16) can be expressed as,

$$R_{i+1} = c_2 R_i + c_1 \tag{18}$$

In order to increase the guarded region (and consequently the protected region's radius), defenders need to move outwards from the center of the protected region and search around a circle having a larger radius. The expansion task progresses by alternating circular sweeps and outward movements until the protected region is bounded by the largest possible circle. During the defenders outward motion phases, it is assumed that invaders are not detected by defenders since their line sensors have zero width. Hence, throughout the outward motions of defenders, the protected region continues to shrink due to the inward progression of invaders into the protected region. Thus, to accommodate invader progression towards the protected region during the outward advancement times of the defenders, defenders are able to protect a marginally smaller protected region than a region with the maximal radius of (6). Let $\varepsilon > 0$, and denote by $\hat{R}_{N_n-1}$ the radius of the protected region that is $\varepsilon$ close to $\bar{R}_{N_c}$. Therefore,

$$\hat{R}_{N_n-1} = R_{max} = \frac{nV_s r}{2\pi V_T} - \varepsilon \tag{19}$$

The number of iterations required for the defender team to expand the protected region to be bounded by a circle with a radius of $\hat{R}_{N_n-1}$ that is $\varepsilon$ close to $\bar{R}_{N_c}$ is calculated by similar steps as the calculation in Appendix $A$ of [23]. It is given by,

$$N_n = \left\lceil \frac{1}{\ln c_2} \ln\left(\frac{\hat{R}_{N_n-1} - \frac{c_1}{1-c_2}}{R_0 - \frac{c_1}{1-c_2}}\right)\right\rceil \tag{20}$$

The radius $R_{max} = \hat{R}_{N_n-1}$ is the maximal radius the protected region expands to, and is used to calculate the number of sweeps required to reach this radius. The actual radius that bounds the protected region after $N_n$ sweeps is denoted by $R_{N_n-1}$ and is computed after $N_n$ is calculated. Replacing the coefficients in (20) yields that the number of sweeps required for the defenders to increase the protected region to be in a circle with the radius of the last sweep around the region, $R_{N_n-1}$, is,

$$N_n = \left\lceil \frac{\ln\left(\frac{-2\pi V_T \varepsilon}{2\pi R_0 V_T - nV_s r}\right)}{\ln\left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)} \right\rceil \tag{21}$$

To determine the number of required sweeps, the ceiling operator is used in order to implicitly demand that the number of iterations is an integer number, thus causing defenders to complete their sweep cycle and meet the defender that searches the adjacent section. This leads defenders to finish sweep $N_n - 1$, even when the protected region's radius is somewhat larger than $\hat{R}_{N_n-1}$.

After completing the last circular sweep, defenders perform the last outward advancement, and defenders continue to circularly patrol around a protected region with a radius of $R_{max}$ indefinitely. This implies that after reaching $R_{max}$, defenders continue to circularly patrol the region at a fixed (and maximal) radius using the circular defense pincer strategy. □

**Theorem 4.** *For a defender team with $n$ defenders, for which $n$ is even, performing the circular defense pincer sweep process, denote by $T_{out}$ the sum of all outward advancement times and by $T_{circular}$ the sum of all circular search times. Hence, the total search time necessary for the defender team to expand the protected region to its maximal size is given by,*

$$T(n) = T_{out}(n) + T_{circular}(n) \tag{22}$$

*$T_{out}(n)$ is given by,*

$$T_{out}(n) = \frac{nr}{2\pi V_T} - \frac{R_0 + \varepsilon}{V_s} \tag{23}$$

*And $T_{circular}(n)$ is given by,*

$$\begin{aligned} T_{circular}(n) &= \frac{R_0(V_s+V_T)}{V_s V_T} - \frac{rn(V_s+V_T)}{2\pi V_T{}^2} \\ &-\left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n} \left(\frac{2\pi R_0 V_T - rnV_s}{V_s V_T}\right)\left(\frac{V_s+V_T}{2\pi V_T}\right) + \frac{N_n r}{V_T} \end{aligned} \tag{24}$$

**Proof.** The total search time required for a defender team of $n$ defenders to enlarge the protected region to its maximal size is obtained by combining the total outward advancement times together with the total circular sweep times around the protected region in all iterations. Denote by $T_{out}(n)$ the sum of all the outward advancement times and by $T_{circular}(n)$ the sum of all circular sweep times. The total search time is given in (22).

### 4.2.2. Outward advancement times calculation

Denote the total outward advancement times until the protected region is within a circle of radius $R_{N_n-1}$ as $\tilde{T}_{out}(n)$. This time is calculated by,

$$\tilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i} \tag{25}$$

The total outward advancement times are given by,

$$T_{out}(n) = \tilde{T}_{out}(n) + T_{out_{last}}(n) \tag{26}$$

Where $T_{out_{last}}(n)$ is the advancement time required for the last outward advancement before expanding the protected region to its maximal size. Throughout the outward advancements phases the defenders do not perform sweeping and detection of invaders and hence invaders are not detected until the defenders finish their outwards motion and resume the sweeping of the protected region. Following a defender's completion of the outwards progression phase, its sensor overlaps both the invader region and the protected region by $r$.

Substituting $T_{out_i}$ in (25) yields that the accumulation of outward advancement times prior to the protected region being within a circle of radius $R_{N-1}$ can be calculated as follows,

$$\tilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i} = \frac{\left(N_n - 1\right) r}{V_s + V_T} - \frac{2\pi V_T \sum_{i=0}^{N_n-2} R_i}{nV_s\left(V_s + V_T\right)} \tag{27}$$

The first outward advancement takes place after the protected region is within a circle of radius $R_0$ and the final outward advancement occurs after iteration number $N_n - 2$, causing the protected region to expand from being inside a circle of radius $R_{N_n-2}$ to being within a circle of radius $R_{N_n-1}$. Following this motion, the defender team circularly sweeps around the protected region with a radius of $R_{N_n-1}$. $R_{N_n-1}$ is calculated using the recursive relation in (18) and is given by,

$$R_{N_n-1} = \frac{c_1}{1-c_2} + c_2{}^{N_n-1}\left(R_0 - \frac{c_1}{1-c_2}\right) \tag{28}$$

Substitution of coefficients in (28) yields,

$$R_{N_n-1} = \frac{nV_s r}{2\pi V_T} + \left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\frac{2\pi R_0 V_T - nV_s r}{2\pi V_T}\right) \tag{29}$$

The full derivation of $\tilde{T}_{out}(n)$ is continued in Appendix $A$. Hence,

$$\begin{aligned} \tilde{T}_{out}(n) &= \textstyle\sum_{i=0}^{N_n-2} T_{out_i} = \\ &-\frac{R_0}{V_s} + \frac{nr}{2\pi V_T} + \left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\frac{2\pi R_0 V_T - nV_s r}{2\pi V_T V_s}\right) \end{aligned} \tag{30}$$

Following the last circular sweep the defenders perform the last outward advancement, until reaching $R_{max}$. The time it takes the defenders to perform this last outward advancement is given by,

$$T_{out_{last}}(n) = \frac{\left(\hat{R}_{N_n-1} - R_{N_n-1}\right)}{V_s} \tag{31}$$

After this last outward sweep, the defenders perform circular sweeps around a protected region of radius $R_{max}$ and continuously protect the region from the entrance of invaders after reaching the maximal protected region the defenders can guard. Summing $\tilde{T}_{out}(n)$ and the last outward advancement time in (31) yields,

$$T_{out}(n) = \frac{nr}{2\pi V_T} - \frac{R_0 + \varepsilon}{V_s} \tag{32}$$

### 4.2.3. Circular sweep times calculation

The time to complete the first circular sweep is $T_0 = \frac{2\pi R_0}{nV_s}$. Similarly, the time it takes to perform a circular motion spanning an angular section of $\frac{2\pi}{n}$ around a circle of radius $R_i$ while moving with a speed of $V_s$ is,

$$T_i = \frac{2\pi R_i}{nV_s} \tag{33}$$

Denote the coefficient $c_3$ by,

$$c_3 = \frac{2\pi r}{n\left(V_s + V_T\right)} \tag{34}$$

It can be noted that by multiplying (18) by $\frac{2\pi}{nV_s}$ one obtains a recursive difference equation that can be utilized to calculate the circular sweep times. Therefore the sweep times can be written as,

$$T_{i+1} = c_2 T_i + c_3 \tag{35}$$

Denote by $T_{circular}(n)$ the total circular sweep times required to expand the protected region to be within a circle having a radius equal to or greater than $\hat{R}_{N_n-1}$. The calculation of $T_{circular}(n)$ follows similar steps as in appendix $C$ of [23]. Hence,

$$T_{circular}(n) = \frac{T_0 - c_2 T_{N_n-1} + \left(N_n - 1\right)c_3}{1 - c_2} \tag{36}$$

The calculation of the last circular sweep time prior to the protected region being within a circle having a radius greater than or equal to $\hat{R}_{N-1}$ follows similar steps as in appendix $D$ of [23]. Hence,

$$T_{N_n-1} = \frac{c_3}{1-c_2} + c_2{}^{N_n-1}\left(T_0 - \frac{c_3}{1-c_2}\right) \tag{37}$$

Substitution of coefficients in (37) results in,

$$T_{N-1} = \frac{r}{V_T} + \left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\frac{2\pi R_0 V_T - rnV_s}{nV_s V_T}\right) \tag{38}$$

Therefore, the total circular sweep times from (36) are,

$$\begin{aligned} T_{circular}(n) &= \frac{R_0(V_s+V_T)}{V_s V_T} - \frac{rn(V_s+V_T)}{2\pi V_T{}^2} \\ &-\left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n}\left(\frac{2\pi R_0 V_T - rnV_s}{V_s V_T}\right)\left(\frac{V_s+V_T}{2\pi V_T}\right) + \frac{N_n r}{V_T} \end{aligned} \tag{39}$$

Subsequently to the completion of sweep $N_n$ the protected region is within a circle of radius $R_{N_n-1}$. □

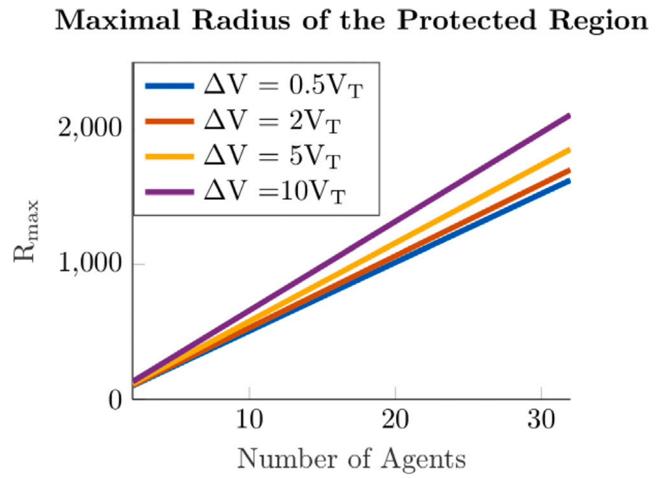


**Fig. 5.** Maximal protected region's radius. We simulated circular defense sweep processes with an even number of agents, ranging from 2 to 32 agents. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $\varepsilon = 0.2$.

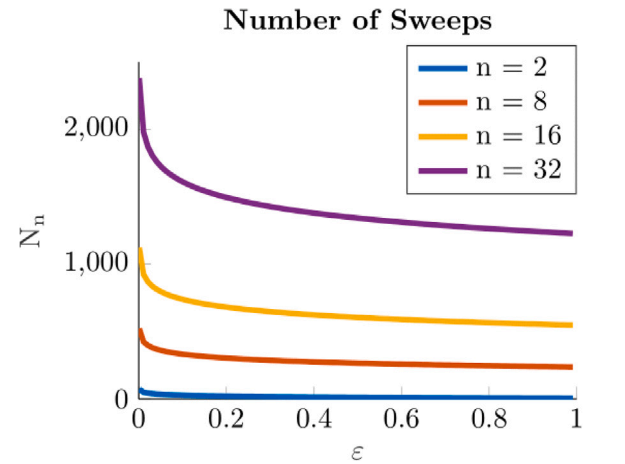


**Fig. 6.** Number of sweeps required to expand the protected region to its maximal size as a function of $\varepsilon$. We plot the results for defenders performing the circular defense pincer sweep processes with 2, 8, 16 and 32 agents. The chosen values of the parameters are $r = 10$, $V_T = 1$, $V_s = 31.9159$.

#### 4.2.4. Numerical experiments

Fig. 5 presents the maximal protected region's radius that the defenders are able to protect. The maximal radius clearly depends on the number of defenders and their speed. Fig. 6 presents the number of sweeps required to expand the protected region to its maximal size as a function of $\varepsilon$. Fig. 7 presents the expansion time of the protected region to $R_{N_c} - \varepsilon$ for a fixed speed exceeding the circular critical speed. Fig. 8 presents the total search times for different numbers of defenders until maximal expansion of the protected region is achieved. In all presented graphs the defenders' speed is equal and is independent of the number of defenders performing the expansion protocols, and is chosen so that search times of defender teams with different number of defenders are correctly compared. The chosen value of $R_0$ in all numerical experiments is $R_0 = 100$. The values of $\Delta V$ mentioned in the plots are speeds exceeding the critical speed of two defenders employing the circular defense pincer sweep process, due to the fact that as the number of defenders increases, the critical speed required for successfully performing the defense task decreases. Hence, defender teams with more defenders can achieve their goal of defending the region while moving at speeds exceeding the critical speed of two defenders, while the contrary argument is false. The second plot from the top of Fig. 8 presents the search time reduction obtained when the number of participating defenders increases.

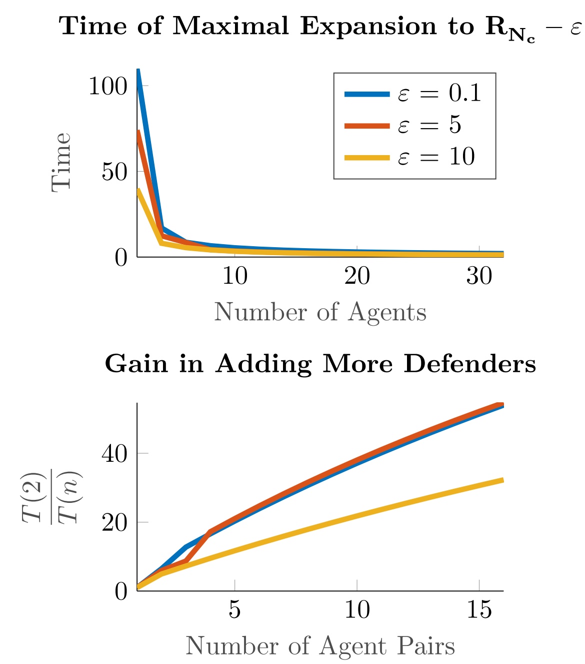


**Fig. 7.** Time of maximal expansion of the protected region to $R_{N_c} - \varepsilon$ and gain in adding more defenders for equal defender speeds. We simulated the circular defense pincer sweep processes for an even number of agents, ranging from 2 to 32 agents. The chosen values of the parameters are $r = 10$, $V_T = 1$, $\Delta V = 10$ and $R_{max} = 120$.

## 5. Spiral defense pincer sweep process

In order to handle the inherent inefficiency of the circular defense pincer sweep protocol, which first and foremost results from the fact that at the beginning of each sweep only half the length of the defenders' sensors are inside the invader region, we propose a modification to the defense process that tackles this inefficiency. This modification strives to increase the part of the defenders' sensors over the invader region so that they can detect invaders further away from the protected region. Therefore, a spiral scan in which the tip of a defender's sensor follows the expanding protected region's boundaries is proposed.

### 5.1. The defense task and critical speed analysis

At the beginning of the defense protocol, each defender's sensor overlaps the protected region by 0 (and consequently a length of $2r$ is over the complementing invader region). Sweeping in a pincer movement enables defenders to have a critical speed that is based only upon the time it takes them to traverse their allocated angular section of $\frac{2\pi}{n}$. In a similar manner as in the circular defense pincer sweep process, defenders sweeping at greater speeds than the corresponding critical speed of the scenario, switch their search direction once they finish their outward advancement phase. At the next iteration defenders sweep around a section having a larger radius.

Defenders' begin their spiral motion with their sensors' tips tangent to the boundary of the protected region. To keep their sensors tangent to the protected region during the spiral sweeping phases, defenders move at an angle $\phi$ with respect to the normal of the protected region. $\phi$ depends on the ratio between defender and invader speeds. Moving by a constant angle $\phi$ with respect to the normal of the protected region allows to preserve the protected region's circular shape and to keep the entire length of the defenders' sensors outside of the protected region, thus enabling detection of invaders at greater distances from

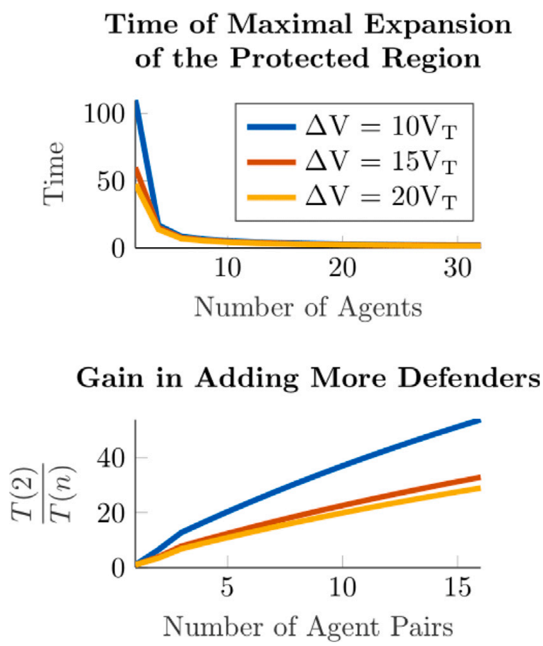


**Fig. 8.** Time of maximal expansion of the protected region and gain in adding more defenders for different defender speeds. We simulated the circular defense pincer sweep processes for an even number of agents, ranging from 2 to 32 agents. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_{max} = 120$.

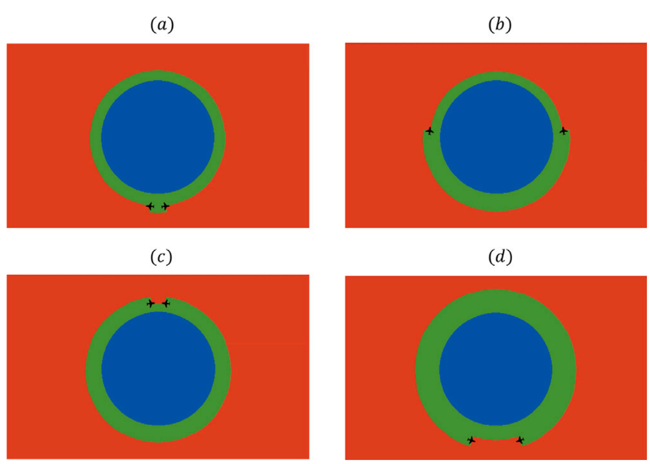


**Fig. 9.** Swept areas and protected region's status for different times in a scenario where 2 defenders perform the spiral defense pincer sweep process. (a) - Beginning of second sweep. (b) - Midway of the second sweep. (c) - End of the second sweep. (d) - End of the third sweep. Green areas show locations that were searched and hence do not contain invaders and red areas indicate locations where potential invaders may be present. Blue areas represent locations that belong to the initial protected region that does not contain invaders. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the boundary of the protected region. Fig. 1(b) depicts the starting locations of 2 defenders performing the spiral defense pincer sweep process. Fig. 9 shows the cleaning progress during the expansion of the protected region when 2 defenders employ the spiral defense pincer sweep protocol. $\phi$ is given by,

$$\sin\phi = \frac{V_T}{V_s} \tag{40}$$

Hence, $\phi = \arcsin\left(\frac{V_T}{V_s}\right)$. The defender's angular speed or rate of change of its angle with respect to the center of the protected region, $\theta_s$, can be expressed by the following function of $\phi$,

$$\frac{d\theta_s}{dt} = \frac{V_s\cos\phi}{R_s(t)} = \frac{\sqrt{{V_s}^2 - {V_T}^2}}{R_s(t)} \tag{41}$$

Integration of (41) between the initial and final search times of a particular sweep leads to,

$$\int_0^{t_\theta} \dot{\theta}(\zeta)d\zeta = \int_0^{t_\theta} \frac{\sqrt{{V_s}^2 - {V_T}^2}}{R_0 + r - V_T\zeta} d\zeta \tag{42}$$

Solving for $\theta\left(t_\theta\right)$ from (42) results in,

$$\theta\left(t_\theta\right) = -\frac{\sqrt{{V_s}^2 - {V_T}^2}}{V_T}\ln\left(\frac{R_0 + r - V_T t_\theta}{R_0 + r}\right) \tag{43}$$

Applying the exponent function to (43) yields,

$$\left(R_0 + r\right)e^{-\frac{V_T\theta(t_\theta)}{\sqrt{{V_s}^2 - {V_T}^2}}} = R_0 + r - V_T t_\theta = R_s(t_\theta) \tag{44}$$

The time required for a defender to complete a spiral scan of the angular section under its responsibility corresponds to changing its angle $\theta$ by $\frac{2\pi}{n}$. The expansion of the invading region at this time must be less than or equal to $2r$, so that defenders will still be able to prevent the entrance of invaders to the protected region. Since during the defenders' outwards movements invaders may continue to progress towards the protected region, defenders' can only move outwards by a somewhat smaller distance to address this concern. If we were to neglect invaders' motion during the outward movement phases, the necessary requirement to ensure invaders cannot enter the protected region without being detected by the defenders is,

$$R_0 - r \le R_s(t_{\frac{2\pi}{n}}) \tag{45}$$

Define,

$$\lambda = e^{-\frac{2\pi V_T}{n\sqrt{{V_s}^2 - {V_T}^2}}} \tag{46}$$

Replacing $R_s(t_{\frac{2\pi}{n}})$ with the expression for the defender's trajectory results in,

$$R_0 - r \le \left(R_0 + r\right)\lambda \tag{47}$$

Hence, to guarantee invaders cannot enter the protected region throughout the spiral scans without being detected, the defenders' speeds has to satisfy,

$$V_S \ge V_T\sqrt{\frac{\left(\frac{2\pi}{n}\right)^2}{\left(\ln\left(\frac{R_0 + r}{R_0 - r}\right)\right)^2} + 1} \tag{48}$$

In order to consider the progression of invaders towards the protected region and modify the critical speed in (48) to cope with this motion, defenders move outwards after completing the spiral sweep with a speed of $V_s$ until the outer tips of their sensors intersect with the inward advancing wavefront of invaders moving at a speed of $V_T$. At this time instance defenders stop their outward advancement and begin a new spiral sweep. In order to construct the critical speed, defenders need to replicate the situation as it was in the beginning of the defense protocol. Thus, in order to so, their speed, $V_s$, must be such that the outer tip of their sensors starts intersecting the approaching intruders once it is at a distance of $R_0 + 2r$ from the protected region's center (consequently the center of a defending agent is located at a distance of $R_0 + r$ from the center and its lower tip at a distance of $R_0$ from the protected region's center). When the defense protocol starts all invaders that are located at a distance of $R_0 + 2r$ from the protected region and are covered by the defenders' sensors are detected, thus the invader wavefront arising from these locations is eliminated and hence does not pass the defenders. Thus, moving at the critical speed ensures that once defenders return to their original locations, other invaders that started to spread from

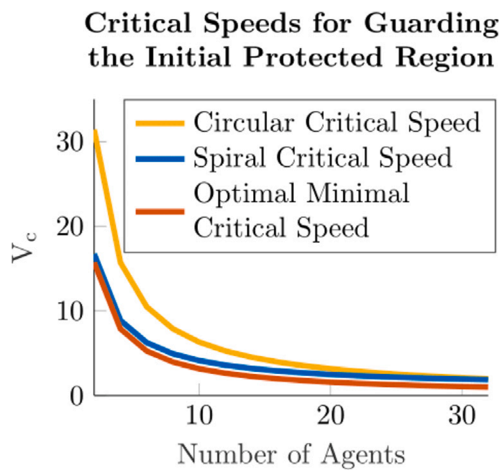


**Fig. 10.** Critical speeds as a function of the number of defenders. The number of defenders is even, and ranges from 2 to 32 defenders, that perform the spiral defense pincer sweep protocol. The optimal lower bound on the critical speeds and the resulting critical speeds of the circular defense pincer sweep protocol are presented for comparison as well. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$.

locations that were not closer to the protected region by more than $2r$ at the beginning of the protocol and are considered as the new invader wavefront are detected again by the defenders' sensors, hence ensuring that all invaders attempting to enter the protected region are detected.

This consideration guarantees that all invaders are detected and that the critical speed of the spiral defense pincer sweep protocol is nearly equal to the optimal lower bound on the defender speed of Theorem 1. Denote the expansion time of the invader region in the first sweep by $T_c$. In order to ensure no invaders enter the region without being detected the following inequality must hold, $V_T T_c \leq \frac{2rV_s}{V_s+V_T}$. Replacing the expression for $T_c$ yields,

$$\left(R_0 + r\right)(1-\lambda) \leq \frac{2rV_s}{V_s + V_T} \tag{49}$$

**Theorem 5.** *In the spiral defense pincer sweep process, the critical speed, $V_s$, enabling the successful completion of the defense task is obtained as the solution of,*

$$V_T T_c = \frac{2rV_s}{V_s + V_T} \tag{50}$$

*Where $T_c$ equals,*

$$T_c = \frac{\left(R_0 + r\right)(1-\lambda)}{V_T} \tag{51}$$

**Proof.** The critical speed of the spiral defense pincer sweep protocol is the lowest speed allowing the success of the defense task. This speed can be derived when (49) is satisfied with equality and solved for $V_s$. This critical speed is computed by solving numerically (50) for $V_s$ with the Newton–Raphson method while using the critical speed in (48) as an initial guess. This speed, that guarantees all invaders are detected, is used in all further calculations of this section. □

As shown in Fig. 10 the spiral critical speed nearly equals the optimal lower bound, specifically for a small number of defenders. For example, when 2 defenders perform the defense protocol the ratio between the spiral critical speed and the optimal lower bound on the speed is 1.06.

### 5.2. The maximal expansion task

#### 5.2.1. Number of sweeps analysis

**Theorem 6.** *The maximal radius that $n$ spiral sweeping defenders, with a linear sensor of length $2r$, a given speed $V_s$ and a maximal invader speed of $V_T$ can safely protect against the entrance of invaders is,*

$$\bar{R}_{N_s} = \frac{2rV_s}{(1-\lambda)\left(V_s + V_T\right)} - r \tag{52}$$

**Theorem 7.** *For a defender team with $n$ defenders for which $n$ is even, performing the spiral defense pincer sweep process, the number of sweeps required for the defender team to expand the protected region to a circle with a radius that is $\epsilon$ close to the maximal boundable radius $\bar{R}_{N_s}$ is given by,*

$$N_n = \left\lceil \frac{\ln\left(\frac{2rV_s(1-V_T)-\epsilon(1-\lambda)(V_s+1)(V_s+V_T)}{(V_s+V_T)((R_0+r)(1-\lambda)(V_s+1)-2rV_s)}\right)}{\ln\left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)} \right\rceil \tag{53}$$

*Denote by $T_{out}(n)$ the sum of all outward advancement times and by $T_{spiral}(n)$ the sum of all the spiral search times. Hence, the total search time necessary for the defender team to expand the protected region to its maximal size is given by,*

$$T(n) = T_{out}(n) + T_{spiral}(n) \tag{54}$$

*$T_{out}(n)$ is given by,*

$$T_{out}(n) = \frac{2r(V_s+1+V_T)}{(1-\lambda)(V_s+V_T)(V_s+1)} - \frac{R_0+2r+\epsilon}{V_s} - \frac{r(V_s-1+\lambda V_s+\lambda)}{V_s(1-\lambda)(V_s+1)} \tag{55}$$

*And $T_{spiral}(n)$ is given by,*

$$\begin{aligned} T_{spiral}(n) = {} & \frac{R_0V_s+R_0V_T+rV_T+2rV_sN_n-rV_s}{V_T(V_s+1)} \\ & - \frac{2rV_s(V_T+V_s\lambda-1+\lambda)}{V_T(V_s+1)^2(1-\lambda)} \\ & - \frac{V_s+V_T}{(1-\lambda)(V_s+1)}\left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)^{N_n}\left(\frac{(R_0+r)(1-\lambda)}{V_T} - \frac{2rV_s}{V_T(V_s+1)}\right) \end{aligned} \tag{56}$$

**Proof.** Denote by $\Delta V > 0$ the excess speed of the defender above the critical speed. Hence, the defender's speed is, $V_s = V_c + \Delta V$. At the start of each sweep the center of the defender's sensor is at a distance of $R_i + r$ from the protected region's center. $\theta\left(t_\theta\right)$ is calculated in (43). Substituting $R_0$ with $R_i$ results in,

$$\theta\left(t_\theta\right) = -\frac{\sqrt{{V_s}^2 - {V_T}^2}}{V_T}\ln\left(\frac{R_i + r - V_T t_\theta}{R_i + r}\right) \tag{57}$$

Denote the time required for a defender to complete the search of an angular section of $\theta\left(t_\theta\right) = \frac{2\pi}{n}$ by $T_{spiral_i}$. It is obtained from (57) and equals,

$$T_{spiral_i} = \frac{\left(R_i + r\right)(1-\lambda)}{V_T} \tag{58}$$

If defenders move with speeds greater than the critical speed required for the scenario, the distance a defender may advance outward from the center of the protected region is $\delta_i(\Delta V)$,

$$\delta_i(\Delta V) = 2r - V_T T_{spiral_i}\left(1 + \frac{1}{V_s}\right), \quad 0 \leq \delta_i(\Delta V) \leq 2r \tag{59}$$

Once defenders finish the outward advancement phase, the protected region expands to an updated circular protected region with a radius of $R_{i+1} = R_i + \delta_i(\Delta V)$. At the end of the spiral maneuver the protected region is again circular, with a larger radius. The proof for this property follows similar steps as provided in Appendix $H$ of [23].

Depending on the number of participating defenders and the iteration number, the distance a defender can advance outwards after completing a sweep is,

$$\delta_i(\Delta V) = 2r - \left(R_i + r\right)(1-\lambda)\left(\frac{V_s + 1}{V_s}\right) \tag{60}$$

Where in the term $\delta_i(\Delta V)$, $\Delta V$ denotes the increase in the agent's speed relative to the critical speed. The number of sweep iterations the defenders performed around the protected region is denoted by $i$, where $i$ starts from sweep number 0.

Since during the time in which defenders move outwards from the protected region, invaders continue to advance towards it, defenders can advance outwards by a slightly lesser distance than $\delta_i(\Delta V)$. The time required for defenders to move outwards until their entire sensors are outside of the protected region depends on the relative speed between the defenders' outward speed and the invader region's inwards expansion speed. Therefore, defenders are able to advance outwards by,

$$\delta_{i_{eff}}(\Delta V) = \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{61}$$

Hence, the radius of the expanded circular protected region is,

$$R_{i+1} = R_i + \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{62}$$

Denote $\widetilde{R}_i = R_i + r$. Replacing $R_i$ with $\widetilde{R}_i$ results in a similar structure of formulas as in the circular defense pincer sweep process and enables to use the same methodology along with the appropriate change of coefficients to solve for the maximal defendable radius of the protected region. Replacing the expression for $\delta_i(\Delta V)$ into (62) yields,

$$\tilde{R}_{i+1} = \tilde{R}_i + \left(2r - \tilde{R}_i(1-\lambda)\left(\frac{V_s + 1}{V_s}\right)\right)\frac{V_s}{V_s + V_T} \tag{63}$$

Rearranging terms results in a difference equation that resembles the equation obtained for the circular defense pincer sweep process,

$$\tilde{R}_{i+1} = \tilde{R}_i\left(\frac{V_T + V_s\lambda - 1 + \lambda}{V_s + V_T}\right) + \frac{2rV_s}{V_s + V_T} \tag{64}$$

Denote the coefficients in (64) by,

$$c_1 = \frac{2rV_s}{V_s + V_T}, c_2 = \frac{V_T + V_s\lambda - 1 + \lambda}{V_s + V_T} \tag{65}$$

Hence, (64) is expressed as,

$$\widetilde{R}_{i+1} = c_2\widetilde{R}_i + c_1 \tag{66}$$

Since the defenders need to move outwards from protecting a region with a smaller radius, and during this outwards movement the protected region continues to shrink due to possible movements of invaders, the defenders can protect a slightly smaller region. For any even number of defenders, $n$, the expansion protocol continues in this way until the protected region is enlarged to the largest possible circle. Let $\varepsilon > 0$ and denote by $\widehat{R}_{N_n-1}$ the radius of the protected region that is $\varepsilon$ close to $\bar{R}_{N_s}$,

$$\widehat{R}_{N_n-1} = R_{max} = \frac{2rV_s}{(1-\lambda)\left(V_s + V_T\right)} - r - \varepsilon \tag{67}$$

Due to the same difference equation structure as in the circular defense pincer sweep protocol, the number of spiral sweeps is calculated similarly. Hence, the number of iterations required for the defender team to expand the protected region to a circle of radius $\widehat{R}_{N_n-1}$ is,

$$N_n = \left\lceil \frac{\ln\left(\frac{2rV_s(1-V_T)-\varepsilon(1-\lambda)(V_s+1)(V_s+V_T)}{(V_s+V_T)((R_0+r)(1-\lambda)(V_s+1)-2rV_s)}\right)}{\ln\left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)} \right\rceil \tag{68}$$

The radius $R_{max} = \widehat{R}_{N_n-1}$ is the maximal radius the protected region expands to, and is used to calculate the number of sweeps required to reach this radius. The actual radius of the protected region after $N_n$ sweeps is denoted by $R_{N_n-1}$ and is computed after $N_n$ is calculated. After the last spiral sweep, the defenders perform the last outward advancement, and the defenders continue to perform spiral sweeps around a protected region with a radius of $R_{max}$.

*5.2.2. Outward advancement times calculation*

The outward advancement time depends on the iteration number. It is denoted by $T_{out_i}$ and is expressed as,

$$T_{out_i} = \frac{\delta_{i_{eff}}(\Delta V)}{V_s} = \frac{2r - \tilde{R}_i(1-\lambda)\left(\frac{V_s+1}{V_s}\right)}{V_s + V_T} \tag{69}$$

Denote the total outward advancement times until the protected region is enlarged to a circle of radius $R_{N_n-1}$ by $\widetilde{T}_{out}$, where $\widetilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i}$. The total outward advancement time is given by the sum of $\widetilde{T}_{out}(n)$ and the last outward advancement time until reaching $R_{max}$ denoted by $T_{out_{last}}$, i.e.,

$$T_{out}(n) = \widetilde{T}_{out}(n) + T_{out_{last}} \tag{70}$$

Throughout the outward advancement phases the defenders do not perform sweeping and detection of invaders and hence invaders are not detected until the defenders finish their outwards motion and resume the sweeping of the protected region. Following a defender's completion of the outwards progression phase, its sensor overlaps the invader region $2r$ (and therefore the footprint of its sensor that is over the protected region is 0). The total search time until the protected region is expanded into a circle of radius $R_{N_n-1}$ is given by the sum of the total spiral and outward advancement sweep times. Hence,

$$T(n) = \widetilde{T}_{out}(n) + T_{spiral}(n) \tag{71}$$

Replacing the expression for $T_{out_i}$ yields that the accumulative outward advancement times before the protected region is enlarged to its maximal size are,

$$\widetilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i} = \frac{2r(N_n-1)}{V_s+V_T} - \frac{(1-\lambda)(V_s+1)\sum_{i=0}^{N_n-2}\tilde{R}_i}{V_s(V_s+V_T)} \tag{72}$$

The full derivation of $\widetilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i}$ is continued in Appendix $B$. Hence,

$$\widetilde{T}_{out}(n) = \frac{2r}{V_s+V_T} - \frac{\tilde{R}_0}{V_s} + \frac{2r(V_T+V_s\lambda-1+\lambda)}{(1-\lambda)(V_s+1)(V_s+V_T)} + \left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)^{N_n-1}\frac{(R_0+r)(1-\lambda)(V_s+1)-2rV_s}{V_s(1-\lambda)(V_s+1)} \tag{73}$$

$R_{N_n-1}$ is computed by the same methodology as in the circular defense sweep process section and is given by,

$$R_{N_n-1} = \frac{r(V_s-1+\lambda V_s+\lambda)}{(1-\lambda)(V_s+1)} + \left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)^{N_n-1}\left(R_0 + r - \frac{2rV_s}{(1-\lambda)(V_s+1)}\right) \tag{74}$$

Following the last spiral sweep, the defenders perform the last outward advancement, until reaching $R_{max}$. The time it takes them to perform this last outward advancement is,

$$T_{out_{last}} = \frac{\left(\widehat{R}_{N_n-1} - R_{N_n-1}\right)}{V_s} \tag{75}$$

After this last outward sweep, the defenders perform spiral sweeps around a protected region of radius $R_{max}$ and continuously protect the region from the entrance of invaders after reaching the maximal protected region they can guard. Summing $\widetilde{T}_{out}(n)$ and the last outward advancement time in (75) yields,

$$T_{out}(n) = \frac{2r(V_s+1+V_T)}{(1-\lambda)(V_s+V_T)(V_s+1)} - \frac{R_0+2r+\varepsilon}{V_s} - \frac{r(V_s-1+\lambda V_s+\lambda)}{V_s(1-\lambda)(V_s+1)} \tag{76}$$

*5.2.3. Spiral sweep times calculation*

The time to perform a spiral sweep around radius $\widetilde{R}_i$ is calculated by multiplying $\widetilde{R}_i$ with $\frac{1-\lambda}{V_T}$. Therefore, by multiplying (64) with $\frac{1-\lambda}{V_T}$ the following difference equation for the sweep spiral times is,

$$T_{i+1} = c_2 T_i + c_3 \tag{77}$$

Where the coefficient $c_3$ is,

$$c_3 = \frac{2rV_s(1-\lambda)}{\left(V_s + V_T\right)V_T} \tag{78}$$

The total spiral sweep times required to expand the protected region into its largest size are calculated by similar steps as the circular sweep times in the previous section. Hence,

$$T_{spiral}(n) = \frac{T_0 - c_2 T_{N_n-1} + \left(N_n - 1\right) c_3}{1 - c_2} \tag{79}$$

The time required for defenders to perform the first spiral sweep is,

$$T_0 = \frac{\left(R_0 + r\right)(1 - \lambda)}{V_T} \tag{80}$$

The time to perform the last spiral sweep before the protected region reaches its maximal radius of $R_{N_n-1}$ is given by,

$$T_{N_n-1} = \frac{c_3}{1 - c_2} + c_2^{N_n-1}\left(T_0 - \frac{c_3}{1 - c_2}\right) \tag{81}$$

Substitution of coefficients results in,

$$T_{N_n-1} = \frac{2rV_s}{V_T(V_s+1)} + \left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)^{N_n-1}\left(\frac{(R_0+r)(1-\lambda)}{V_T} - \frac{2rV_s}{V_T(V_s+1)}\right) \tag{82}$$

Yielding that the total spiral sweep times are,

$$T_{spiral}(n) = \frac{R_0V_s+R_0V_T+rV_T+2rV_sN_n-rV_s}{V_T(V_s+1)} - \frac{2rV_s(V_T+V_s\lambda-1+\lambda)}{V_T(V_s+1)^2(1-\lambda)} - \frac{V_s+V_T}{(1-\lambda)(V_s+1)}\left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)^{N_n}\left(\frac{(R_0+r)(1-\lambda)}{V_T} - \frac{2rV_s}{V_T(V_s+1)}\right) \quad \square \tag{83}$$

*5.2.4. Numerical experiments*

Fig. 11 presents the maximal protected region's radius that the defenders are able to protect. The maximal radius clearly depends on the number of defenders and their speeds. Fig. 12 presents the number of sweeps required to expand the protected region to its maximal size as a function of $\varepsilon$. Fig. 13 presents the expansion time of the protected region to $R_{N_s} - \varepsilon$ for a fixed speed exceeding the spiral critical speed of 2 defenders that perform the spiral defense pincer sweep protocol. Fig. 14 presents the total search times for different numbers of defenders. In all presented graphs the defenders' speed is equal and is independent of the number of defenders performing the expansion protocol, and is chosen so that search times of defender teams with different number of defenders are correctly compared. The chosen value of $R_0$ in all numerical experiments is $R_0 = 100$. The values of $\Delta V$ mentioned in the plots are speeds above the critical speed of two defenders employing the spiral defense pincer sweep process. The second plot from the top of Fig. 14 presents the search time reduction obtained when the number of participating defenders increases. The critical speed required for the defender team to perform the defense task is determined by solving numerically the equation presented in Theorem 5, consequently ensuring invaders cannot enter the protected region undetected.

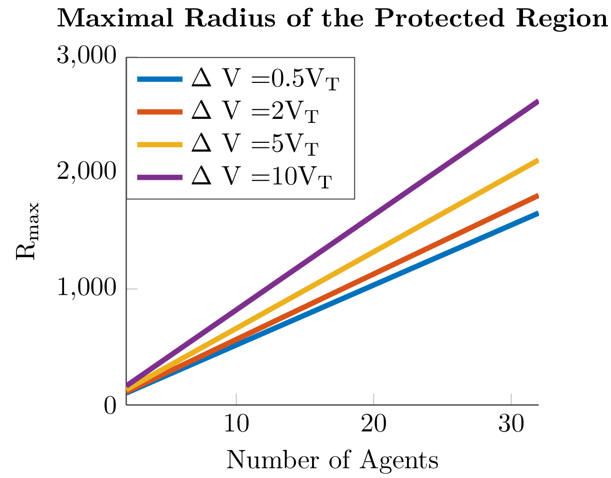


**Fig. 11.** Maximal protected region's radius. The spiral defense pincer sweep processes were simulated with an even number of defenders, ranging from 2 to 32 defenders. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $\varepsilon = 0.2$.

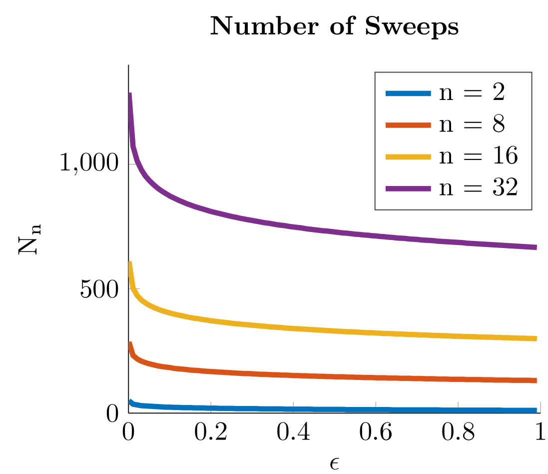


**Fig. 12.** Number of sweeps required to expand the protected region to its maximal size as a function of $\varepsilon$. We plot the results for defenders performing the spiral defense pincer sweep processes with 2, 8, 16 and 32 defenders. The chosen values of the parameters are $r = 10$, $V_T = 1$, $V_s = 17.7219$.

## 6. Comparative analysis between circular and spiral defense pincer sweep strategies

The purpose of this section is to compare between the attained results for the circular and spiral defense pincer sweep processes using the relevant performance metrics. These metrics constitute the minimal defender speed required for successful defense of the initial protected region, the time to expand the protected region to the maximal defendable area and the maximal feasible protected region's radius resulting from the defense protocol. To accurately compare between the total search times of defender swarms that can perform both the circular and spiral defense pincer sweep processes, the number of defenders as well as the defenders' speed has to be equal in the compared circular and spiral defender swarms.

Defenders performing the circular defense pincer sweep process require a higher critical speed compared to defenders performing the spiral defense pincer sweep process. Therefore, Fig. 15 presents the maximal protected region's radius that the defender team is able to expand the region into, when defenders employ the spiral defense pincer sweep process. The results are obtained for different speeds above the circular critical speed. The resulting maximal radius is clearly larger compared to the maximal protected region's radius that is achieved with a defender team that employs the circular defense pincer sweep process in Fig. 5.

Fig. 16 shows the spiral defense pincer sweep process's total search times obtained for different speeds above the circular critical speed of 2 defenders. This implies that values of $\Delta V$ shown in the plots correspond to defender speeds that equal nearly twice the spiral critical speeds. Requiring a higher critical speed means defender teams performing the circular defense pincer sweep process can expand the protected region to a smaller area compared to a defender team with the same capabilities performing the spiral protocol.

Fig. 17 compares the search times until the maximal expansion of the protected region is obtained and the gain in adding more defenders for circular sweeping swarms and spiral sweeping swarms. The results are computed with the same defender speeds for both the circular and spiral defense pincer sweep processes. The reduction in

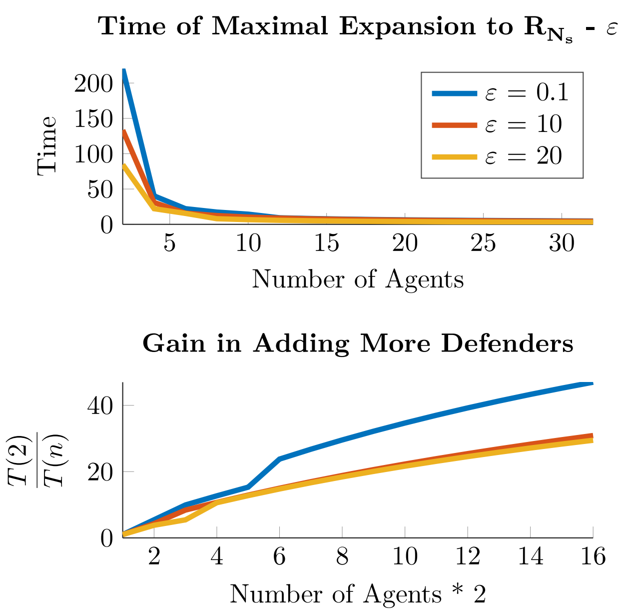


**Fig. 13.** Time of maximal expansion of the protected region to $R_{N_s} - \varepsilon$ and gain in adding more defenders for equal defender speeds. We simulated the spiral defense pincer sweep processes for an even number of defenders, ranging 2 to 32 defenders. The chosen values of the parameters are $r = 10$, $V_T = 1$, $\Delta V = 10$ and $R_{max} = 150$.

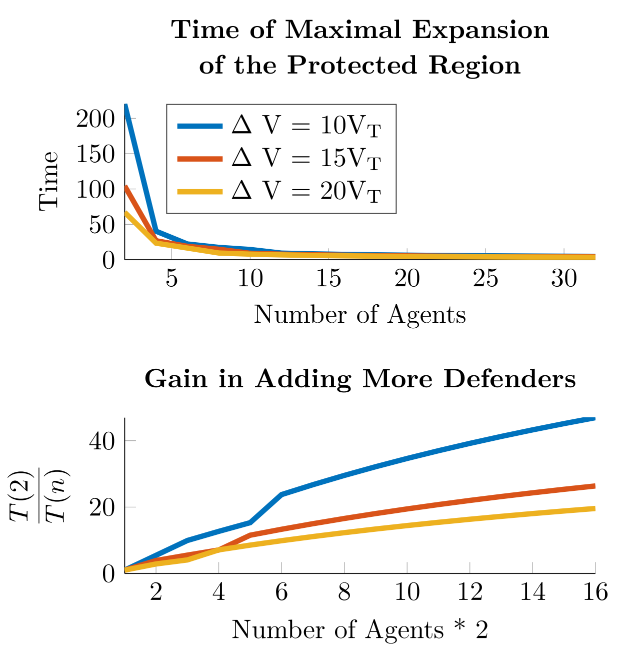


**Fig. 14.** Time of maximal expansion of the protected region and gain in adding more defenders for different defender speeds. We simulated the spiral defense pincer sweep processes for an even number of defenders, ranging from 2 to 32 defenders. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_{max} = 150$.

total search time achieved when defenders perform the spiral defense pincer sweep process are clearly noticeable. This result holds regardless to the number of defenders that perform the defense protocols or to their speeds.

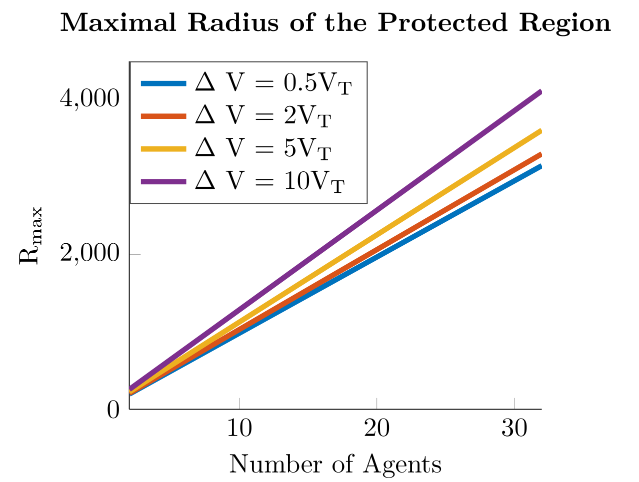


**Fig. 15.** Maximal protected region's radius. We simulated the spiral sweep protocols with an even number of defenders, ranging from 2 to 32 defenders. We show results obtained for different values of speeds above the critical speed of 2 defenders that employ the circular defense pincer sweep process. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $\varepsilon = 0.2$.

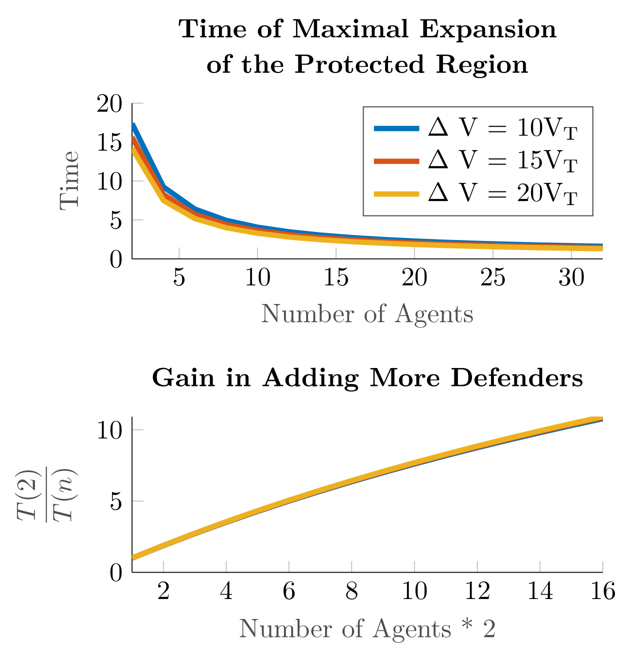


**Fig. 16.** Time of maximal expansion of the protected region and gain in adding more defenders for different defender speeds above the critical speed of 2 defenders employing the circular defense pincer sweep process. We simulated the spiral pincer sweep protocols for an even number of defenders, ranging from 2 to 32 defenders. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_{max} = 120$.

## 7. Comparison to state-of-the-art same-direction defense strategies

The purpose of this section is to compare the developed circular and spiral pincer sweep guarding and expansion strategies to prevalent approaches for defense against smart invaders which are considered as the state-of-the-art in defense against smart invaders. Such approaches usually distribute the defending agents equally around the protected region and require that all defenders move in the same direction. Such an approach is presented in [7], although the authors are interested only on solving the defense task and do not provide explicit expansion protocols that allow to achieve a maximal protected region or a detailed

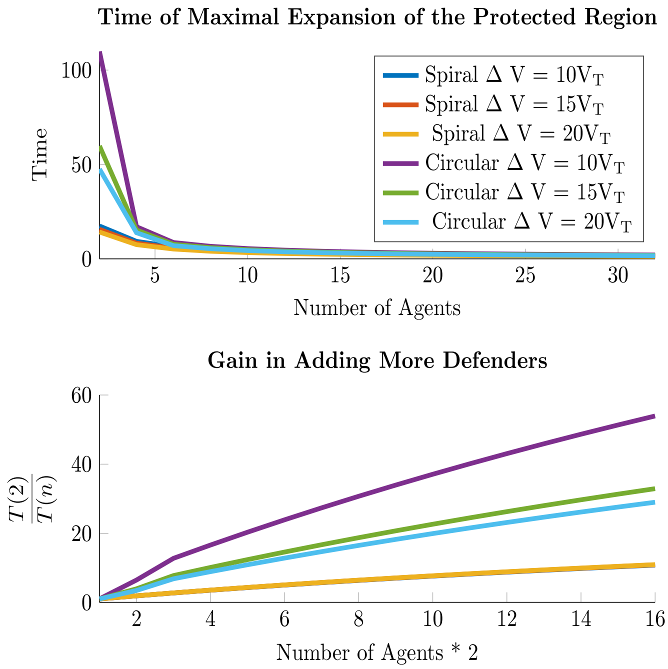


**Fig. 17.** Time of maximal expansion of the protected region and gain in adding more defenders for the circular and spiral defense sweep protocols. We simulated sweep protocols with an even number of defenders, ranging from 2 to 32 defenders, that perform the circular and spiral defense pincer sweep protocols at speeds above the critical speed of 2 defenders that perform the circular defense pincer sweep process. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_{max} = 120$.

analytical analysis of sweep times. Hence, we develop two alternative same-direction defense protocols, circular and spiral, that enable the comparison of pincer-based and same-direction defense protocols against smart invaders.

We provide a quantitative comparison between the discussed 3 metrics: critical speeds, maximal defendable area and the time required to reach maximal expansion. Circular and spiral defense pincer sweep protocols and circular and spiral defense same-direction sweep protocols are compared, proving the superiority of pincer-based approaches across all 3 metrics. We prove that the corresponding pincer-based protocols yield lower critical speeds, shorter time to increase the protected region to its maximal size as well as the ability to expand the protected region to a larger area compared to same-direction protocols.

These results are expected since defenders implementing same-direction protocols need to scan additional angular portions of the environment in each sweep around the region, to ensure no invader enters the protected region undetected. However, in pincer-based defense protocols, scanning such additional sectors is not required as a result of the complementary trajectories implemented by the defenders.

The critical speed necessary for defenders that perform the same-direction circular or spiral defense sweep protocols is higher compared to the minimal critical speed of their pincer-based counterparts. This can be observed in Fig. 18. The same-direction defense protocols are developed by using similar considerations as the same-direction protocols in [24]. These considerations lead to a same-direction circular protocol speed that equals,

$$V_{c_{circ\_same}} = \frac{2\pi R_0 V_T}{nr} + V_T \tag{84}$$

The solution for the spiral same-direction defense protocol critical speed is solved numerically using the Newton–Raphson method from the equation below while using the spiral pincer-based critical speed

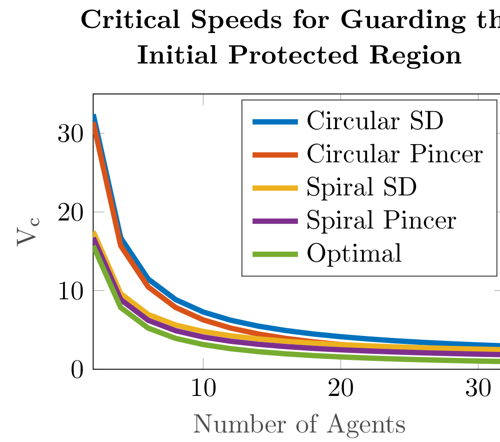


**Fig. 18.** Critical speeds as a function of the number of defenders. The number of defenders is even, and ranges from 2 to 32 defenders, that perform the spiral and circular defense pincer sweep protocols as well as the spiral and circular defense same-direction sweep protocols. The optimal lower bound on the critical speeds is presented for comparison as well. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$.

as an initial guess.

$$F\left(V_s\right) = \frac{2rV_s}{V_s+V_T} - \left(R_0+r\right)\left(1 - e^{-\frac{\left(\frac{2\pi}{n}+\arcsin\left(\frac{2rV_s}{\left(V_s+V_T\right)\left(R_0+2r\right)}\right)\right)V_T}{\sqrt{{V_s}^2-{V_T}^2}}}\right) \tag{85}$$

The angle $\beta_0$ denotes an additional angular sector that needs to be guarded in addition to the $\frac{2\pi}{n}$ angular sector that is swept when performing the spiral defense pincer protocols. $\beta_0$ is given by,

$$\beta_0 = \arcsin\left(\frac{2rV_s}{\left(V_s+V_T\right)\left(R_0+2r\right)}\right) \tag{86}$$

Because pincer-based defense sweep protocols require a lower critical speed compared to same-direction defense strategies, to fairly compare the performance of the different defense protocols, all defenders in each of the compared swarms is required to move at speeds above the critical speed of 2 defenders that perform the same-direction circular defense protocol since it has the highest critical speed compared to the circular pincer, spiral pincer and spiral same-direction defense protocols.

The necessity to have a higher critical speed means that there are domains that can be successfully guarded using a team of defenders implementing pincer-based defense protocols but cannot be defended with a team of equal capabilities that performs same-direction defense protocols. Additionally, this means that defender teams performing pincer-based defense protocols are able to expand the protected region into a larger area compared to their same-direction alternative protocols.

Fig. 19 shows the maximal defendable protected region's radius attained for each defense protocol. The results show that the spiral pincer-based approaches are best while circular same-direction defense protocols allow defenders to expand the protected region to the smallest area compared to the other expansion algorithms.

Fig. 20 shows the time until maximal expansion of the protected region to a radius of $R_{max} = 120$ for circular and spiral same-direction and circular and spiral pincer-based protocols. The value of $R_{max} = 120$ was chosen since a region with this radius can be successfully guarded by all 4 protocols. All compared swarms have equal number of defenders and move at speeds that are $10V_T$ above the critical speed

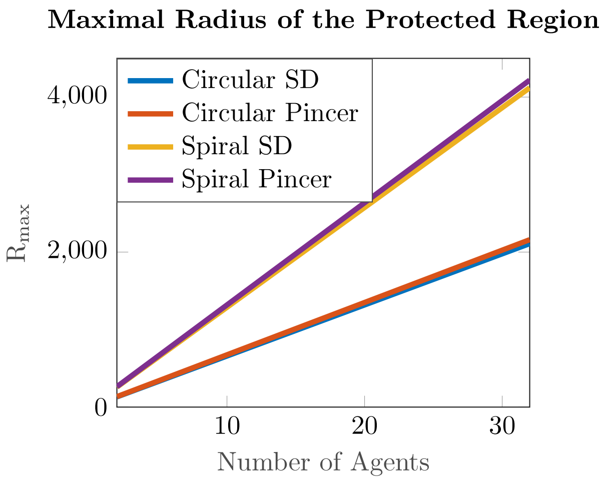


**Fig. 19.** Maximal protected region's radius. We simulated the spiral and circular defense pincer sweep protocols as well as the spiral and circular defense same-direction sweep protocols with an even number of defenders, ranging from 2 to 32 defenders. We show results obtained for $\Delta V = 10V_T$ above the critical speed of 2 defenders that employ the circular defense same-direction sweep process. The chosen values of the parameters are $r = 10$, $V_T = 10$ and $\varepsilon = 0.2$.

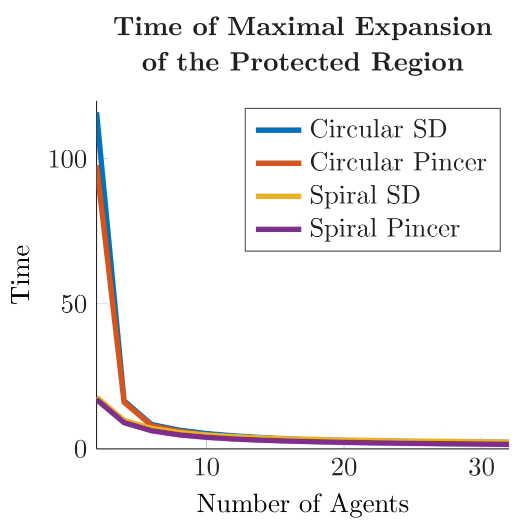


**Fig. 20.** Time of maximal expansion of the protected region for the circular and spiral defense sweep protocols as well as the spiral and circular defense same-direction sweep protocols. We simulated sweep protocols with an even number of defenders, ranging from 2 to 32 defenders, that perform the defense sweep protocols at speeds of $10V_T$ above the critical speed of 2 defenders that perform the circular defense same-direction sweep process. The chosen values of the parameters are $r = 10$, and $R_{max} = 120$.

of 2 defenders that perform the circular defense same-direction sweep process. Results show that the spiral defense pincer sweep protocol results in the fastest expansion time of the protected region to a given radius.

Fig. 21 shows a zoomed-in plot of Fig. 20 that displays the time until maximal expansion of the protected region, for swarms of defenders with 4 to 22 defenders. Results show that the spiral pincer defense protocols enables the defending team to expand the protected region to an area of a certain size more quickly compared to the circular same-direction, circular pincer and spiral same-direction defense protocols. Additionally, results show that for increasing number of defenders, circular pincer-based protocols lead to shorter sweep times even when compared to spiral same-direction defense protocols. This implies that despite the fact that pincer-based circular defense protocols may be

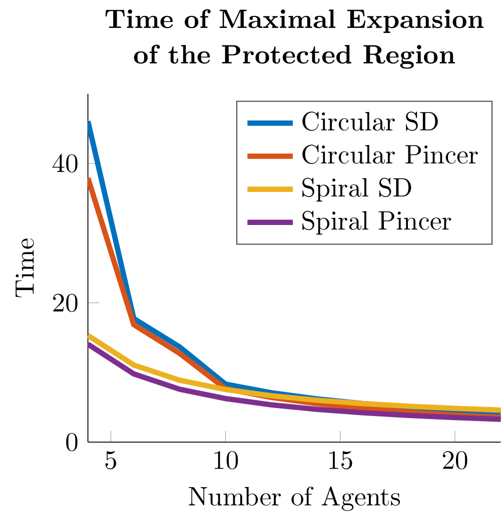


**Fig. 21.** Zoom in on the time of maximal expansion of the protected region for the circular and spiral defense sweep protocols as well as the spiral and circular defense same-direction sweep protocols. We simulated sweep protocols with an even number of defenders, ranging from 4 to 22 defenders, that perform the defense sweep protocols at speeds of $10V_T$ above the critical speed of 2 defenders that perform the circular defense same-direction sweep process. The chosen values of the parameters are $r = 10$, and $R_{max} = 120$.

implemented with defenders possessing more basic capabilities compared to defenders executing spiral strategies, the cooperation among defenders greatly improves the performance of the defender team.

## 8. Conclusions and future research directions

This research studies the problem of guaranteeing defense of an initial region against smart mobile invaders by a swarm of defending agents that act as visual sensors. Invaders are initially located outside a known circular environment which they try to enter without being detected by the defenders. Two novel algorithms that guarantee no intruder enters the region without being detected by a defender team that uses pincer movements between defending pairs are developed and compared to state-of-the-art approaches, proving the superiority of pincer-based defense protocols. Having a speed that exceeds the critical speed that allows defending the initial region, allows the defenders to gradually expand the protected region as well. Numerical and illustrative simulations using MATLAB and NetLogo demonstrate the performance of the proposed algorithms.

A possible extension to this work is to generalize the results for environments with more complex geometries, possibly in the presence of obstacles and apply the pincer expansion protocols in such settings. An additional interesting research avenue is to develop an algorithm that will be robust to failures of defenders and will allow to reorganize the defender team and enable it to continue the defense and expansion tasks with less defenders. Another possible research direction is to consider the solution to pincer-based defense tasks using defenders with different maximal speeds.

## CRediT authorship contribution statement

**Roee M. Francos:** Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Alfred M. Bruckstein:** Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Validation, Writing – review & editing.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## Appendix A

This appendix provides an analytical computation of the total outward advancement times required in order for a defender team performing the circular defense pincer sweep process to expand the protected region to the maximal defendable area and completes the calculation from Section 4. Denote the total outward advancement time by $\widetilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i}$. Eq. (30) is expressed as,

$$\widetilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i} = \frac{\left(N_n - 1\right) r}{V_s + V_T} - \frac{2\pi V_T \sum_{i=0}^{N_n-2} R_i}{n V_s \left(V_s + V_T\right)} \tag{A.1}$$

The method to calculate $\sum_{i=0}^{N_n-2} R_i$ is developed in Appendix *E* of [23] and equals,

$$\sum_{i=0}^{N_n-2} R_i = \frac{R_0 - c_2 R_{N_n-2} + (N_n - 2)c_1}{1 - c_2} \tag{A.2}$$

The calculation of $R_{N_n-2}$ is provided in Appendix *B* of [23]. It is given by,

$$R_{N_n-2} = \frac{c_1}{1 - c_2} + c_2{}^{N_n-2}\left(R_0 - \frac{c_1}{1 - c_2}\right) \tag{A.3}$$

Replacement of coefficients in (A.3) results in,

$$R_{N_n-2} = \frac{nrV_s}{2\pi V_T} + \left(1 + \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-2}\left(\frac{2\pi R_0 V_T - nrV_s}{2\pi V_T}\right) \tag{A.4}$$

exchanging the coefficients in (A.2) leads to,

$$\begin{aligned}\sum_{i=0}^{N_n-2} R_i &= \frac{R_0 n(V_s+V_T)}{2\pi V_T} - \frac{n^2 V_s r(V_s+V_T)}{(2\pi V_T)^2} + \frac{nrV_s(N_n-1)}{2\pi V_T} \\ &-\left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\frac{n(2\pi R_0 V_T - nV_s r)(V_s+V_T)}{(2\pi V_T)^2}\right)\end{aligned} \tag{A.5}$$

Replacing the expression for $\sum_{i=0}^{N_n-2} R_i$ from (A.5) to Eq. (A.1) yields,

$$\begin{aligned}\widetilde{T}_{out}(n) &= \sum_{i=0}^{N_n-2} T_{out_i} = \\ &-\frac{R_0}{V_s} + \frac{nr}{2\pi V_T} + \left(1 - \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\frac{2\pi R_0 V_T - nV_s r}{2\pi V_T V_s}\right)\end{aligned} \tag{A.6}$$

## Appendix B

This appendix provides an analytical computation of the total outward advancement times required in order for a defender team performing the spiral defense pincer sweep process to expand the protected region to the maximal defendable area and completes the calculation from Section 5. Denote the total outward advancement time by $\widetilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i}$. Eq. (72) is expressed as,

$$\widetilde{T}_{out}(n) = \sum_{i=0}^{N_n-2} T_{out_i} = \frac{2r(N_n-1)}{V_s+V_T} - \frac{(1-\lambda)(V_s+1)\sum_{i=0}^{N_n-2}\tilde{R}_i}{V_s(V_s+V_T)} \tag{B.1}$$

The method to calculate $\sum_{i=0}^{N_n-2} R_i$ is developed in Appendix *E* of [23] and equals,

$$\sum_{i=0}^{N_n-2} \widetilde{R}_i = \frac{\widetilde{R}_0 - c_2 \widetilde{R}_{N_n-2} + (N_n - 2)c_1}{1 - c_2} \tag{B.2}$$

The calculation of $R_{N_n-2}$ is provided in Appendix *B* of [23]. It is given by,

$$\widetilde{R}_{N_n-2} = \frac{c_1}{1 - c_2} + c_2{}^{N_n-2}\left(\widetilde{R}_0 - \frac{c_1}{1 - c_2}\right) \tag{B.3}$$

Replacement of coefficients in (B.3) results in,

$$\begin{aligned}\tilde{R}_{N_n-2} &= \frac{2rV_s}{(1-\lambda)(V_s+1)} + \\ &\left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)^{N_n-2}\frac{(R_0+r)(1-\lambda)(V_s+1)-2rV_s}{(1-\lambda)(V_s+1)}\end{aligned} \tag{B.4}$$

exchanging the coefficients in (B.2) leads to,

$$\begin{aligned}\sum_{i=0}^{N_n-2}\tilde{R}_i &= \tilde{R}_0\frac{V_s+V_T}{V_s(1-\lambda)} - \frac{2r(V_T+V_s\lambda)}{V_s(1-\lambda)^2} - \\ &\frac{V_s+V_T}{V_s(1-\lambda)}\left(\frac{V_T+V_s\lambda}{V_s+V_T}\right)^{N_n-1}\left(\tilde{R}_0 - \frac{2r}{1-\lambda}\right) + \frac{2r(N_n-2)}{1-\lambda}\end{aligned} \tag{B.5}$$

Replacing the expression for $\sum_{i=0}^{N_n-2}\tilde{R}_i$ from (B.5) to Eq. (B.1) yields,

$$\begin{aligned}\widetilde{T}_{out}(n) &= \frac{2r}{V_s+V_T} - \frac{\tilde{R}_0}{V_s} + \frac{2r(V_T+V_s\lambda-1+\lambda)}{(1-\lambda)(V_s+1)(V_s+V_T)} + \\ &\left(\frac{V_T+V_s\lambda-1+\lambda}{V_s+V_T}\right)^{N_n-1}\frac{(R_0+r)(1-\lambda)(V_s+1)-2rV_s}{V_s(1-\lambda)(V_s+1)}\end{aligned} \tag{B.6}$$

## Appendix C. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.robot.2024.104620.

## References


[1] B.O. Koopman, Search and Screening: General Principles with Historical Applications, Pergamon Press, 1980.

[2] L.D. Stone, J.O. Royset, A.R. Washburn, Optimal Search for Moving Targets (International Series in Operations Research & Management Science 237), Springer, Cham, Switzerland, 2016.

[3] I. Rekleitis, V. Lee-Shue, A.P. New, H. Choset, Limited communication, multi-robot team based coverage, in: Proceeding of the IEEE International Conference on Robotics and Automation (ICRA) 2004, Vol. 4, IEEE, 2004, pp. 3462–3468.

[4] S. Alpern, S. Gal, The Theory of Search Games and Rendezvous, Vol. 55, Springer Science & Business Media, 2006.

[5] P. Vincent, I. Rubin, A framework and analysis for cooperative search using UAV swarms, in: Proceedings of the 2004 ACM Symposium on Applied Computing, ACM, 2004, pp. 79–86.

[6] Y. Altshuler, V. Yanovsky, I.A. Wagner, A.M. Bruckstein, Efficient cooperative search of smart targets using uav swarms, Robotica 26 (4) (2008) 551–557.

[7] T.G. McGee, J.K. Hedrick, Guaranteed strategies to search for mobile evaders in the plane, in: 2006 American Control Conference, IEEE, 2006, pp. 6–pp.

[8] D. Shishika, V. Kumar, Local-game decomposition for multiplayer perimeter-defense problem, in: 2018 IEEE Conference on Decision and Control, CDC, IEEE, 2018, pp. 2093–2100.

[9] D. Shishika, J. Paulos, M.R. Dorothy, M.A. Hsieh, V. Kumar, Team composition for perimeter defense with patrollers and defenders, in: 2019 IEEE 58th Conference on Decision and Control, CDC, IEEE, 2019, pp. 7325–7332.

[10] D. Shishika, J. Paulos, V. Kumar, Cooperative team strategies for multi-player perimeter-defense games, IEEE Robot. Autom. Lett. 5 (2) (2020) 2738–2745.

[11] V.R. Makkapati, P. Tsiotras, Optimal evading strategies and task allocation in multi-player pursuit–evasion problems, Dynam. Games Appl. 9 (4) (2019) 1168–1187.

[12] T.H. Chung, G.A. Hollinger, V. Isler, Search and pursuit-evasion in mobile robotics, Auton. Robots 31 (4) (2011) 299–316.

[13] S.S. Kumkov, S. Le Ménec, V.S. Patsko, Zero-sum pursuit-evasion differential games with many objects: survey of publications, Dynam. Games Appl. 7 (4) (2017) 609–633.

[14] I.E. Weintraub, M. Pachter, E. Garcia, An introduction to pursuit-evasion differential games, in: 2020 American Control Conference, ACC, IEEE, 2020, pp. 1049–1066.

[15] E. Garcia, D.W. Casbeer, M. Pachter, Optimal strategies of the differential game in a circular region, IEEE Control Syst. Lett. 4 (2) (2019) 492–497.

[16] E. Garcia, D.W. Casbeer, A. Von Moll, M. Pachter, Multiple pursuer multiple evader differential games, IEEE Trans. Automat. Control 66 (5) (2020) 2345–2350.

[17] N. Agmon, S. Kraus, G.A. Kaminka, Multi-robot perimeter patrol in adversarial settings, in: 2008 IEEE International Conference on Robotics and Automation, IEEE, 2008, pp. 2339–2345.

[18] N. Agmon, G.A. Kaminka, S. Kraus, Multi-robot adversarial patrolling: facing a full-knowledge opponent, J. Artificial Intelligence Res. 42 (2011) 887–916.

[19] J. Czyzowicz, L. Gasieniec, A. Kosowski, E. Kranakis, Boundary patrolling by mobile agents with distinct maximal speeds, in: Algorithms–ESA 2011: 19th Annual European Symposium, SaarbrÜCken, Germany, September 5-9, 2011. Proceedings 19, Springer, 2011, pp. 701–712.

[20] J. Czyzowicz, E. Kranakis, D. Pajak, N. Taleb, Patrolling by robots equipped with visibility, in: Structural Information and Communication Complexity: 21st International Colloquium, SIROCCO 2014, Takayama, Japan, July 23-25, 2014. Proceedings 21, Springer, 2014, pp. 224–234.

[21] S. Bajaj, S.D. Bopardikar, Dynamic boundary guarding against radially incoming targets, in: 2019 IEEE 58th Conference on Decision and Control, CDC, IEEE, 2019, pp. 4804–4809.

[22] S. Bajaj, E. Torng, S.D. Bopardikar, Competitive perimeter defense on a line, in: 2021 American Control Conference, ACC, IEEE, 2021, pp. 3196–3201.

[23] R.M. Francos, A. M. Bruckstein, Search for smart evaders with swarms of sweeping agents, IEEE Trans. Robot. 38 (2) (2021) 1080–1100.

[24] R.M. Francos, A.M. Bruckstein, Pincer movements are always better than same-direction search, in: 2023 20th International Conference on Ubiquitous Robots, UR, IEEE, 2023, pp. 19–26.

[25] S. Tisue, U. Wilensky, Netlogo: A simple environment for modeling complexity, in: International Conference on Complex Systems, Vol. 21, Boston, MA, 2004, pp. 16–21.

[26] R.M. Francos, A.M. Bruckstein, Search for smart evaders with sweeping agents, Robotica 39 (12) (2021) 2210–2245.

**Roee M. Francos** received the B.Sc. degree in Electrical and Computer Engineering from Ben-Gurion University, Beer Sheva, Israel in 2016. Currently he is in a direct track towards a Ph.D. degree in Computer Science at the Technion-Israel Institute of Technology, Haifa, Israel. His current research interests include multi-agent systems, autonomous systems, distributed control, intelligent transportation, estimation theory and computer vision.

**Alfred M. Bruckstein** received his B.Sc. and M.Sc. degrees at the Technion, Haifa, in 1976 and 1980, respectively, and earned a Ph.D. degree in Electrical Engineering in Stanford University, California, in 1984. From October 1984, he has been with the Technion, where he holds the Ollendorff Chair in Science. His present research interests are in swarm/ant robotics, image and signal processing, analysis and synthesis, pattern recognition and applied geometry. Professor Bruckstein authored and co-authored over one hundred and fifty journal papers in the fields of interest mentioned. He is a SIAM Fellow for contributions to Signal Processing, Image Analysis and Ant Robotics.