

DIVIDE AND ABSTRACT: AUTOFORMALIZATION VIA DECOMPOSITION AND ABSTRACTION LEARNING

Anonymous authors

Paper under double-blind review

ABSTRACT

Autoformalization, the task of translating informal mathematics into formal machine-verifiable languages, remains challenging due to three fundamental limitations: bottleneck by existing abstractions, difficulty of complex statements, and poor transferability across languages. Existing approaches rely heavily on pre-defined libraries and expect LLMs to directly generate complete formalizations, limiting their generalizability to complex statements and new formal languages. We propose *Divide and Abstract (DNA)*, a zero-training framework that addresses these challenges through a two-phase approach. First, *DNA* extracts common mathematical concepts from the entire corpus and formalizes them as reusable abstractions, extending the target language’s capability. Second, *DNA* hierarchically decomposes each statement into structured informal clauses, translates each clause using the learned abstractions, and composes them into complete formalizations. Our evaluation on the LeanEuclidPlus and ProofNet-Hard benchmarks demonstrates consistent improvements across multiple model families, achieving up to $8.60\times$ performance gains over baseline approaches. Notably, *DNA* enables smaller models to match larger baseline models and shows particularly strong performance on complex mathematical statements requiring nested reasoning. The framework requires no training on target languages, making it effective for low-resource domain-specific languages. Our code is available at <https://github.com/anonymousauthor567/DivdedAndAbstract>.

1 INTRODUCTION

Autoformalization is the task of automatically translating informal mathematics into formal languages designed for theorem provers such as Lean (Moura & Ullrich, 2021) and Z3 (De Moura & Bjørner, 2008), so that theorems and proofs written in natural languages can be mechanically verified. The success of autoformalization will significantly reduce the amount of labor from human experts, and thus empower three transformative applications:

1. Synthesizing informal-formal parallel data for training neural theorem provers (Xin et al., 2024).
2. Grounding and guiding reasoning processes in natural languages (Yang et al., 2022).
3. Accelerating verification of important mathematical theorems (Gonthier et al., 2013; Hales et al., 2015) or engineering systems (Zhao et al., 2012; Reid et al., 2016) not only to identify potential gaps and mistakes, but also to facilitate certified extensions and future theoretical discovery.

Inspired by the breakthrough of Large Language Models (LLMs) in machine translation between natural languages, researchers have increasingly applied LLMs to autoformalization (Weng et al., 2025). Even though the ultimate vision is to automatically formalize entire theories that include axioms, definitions, notations, theorems, and proofs, current LLMs struggle to even formalize an individual statement (Murphy et al., 2024; Liu et al., 2025a).

Previous work in LLM-based statement autoformalization has explored in-context learning (Wu et al., 2022), fine-tuning specialized autoformalizer models (Jiang et al., 2023a), invoking retrieval before LLM inference (Liu et al., 2025a), or employing post-inference techniques such as majority voting and semantic consistency ranking by an embedding model (Li et al., 2024a). However, all these approaches suffer from limited generalizability due to three fundamental challenges:

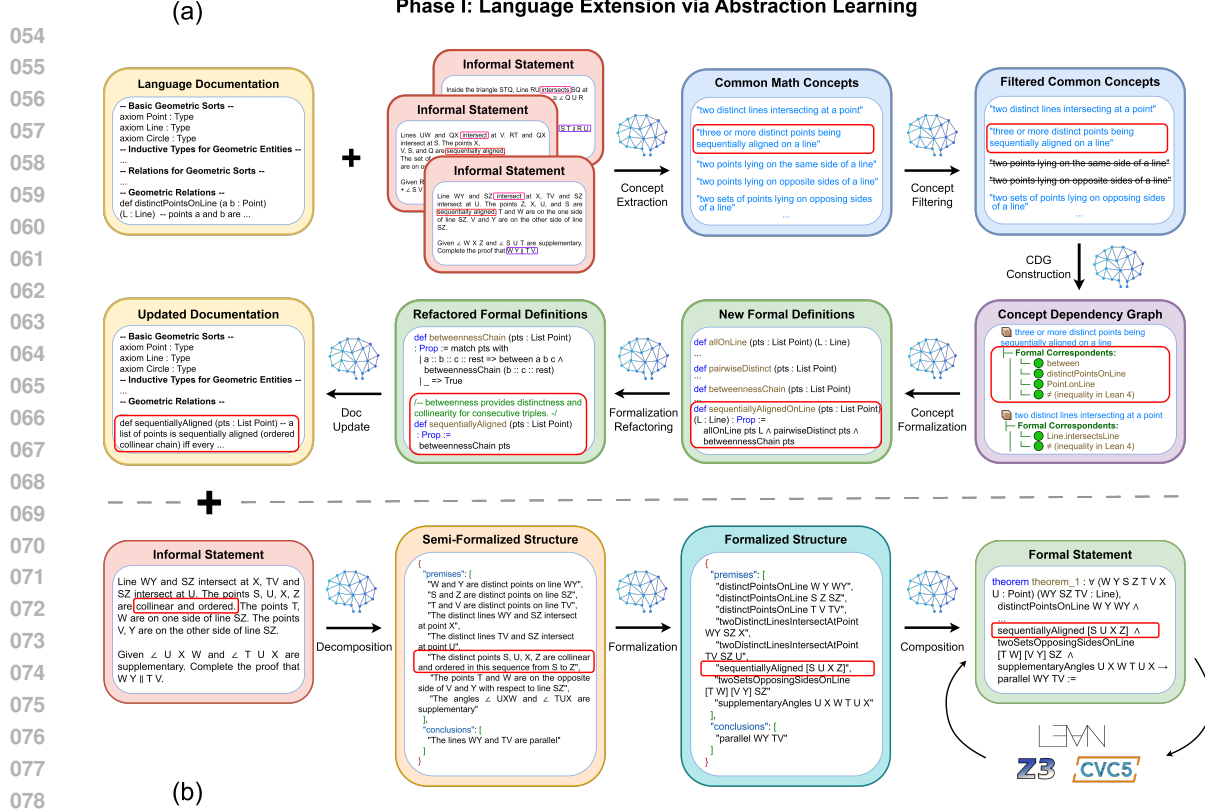


Figure 1: Overview of the *Divide and Abstract (DNA)* framework. **(a) Phase I (Language Extension):** Given an informal corpus, we extract common mathematical concepts, filter out those already present in the target language, construct a concept dependency graph (CDG), and systematically formalize concepts following topological ordering to extend the DSL. **(b) Phase II (Statement Formalization):** For each statement, we hierarchically decompose it into semi-formalized components, translate each component using the learned abstractions, compose them into complete formal statements, and refine using symbolic validator feedback.

1. Bottleneck by Existing Abstractions: The performance of formalizing the same informal statement is limited by the richness of existing abstractions in the target formal language. For example, if the target language only contains basic formal definitions of sets and real numbers, then formalizing the relation “a topological space being a manifold” would be difficult even for human experts. In contrast, if the target language already provides a formal relation “isManifold”, then the translation is simply a one-to-one mapping. However, all existing methods rely heavily on predefined abstractions in existing libraries like Mathlib in Lean (Mathlib Community, 2020) and HOL sessions in Isabelle (Nipkow et al., 2002), which fail to generalize to the cases where the target language doesn’t provide an extensive coverage of high-level abstractions.

2. Difficulty of Complex Statements: Regardless of the availability of relevant abstractions, the difficulty of formalization also grows proportionally to the complexity of statements. For example, formalizing a simple statement like “function f is continuous” is straightforward when the formal definition of continuity exists. However, formalizing a nested statement such as “the space of continuous functions from a compact manifold to a Banach space forms a complete metric space under the supremum norm” requires understanding and correctly composing multiple layers of mathematical abstractions. Nonetheless, all existing approaches expect the LLMs to directly generate the final formalization at inference time, which does not generalize well to complex statements that involve nested quantifiers, higher-order objects, and composite relations.

3. Transferability to Different Languages: Orthogonal to abstraction availability and statement complexity, the choice of target formal language constitutes a third dimension of the generalizability challenge. Most fine-tuned autoformalizer models (Gao et al., 2025; Wang et al., 2025; Xuejun

et al., 2025) are heavily trained on Lean Mathlib and formal statements that directly use Mathlib abstractions. However, as shown in Table 5, these models perform poorly when the target is a domain-specific language (DSL) that is not in the training data, or even when Mathlib evolves to newer versions with updated syntax and library structures.

To address these limitations in generalizability, we propose an end-to-end framework *Divide and Abstract (DNA)* with three key designs that directly correspond to each challenge:

1. Abstraction Learning: While previous methods treat the formalization of each statement independently, we leverage the insight that statements within the same corpus often share common mathematical concepts like definitions of objects and relations. As shown in Figure 1(a), given the current language documentation and a target informal corpus, *DNA* first extracts these shared concepts from the entire corpus and formalizes them as a collection of reusable formal abstractions. This phase essentially extends the target language by enriching the library of available abstractions, thereby facilitating the statement formalization both within and beyond the corpus.

2. Hierarchical Decomposition: Unlike existing approaches that let LLMs directly generate the formal statement, we take advantage of the fact that mathematical statements are semi-structured data — a statement consists of the quantifications and the body, the body can be broken down into a list of premises and a list of conclusions, and quantified statements nested in the premises or conclusion can be recursively decomposed in the same manner. As shown in Figure 1, *DNA* hierarchically decomposes a statement into a structure of informal clauses, translates each clause into its formal correspondents, and composes the formal clauses back into a complete formalization. Since we have learned a rich library of reusable abstractions, the formalization of each individual clause becomes significantly more tractable than formalizing the entire complex statement at once.

3. Zero-Training Framework: *DNA* is designed to be a zero-training, plug-and-play framework. This is achieved by simply providing the target language’s documentation in the context of the LLM. After the abstraction learning phase, the documentation is updated with the newly learned formal definitions, and then passed as context to the statement formalization phase, conditioning the LLM on both the original language specifications and a library of directly usable high-level definitions. The entire framework requires no training on the target formal language, and is thus particularly effective for low-resource DSLs.

We evaluate *DNA* on LeanEuclidPlus and ProofNet-Hard against fine-tuned autoformalizers and general-purpose LLMs including GPT-4.1/5, Claude-4-Sonnet, and Qwen3 variants. *DNA* consistently outperforms baselines across all model families, with the most dramatic improvement being Qwen3-14B advancing from 1.0 to 9.6 success rate ($9.6\times$ gain), while GPT-4.1-mini improves from 4.8 to near GPT-4.1 levels. On ProofNet-Hard, all baselines achieve zero success while *DNA* enables successful formalization, demonstrating its effectiveness for domain-specific languages lacking extensive training data. To summarize, our offer the following contributions in this paper:

(1) identifying three generalizability challenges in autoformalization, (2) proposing a zero-training framework combining abstraction learning and hierarchical decomposition, (3) demonstrating $1.17\times$ to $9.6\times$ improvements across diverse models and benchmarks, and (4) enabling effective transfer to challenging domain-specific languages where specialized models fail entirely.

2 THE *Divide and Abstract* FRAMEWORK

2.1 PHASE I: LANGUAGE EXTENSION

As illustrated in Figure 1(a), the language extension process unfolds through six coordinated steps. The first three steps focus on abstraction learning and dependency analysis, while the latter three handle engineering concerns to transform the learned concepts into a downstream-user-friendly library of plug-and-play abstractions.

Step 1: Concept Extraction. Given the informal corpus as input, we systematically extract common mathematical concepts using an LLM. We identify three types of concepts: definitions of mathematical objects such as points, groups, and series; relations between objects such as “a point being on a line” or “a group being a subgroup of another”; and functions mapping objects to objects such as Euclidean distance or determinant. Our extraction process enforces precision by specifying argument

Step 1: Concept Extraction			Step 2: Concept Filtering	
Run	Recall	Correctness	Run	Correctness
Average	94.44%	100.00%	Average	100.00%

Table 1: Performance analysis for concept extraction and filtering steps.

types and counts, well-definedness by ensuring conventional mathematical validity, and abstractness by avoiding particular variable names or specific instances.

Step 2: Concept Filtering. Using the extracted concepts and current DSL documentation as input, we filter out concepts that already have direct formal correspondents in the target language. This prevents reinventing existing abstractions while focusing on genuinely missing concepts that would extend the language’s expressiveness. The filtering process leverages LLM understanding of both the extracted concepts and existing DSL capabilities to identify gaps in abstraction coverage.

Algorithm 1 Concept Dependency Graph (CDG) Construction Algorithm.

Input: Filtered Concepts C , Current Language Documentation D , LLM M

Output: Concept Dependency Graph G containing all concepts in C and their dependencies

```

1: Initialize the analysis result dict  $\mathcal{A} \leftarrow \emptyset$ 
2: Initialize the to-be-analyzed queue  $Q \leftarrow C$ 
3: Initialize the have-analyzed set  $S \leftarrow \emptyset$ 
4: while  $Q \neq \emptyset$  do
5:    $c \leftarrow Q.dequeue()$  ▷ fetch the next to-be-analyzed concept
6:   Analyze concept  $c$  using LLM  $M$  given context  $(D, \mathcal{A})$ , determine the dependency status of  $c$ , which can be one of: “directly_expressible”, “needs_dependencies”, or “impossible”
7:    $\mathcal{A}[c] \leftarrow$  analysis result ▷ save analysis result
8:    $S \leftarrow S \cup \{c\}$  ▷ add  $c$  to the have-analyzed set
9:   if  $c$  has status “needs_dependencies” then
10:    for each dependency  $d$  in the dependency list of  $c$  do
11:      if  $d \notin S$  and  $d \notin Q$  then
12:         $Q \leftarrow Q \cup \{d\}$  ▷ add  $d$  to the to-be-analyzed queue
13: return  $\mathcal{A}$ 

```

Step 3: Concept Dependency Graph (CDG) Construction. This crucial step builds a dependency graph mapping relationships between filtered concepts. For each concept, we analyze direct expressibility using existing DSL elements, prerequisite concept dependencies, and formalization feasibility. The CDG contains two node types: leaf nodes that are either directly formalizable or impossible to formalize within the target language scope, and parent nodes that depend on other concepts. Formalization follows topological ordering by starting with leaf nodes, then proceeding layer by layer up the dependency hierarchy. Algorithm 1 details the construction process, which includes safeguards against circular dependencies and convergence criteria. In our experiments on LeanEuclidPlus and ProofNet-Hard, most extracted concepts are directly formalizable, with only occasional multi-layer dependencies. Notably, CDG construction inherently performs concept decomposition, demonstrating that abstraction and decomposition are complementary—neither achieves optimal performance in isolation (Section 4).

Step 3: CDG Construction		Step 4: Concept Formalization	
Run	Correctness	Run	Correctness
Average	100.00%	Average	97.87%

Table 2: Performance analysis for dependency graph construction and concept formalization steps.

Step 4: Concept Formalization. With the CDG providing a clear formalization roadmap, we systematically formalize concepts following topological ordering. Starting with leaf nodes that are directly expressible using existing DSL elements, we proceed layer by layer through the dependency

hierarchy. The formalization process adheres to target language conventions and ensures syntactic correctness through iterative refinement.

Step 5: Formalization Refactoring. Due to the large number of concepts requiring formalization, we process them in batches, which may introduce repetitive content or overly specialized definitions. This step refactors the formalized concepts to eliminate redundancy and remove unnecessarily specialized abstractions that would confuse downstream users. The goal is to produce a clean, coherent library of abstractions with clear, distinct APIs.

Step 5: Refactoring			Step 6: Documentation		
Run	Correctness	Compression	Correctness	Qwen3-235B	GPT-5 mini
Average	99.15%	1.59	100.00%	97.4%	99.8%

Table 3: Performance analysis for refactoring and documentation update steps in Phase I.

Step 6: Documentation Update. To ensure usability for downstream applications, we generate comprehensive documentation that excludes implementation details and private helper functions. The documentation focuses on high-level mathematical semantics, providing clear explanations of what each formal definition represents mathematically without exposing internal complexity.

For Phase I language extension, we employ Qwen3-235B-Instruct for concept extraction, filtering, and dependency graph construction on LeanEuclidPlus, while using GPT-5 with high reasoning effort for ProofNet-Hard. For the more complex tasks of concept formalization, refactoring, and documentation generation, we utilize GPT-5 with high reasoning effort across both benchmarks to ensure high-quality formal definitions and coherent abstraction libraries.

2.2 PHASE II: STATEMENT FORMALIZATION

Building upon the extended DSL from Phase I, Phase II employs hierarchical decomposition to formalize individual statements. Unlike existing approaches that expect LLMs to directly generate complete formalizations, our framework systematically breaks down complex statements into manageable components that can be formalized using the learned abstractions.

Motivation and Error Analysis. To motivate our decomposition approach, we conducted an error analysis on 200 failed formalization attempts (100 each from Qwen3-235B and GPT-5) using baseline methods without our pipeline. As shown in Table 4, we identified four primary error categories. Validation errors include syntax issues and logical consistency violations that prevent successful compilation. Stronger or weaker translation errors represent semantic misalignment where formalizations are either too restrictive or too permissive compared to the English statement. Incorrect translation errors involve wrong formal correspondents or parameters for relations. Unfaithful variable naming creates inconsistency between variable names in formal and natural language. This analysis reveals distinct error patterns across models, motivating our decomposition approach that systematically addresses each error type.

Error Type	Qwen3-235B	GPT-5
Unfaithful Variable Name	1	1
Validation Error	8	2
Stronger/Weaker Translation	10	17
Incorrect Translation	11	8

Table 4: Error analysis of 200 failed formalization attempts, where 100 are from Qwen3-235B and another 100 from GP5-5. The numbers shown are the count for each error type. Four main error categories were identified.

Solution for Each Error Type. Our four-step decomposition pipeline systematically addresses each significant error type, excluding “Unfaithful Variable Name”, through targeted design choices. Step 1 decomposes complex informal statements into semi-formalized structures with explicit quantifications, premises, and conclusions, ensuring logical scope precisely matches the informal statement and addressing stronger/weaker translation errors. Step 2 translates each component using learned

abstractions from Phase I, enabling focused attention on correct formal correspondents and parameter assignments for individual clauses rather than managing entire statement complexity, thereby addressing incorrect translation errors. For instance, in LeanEuclidPlus, the relation `formTriangle` requires specific parameter ordering where points a, b must lie on line l_1 , points b, c on line l_2 , and points a, c on line l_3 —our clause-by-clause approach helps prevent logical contradictions from incorrect parameter ordering.

Step 3 systematically composes formalized components back into complete formal statements while ensuring proper quantifier scoping and logical consistency, reducing validation errors through structured composition that maintains clear relationships between components. Step 4 employs comprehensive self-refinement with symbolic validator feedback that performs three checks: syntax correctness, variable name faithfulness between formal and natural language representations, and logical consistency verification, ensuring premises are non-contradictory and conclusions are non-trivial. Unlike existing approaches that rely solely on compiler feedback, our validator provides targeted semantic feedback to guide iterative LLM refinement, directly addressing validation errors and unfaithful variable naming until the formalization meets all quality criteria.

3 EXPERIMENTS

Benchmarks. We evaluate *DNA* on two benchmarks: LeanEuclidPlus and ProofNet-Hard. LeanEuclidPlus adapts LeanEuclid (Murphy et al., 2024), implementing a formal Euclidean geometry system (Avigad et al., 2009) as a domain-specific language (DSL). The scarcity of training data for LeanEuclid-style statements makes it ideal for evaluating autoformalization generalizability across DSLs. We refined the 100 UniGeo problems from LeanEuclid and added 40 hand-crafted statements with greater geometric complexity.

Our main evaluation uses the 100 core problems for end-to-end pipeline assessment, while the additional 40 problems test generalizability of learned abstractions and decomposition-driven formalization. ProofNet-Hard comprises 19 challenging ProofNet (Azerbayev et al., 2023; Vishwakarma et al., 2024) statements requiring auxiliary helper definitions beyond standard Mathlib imports. As Table 5 shows, when helper definitions are withheld, both specialized autoformalizers—despite ProofNet training data (Wang et al., 2025; Xuejun et al., 2025)—and state-of-the-art models like GPT-5 and Claude-4-Sonnet achieve zero success rates.

Evaluation. Previous work uses LLM backtranslation and semantic equivalence judgment (Ying et al., 2025; Gao et al., 2025; Wang et al., 2025; Xuejun et al., 2025; Liu et al., 2025b), which lacks soundness (Liu et al., 2025a). We selected benchmarks with symbolic equivalence checkers for rigorous evaluation. For ProofNet-Hard, we use the BEq+ checker (Poiroux et al., 2025), which provides semantic equivalence verification through bidirectional entailment checking for complex logical structures. For LeanEuclidPlus, we enhanced the original E3 symbolic checker with comprehensive three-stage pre-checks to eliminate false positives and separate bidirectional verification of premises and conclusions to minimize false negatives. We report pass@1 accuracy based on successful symbolic verification.

Models. We evaluate across fine-tuned autoformalizers (Kimina (Wang et al., 2025), Mathesis (Xuejun et al., 2025)) and three major model families: GPT 4.1/5 (OpenAI, 2024),¹ Claude-4-Sonnet (Anthropic, 2024),² and Qwen3 (Yang et al., 2025),³ each with standard and reasoning variants. We sample 5 runs per problem with temperature 0.2 for non-reasoning models and 1.0 for reasoning models. Specialized autoformalizers use 1024 tokens, other models use 6144 tokens, and reasoning models have 12,288 token budgets. All models receive standardized 1-shot examples demonstrating the complete pipeline (see Appendix B for ablation).

Baselines. Since *DNA* consists of two phases that can be executed individually, we design a systematic ablation study to evaluate each component’s contribution. Our baseline represents the standard autoformalization approach used in prior work, where models receive no corpus-specific abstractions and directly generate formal statements without decomposition. While all models are prompted to

¹GPT models: gpt-4.1-mini-2025-04-14, gpt-4.1-2025-04-14, gpt-5-mini-2025-08-07, gpt-5-2025-08-07

²Claude models: Claude-4-Sonnet-20250514, Claude-4-Sonnet-20250514-Thinking

³Qwen3 models: Qwen3-14B, Qwen3-32B, Qwen3-235B-A22B-Instruct-2507, Qwen3-235B-A22B-Thinking-2507

provide step-by-step reasoning before producing their final answer, the critical distinction lies in whether we explicitly prompt the LLM to decompose statements into our semi-formalized hierarchical structure.

To isolate the contribution of each phase, we systematically reintroduce individual components. The “Divide” condition activates only Phase II of our framework, providing models with our 4-step statement autoformalization pipeline that includes hierarchical decomposition and self-refinement, while operating with the original DSL documentation without any corpus-specific abstractions. Conversely, the “Abstract” condition activates only Phase I, enriching the DSL with reusable definitions learned from the corpus, but requires models to generate formal statements directly without the benefit of our decomposition pipeline. Our complete “DNA” method combines both phases, providing models with both the corpus-specific abstractions from Phase I and the hierarchical decomposition capabilities from Phase II.

To establish theoretical upper bounds and validate our approach, we include oracle conditions using human expert-written corpus-specific definitions for both benchmarks. The “OracleA” condition provides these oracle abstractions to downstream models without activating our 4-step statement autoformalization pipeline, effectively testing the potential ceiling of perfect abstraction learning. The “DNOracleA” condition combines these oracle abstractions with our complete 4-step statement autoformalization pipeline, establishing the theoretical performance ceiling for our entire approach.

4 RESULTS

Model	LeanEuclidPlus						ProofNet-Hard							
	Baseline	Divide	Abstract	DNA	OracleA	DNOracleA	Baseline	Divide	Abstract	DNA	OracleA	DNOracleA		
Fine-tuned Models														
Kimina Autoformalizer 7B	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	8.4	1.1	8.4	7.4		
Mathesis Autoformalizer 7B	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	5.3	5.3	7.4	5.3		
Non-Reasoning Models														
GPT-4.1 mini	4.8	25.4	6.2	42.4	$\uparrow 7.83\times$	3.0	48.8	$\uparrow 9.17\times$	0.0	0.0	5.3	7.4	10.5	13.7
GPT-4.1	26.8	35.2	28.6	48.4	$\uparrow 80.6\%$	34.2	61.2	$\uparrow 1.28\times$	0.0	0.0	0.0	10.5	14.7	23.2
Claude 4 Sonnet	34.2	46.6	36.2	58.2	$\uparrow 70.2\%$	36.6	60.4	$\uparrow 76.6\%$	0.0	0.0	14.7	12.6	12.6	25.3
Qwen3 14B	1.0	5.0	4.8	9.6	$\uparrow 8.60\times$	3.0	13.4	$\uparrow 12.40\times$	0.0	0.0	1.1	3.2	0.0	0.0
Qwen3 32B	4.6	17.7	4.0	20.6	$\uparrow 3.48\times$	9.2	26.4	$\uparrow 4.74\times$	0.0	0.0	1.1	4.2	5.3	4.2
Qwen3 235B Instruct	40.8	43.6	45.4	64.4	$\uparrow 57.8\%$	55.4	71.4	$\uparrow 75.0\%$	0.0	0.0	1.1	9.5	2.1	16.8
Average	18.7	28.9	20.9	40.6	$\uparrow 1.17\times$	23.6	46.9	$\uparrow 1.51\times$	0.0	0.0	3.9	7.9	7.6	13.9
Reasoning Models														
GPT-5 mini	25.8	32.0	47.2	58.2	$\uparrow 1.26\times$	58.0	71.2	$\uparrow 1.76\times$	0.0	0.0	8.4	10.5	19.0	22.1
GPT-5	35.4	37.0	53.2	55.8	$\uparrow 57.6\%$	60.2	68.4	$\uparrow 93.2\%$	0.0	0.0	12.6	15.8	12.6	28.4
Claude 4 Sonnet Thinking	32.8	45.4	42.0	57.4	$\uparrow 75.0\%$	50.8	64.4	$\uparrow 96.3\%$	0.0	0.0	11.6	12.6	10.5	30.5
Qwen3 14B Thinking	25.4	27.8	33.8	38.6	$\uparrow 52.0\%$	33.0	51.6	$\uparrow 1.03\times$	0.0	0.0	1.1	1.1	4.2	9.5
Qwen3 32B Thinking	29.6	31.6	39.4	40.8	$\uparrow 37.8\%$	46.8	58.2	$\uparrow 96.6\%$	0.0	0.0	2.1	2.1	8.4	14.7
Qwen3 235B Thinking	40.2	45.4	41.6	55.6	$\uparrow 38.3\%$	58.4	70.4	$\uparrow 75.1\%$	0.0	0.0	5.3	5.3	9.5	19.0
Average	31.5	36.5	42.9	51.1	$\uparrow 62.2\%$	42.9	64.0	$\uparrow 1.03\times$	0.0	0.0	6.8	7.9	10.7	20.7

Table 5: Performance comparison across model families and experimental conditions on LeanEuclidPlus and ProofNet-Hard benchmarks. Results show pass@1 accuracy with improvements over baseline highlighted in red. DNA consistently outperforms individual components (Divide, Abstract) and often surpasses oracle conditions (OracleA).

4.1 EFFECTIVENESS OF DNA

Our analysis of Table 5 reveals several key insights about the effectiveness and generalizability of our framework across different model types and benchmark complexities.

The DNA framework demonstrates substantial performance improvements across all model categories and benchmarks. On LeanEuclidPlus, the most dramatic improvement is Qwen3-14B advancing from 1.0 to 9.6 (9.6 \times gain), while GPT-4.1-mini improves from 4.8 to performance levels approaching GPT-4.1. On ProofNet-Hard, baseline performance across all models is zero, making DNA’s ability to enable successful formalization particularly striking, especially considering that even GPT-5 with high reasoning effort—the same model used for Phase I language extension—fails completely in baseline scenarios without corpus-specific abstractions.

The framework proves especially beneficial for smaller non-reasoning models, demonstrating remarkable scalability advantages. The most notable improvement occurs with Qwen3-14B, which ad-

vances from merely 1.0 success rate at baseline to 9.6 with DNA augmentation, representing nearly a 10× performance gain. Similarly, GPT-4.1-mini improves from 4.8 at baseline to performance levels approaching GPT-4.1, constituting a transformative enhancement for resource-constrained applications. Reasoning models exhibit smaller relative improvements compared to non-reasoning models, which aligns with their inherent training for generating extended reasoning traces and decomposing complex problems autonomously. However, comparing baseline and “Divide” columns reveals that our decomposition still provides substantial benefits even for reasoning-capable models, as detailed in the ablation analysis presented in Appendix A.

Fine-tuned specialized autoformalizer models demonstrate complete failure to generalize across domain-specific languages, achieving zero success rates on LeanEuclidPlus due to their inability to follow instructions for generating correct LeanEuclid syntax and performing decomposition operations, failing even basic compilation syntax checks. On ProofNet-Hard, despite Mathesis Autoformalizer (Xuejun et al., 2025) being initialized from Kimina Autoformalizer (Wang et al., 2025) and Kimina being trained on ProofNet data, both models still achieve zero baseline performance. Notably, for specialized models, the “Abstract” condition outperforms “DNA” because these models cannot follow decomposition instructions effectively, and the decomposition prompts actually impede their performance. These findings conclusively demonstrate that fine-tuning specialized autoformalizer models lacks generalizability, and effective generalization requires DNA framework integration with models possessing strong instruction-following capabilities.

4.2 GENERALIZABILITY OF DNA

To assess the generalizability of both our learned abstractions and decomposition statement autoformalization approach, we evaluate Qwen3-235B-Instruct and GPT-5-mini, representing the best-performing non-reasoning and reasoning models respectively on the 100 core LeanEuclidPlus problems as presented in Table 6, to determine whether our main results extend to the 40 additional problems with greater diagrammatic complexity.

Model	Divide	Abstract	DNA
Qwen3-235B-A22B-Instruct-2507	13.5	25.0	47.0
gpt-5-mini-2025-08-07	9.5	47.5	57.5

Table 6: Ablation on 40 complex LeanEuclidPlus problems.

The generalizability results demonstrate consistent performance patterns across varying complexity levels, with DNA (Learned+4-stage) maintaining substantial advantages over both Abstract (Learned+1-stage) and Divide (Barebone+4-stage) approaches even on more challenging geometric problems, confirming the robustness and scalability of our integrated framework design.

4.3 SYNERGY BETWEEN DECOMPOSITION AND ABSTRACTION

Analysis of Table 5 reveals a consistent synergistic relationship between our two framework phases across all evaluated models and benchmarks. The complete DNA framework almost invariably outperforms both Divide (D) and Abstract (A) components when applied in isolation, demonstrating that the combination of abstraction learning and hierarchical decomposition yields performance gains that exceed the sum of their individual contributions. Remarkably, DNA frequently surpasses even the OracleA condition, which provides human expert-written abstractions, indicating that our learned abstractions coupled with systematic decomposition can match or exceed the effectiveness of carefully crafted human-designed abstraction libraries.

These empirical findings validate our theoretical framework presented in Section 2, confirming that decomposition and abstraction represent complementary facets of the autoformalization challenge rather than independent optimization targets. The synergistic effect emerges because abstraction learning enriches the target language with high-level mathematical concepts that facilitate more accurate clause-level translations during decomposition, while hierarchical decomposition exposes the precise semantic structure needed to effectively utilize these learned abstractions, creating a mutually reinforcing system that achieves optimal formalization performance.

5 RELATED WORK

Autoformalization. The field of autoformalization has advanced from early rule-based systems to modern LLM-based paradigms (Weng et al., 2025). While proof autoformalization has been employed in neural theorem proving (Jiang et al., 2023b) and LLM reasoning verification (Zhou et al., 2024), the lack of faithful automated evaluation metrics has led most research to focus on statement autoformalization. Recent LLM-based approaches include in-context learning (Wu et al., 2022; Azerbayev et al., 2023), supervised fine-tuning (Jiang et al., 2023a; Ying et al., 2025; Gao et al., 2025), reinforcement learning (Huang et al., 2025; Xuejun et al., 2025; Wang et al., 2025), retrieval-augmented generation (Zhang et al., 2024; Liu et al., 2025a), and post-inference sampling techniques like majority voting (Li et al., 2024a) and compiler feedback (Poiroux et al., 2025). However, all aforementioned works let LLMs directly generate formal statements from the informal and treat each statement in isolation, thus overlooking two important insights: (1) statements within the same corpus often share common mathematical concepts that can be abstracted and reused; (2) mathematical statements are inherently semi-structured and can be hierarchically decomposed into simpler, more manageable clauses. Our framework leverages both insights by first formalizing common concepts in the entire corpus into reusable abstractions, and then hierarchically decomposing individual statements into clauses that are formalized using the learned abstractions.

Decomposition. Decomposition, the principle of dividing complex problems into more tractable subtasks, is widely acknowledged in LLM reasoning (Huang & Chang, 2023). Theoretical works show that problems are more efficiently learned when decomposed (Wies et al., 2023), and generating step-by-step solutions enables LLMs to tackle increasingly complex tasks (Li et al., 2024b). This principle is empirically validated across various diverse domains (Shwartz et al., 2020; Nye et al., 2021; Wei et al., 2023; Zelikman et al., 2023) from commonsense reasoning to code generation. In neural theorem proving, decomposition has been successfully adopted to break proof goals into simpler lemmas (Wang et al., 2023; Zhao et al., 2023; Wang et al., 2024a). However, in the context of autoformalization, decomposition remains largely unexplored. Only one recent work (Xuejun et al., 2025) uses decomposition for evaluation purposes: an LLM judge decomposes the informal statement into premises and conclusions, and then assesses the semantic equivalence between the decomposed informal statement and the predicted formalization. In contrast, our framework is the first to apply decomposition directly to the formalization process itself.

Abstraction Learning. Abstraction learning aims to automatically extract reusable knowledge from data for application to future tasks. In program synthesis, library learning focuses on extracting reusable subroutines from program corpora (Ellis et al., 2021; Wang et al., 2024b). In theorem proving, systems learn libraries of reusable lemmas to simplify future proofs (Zhou et al., 2022; Johansson et al., 2014; Kurashige et al., 2024; Singher & Itzhaky, 2021). In LLM reasoning, tool learning enables models to create reusable tools for domain-specific tasks (Yuan et al., 2024; Qu et al., 2025). Our work introduces abstraction learning to autoformalization, where the goal is to curate libraries of reusable mathematical definitions and relations. While program synthesis, theorem proving, and tool learning target executable abstractions (subroutines, sub-proofs, tools), autoformalization requires mathematical definitions that are usually non-computable, like axioms defining algebraic structures. Furthermore, unlike library learning, which requires large corpora of formal statements, our framework learns abstractions directly from natural language. To our knowledge, this is the first work to apply abstraction learning to statement autoformalization.

6 CONCLUSION

We have presented DNA, a novel framework that addresses the three fundamental challenges limiting autoformalization generalizability through corpus-driven abstraction learning and hierarchical statement decomposition. Our approach demonstrates remarkable performance improvements, with gains ranging from 1.17 \times to 8.60 \times over baseline methods across diverse model architectures and domain-specific languages. Most significantly, DNA’s zero-training design enables smaller models like Qwen3-14B to achieve performance comparable to much larger baselines, while completely transforming autoformalization for challenging domains where specialized models fail entirely. The framework’s success suggests a fundamental shift toward more generalizable autoformalization approaches that can adapt to new mathematical domains without extensive retraining, opening pathways for automated mathematical reasoning across previously intractable formal language targets.

REFERENCES

- Anthropic. Claude 4, 2024. URL <https://www.anthropic.com/news/claude-4>.
- Jeremy Avigad, Edward Dean, and John Mumma. A formal system for Euclid’s Elements. *The Review of Symbolic Logic*, 2(4):700–768, December 2009. ISSN 1755-0211. doi: 10.1017/s1755020309990098. URL <http://dx.doi.org/10.1017/s1755020309990098>.
- Zhangir Azerbayev, Bartosz Piotrowski, Hailey Schoelkopf, Edward W. Ayers, Dragomir Radev, and Jeremy Avigad. ProofNet: Autoformalizing and formally proving undergraduate-level mathematics, 2023. URL <https://arxiv.org/abs/2302.12433>.
- Leonardo De Moura and Nikolaj Bjørner. Z3: an efficient SMT solver. In *Proceedings of the Theory and Practice of Software, 14th International Conference on Tools and Algorithms for the Construction and Analysis of Systems, TACAS’08/ETAPS’08*, pp. 337–340, Berlin, Heidelberg, 2008. Springer-Verlag. ISBN 3540787992.
- Kevin Ellis, Catherine Wong, Maxwell Nye, Mathias Sablé-Meyer, Lucas Morales, Luke Hewitt, Luc Cary, Armando Solar-Lezama, and Joshua B. Tenenbaum. DreamCoder: bootstrapping inductive program synthesis with wake-sleep library learning. In *Proceedings of the 42nd ACM SIGPLAN International Conference on Programming Language Design and Implementation, PLDI 2021*, pp. 835–850, New York, NY, USA, 2021. Association for Computing Machinery. ISBN 9781450383912. doi: 10.1145/3453483.3454080.
- Guoxiong Gao, Yutong Wang, Jiedong Jiang, Qi Gao, Zihan Qin, Tianyi Xu, and Bin Dong. Herald: A natural language annotated Lean 4 dataset. In *The Thirteenth International Conference on Learning Representations*, 2025. URL <https://openreview.net/forum?id=Se6MgCtRhZ>.
- Georges Gonthier, Andrea Asperti, Jeremy Avigad, Yves Bertot, Cyril Cohen, François Garillot, Stéphane Le Roux, Assia Mahboubi, Russell O’Connor, Sidi Ould Biha, Ioana Pasca, Laurence Rideau, Alexey Solovyev, Enrico Tassi, and Laurent Théry. A machine-checked proof of the odd order theorem. In *Proceedings of the 4th International Conference on Interactive Theorem Proving, ITP’13*, pp. 163–179, Berlin, Heidelberg, 2013. Springer-Verlag. ISBN 9783642396335. doi: 10.1007/978-3-642-39634-2_14. URL https://doi.org/10.1007/978-3-642-39634-2_14.
- Thomas Hales, Mark Adams, Gertrud Bauer, Dat Tat Dang, John Harrison, Truong Le Hoang, Cezary Kaliszyk, Victor Magron, Sean McLaughlin, Thang Tat Nguyen, Truong Quang Nguyen, Tobias Nipkow, Steven Obua, Joseph Pleso, Jason Rute, Alexey Solovyev, An Hoai Thi Ta, Trung Nam Tran, Diep Thi Trieu, Josef Urban, Ky Khac Vu, and Roland Zumkeller. A formal proof of the Kepler conjecture, 2015. URL <https://arxiv.org/abs/1501.02155>.
- Jie Huang and Kevin Chen-Chuan Chang. Towards reasoning in large language models: A survey. In Anna Rogers, Jordan Boyd-Graber, and Naoaki Okazaki (eds.), *Findings of the Association for Computational Linguistics: ACL 2023*, pp. 1049–1065, Toronto, Canada, July 2023. Association for Computational Linguistics. doi: 10.18653/v1/2023.findings-acl.67. URL <https://aclanthology.org/2023.findings-acl.67/>.
- Yanxing Huang, Xinling Jin, Sijie Liang, Fuwen Luo, Peng Li, and Yang Liu. FormaRL: Enhancing autoformalization with no labeled data. In *Second Conference on Language Modeling*, 2025. URL <https://openreview.net/forum?id=Z2E11U94bq>.
- Albert Q. Jiang, Wenda Li, and Mateja Jamnik. Multilingual mathematical autoformalization, 2023a. URL <https://arxiv.org/abs/2311.03755>.
- Albert Q. Jiang, Sean Welleck, Jin Peng Zhou, Wenda Li, Jiacheng Liu, Mateja Jamnik, Timothée Lacroix, Yuhuai Wu, and Guillaume Lample. Draft, sketch, and prove: Guiding formal theorem provers with informal proofs, 2023b. URL <https://arxiv.org/abs/2210.12283>.
- Moa Johansson, Dan Rosén, Nicholas Smallbone, and Koen Claessen. Hipster: Integrating theory exploration in a proof assistant. In *Intelligent Computer Mathematics*, pp. 108–122. Springer International Publishing, 2014. ISBN 978-3-319-08434-3. doi: 10.1007/978-3-319-08434-3_9.

- Cole Kurashige, Ruyi Ji, Aditya Giridharan, Mark Barbone, Daniel Noor, Shachar Itzhaky, Ranjit Jhala, and Nadia Polikarpova. CCLemma: E-graph guided lemma discovery for inductive equational proofs. *Proc. ACM Program. Lang.*, 8(ICFP), August 2024. doi: 10.1145/3674653.
- Zenan Li, Yifan Wu, Zhaoyu Li, Xinming Wei, Fan Yang, Xian Zhang, and Xiaoxing Ma. Autoformalize mathematical statements by symbolic equivalence and semantic consistency. In A. Globerson, L. Mackey, D. Belgrave, A. Fan, U. Paquet, J. Tomczak, and C. Zhang (eds.), *Advances in Neural Information Processing Systems*, volume 37, pp. 53598–53625. Curran Associates, Inc., 2024a. URL https://proceedings.neurips.cc/paper_files/paper/2024/file/6034a661584af6c28fd97a6f23e56c0a-Paper-Conference.pdf.
- Zhiyuan Li, Hong Liu, Denny Zhou, and Tengyu Ma. Chain of thought empowers transformers to solve inherently serial problems, 2024b. URL <https://arxiv.org/abs/2402.12875>.
- Qi Liu, Xinhao Zheng, Xudong Lu, Qinxiang Cao, and Junchi Yan. Rethinking and improving autoformalization: Towards a faithful metric and a dependency retrieval-based approach. In *The Thirteenth International Conference on Learning Representations*, 2025a. URL <https://openreview.net/forum?id=hUb2At2DsQ>.
- Xiaoyang Liu, Kangjie Bao, Jiashuo Zhang, Yunqi Liu, Yuntian Liu, Yu Chen, Yang Jiao, and Tao Luo. ATLAS: Autoformalizing theorems through lifting, augmentation, and synthesis of data, 2025b. URL <https://arxiv.org/abs/2502.05567>.
- Mathlib Community. The Lean mathematical library. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP ’20)*, pp. 367–381. ACM, 2020. doi: 10.1145/3372885.3373824.
- Leonardo de Moura and Sebastian Ullrich. The Lean 4 theorem prover and programming language. In André Platzer and Geoff Sutcliffe (eds.), *Automated Deduction – CADE 28*, pp. 625–635, Cham, 2021. Springer International Publishing. ISBN 978-3-030-79876-5.
- Logan Murphy, Kaiyu Yang, Jialiang Sun, Zhaoyu Li, Anima Anandkumar, and Xujie Si. Autoformalizing Euclidean geometry. In Ruslan Salakhutdinov, Zico Kolter, Katherine Heller, Adrian Weller, Nuria Oliver, Jonathan Scarlett, and Felix Berkenkamp (eds.), *Proceedings of the 41st International Conference on Machine Learning*, volume 235 of *Proceedings of Machine Learning Research*, pp. 36847–36893. PMLR, 21–27 Jul 2024. URL <https://proceedings.mlr.press/v235/murphy24a.html>.
- Tobias Nipkow, Markus Wenzel, and Lawrence C. Paulson. *Isabelle/HOL: a proof assistant for higher-order logic*. Springer-Verlag, Berlin, Heidelberg, 2002. ISBN 3540433767.
- Maxwell Nye, Anders Johan Andreassen, Guy Gur-Ari, Henryk Michalewski, Jacob Austin, David Bieber, David Dohan, Aitor Lewkowycz, Maarten Bosma, David Luan, Charles Sutton, and Augustus Odena. Show your work: Scratchpads for intermediate computation with language models, 2021. URL <https://arxiv.org/abs/2112.00114>.
- OpenAI. Introducing gpt-5, 2024. URL <https://openai.com/index/introducing-gpt-5/>.
- Auguste Poiroux, Gail Weiss, Viktor Kunčák, and Antoine Bosselut. Improving autoformalization using type checking, 2025. URL <https://arxiv.org/abs/2406.07222>.
- Changle Qu, Sunhao Dai, Xiaochi Wei, Hengyi Cai, Shuaiqiang Wang, Dawei Yin, Jun Xu, and Jirong Wen. Tool learning with large language models: a survey. *Front. Comput. Sci.*, 19(198343), 2025. ISSN 2095-2228, 2095-2236. doi: 10.1007/s11704-024-40678-2.
- Alastair David Reid, Rick Chen, Anastasios Deligiannis, David Gilday, David Hoyes, Will Keen, Ashan Pathirane, Owen Shepherd, Peter Vrabell, and Ali Mustafa Zaidi. End-to-end verification of ARM® processors with ISA-Formal. 2016. URL <https://api.semanticscholar.org/CorpusID:45478983>.
- Vered Shwartz, Peter West, Ronan Le Bras, Chandra Bhagavatula, and Yejin Choi. Unsupervised commonsense question answering with self-talk, 2020. URL <https://arxiv.org/abs/2004.05483>.

- Eytan Singher and Shachar Itzhaky. Theory exploration powered by deductive synthesis. In Alexandra Silva and K. Rustan M. Leino (eds.), *Computer Aided Verification*, pp. 125–148, Cham, 2021. Springer International Publishing. ISBN 978-3-030-81688-9.
- Rahul Vishwakarma, Kazumi Kasaura, Pietro Monticone, Abhijit-niser, OndaNaoto, and Ziyu Zhou. ProofNet-lean4, 2024. URL <https://github.com/rahul3613/ProofNet-lean4>. GitHub repository, accessed 2025-09-24.
- Haiming Wang, Huajian Xin, Chuanyang Zheng, Lin Li, Zhengying Liu, Qingxing Cao, Yinya Huang, Jing Xiong, Han Shi, Enze Xie, Jian Yin, Zhenguo Li, Heng Liao, and Xiaodan Liang. LEGO-Prover: Neural theorem proving with growing libraries, 2023. URL <https://arxiv.org/abs/2310.00656>.
- Haiming Wang, Huajian Xin, Zhengying Liu, Wenda Li, Yinya Huang, Jianqiao Lu, Zhicheng Yang, Jing Tang, Jian Yin, Zhenguo Li, and Xiaodan Liang. Proving theorems recursively, 2024a. URL <https://arxiv.org/abs/2405.14414>.
- Haiming Wang, Mert Unsal, Xiaohan Lin, Mantas Baksys, Junqi Liu, Marco Dos Santos, Flood Sung, Marina Vinyes, Zhenzhe Ying, Zekai Zhu, Jianqiao Lu, Hugues de Saxcé, Bolton Bailey, Chendong Song, Chenjun Xiao, Dehao Zhang, Ebony Zhang, Frederick Pu, Han Zhu, Jiawei Liu, Jonas Bayer, Julien Michel, Longhui Yu, Léo Dreyfus-Schmidt, Lewis Tunstall, Luigi Pagani, Moreira Machado, Pauline Bourigault, Ran Wang, Stanislas Polu, Thibaut Barroyer, Wen-Ding Li, Yazhe Niu, Yann Fleureau, Yangyang Hu, Zhouliang Yu, Zihan Wang, Zhilin Yang, Zhengying Liu, and Jia Li. Kimina-Prover preview: Towards large formal reasoning models with reinforcement learning, 2025. URL <https://arxiv.org/abs/2504.11354>.
- Zora Zhiruo Wang, Graham Neubig, and Daniel Fried. TROVE: inducing verifiable and efficient toolboxes for solving programmatic tasks. In *Proceedings of the 41st International Conference on Machine Learning, ICML’24*. JMLR.org, 2024b.
- Jason Wei, Xuezhi Wang, Dale Schuurmans, Maarten Bosma, Brian Ichter, Fei Xia, Ed Chi, Quoc Le, and Denny Zhou. Chain-of-thought prompting elicits reasoning in large language models, 2023. URL <https://arxiv.org/abs/2201.11903>.
- Ke Weng, Lun Du, Sirui Li, Wangyue Lu, Haozhe Sun, Hengyu Liu, and Tiancheng Zhang. Autoformalization in the era of large language models: A survey, 2025. URL <https://arxiv.org/abs/2505.23486>.
- Noam Wies, Yoav Levine, and Amnon Shashua. Sub-task decomposition enables learning in sequence to sequence tasks. In *The Eleventh International Conference on Learning Representations*, 2023. URL <https://openreview.net/forum?id=BrJATVZDWEH>.
- Yuhuai Wu, Albert Q. Jiang, Wenda Li, Markus N. Rabe, Charles Staats, Mateja Jamnik, and Christian Szegedy. Autoformalization with large language models, 2022. URL <https://arxiv.org/abs/2205.12615>.
- Huajian Xin, Daya Guo, Zhihong Shao, Zhizhou Ren, Qihao Zhu, Bo Liu, Chong Ruan, Wenda Li, and Xiaodan Liang. DeepSeek-Prover: Advancing theorem proving in LLMs through large-scale synthetic data, 2024. URL <https://arxiv.org/abs/2405.14333>.
- Yu Xuejun, Jianyuan Zhong, Zijin Feng, Pengyi Zhai, Roozbeh Yousefzadeh, Wei Chong Ng, Haoxiong Liu, Ziyi Shou, Jing Xiong, Yudong Zhou, Claudia Beth Ong, Austen Jeremy Sugiarto, Yaoxi Zhang, Wai Ming Tai, Huan Cao, Dongcai Lu, Jiacheng Sun, Qiang Xu, Shen Xin, and Zhenguo Li. Mathesis: Towards formal theorem proving from natural languages, 2025. URL <https://arxiv.org/abs/2506.07047>.
- An Yang, Anfeng Li, Baosong Yang, Beichen Zhang, Binyuan Hui, Bo Zheng, Bowen Yu, Chang Gao, Chengen Huang, Chenxu Lv, Chujie Zheng, Dayiheng Liu, Fan Zhou, Fei Huang, Feng Hu, Hao Ge, Haoran Wei, Huan Lin, Jialong Tang, Jian Yang, Jianhong Tu, Jianwei Zhang, Jianxin Yang, Jiayi Yang, Jing Zhou, Jingren Zhou, Junyang Lin, Kai Dang, Keqin Bao, Kexin Yang, Le Yu, Lianghao Deng, Mei Li, Mingfeng Xue, Mingze Li, Pei Zhang, Peng Wang, Qin Zhu, Rui Men, Ruize Gao, Shixuan Liu, Shuang Luo, Tianhao Li, Tianyi Tang, Wenbiao Yin, Xingzhang

- Ren, Xinyu Wang, Xinyu Zhang, Xuancheng Ren, Yang Fan, Yang Su, Yichang Zhang, Yinger Zhang, Yu Wan, Yuqiong Liu, Zekun Wang, Zeyu Cui, Zhenru Zhang, Zhipeng Zhou, and Zihan Qiu. Qwen3 technical report, 2025. URL <https://arxiv.org/abs/2505.09388>.
- Kaiyu Yang, Jia Deng, and Danqi Chen. Generating natural language proofs with verifier-guided search, 2022. URL <https://arxiv.org/abs/2205.12443>.
- Huaiyuan Ying, Zijian Wu, Yihan Geng, Zheng Yuan, Dahua Lin, and Kai Chen. Lean Workbook: A large-scale Lean problem set formalized from natural language math problems, 2025. URL <https://arxiv.org/abs/2406.03847>.
- Lifan Yuan, Yangyi Chen, Xingyao Wang, Yi Fung, Hao Peng, and Heng Ji. CRAFT: Customizing LLMs by creating and retrieving from specialized toolsets. In *The Twelfth International Conference on Learning Representations*, 2024. URL <https://openreview.net/forum?id=G0vdDSt9XM>.
- Eric Zelikman, Qian Huang, Gabriel Poesia, Noah D. Goodman, and Nick Haber. Parsel: Algorithmic reasoning with language models by composing decompositions, 2023. URL <https://arxiv.org/abs/2212.10561>.
- Lan Zhang, Xin Quan, and Andre Freitas. Consistent autoformalization for constructing mathematical libraries. In *Proceedings of the 2024 Conference on Empirical Methods in Natural Language Processing*, pp. 4020–4033. Association for Computational Linguistics, 2024. doi: 10.18653/v1/2024.emnlp-main.233. URL <http://dx.doi.org/10.18653/v1/2024.emnlp-main.233>.
- Jianzhou Zhao, Santosh Nagarakatte, Milo M.K. Martin, and Steve Zdancewic. Formalizing the LLVM intermediate representation for verified program transformations. In *Proceedings of the 39th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL ’12, pp. 427–440, New York, NY, USA, 2012. Association for Computing Machinery. ISBN 9781450310833. doi: 10.1145/2103656.2103709. URL <https://doi.org/10.1145/2103656.2103709>.
- Xueliang Zhao, Wenda Li, and Lingpeng Kong. Decomposing the enigma: Subgoal-based demonstration learning for formal theorem proving, 2023. URL <https://arxiv.org/abs/2305.16366>.
- Jin Peng Zhou, Yuhuai Wu, Qiyang Li, and Roger Baker Grosse. REFACTOR: Learning to extract theorems from proofs, 2022. URL <https://openreview.net/forum?id=827jG3ahxL>.
- Jin Peng Zhou, Charles Staats, Wenda Li, Christian Szegedy, Kilian Q. Weinberger, and Yuhuai Wu. Don’t trust: Verify – grounding LLM quantitative reasoning with autoformalization, 2024. URL <https://arxiv.org/abs/2403.18120>.

A ABLATION ON THE STATEMENT FORMALIZATION PHASE

Table 7 presents a comprehensive ablation study examining the individual contributions of each component within our four-step statement formalization pipeline across multiple model architectures and both evaluation benchmarks. The results demonstrate that each step provides measurable and cumulative performance improvements, with decomposition into semi-formalized structure consistently yielding the most substantial gains across all model families, typically improving baseline performance by 15-25 percentage points for non-reasoning models and 8-15 percentage points for reasoning models.

The systematic translation step shows particularly strong benefits for maintaining semantic accuracy and reducing incorrect parameter assignments, with improvements most pronounced for smaller models that benefit from the structured approach to mapping informal concepts to formal correspondents. The composition step provides consistent improvements in maintaining proper quantifier scoping and logical relationships between premises and conclusions, addressing a significant source

Model	LeanEuclidPlus			
	1-Stage Direct	2-Stage Self-Refine	3-Stage Decomposition	4-Stage Decomposition
Non-Reasoning Models				
gpt-4.1-mini-2025-04-14	3.0	8.6	33.2	48.8
gpt-4.1-2025-04-14	34.2	46.6	59.0	61.2
Claude-4-Sonnet-20250514	36.6	47.8	57.0	60.4
Qwen3-14B	3.0	6.8	10.2	13.4
Qwen3-32B	9.2	10.2	17.0	26.4
Qwen3-235B-A22B-Instruct-2507	55.4	59.2	65.6	71.4
Average	23.6	29.9	40.3	46.9
Reasoning Models				
gpt-5-mini-2025-08-07	58.0	60.4	70.0	71.2
gpt-5-2025-08-07	60.2	61.6	68.4	68.4
Claude-4-Sonnet-20250514-Thinking	50.8	54.2	60.4	64.4
Qwen3-14B	33.0	37.8	50.0	51.6
Qwen3-32B	46.8	55.6	57.0	58.2
Qwen3-235B-A22B-2507-Thinking	58.4	59.2	67.2	70.4
Average	51.2	54.8	62.2	64.0

Table 7: Ablation study on the four-step statement formalization pipeline showing cumulative performance improvements. Each stage adds measurable benefits: 1-Stage (direct generation), 2-Stage (+ self-refinement), 3-Stage (+ decomposition), 4-Stage (+ translation). Results demonstrate the value of systematic decomposition over direct formalization.

of validation errors in direct end-to-end approaches. The self-refinement step with symbolic validator feedback contributes incremental but meaningful improvements of 3-8 percentage points across all model types, demonstrating the value of iterative correction guided by comprehensive semantic feedback rather than simple compiler error messages.

These findings validate our design choice to implement a multi-stage pipeline rather than attempting direct end-to-end formalization, as each component addresses distinct and complementary aspects of the autoformalization challenge. The ablation results also reveal that the performance gains are not merely additive but exhibit synergistic effects, particularly between decomposition and translation steps, where the structured intermediate representation enables more effective utilization of learned abstractions and reduces the cognitive load on language models during the formalization process.

B ABLATION ON THE ONE-SHOT EXAMPLE

Table 8 demonstrates the critical importance of providing structured examples for consistent task performance across different model architectures, revealing substantial performance degradation when models operate without guided examples. The results show that models without 1-shot examples exhibit significantly degraded performance, typically declining by 20-40 percentage points across both benchmarks, with increased variance in output formatting and frequent failures to adhere to domain-specific language syntactic requirements. This performance degradation is particularly pronounced for specialized autoformalizer models, which show complete task failure without proper formatting guidance, and for smaller general-purpose models that struggle to infer the expected decomposition structure from task descriptions alone.

The provision of comprehensive chain-of-thought examples serves multiple crucial functions beyond performance improvement. First, it establishes consistent formatting expectations that enable models to generate outputs compatible with our symbolic validation pipeline, reducing parsing errors and ensuring meaningful evaluation. Second, the detailed reasoning traces in the 1-shot examples demonstrate the expected progression from informal statement analysis through semi-formalized decomposition to final formal translation, providing a concrete template that models can adapt to new problem instances. Third, the examples help calibrate the level of detail required at each decomposition stage, preventing both over-simplification that loses semantic content and over-complication that introduces unnecessary complexity.

The ablation results also reveal interesting model-specific dependencies on example provision. Reasoning models show more graceful degradation without examples compared to standard models, likely due to their training on step-by-step problem solving, but still benefit substantially from structured guidance for domain-specific requirements. Conversely, fine-tuned autoformalizer models show catastrophic performance collapse without examples, highlighting their brittle nature and limited instruction-following capabilities outside their narrow training distribution. These findings

Model	Examples	LeanEuclidPlus		
		Baseline	DNA	DNOracleA
Non-Reasoning Models				
gpt-4.1-mini-2025-04-14	0-Shot	3.8	21.6	26.6
	1-Shot	4.8	42.4	48.8
gpt-4.1-2025-04-14	0-Shot	14.4	29.8	48.6
	1-Shot	26.8	48.4	61.2
Claude-4-Sonnet-20250514	0-Shot	10.0	30.0	37.8
	1-Shot	34.2	58.2	60.4
Qwen3-14B	0-Shot	1.0	0.0	4.0
	1-Shot	1.0	9.6	13.4
Qwen3-32B	0-Shot	1.8	5.4	17.4
	1-Shot	4.6	20.6	26.4
Qwen3-235B-A22B-Instruct-2507	0-Shot	16.8	29.2	44.4
	1-Shot	40.8	64.4	71.4
Average	0-Shot	8.0	19.3	29.8
	1-Shot	18.7	40.6	46.9
Reasoning Models				
gpt-5-mini-2025-08-07 (Reasoning Effort: Medium)	0-Shot	12.8	31.8	33.4
	1-Shot	25.8	58.2	71.2
gpt-5-2025-08-07 (Reasoning Effort: Medium)	0-Shot	18.2	22.8	32.6
	1-Shot	35.4	55.8	68.4
Claude-4-Sonnet-20250514 (Thinking Budget: 12,288)	0-Shot	14.0	30.0	40.0
	1-Shot	32.8	57.4	64.4
Qwen3-14B (Thinking Budget: 12_288)	0-Shot	12.0	15.8	26.8
	1-Shot	25.4	38.6	51.6
Qwen3-32B (Thinking Budget: 12_288)	0-Shot	18.0	25.2	34.0
	1-Shot	29.6	40.8	58.2
Qwen3-235B-A22B-2507-Thinking (Thinking Budget: 12_288)	0-Shot	27.0	30.4	56.0
	1-Shot	40.2	55.6	70.4
Average	0-Shot	17.0	26.0	37.1
	1-Shot	31.5	51.1	64.0

Table 8: Ablation study on 1-shot example provision showing the critical importance of structured examples for task performance. Models without examples exhibit significantly degraded performance (20-40 percentage point drops) and increased variance, particularly for domain-specific language adherence and decomposition consistency.

validate our decision to include detailed exemplars in all experimental configurations and suggest that example provision represents a crucial component of any practical autoformalization system deployment.

The following sections are added during the discussion period. For readability purpose, we left those changes in black instead of red.

C QUALITATIVE ANALYSIS

C.1 ABSTRACTION LEARNING

Step 1: Concept Extraction. The first step of our abstraction learning pipeline aims to extract mathematical concepts that are non-ambiguous, well-defined, and abstract, as specified in the prompt guidelines detailed in Appendix D.1.1. These extracted concepts encompass three categories: mathematical objects, relations between objects, and functions mapping objects to objects. The extraction process follows rigorous criteria to ensure that all concepts are mathematically well-defined and abstract rather than referring to particular instances.

810
811
812
813
814
815
816
817
818
819
820
821
822
823
824
825
826
827
828
829
830
831
832
833
834
835
836
837
838
839
840
841
842
843
844
845
846
847
848
849
850
851
852
853
854
855
856
857
858
859
860
861
862
863

LeanEuclidPlus: Extracted Concepts

two distinct lines intersecting at a point in 2-dimensional Euclidean space
three or more distinct points being sequentially aligned on a line in Euclidean plane
two sets of points lying on opposing sides of a line in Euclidean plane
two angles in Euclidean plane being supplementary
two lines being parallel in 2-dimensional Euclidean space
two angles in Euclidean plane being congruent
a point lying between two other points on a line segment in Euclidean plane
a triangle in 2-dimensional Euclidean space defined by three non-collinear points
two triangles sharing a common vertex in Euclidean plane
two points lying on the same side of a line in Euclidean plane
two points lying on opposite sides of a line in Euclidean plane
...

ProofNet-Hard: Extracted Concepts

square root function mapping nonnegative real numbers to nonnegative real numbers
a sequence of real numbers being convergent
limit function mapping a convergent sequence of real numbers to a real number
topology on a set X
family of topologies on a fixed set X
intersection of a family of topologies on X being a topology on X
union of a family of topologies on X not necessarily being a topology on X
one topology on X being finer than another topology on X (inclusion of collections)
two topologies on the same set being comparable under inclusion
two topologies on the same set being not comparable under inclusion
basis for a topology on a set
topology on \mathbb{R} generated by the basis of intervals with rational endpoints
...

Figure 2: Example result from Step 1 Concept Extraction.

To illustrate the types of concepts extracted, we examine representative examples from both benchmarks shown in Figure 2. For LeanEuclidPlus, the extracted concepts include objects such as non-degenerate triangles in 2-dimensional Euclidean space, relations such as two distinct points being on a line, and various geometric functions. For ProofNetHard, which addresses more advanced mathematical domains, the extracted concepts span a broader range: objects include topologies on a set X , families of topologies on a set X , and bases for topologies on a set X ; relations include predicates such as a sequence of real numbers being convergent and the intersection of a family of topologies on X being a topology on X ; functions include the square root function mapping nonnegative real numbers to nonnegative real numbers and the limit function mapping a convergent sequence of real numbers to a real number.

We evaluate the quality of this extraction step using two metrics reported in Table 1, both averaged across 5 runs. The correctness metric measures what percentage of the extracted concepts are indeed non-ambiguous, well-defined, and abstract out of all extracted concepts. The recall metric quantitatively analyzes how many concepts from the oracle abstractions written by human experts are successfully identified and extracted by our system. Both metrics are evaluated manually because the output of this step consists of freeform natural language descriptions, and no reliable automated

864
865
866
867
868
869
870
871
872
873
874
875
876
877
878
879
880
881
882
883
884
885
886
887
888
889
890
891
892
893
894
895
896
897
898
899
900
901
902
903
904
905
906
907
908
909
910
911
912
913
914
915
916
917

LeanEuclidPlus: Filtered Out Concepts

a point lying between two other points on a line segment in Euclidean plane
two points lying on the same side of a line in Euclidean plane
two points lying on opposite sides of a line in Euclidean plane
the measure of an angle in Euclidean plane mapping an angle to a real number in degrees
a convex quadrilateral in 2-dimensional Euclidean space being a parallelogram

LeanEuclidPlus: Kept Concepts

two distinct lines intersecting at a point in 2-dimensional Euclidean space
three or more distinct points being sequentially aligned on a line in Euclidean plane
two sets of points lying on opposing sides of a line in Euclidean plane
two angles in Euclidean plane being supplementary
two lines being parallel in 2-dimensional Euclidean space
two angles in Euclidean plane being congruent
two triangles sharing a common vertex in Euclidean plane
an angle at a vertex of a triangle in Euclidean plane
...

ProofNet-Hard: Filtered Out Concepts

square root function mapping nonnegative real numbers to nonnegative real numbers
a sequence of real numbers being convergent
limit function mapping a convergent sequence of real numbers to a real number
family of topologies on a fixed set X
intersection of a family of topologies on X being a topology on X
union of a family of topologies on X not necessarily being a topology on X
basis for a topology on a set X
connected topological space (no separation into two disjoint nonempty open sets)
Stone-Ćech compactification βX of a completely regular space X
...

ProofNet-Hard: Kept Concepts

sequence of real numbers defined by $a_i = \sqrt{i+1} - \sqrt{i}$ for i in the natural numbers
subbasis for a topology on a set X
lower limit topology \mathbb{R}_l on the real line (topology with basis of half-open intervals $[a, b)$)
 K -topology \mathbb{R}_K on the real line
topology on \mathbb{R} generated by the basis of intervals with rational endpoints
sequence of functions $f_n: [0, 1] \rightarrow \mathbb{R}$ defined by $f_n(x) = x^n$
pointwise limit function $g: [0, 1] \rightarrow \mathbb{R}$ with $g(x) = 0$ for $0 \leq x < 1$ and $g(1) = 1$
limit point compactness (every infinite subset of X has a limit point in X)
uniform topology on $[0, 1]^\omega$ induced by the supremum metric
...

Figure 3: Example result from Step 2 Concept Filtering.

evaluation alternative exists for assessing the semantic correctness and completeness of concept extraction at this level.

Step 2: Concept Filtering. The second step of our abstraction learning pipeline performs concept filtering to remove duplicates and concepts already formalized in the target language, as specified by the guidelines in Appendix D.1.2. The filtering behavior and results vary significantly between our two benchmarks due to differences in the richness of their underlying formal libraries.

For LeanEuclidPlus, shown in Figure 3, the minimal nature of the target DSL results in relatively few concepts being filtered out. Representative examples include the concept of a point lying between two other points on a line segment in Euclidean space, which is filtered because it corresponds to the primitive relation `between` axiomatized in the target DSL. Similarly, the concept of two points lying on the same side of a line is filtered because it corresponds to the primitive relation

`Point.sameSide`. The relation describing two points being on opposite sides of a line is also filtered because it already exists in the target DSL as an abbreviation expressed as $\neg a.onLine\ l \wedge \neg b.onLine\ l \wedge \neg sameSide\ a\ b\ l$.

In contrast, `ProofNetHard` exhibits substantially more filtering activity due to the extensive coverage of Lean Mathlib, as illustrated in Figure 3. Concepts such as a sequence of real numbers being convergent are filtered because they have direct formal correspondents in Mathlib, specifically the filter `limit Filter.Tendsto f atTop (nhds l)` defined in `Mathlib/Topology/Instances/Real`. The concept of a family of topologies on a set X is filtered because it is already represented as `Set (TopologicalSpace X)` or as an indexed family $\iota \rightarrow TopologicalSpace\ X$ with complete lattice operations in `Mathlib/Topology/Basic`. Even sophisticated mathematical concepts such as the Stone-Cech compactification of a completely regular space X are filtered because Mathlib provides comprehensive support through `StoneCech X` along with the unit map `stoneCechUnit` and extension operation `stoneCechExtend` in `Mathlib/Topology/StoneCech`.

We evaluate this filtering step using the correctness (precision) metric reported in Table 1 for the filtering step, which measures what percentage of filtered-out concepts are indeed duplicates or already-formalized out of all filtered-out concepts, averaged across 5 runs. We deliberately focus on precision rather than recall for two important reasons. First, in the abstraction learning phase, our priority is to formalize reusable abstractions without missing potentially useful concepts, making filtering precision naturally more important than recall. Second, borderline cases exist where a natural language concept does not directly correspond to exactly one formal definition but can be easily formalized by slightly adapting an existing definition. For instance, the concept of two triangles sharing a common vertex in Euclidean space has no direct corresponding formal definition, but it can be straightforwardly expressed by specifying the three vertices of each triangle. We design the filtering step to remove only those cases that are absolutely obvious duplicates or already-formalized concepts, erring on the side of caution to preserve potentially useful abstractions. The correctness metric is evaluated manually for the same reason as Step 1, since the output consists of freeform natural language and no reliable automated evaluation alternative exists.

Step 3: Concept Dependency Graph (CDG) Construction. The third step of our abstraction learning pipeline constructs concept dependency graphs that analyze how each extracted concept can be formalized using existing formal definitions and other concepts, following the guidelines in Appendix D.1.3. The depth and complexity of these dependency graphs vary across benchmarks and concepts.

For `LeanEuclidPlus`, Figure 4 presents an example of a depth-3 CDG where the target concept to be formalized depends on both already-available formal definitions in the target language, such as `Triangle.ofPoints` and `Line.intersectsLine`, as well as other concepts that do not yet have single direct formal correspondents and must themselves be formalized first. An illustrative example of the nuanced dependency analysis performed in this step involves the concept of a point lying on a line segment including endpoints in the Euclidean plane. This concept is superficially similar to the previously filtered concept of a point lying between two other points on a line segment in Euclidean space, which was removed because it has a direct formal correspondent in the primitive relation `between`. However, these concepts are semantically distinct: the former permits the point to be an endpoint of the line segment, while the latter does not. Consequently, the former concept is retained and can be formalized using `between`, logical disjunction, and equality. The dependency analysis also correctly identifies that the concept of the line through two distinct points in Euclidean plane is directly expressible using `Point.onLine`, inequality, and existential quantification. The concept dependency for a line intersecting two sides of a triangle in Euclidean plane is similarly constructed correctly with appropriate dependencies identified.

For `ProofNetHard`, Figure 4 shows a simpler depth-2 CDG. The example concept of the lower limit topology on the real line, which is the topology with a basis of half-open intervals $[a, b)$, demonstrates correct dependency construction. The analysis correctly identifies that this concept is directly expressible using the following formal correspondents: `Set (Set \mathbb{R})`, `Ico : $\mathbb{R} \rightarrow \mathbb{R} \rightarrow Set\ \mathbb{R}$` representing half-open intervals $[a, b)$, `TopologicalSpace.GenerateFrom : Set (Set \mathbb{R}) $\rightarrow TopologicalSpace\ \mathbb{R}$` , existential quantification, conjunction, and equality to describe the generating family. This concept dependency is correctly constructed and

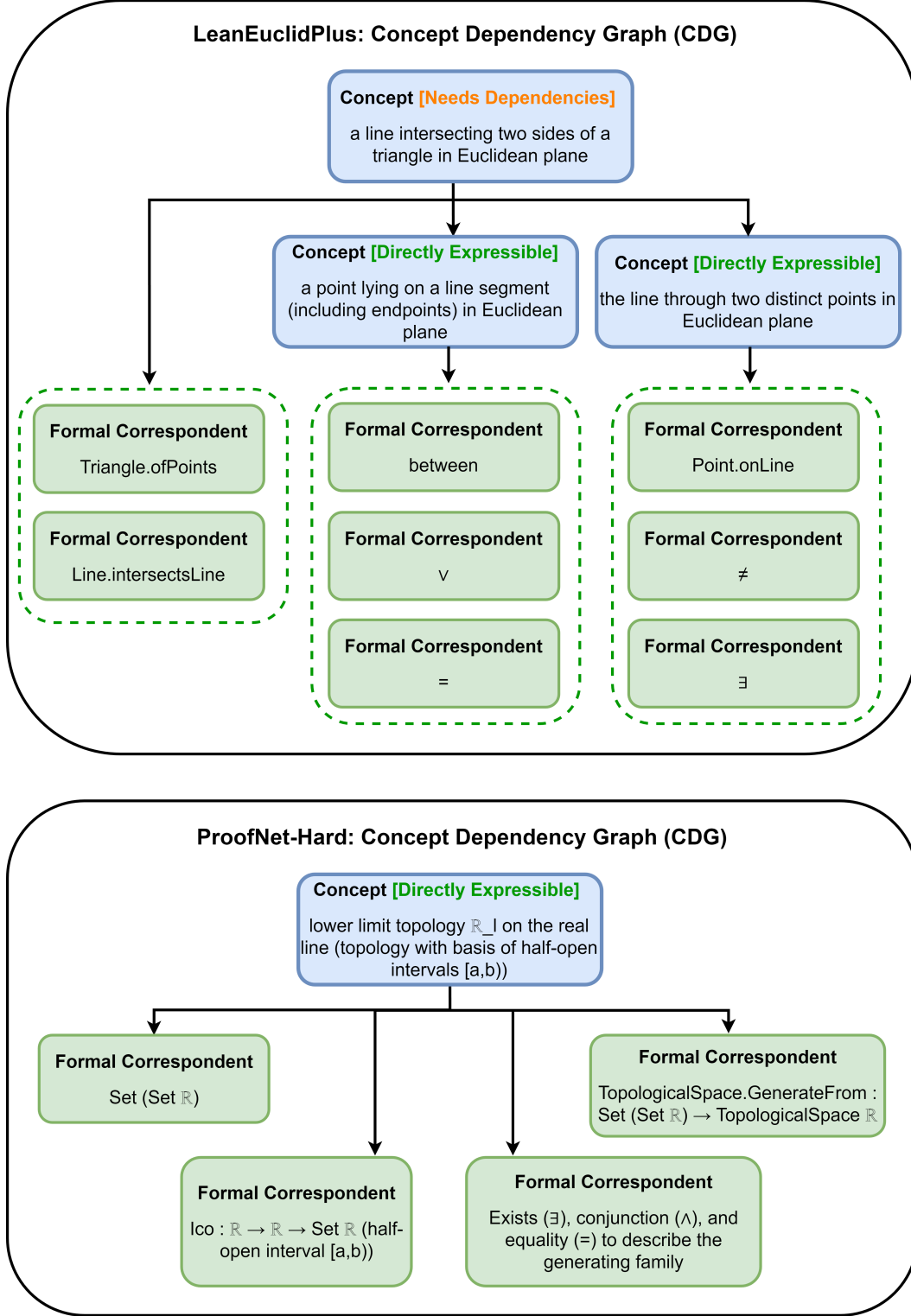


Figure 4: Example result from Step 3 CDG Construction.

precisely matches the oracle definition of the lower limit topology written by human experts, validating the accuracy of our dependency analysis.

We evaluate this CDG construction step using the correctness metric reported in Table 2, which measures what percentage of the dependency analyses for each concept are correctly constructed out of all dependency analyses, averaged across 5 runs. The correctness metric is evaluated manually for

1026
1027
1028
1029
1030
1031
1032
1033
1034
1035
1036
1037
1038
1039
1040
1041
1042
1043
1044
1045
1046
1047
1048
1049
1050
1051
1052
1053
1054
1055
1056
1057
1058
1059
1060
1061
1062
1063
1064
1065
1066
1067
1068
1069
1070
1071
1072
1073
1074
1075
1076
1077
1078
1079

LeanEuclidPlus: Formalized Concepts

```
@[simp]
def distinctFromList (x : Point) (ys : List Point) : Prop :=
  match ys with
  | [] => True
  | y :: ys => x ≠ y ∧ distinctFromList x ys

@[simp]
def pairwiseDistinct (pts : List Point) : Prop :=
  match pts with
  | [] => True
  | x :: xs => distinctFromList x xs ∧ pairwiseDistinct xs

@[simp]
def allOnLine (pts : List Point) (L : Line) : Prop :=
  match pts with
  | [] => True
  | x :: xs => x.onLine L ∧ allOnLine xs L

@[simp]
def betweennessChain (pts : List Point) : Prop :=
  match pts with
  | a :: b :: c :: rest => between a b c ∧ betweennessChain (b :: c :: rest)
  | _ => True

@[simp]
def sequentiallyAlignedOnLine (pts : List Point) (L : Line) : Prop :=
  allOnLine pts L ∧ pairwiseDistinct pts ∧ betweennessChain pts
```

...

ProofNet-Hard: Formalized Concepts

```
@[simp]
def TopologicalSpace.comparable {α : Type*} (t u : TopologicalSpace α) : Prop :=
  t ≤ u ∨ u ≤ t

@[simp]
def TopologicalSpace.notComparable {α : Type*} (t u : TopologicalSpace α) : Prop :=
  ¬ TopologicalSpace.comparable t u

@[simp]
def IsSubbasisFor {X : Type*} (S : Set (Set X)) (t : TopologicalSpace X) : Prop :=
  t = TopologicalSpace.generateFrom S

@[simp]
def lowerLimitGeneratingSets : Set (Set ℝ) :=
  {U | ∃ a b : ℝ, a < b ∧ U = Set.lco a b}

@[simp]
def realLowerLimitTopology : TopologicalSpace ℝ :=
  TopologicalSpace.generateFrom lowerLimitGeneratingSets
```

...

Figure 5: Example result from Step 4 Concept Formalization.

LeanEuclidPlus: Refactored Formalization

```

@[simp]
def betweennessChain (pts : List Point) : Prop :=
  match pts with
  | a :: b :: c :: rest => between a b c ∧ betweennessChain (b :: c :: rest)
  | _                    => True

@[simp]
def sequentiallyAligned (pts : List Point) : Prop :=
  betweennessChain pts
...

```

ProofNet-Hard: Refactored Formalization

```

@[simp]
def TopologiesComparable {α : Type*} (t u : TopologicalSpace α) : Prop :=
  t ≤ u ∨ u ≤ t

@[simp]
def IsSubbasisFor {X : Type*} (S : Set (Set X)) (t : TopologicalSpace X) : Prop :=
  t = TopologicalSpace.generateFrom S

@[simp]
def LowerLimitSubbasis : Set (Set ℝ) :=
  {U | ∃ a b : ℝ, a < b ∧ U = Set.Ico a b}

@[simp]
def SorgenfreyTopology : TopologicalSpace ℝ :=
  TopologicalSpace.generateFrom LowerLimitSubbasis
...

```

Figure 6: Example result from Step 5 Formalization Refactoring.

the same reason as Steps 1 and 2, since the output of this step consists of freeform natural language descriptions of dependencies and formal correspondents.

Step 4: Concept Formalization. The fourth step of our abstraction learning pipeline translates the concepts identified in the CDG into actual formal definitions in the target language, following the guidelines in Appendix D.1.4. This step involves generating both the main target definitions and any necessary helper definitions to support them.

For LeanEuclidPlus, Figure 5 illustrates the formalization of the concept of three or more distinct points being sequentially aligned on a line in Euclidean plane, along with several helper definitions. The natural language description of this concept encompasses multiple sub-concepts: mutually distinct points, collinear points, and points aligned in sequential order. The language model correctly decomposes the formalization task by first formalizing each of these sub-concepts as helper definitions, then composing them to formalize the target concept. While this approach is correct and faithful to the original natural language concept, it introduces some redundancy that will be addressed in the subsequent refactoring phase.

For ProofNetHard, Figure 5 demonstrates the formalization of concepts with their helper definition dependencies. The formalization of `TopologicalSpace.notComparable` depends on and utilizes `TopologicalSpace.Comparable` as a helper definition. Similarly, `realLowerLimitTopology` depends on and uses `lowerLimitGeneratingSets` as a helper definition. Both the main concepts and their helper definitions are correctly formalized, demonstrating the system’s ability to manage dependencies during formalization.

We evaluate this formalization step using the correctness metric reported in Table 2, which measures what percentage of formal definitions are correct out of all formal definitions, with helper definitions counted separately (essentially counting the number of function definitions), averaged across 5 runs. Importantly, concepts do not need to be formalized by strictly following the previously analyzed concept dependency graph; a formalization is considered correct as long as it can be interpreted as faithful to the original natural language concept. The correctness metric is evaluated manually despite the output being in formal language because the same natural language concept can be formalized in multiple ways that are all acceptable to human experts. These alternative formalizations may be non-definitionally equivalent or may not be recognized as equivalent by any existing automated checker. As discussed in Section 3, no symbolic equivalence checker is complete, and some are not even sound, necessitating manual evaluation by human experts to assess faithfulness.

Step 5: Formalization Refactoring. The fifth step of our abstraction learning pipeline performs formalization refactoring to eliminate redundant definitions and extract common helper definitions, thereby creating a more concise and general library of learned abstractions that downstream users, whether human or language models, can better understand and employ. This refactoring process follows the guidelines specified in Appendix D.1.5.

For LeanEuclidPlus, Figure 6 demonstrates significant refactoring opportunities. As shown in the initial formalization in Figure 5, the concept was implemented using multiple helper definitions for mutually distinct points, collinear points, and sequential alignment. However, careful analysis reveals that the chain of betweenness relations already guarantees that the list of points must lie on the same line and must be mutually distinct by virtue of the `between` relation’s semantics. Consequently, the two separate helper definitions for collinearity and mutual distinctness can be eliminated as redundant. The refactored version shown in Figure 6 consolidates these concepts by retaining only the essential `betweennessChain` definition, with `sequentiallyAligned` serving as a more intuitive alias for the same concept.

For ProofNetHard, Figure 6 shows more conservative refactoring activity. The definition `TopologicalSpace.notComparable` is removed because it is simply the logical negation concatenated with `TopologicalSpace.Comparable`, making a separate API unnecessary and potentially confusing. The refactoring also includes two renamings that improve domain specificity and clarity: `LowerLimitGeneratingSets` is renamed to `LowerLimitSubbasis` to better reflect its mathematical role, and `realLowerLimitTopology` is renamed to `sorgenfreyTopology` using the conventional mathematical terminology. While these refactorings are not strictly necessary for correctness, they enhance readability and align with domain conventions, and we therefore consider them correct refactorings.

We evaluate this refactoring step using two metrics reported in Table 3. The correctness metric measures what percentage of refactored formal definitions are correct out of all definitions that were refactored, with helper definitions counted separately (essentially counting the number of function definitions), averaged across 5 runs. Additionally, we quantitatively analyze the compression ratio, which is the ratio of the number of formal definitions before refactoring to the number after refactoring, averaged across 5 runs. This compression ratio metric is important because it indicates how effectively common helper functions are reused and how many unnecessary formal definitions are eliminated. A high compression ratio facilitates better understanding and utilization of the learned abstractions by both humans and language models, as it reduces cognitive load and clarifies the essential conceptual structure. The correctness metric is evaluated manually for the same reason as Step 4, since multiple semantically equivalent refactorings may be acceptable to human experts.

Step 6: Documentation Update. The sixth and final step of our abstraction learning pipeline generates clear and concise natural language documentation for the learned abstractions, abstracted away from low-level implementation details such as internal helper definitions, to serve both human and language model users. This step is critical because excessive definitions in the documentation would inevitably confuse both language models as the context grows and human users attempting to understand and use the learned abstractions. The documentation generation process follows the guidelines specified in Appendix D.1.6.

For LeanEuclidPlus, Figure 7 illustrates the selective documentation approach employed by this step. The internal helper definition `betweennessChain` is not directly exposed in the documentation; instead, its more intuitively-named alias `sequentiallyAligned` is pre-

LeanEuclidPlus: Updated Documentation

```

def twoDistinctLinesIntersectAtPoint (L M : Line) (i : Point) --
two distinct lines L and M intersect at point i, i.e.  $L \neq M$ ,
i lies on both L and M, and  $L.intersectsLine\ M$ .
The syntax is `twoDistinctLinesIntersectAtPoint L M i`.

def sequentiallyAligned (pts : List Point) --
a list of points is sequentially aligned (ordered collinear chain) iff every consecutive
triple  $(p_i, p_{i+1}, p_{i+2})$  satisfies `between pi pi+1 pi+2`.
No extra line or global distinctness is required beyond betweenness.
The syntax is `sequentiallyAligned [p0, p1, p2, ...]`.

def sequentiallyAlignedThreeOrMore (pts : List Point) --
like `sequentiallyAligned`, additionally requiring the list to contain at least three points.
The syntax is `sequentiallyAlignedThreeOrMore [p0, p1, p2, ...]`.

def twoSetsOpposingSidesOnLine (xs ys : List Point) (L : Line) --
two lists of points lie on opposite sides of line L: (1) no point from either list is on L;
(2) every cross-pair  $(x \in xs, y \in ys)$  is on opposing sides of L.
The syntax is `twoSetsOpposingSidesOnLine [x1, x2, ...] [y1, y2, ...] L`.

def supplementaryAngles (a b c d e f : Point) --
angles  $\angle a:b:c$  and  $\angle d:e:f$  are supplementary iff  $(\angle a:b:c) + (\angle d:e:f) = \angle L + \angle L$ .
The syntax is `supplementaryAngles a b c d e f`.

...

```

Figure 7: Example result from Step 6 Documentation Update.

sented to users. The natural language description correctly captures the formal semantics of `sequentiallyAligned`, attending to subtle but important details such as the fact that the points are mutually distinct. This demonstrates the system’s ability to abstract away implementation details while preserving semantic precision in user-facing documentation.

For ProofNetHard, we adopt a different approach due to the scale of the original Lean Mathlib documentation, which exceeds far beyond the context limit of any current LLMs. Instead, we skip the documentation update step and directly use the compact refactored code (comprising just 21 formal definitions) as documentation for downstream tasks in the decomposition phase.

We evaluate the documentation update step using quantitative metrics reported in Table 3. The correctness metric measures what percentage of documented formal definitions correctly convey the meaning of the definition without missing any essential details or introducing ambiguity, out of all documented definitions, averaged across 5 runs. To quantitatively validate the quality of the generated documentation, we also evaluate downstream task performance on LeanEuclidPlus using two strong models, Qwen3-235B and GPT-5, with our four-stage hierarchical decomposition pipeline. Under the hypothesis that good documentation should enable strong models to generate syntactically correct formalizations with high reliability, we report the compilation rate of the two models averaged across 5 runs. To further demonstrate the stability and robustness of the documentation produced in Step 6, we also report detailed statistics across runs: for Qwen3-235B, the maximum compilation rate is 98.2%, minimum is 96.8%, mean is 97.4%, and standard deviation is 0.47%; for GPT-5, the maximum is 100.0%, minimum is 99.0%, mean is 99.8%, and standard deviation is 0.45%. These consistently high compilation rates with low variance validate the quality and stability of the generated documentation. The correctness metric is evaluated manually for the same reason as Steps 1, 2, and 3, since the output consists of freeform natural language descriptions. The downstream task performance metrics are evaluated automatically using the same evaluation pipeline as the main experiments on LeanEuclidPlus.

C.2 COMPARISON BETWEEN LEARNED AND ORACLE ABSTRACTIONS

We present a qualitative comparison between the abstractions learned by our system and the oracle abstractions hand-crafted by human experts. Figure 8 illustrates representative examples from LeanEuclidPlus, and Figure 9 presents corresponding examples from ProofNet-Hard. Our analysis reveals three distinct categories of learned abstractions.

Correct Overlapping Abstractions. The first category comprises abstractions where both the learned and oracle definitions capture semantically equivalent concepts, albeit with different naming conventions or slightly different formulations.

For LeanEuclidPlus, the learned abstraction `betweennessChain` and its alias `sequentiallyAligned` correspond directly to the oracle’s `sequentiallyAlignedList`, both encoding the property that consecutive points in a list satisfy the betweenness relation. Similarly, `twoDistinctLinesIntersectAtPoint` mirrors the oracle’s `twoLinesIntersectAtPoint`, and the learned `parallel` and `perpendicularAt` definitions align precisely with their oracle counterparts. The learned `trianglesSimilar` captures the same geometric concept as the oracle’s `similar` abbreviation, encompassing both angle equality and proportional side lengths.

For ProofNet-Hard, the learned `sqrtSuccDiff` exactly matches the oracle’s function `g`, both computing the difference between consecutive square roots. The learned `LowerLimitSubbasis`, `sorgenfreyTopology`, `KNatRecip`, `KTopologySubbasis`, and `kTopology` correspond semantically to the oracle’s `lower_limit_topology`, `K`, and `K_topology`.

These examples demonstrate that our abstraction learning phase is able to recover most of the common mathematical concepts identified by human experts, validating the effectiveness of our automated approach for discovering domain-specific abstractions without manual intervention.

Correct Non-Overlapping Abstractions. The second category consists of abstractions that are correct formalizations of the natural language concepts but do not overlap with the oracle.

For LeanEuclidPlus, the learned abstractions include `noneOnLine`, which recursively verifies that no point in a list lies on a given line, and `allOpposingSidesToPoint`, which checks whether all points in a list are on opposite sides of a line relative to a reference point. The learned `pairwiseAcrossOpposing` and `twoSetsOpposingSidesOnLine` provide compositional predicates for reasoning about point configurations across lines, while `lineCutsTriangleOnABandACat` encapsulates the specific geometric configuration where a line intersects two sides of a triangle. In contrast, the oracle for LeanEuclidPlus contains different abstractions such as `pointDistinctFromList`, `mutuallyDistinctPointsList`, `sameSideList`, and `sameSideDistinctList`, which focus on distinctness and same-side predicates over point lists.

For ProofNet-Hard, the learned abstractions include `StrictlyBoundedAboveBy`, a generic predicate for sequences bounded by a constant, `TopologiesComparable` for comparing topologies, and `NestedClosedNonempty` along with `NestedClosedNonemptyInterNonempty` for reasoning about nested sequences of closed sets. The oracle involves different abstractions such as `is_topology`, a first-principles definition of topological structure, and `countably_compact`, which directly encodes the definition of countable compactness.

These non-overlapping abstractions illustrate that concept extraction by humans and LLMs from the same natural language corpus can be complementary: each identifies valid abstractions that the other may overlook, suggesting that combining both approaches could yield richer and more comprehensive libraries of reusable abstractions.

Correct Learned Abstractions, Wrong Oracle. The third category, observed exclusively in ProofNet-Hard, consists of cases where the learned abstractions are correct while the oracle abstractions contain semantic errors. The learned `IsLimitPoint` correctly defines a limit point (accumulation point) of a set A as a point x for which every neighborhood intersects $A \setminus \{x\}$, and `LimitPointCompact` correctly states that every infinite subset has such a limit point. However,

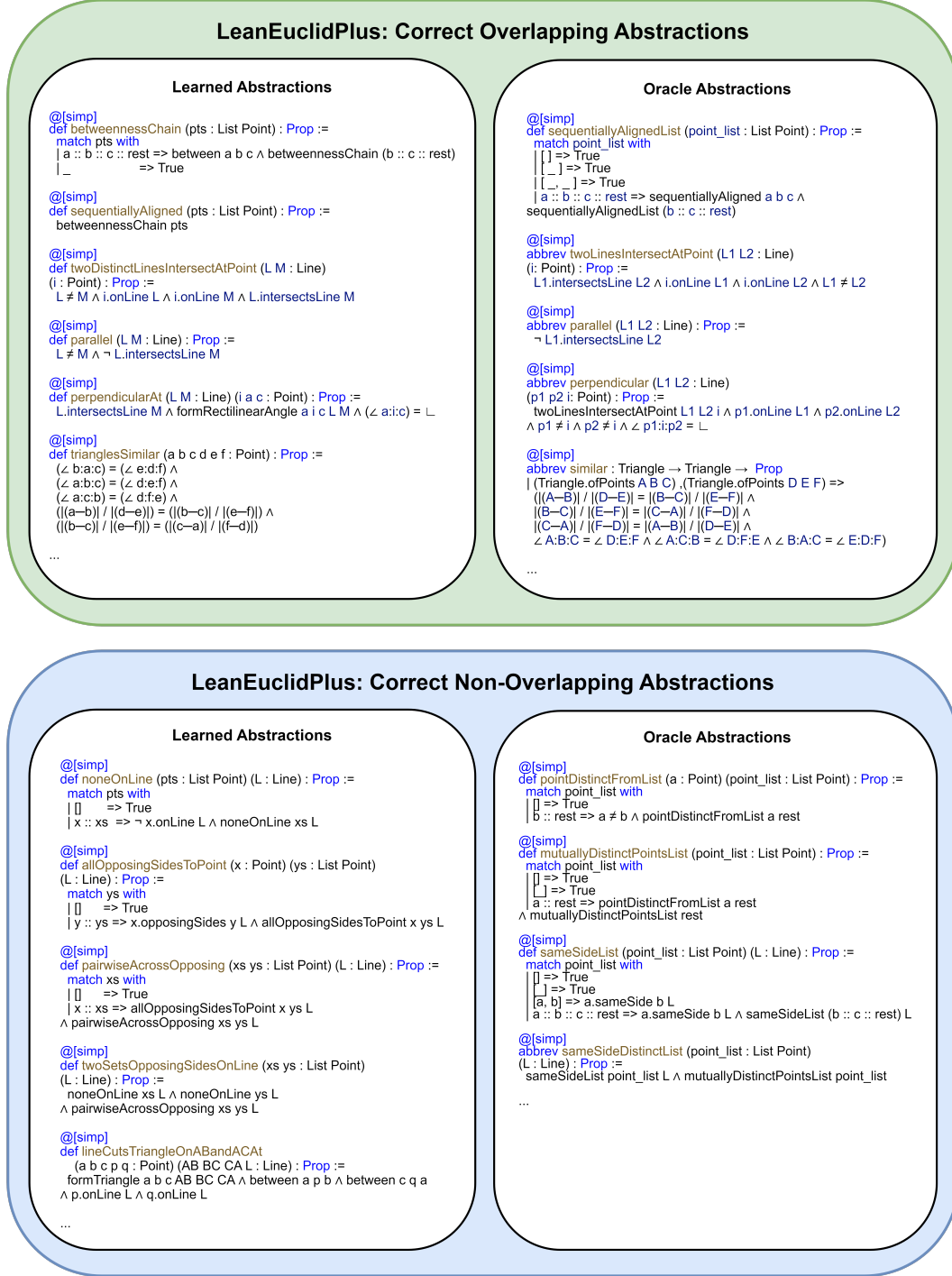


Figure 8: Comparison between Learned and Oracle Abstractions for LeanEuclidPlus.

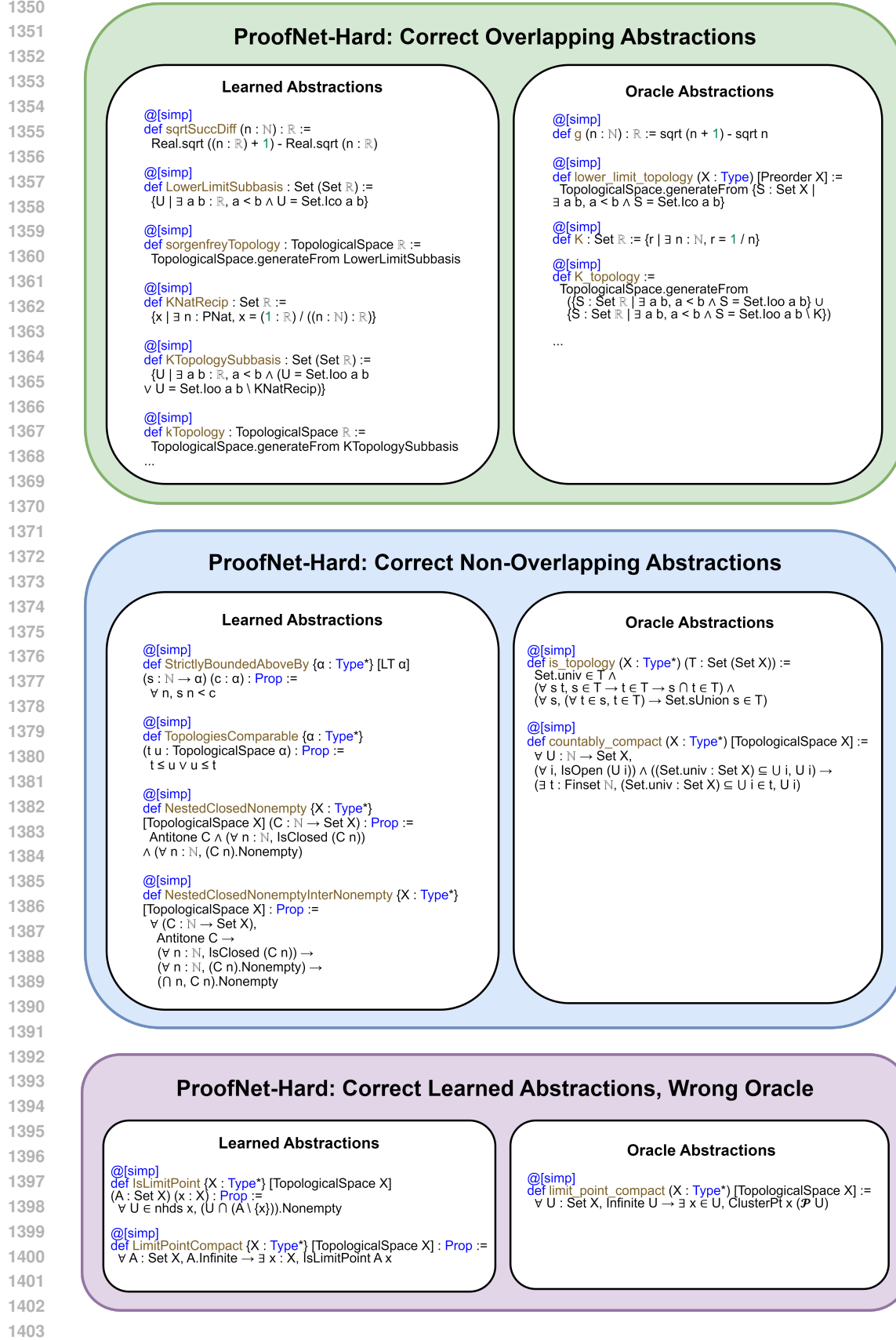


Figure 9: Comparison between Learned and Oracle Abstractions for ProofNet-Hard.

the oracle’s `limit_point_compact` incorrectly uses `ClusterPt x (P U)`, conflating the notions of cluster point and accumulation point.

In Mathlib, `ClusterPt x F` is defined as `NeBot (N x Π F)`, which for a principal filter $\mathcal{P} U$ reduces to checking whether $x \in \overline{U}$. Crucially, since any point $x \in U$ trivially satisfies `ClusterPt x (P U)` (every neighborhood of x contains x itself and hence meets U), the oracle’s definition degenerates to merely asserting that U is nonempty—a property that holds vacuously for any infinite set. This renders the oracle definition trivially true in every topological space, including infinite discrete spaces where the correct limit-point compactness property provably fails.

This example demonstrates that automated abstraction learning by LLMs can sometimes produce more reliable results than manually constructed oracles, particularly in domains where the formalizations contain subtle semantic distinctions that may escape human review.

C.3 HIERARCHICAL DECOMPOSITION

We present a qualitative analysis demonstrating the effectiveness of our hierarchical decomposition approach (Section 2.2) for autoformalization. Using Qwen3-235B-A22B-Instruct-2507 with learned abstractions on LeanEuclidPlus, we compare direct formalization (Abstract column in Table 5) against our four-stage decomposition pipeline (DNA column in Table 5) on a representative problem.

Informal Statement. The English statement describes two triangles TUV and VSW sharing vertex W, with lines SU and TV intersecting at W. Given that W lies between S and U, and between T and V, along with the premises $\angle TUV = \angle UTW$ and $SV \parallel TU$, the goal is to prove $\angle V = \angle S$.

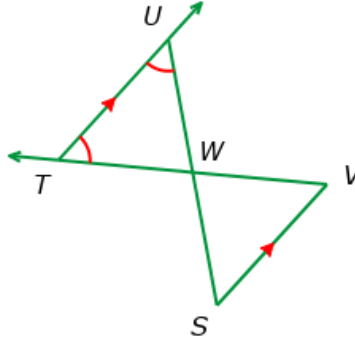


Figure 10: Diagram for Parallel-17 in LeanEuclidPlus.

The following is the complete reasoning trace produced by Qwen3-235B when formalizing directly without hierarchical decomposition:

```
We are given an English statement involving two triangles, TUV and VSW, sharing a point W. The lines SU and TV
intersect at W, and W lies between S and U, and also between T and V. We are given that \angle TUV =
\angle UTW (i.e., two angles in triangle TUV are congruent), and that SV is parallel to TU. We are to
prove that \angle V = \angle S.
```

```
Let's break this down step by step.
```

```
---
```

```
### Step 1: Identify all points, lines, and geometric entities
```

```
Points mentioned:
```

- T, U, W -- form triangle TUV
- V, S, W -- form triangle VSW
- W is the shared point
- Lines: SU and TV intersect at W
- W lies between S and U -> between S W U
- W lies between T and V -> between T W V

```
Lines to define:
```

- For triangle TUV: sides TU, UW, WT
- For triangle VSW: sides VS, SW, WV
- Line SU: contains S and U
- Line TV: contains T and V

1458

1459 We are given:

1460 1. $\angle TUV = \angle UTW$

1461 - $\angle TUV$ is the angle at U in triangle TUV: $\angle T:U:W$

1462 - $\angle UTW$ is the angle at T in triangle TUV: $\angle U:T:W$

1463 - So: $(\angle T:U:W) = (\angle U:T:W)$

1464 2. $SV \parallel TU$

1465 - SV is the line through S and V

1466 - TU is the line through T and U

1467 - So: parallel (line SV) (line TU)

1468 3. We are to prove: $\angle V = \angle S$

1469 - $\angle V$ likely means the angle at V in triangle VSW: $\angle S:V:W$ or $\angle W:V:S$?

1470 - $\angle S$ likely means the angle at S in triangle VSW: $\angle V:S:W$ or $\angle W:S:V$?

1471 But note: triangle VSW has vertices V, S, W. So:

1472 - $\angle V$ in triangle VSW is $\angle S:V:W$

1473 - $\angle S$ in triangle VSW is $\angle V:S:W$

1474 So we are to prove: $(\angle S:V:W) = (\angle V:S:W)$

1475 But wait -- is that necessarily true? Let's see.

1476 Alternatively, could $\angle V$ and $\angle S$ refer to angles in different triangles? But the only triangles

1477 are TUV and VSW.

1478 Given that $SV \parallel TU$, and lines SU and TV intersect at W, this looks like a configuration where two lines cross

1479 at W, and we have a pair of parallel lines cutting across them -- possibly forming similar triangles or

1480 using properties of parallel lines and transversals.

1481 But the conclusion is $\angle V = \angle S$ -- i.e., two angles at V and S, likely in triangle VSW.

1482 Wait -- if $SV \parallel TU$, and TV and SU are transversals intersecting at W, then we might use alternate interior

1483 angles or corresponding angles.

1484 But $\angle V$ and $\angle S$ are both in triangle VSW -- if $\angle S:V:W = \angle V:S:W$, that would mean

1485 triangle VSW is isosceles at W.

1486 But let's reexamine.

1487 Alternatively, could $\angle V$ mean $\angle WVS$ and $\angle S$ mean $\angle WSV$? Yes -- that's the same as

1488 above.

1489 So the conclusion is: in triangle VSW, \angle at V equals \angle at S \rightarrow triangle VSW is isosceles with $|$

1490 $SW| = |VW|$?

1491 But the conclusion is just $\angle V = \angle S$, i.e., angle equality.

1492 So we are to prove: $(\angle S:V:W) = (\angle V:S:W)$

1493 But we are given:

1494 - $\angle T:U:W = \angle U:T:W \rightarrow$ triangle TUV is isosceles at W? Let's see:

1495 - \angle at U: $\angle T:U:W$

1496 - \angle at T: $\angle U:T:W$

1497 - If these are equal, then in triangle TUV, angles at T and U are equal \rightarrow so sides opposite are equal $\rightarrow |$

1498 $TW| = |UW| \rightarrow$ isosceles at W.

1499 Also given: $SV \parallel TU$

1500 Lines SU and TV intersect at W, with W between S and U, and between T and V.

1501 So the configuration is: two lines (SU and TV) intersect at W, forming an "X", and two segments: TU and SV are

1502 parallel, with T,U on one side, S,V on the other.

1503 This is a classic setup for similar triangles or using parallel line angle properties.

1504 Now, since $SV \parallel TU$, and TV is a transversal (from T to V, passing through W), then:

1505 - $\angle TUV$ and $\angle VSW$ might be alternate interior angles? Let's see.

1506 Wait -- transversal TV crosses parallel lines TU and SV.

1507 But TU and SV are the parallel lines.

1508 Transversal TV: intersects TU at T, and SV at V.

1509 So at point T: angle between TU and TV $\rightarrow \angle U:T:W$ (since W is on TV beyond T? Wait -- W is between T and

1510 V, so from T to V: T-W-V.

1511 So line TV: T -- W -- V

Similarly, SU: S -- W -- U

So TU is from T to U, SV is from S to V.

Given $SV \parallel TU$.

Now, transversal TV crosses TU at T and SV at V.

So the angle at T between TU and TV (i.e., \angle between TU and TW) is $\angle U:T:W$

At V, the angle between SV and TV (i.e., between SV and VW) -- but direction matters.

Since $SV \parallel TU$, and TV is a transversal, then:

1512 - \angle UTW (at T) and \angle SVW (at V) are corresponding angles? Or alternate?
 1513 Let's define directions.
 1514 Assume standard configuration: lines TU and SV are parallel, transversal TV goes from T to V, crossing both.
 1515 Then \angle between TU and TV at T, and \angle between SV and TV at V -- if both are on the same side,
 1516 they are corresponding.
 1517 But we need to relate to the angles.
 1518 Alternatively, transversal SU: from S to U, crosses SV at S and TU at U.
 1519 So at U: angle between TU and SU $\rightarrow \angle$ T:U:W (since W is on SU beyond U? No -- W is between S and U, so
 1520 from S to U: S -- W -- U
 1521 So at U: line TU and line SU meet at U -- angle \angle T:U:W
 1522 At S: line SV and line SU meet at S -- angle \angle V:S:W
 1523 Since SV \parallel TU, and SU is a transversal, then:
 1524 - \angle T:U:W and \angle V:S:W are corresponding angles \rightarrow so they are equal.
 1525 So: \angle T:U:W = \angle V:S:W
 1526 But we are given: \angle T:U:W = \angle U:T:W
 1527 So: \angle U:T:W = \angle V:S:W
 1528 Now, look at transversal TV: from T to V, crosses TU at T and SV at V.
 1529 At T: angle between TU and TV is \angle U:T:W
 1530 At V: angle between SV and TV -- now, depending on direction.
 1531 Since SV \parallel TU, and TV is transversal, then \angle U:T:W and \angle S:V:W are alternate interior angles? Or
 1532 corresponding?
 1533 Let's suppose the lines are oriented so that TU and SV are horizontal, TU above, SV below, TV going down from
 1534 T to V.
 1535 Then at T: angle between TU (left-right) and TV (down) -- say, lower right angle.
 1536 At V: angle between SV (left-right) and TV (up) -- upper right angle.
 1537 These are alternate interior angles \rightarrow equal.
 1538 So \angle U:T:W = \angle S:V:W
 1539 Therefore: \angle S:V:W = \angle U:T:W = \angle T:U:W = \angle V:S:W
 1540 So \angle S:V:W = \angle V:S:W
 1541 Thus, in triangle VSW, angles at V and S are equal $\rightarrow \angle$ V = \angle S
 1542 So the conclusion is: (\angle S:V:W) = (\angle V:S:W)
 1543 Now we formalize.
 1544 ---
 1545 ### Step 2: Declare all variables
 1546 Points: T, U, W, V, S
 1547 Lines:
 1548 - For triangle TUW: TU, UW, WT
 1549 - For triangle VSW: VS, SW, WV
 1550 - Line SU: contains S and U
 1551 - Line TV: contains T and V
 1552 We need to define lines:
 1553 - TU: line through T and U
 1554 - UW: line through U and W
 1555 - WT: line through W and T
 1556 - VS: line through V and S
 1557 - SW: line through S and W
 1558 - WV: line through W and V
 1559 - SU: line through S and U
 1560 - TV: line through T and V
 1561 But note: SU and TV are already mentioned.
 1562 We also need to express:
 1563 - formTriangle T U W TU UW WT
 1564 - formTriangle V S W VS SW WV
 1565 - twoDistinctLinesIntersectAtPoint SU TV W \rightarrow but SU and TV intersect at W, and are distinct
 - between S W U
 - between T W V
 - \angle T:U:W = \angle U:T:W \rightarrow (\angle T:U:W) = (\angle U:T:W)
 - SV \parallel TU \rightarrow parallel (line through S and V) (line through T and U)
 - Conclusion: (\angle S:V:W) = (\angle V:S:W)

1566 But we must define the lines.
1567
1568 Let's assign:
1569 - (T U W V S : Point)
1570 - (TU UW WT : Line) for triangle TUV
1571 - (VS SW WV : Line) for triangle VSW
1572 - (SU TV : Line) for the intersecting lines
1573
1574 But note: SU must be the line through S and U -- but we also have SW and WU? Not necessarily.
1575
1576 We need to ensure that S, W, U are collinear on SU, and T, W, V are collinear on TV.
1577
1578 But we are given that SU and TV intersect at W, and W is between S and U, and between T and V.
1579
1580 So we can define:
1581 - SU: line through S and U -- and W is on it
1582 - TV: line through T and V -- and W is on it
1583
1584 But in the triangle definitions:
1585 - For triangle TUV: we need lines TU, UW, WT
1586 - But UW is part of SU? Not necessarily -- unless U, W, S are collinear, which they are -- since W is on SU
1587 and between S and U.
1588
1589 So points S, W, U are collinear on line SU.
1590
1591 Similarly, T, W, V are collinear on line TV.
1592
1593 So we can define:
1594 - Line TU: through T and U
1595 - Line UW: but U and W are on SU -> so UW is part of SU? But we need a line for side UW of triangle TUV.
1596
1597 But in `formTriangle`, we need three distinct lines: AB, BC, CA.
1598
1599 So for triangle TUV:
1600 - side TU: line TU
1601 - side UW: line UW -- but U and W are on SU -> so line UW must be the same as line SU? Only if S, U, W are
1602 collinear -- which they are.
1603
1604 But we cannot assume that line UW is the same as SU unless we define it.
1605
1606 But in formalization, we can let:
1607 - line UW be the same as line SU? But SU contains S, W, U -- so yes, line through U and W is the same as line
1608 SU.
1609
1610 But in Lean, lines are extensional -- if two lines have two points in common, they are the same.
1611
1612 But in our formal system, we are assigning labels.
1613
1614 So to avoid confusion, we should define:
1615 - Let line SU be the line containing S and U -- and since W is between S and U, W is on SU.
1616 - Then for triangle TUV, side UW is on line SU -- so we can use SU as the line for UW? But in `formTriangle` T
1617 U W TU UW WT, we need:
1618 - T and U on TU
1619 - U and W on UW
1620 - W and T on WT
1621
1622 So UW must be a line containing U and W -- which SU does.
1623
1624 Similarly, for triangle VSW:
1625 - V and S on VS
1626 - S and W on SW -- again, S and W are on SU? No -- S and W are on SU, but W and S are on SU.
1627
1628 But VS is a different line.
1629
1630 Wait -- triangle VSW:
1631 - V and S on line VS
1632 - S and W on line SW -- but S and W are on SU -> so line SW is part of SU
1633 - W and V on line WV
1634
1635 So line SW can be taken as SU? But only if we are allowed to reuse.
1636
1637 But in the formal system, we can have multiple names for the same line, but in `formTriangle`, we need to
1638 assign specific lines.
1639
1640 So to be safe, we define:
1641 - Line SU: contains S, W, U
1642 - Line TV: contains T, W, V
1643 - Line TU: contains T, U
1644 - Line VS: contains V, S
1645 - Line WT: contains W, T -- but W and T are on TV -> so WT is part of TV
1646 - Line WV: contains W, V -- also on TV
1647
1648 So for triangle TUV:
1649 - TU: line TU
1650 - UW: line SU (since U and W on SU)
1651 - WT: line TV (since W and T on TV)
1652
1653 Similarly, for triangle VSW:
1654 - VS: line VS
1655 - SW: line SU (S and W on SU)
1656 - WV: line TV (W and V on TV)

1620 So we can reuse lines SU and TV in both triangles.
1621
1622 Now, we must ensure that the lines are distinct in each triangle.
1623 For triangle TUV: lines TU, SU, TV -- must be distinct
1624 For triangle VSW: lines VS, SU, TV -- must be distinct
1625 We are not told they are distinct, but in a typical configuration they are.
1626 But the `formTriangle` predicate requires the three lines to be distinct.
1627 So we must assume that.
1628 Similarly, `twoDistinctLinesIntersectAtPoint SU TV W` requires `SU != TV`.
1629 Now, let's write the formal statement.
1630 ---
1631 **### Step 3: List all premises**
1632 We need:
1633
1634 1. `formTriangle T U W TU UW WT`
1635 But UW is SU, WT is TV -> so:
1636 -> `formTriangle T U W TU SU TV`
1637 But the arguments are: a b c AB BC CA
1638 So: T, U, W, TU, SU, TV
1639 Meaning:
1640 - T and U on TU
1641 - U and W on SU
1642 - W and T on TV
1643 And TU, SU, TV distinct.
1644
1645 2. `formTriangle V S W VS SW WV`
1646 But SW is SU, WV is TV -> so:
1647 -> `formTriangle V S W VS SU TV`
1648 Arguments: V, S, W, VS, SU, TV
1649 Meaning:
1650 - V and S on VS
1651 - S and W on SU
1652 - W and V on TV
1653 And VS, SU, TV distinct.
1654
1655 3. `distinctPointsOnLine S U SU` -- to ensure S and U are distinct and on SU
1656 But `formTriangle` already requires `distinctPointsOnLine` for each side?
1657 Looking back: `formTriangle` requires distinct points on each line:
1658 - `distinctPointsOnLine a b AB`
1659 - `distinctPointsOnLine b c BC`
1660 - `distinctPointsOnLine a c CA`
1661
1662 So in `formTriangle T U W TU SU TV`:
1663 - `distinctPointsOnLine T U TU`
1664 - `distinctPointsOnLine U W SU`
1665 - `distinctPointsOnLine W T TV`
1666
1667 Similarly for `V S W VS SU TV`:
1668 - `distinctPointsOnLine V S VS`
1669 - `distinctPointsOnLine S W SU`
1670 - `distinctPointsOnLine W V TV`
1671
1672 So we get distinctness of points on lines.
1673
1674 But we also need that W is between S and U, and between T and V.
1675
1676 So:
1677
1678 4. `between S W U`
1679 5. `between T W V`
1680
1681 6. `twoDistinctLinesIntersectAtPoint SU TV W`
1682 This gives:
1683 - `SU != TV`
1684 - W on SU, W on TV
1685 - SU intersects TV at W
1686
1687 But `formTriangle` already requires W on SU and W on TV?
1688 Yes -- from `distinctPointsOnLine U W SU` and `W T TV`, etc.
1689 But `twoDistinctLinesIntersectAtPoint` also gives that they are distinct and intersect at W -- which we
1690 need.
1691
1692 But is it redundant? Possibly, but we need to ensure `SU != TV`, which might not be implied otherwise.
1693
1694 So include it.
1695
1696 7. `\angle T:U:W = \angle U:T:W`
1697 -> `(\angle T:U:W) = (\angle U:T:W)`
1698
1699 8. `SV || TU`
1700 But SV is the line VS (same line), TU is line TU
1701 -> `parallel VS TU`

1674 Conclusion: $\angle S:V:W = \angle V:S:W$
 1675 $\rightarrow (\angle S:V:W) = (\angle V:S:W)$
 1676 Note: $\angle S:V:W$ is the angle at V in triangle VSW: points S, V, W -- so vertex V, with S and W on the
 1677 sides.
 1678 Similarly, $\angle V:S:W$ is angle at S: points V, S, W -- vertex S.
 1679 So we need to output:
 1680 $\llcorner \forall \text{forall } (T\ U\ W\ V\ S : \text{Point}) (TU\ SU\ TV\ VS : \text{Line}), [\text{premises}] \rightarrow (\angle S:V:W) = (\angle V:S:W) \gg\rangle$
 1681 But wait -- in triangle VSW, we used lines VS, SU, TV.
 1682 But SU is used for SW, TV for WV.
 1683 But we also need to refer to angle at V: $\angle S:V:W$ -- this requires points S, V, W.
 1684 Similarly, angle at S: $\angle V:S:W$ -- points V, S, W.
 1685 We have all points.
 1686 But in the premises, we have formTriangle V S W VS SU TV -- this ensures that:
 1687 - V and S on VS
 1688 - S and W on SU
 1689 - W and V on TV
 1689 So the triangle is well-formed.
 1690 Now, write the full formalization.
 1691 ---
 1692 ### Final Formalized Statement
 1693 Variables:
 1694 - Points: T, U, W, V, S
 1695 - Lines: TU (for side TU), SU (for side UW and SW), TV (for side WT and WV), VS (for side VS)
 1696 We use:
 1697 - formTriangle T U W TU SU TV
 1698 - formTriangle V S W VS SU TV
 1699 - between S W U
 1700 - between T W V
 1701 - twoDistinctLinesIntersectAtPoint SU TV W
 1702 - $(\angle T:U:W) = (\angle U:T:W)$
 1703 - parallel VS TU
 1704 Conclusion: $(\angle S:V:W) = (\angle V:S:W)$
 1705 Note: The line VS is the same as SV, TU is the same as UT -- direction doesn't matter for lines or parallel.
 1706 So:
 1707 $\llcorner \forall \text{forall } (T\ U\ W\ V\ S : \text{Point}) (TU\ SU\ TV\ VS : \text{Line}), \text{formTriangle } T\ U\ W\ TU\ SU\ TV \ \&\& \ \text{formTriangle } V\ S\ W\ VS\ SU\ TV$
 1708 $\ \&\& \ \text{between } S\ W\ U \ \&\& \ \text{between } T\ W\ V \ \&\& \ \text{twoDistinctLinesIntersectAtPoint } SU\ TV\ W$
 1709 $\ \&\& \ (\angle T:U:W) = (\angle U:T:W) \ \&\& \ \text{parallel } VS\ TU \rightarrow (\angle S:V:W) = (\angle V:S:W)$
 1710 $\gg\rangle$

The following is the complete reasoning trace produced by Qwen3-235B when using our four-stage hierarchical decomposition pipeline:

```
1711 /*
1712   To semi-formalize the English Statement into a structure of premises and conclusions, we will examine the
1713   English Statement sentence by sentence, and break down each sentence into a natural language clause
1714   that is as atomic as possible.
1715 */
1716 {
1717   "premises": [
1718     /*
1719     The sentence "Two triangles TUV and VSW share a point W." declares two triangles.
1720     */
1721     "The distinct points T, U, and W form a triangle \triangle T U W",
1722     "The distinct points V, S, and W form a triangle \triangle V S W",
1723
1724     /*
1725     The sentence "The lines SU and TV intersect at point W." declares two lines and their intersection.
1726     */
1727     "S and U are distinct points on line SU",
1728     "T and V are distinct points on line TV",
1729     "The distinct lines SU and TV intersect at point W",
1730
1731     /*
1732     The sentence "The point W lies between S and U, and also lies between T and V."
1733     */
1734     "Among the distinct points S, W, U, the point W is between S and U",
1735     "Among the distinct points T, W, V, the point W is between T and V",
1736
1737     /*
1738     The sentence "Given \angle T U W = \angle U T W"
1739     */
1740     "The measure of angle T U W equals the measure of angle U T W",
```



```

1728
1729      /*
1730      The sentence "S V || T U"
1731      */
1732      "S and V are distinct points on line SV",
1733      "T and U are distinct points on line TU",
1734      "Line SV is parallel to line TU"
1735    ],
1736    "conclusions": [
1737      "The measure of angle S V W equals the measure of angle V S W"
1738    ]
1739  }
1740
1741  /*
1742  To convert the Semi-Formalized Structure into a Formalized Structure, we now formalize each clause.
1743  */
1744  {
1745    "premises": [
1746      "formTriangle T U W TU UW WT",
1747      "formTriangle V S W VS SW WV",
1748      "distinctPointsOnLine S U SU",
1749      "distinctPointsOnLine T V TV",
1750      "twoDistinctLinesIntersectAtPoint SU TV W",
1751      "between S W U",
1752      "between T W V",
1753      "(\angle T:U:W) = (\angle U:T:W)",
1754      "distinctPointsOnLine S V VS",
1755      "distinctPointsOnLine T U TU",
1756      "parallel VS TU"
1757    ],
1758    "conclusions": [
1759      "(\angle S:V:W) = (\angle V:S:W)"
1760    ]
1761  }
1762
1763  /*
1764  Variables: (T U W V S : Point) (TU UW WT VS SW WV SU TV : Line)
1765  */
1766
1767  <<< \forall (T U W V S : Point) (TU UW WT VS SW WV SU TV : Line),
1768    formTriangle T U W TU UW WT \land
1769    formTriangle V S W VS SW WV \land
1770    distinctPointsOnLine S U SU \land
1771    distinctPointsOnLine T V TV \land
1772    twoDistinctLinesIntersectAtPoint SU TV W \land
1773    between S W U \land
1774    between T W V \land
1775    (\angle T:U:W) = (\angle U:T:W) \land
1776    distinctPointsOnLine S V VS \land
1777    distinctPointsOnLine T U TU \land
1778    parallel VS TU
1779  ->
1780  (\angle S:V:W) = (\angle V:S:W) >>>

```

Direct Formalization Without Decomposition. When prompted to formalize directly, Qwen3-235B produces the following output:

```

1763  \forall (T U W V S : Point) (TU SU TV VS : Line),
1764    formTriangle T U W TU SU TV \land formTriangle V S W VS SU TV \land
1765    between S W U \land between T W V \land
1766    twoDistinctLinesIntersectAtPoint SU TV W \land
1767    (\angle T:U:W) = (\angle U:T:W) \land parallel VS TU
1768  -> (\angle S:V:W) = (\angle V:S:W)

```

This formalization is missing two critical premises: `distinctPointsOnLine S U SU` and `distinctPointsOnLine T V TV`. Without these, the statement is weaker than intended because it does not enforce that S and U are distinct points on line SU, nor that T and V are distinct points on line TV. This error type corresponds to the “Stronger/Weaker Translation” category in our error analysis (Table 4).

Notably, the model’s reasoning trace explicitly identifies these requirements. In Step 3 of its chain-of-thought, the model states: “`distinctPointsOnLine S U SU` – to ensure S and U are distinct and on SU” and initially considers adding this premise. However, the model then incorrectly reasons: “But `formTriangle` already requires `distinctPointsOnLine` for each side?” and lists that `formTriangle T U W TU SU TV` provides `distinctPointsOnLine U W SU`. This reasoning is flawed: in the formal language, SU is simply a variable name for a line—it could equivalently be called l_1 or any other identifier—and has no inherent semantic connection to the points S and U. The only way to establish that S and U are actually distinct points on this line is by explicitly stating `distinctPointsOnLine S U SU`. While `formTriangle` does constrain U and W to lie on the line named SU, it says nothing about whether S lies on this line. This

illustrates a common failure mode in direct formalization: the extended free-form reasoning process can lead to flawed logical conclusions where the model is misled by suggestive variable naming into believing semantic relationships are already captured when they are not.

Formalization With Hierarchical Decomposition. In contrast, our four-stage decomposition pipeline guides the model through a structured process that prevents such errors. When processing the sentence “The lines SU and TV intersect at point W” in Stage 2 (Semi-Formalized Structure), the model is required to break it down into atomic clauses. This forces explicit representation of implicit assumptions: the model produces three separate clauses—“S and U are distinct points on line SU”, “T and V are distinct points on line TV”, and “The distinct lines SU and TV intersect at point W”. In Stage 3 (Formalized Structure), each atomic clause is independently translated to its formal correspondent. The clause “S and U are distinct points on line SU” simply becomes `distinctPointsOnLine S U SU`, regardless of what other predicates might or might not imply. This one-to-one translation prevents information loss and eliminates the flawed dependency reasoning that caused the direct formalization to fail, yielding:

```
\\forall (T U W V S : Point) (TU UW WT VS SW WV SU TV : Line),
  formTriangle T U W TU UW WT \\land formTriangle V S W VS SW WV \\land
  distinctPointsOnLine S U SU \\land distinctPointsOnLine T V TV \\land
  twoDistinctLinesIntersectAtPoint SU TV W \\land
  between S W U \\land between T W V \\land
  (\\angle T:U:W) = (\\angle U:T:W) \\land
  distinctPointsOnLine S V VS \\land distinctPointsOnLine T U TU \\land
  parallel VS TU
-> (\\angle S:V:W) = (\\angle V:S:W)
```

The decomposed output correctly includes all necessary premises. The hierarchical structure ensures that each semantic unit from the English statement is explicitly represented as an atomic clause, preventing information loss during the translation process.

Token Efficiency. Beyond correctness, our decomposition approach also demonstrates improved token efficiency. Using OpenAI’s BPE tokenizer, the direct formalization consumed 4,661 output tokens for the free-form reasoning trace, whereas the decomposed approach used only 1,892 output tokens, which is a 59.4% significant reduction. The structured intermediate representations serve as a more compact and organized “working memory” compared to verbose natural language reasoning, reducing the cost, inference time, and most importantly the long-context challenge for the model.

D PROMPT TEMPLATES

D.1 ABSTRACTION LEARNING

D.1.1 STEP 1: CONCEPT EXTRACTION

You are an expert in mathematics and logic with deep knowledge of all fields of mathematics. You are given some English mathematical statements and a list of previously extracted mathematical concepts. Your task is to extract ALL mathematical concepts mentioned in the current statements. {additional_specs}

Mathematical Concepts

There are 3 types of mathematical concepts to extract:

1. **Definitions of Mathematical Objects**: Point, Line, Integer, Real Number, Series, Group, Ring, etc.
2. **Relations between Mathematical Objects**: 'a point being on a line', 'an integer being even', 'a group being a subgroup of another group', etc.
3. **Functions mapping Mathematical Objects to Mathematical Objects**: 'the Euclidean distance function mapping two points in Euclidean space to a real number', 'the Determinant function mapping a square matrix to a real number', etc.

Guidelines

1. **Precision**: When extracting definitions, relations and functions, you MUST be as precise and detailed as possible. For example 'congruent' is not a precise mathematical relation! You MUST specify how many arguments and what type of arguments the relation or function takes like 'two integers being congruent modulo 3', 'two triangles being congruent', 'a list of line segments being congruent to each other', etc.
2. **Well-Definedness**: Please **MAKE SURE** that each mathematical object, relation, and function you extract is well-defined, or has a conventionally well-accepted definition. For example, 'two lines being on opposing sides of a point' is not well-defined because a point can't partition a plane into sides.
3. **Abstractness**: The definitions, relations and functions you extract MUST be abstract i.e. not involving any particular objects, variables, names in the current statements. For example, 'line AB being parallel to line CD' is not abstract because it involves the names 'AB' and 'CD'.
4. **Naming Consistency**: When naming concepts, you MUST align with the previously extracted concepts if they refer to the same thing. For example, if previous concepts contain 'two lines being parallel in euclidean space' and the current statements mention 'parallel lines in euclidean space', you should use exactly the name 'two lines being parallel in euclidean space' for this concept extracted from the

```

current statements. Only add new concept names if they are truly new and not covered by previously
extracted concepts.

4. Reasoning: Please carefully analyze the all English statements sentence by sentence, identify all
potential definitions, relations and functions, and double-check that they are well-defined. Provide
your detailed step-by-step reasoning BEFORE outputting your final extraction answer.

Output Format
You MUST provide your final extraction answer i.e. a list of extracted concepts within triple angle
brackets and separated by semi-colons. Each concept is a short phrase containing any necessary
characters except semi-colon!!!

<<< concept1; concept2; ... >>>

Task Context:
Previously Extracted Concepts:

{previous_concepts}

Current English Statements:

{english_statements}

```

D.1.2 STEP 2: CONCEPT FILTERING

```

You are an expert in mathematics, logic, programming lanaguages, and formal verification with deep knowledge
of all fields of mathematics and formalization of mathematics. You are given a list of mathematical
concepts in English, and the documentation of a Domain-Specific Language (DSL) for formal mathematics. {
additional_specs}

I want to include and formalize some of these concepts into the current formal DSL. Your task is to filter out
concepts that do NOT meet the criteria for inclusion.

Filtering Criteria
You MUST filter out concepts that satisfy ANY of the following criteria:
1. Duplication of Other Concepts: The concept is a duplication of another concept. For example, the
concepts 'a point being the midpoint of a line segment' and 'a point dividing a line segment into two
equal parts' are duplications because they describe the exact same mathematical relationship using
different wording. Please keep the one with a clearer and more conventional wording. In this case,
please keep the concept 'a point being the midpoint of a line segment' and filter out the other one.

2. Already Defined in Current DSL: The concept has a direct corresponding formal relation/function/type
already defined in the current DSL. For example, there is a concept 'two distinct points being on a line',
and in the DSL there is a formal relation 'distinctPointsOnLine' with the description: def
distinctPointsOnLine (a b : Point) (L : Line) -- points a and b are distinct and on line L. The syntax
is 'distinctPointsOnLine a b L'. Then the concept 'two distinct points lies on a line' should be
filtered out because it is already defined in the current DSL.

Guidelines
1. Conservative Filtering: When in doubt, KEEP the concept. Only filter out concepts that clearly
violate one of the criteria.

2. DSL Comparison: Carefully compare each concept against the current DSL to identify redundancy or easy
expressibility. Don't filter out concepts by mistake! Note that some concepts might be very similar to a
formal relation/function/type in the current DSL, but are different in some subtle aspects. For example
, the concept 'two points being on a line' is very similar to the formal relation 'def
distinctPointsOnLine (a b : Point) (L : Line)', but the former allows the two points to be the same
point, while the latter requires the two points to be distinct. For another example, the concept 'a set
of points being on the same side of a line' is very similar to the formal relation 'def sameSide (a b :
Point) (L : Line)', but the latter only formalizes the concept that 'two points being on the same side
of a line', while the former allows the set of points to be more than two.

3. Reasoning: Please carefully analyze concept by concept, compare them with other concepts and the
current DSL to identify redundancy, determine which concepts should be FILTERED OUT (NOT KEEPING), and
provide your detailed step-by-step reasoning BEFORE outputting your final filtering answer.

Output Format
You MUST provide your final filtering decision as a list of concepts to FILTER OUT (NOT KEEPING) within
triple angle brackets and separated by semi-colons. The name of the concepts you provide should be 
EXACTLY THE SAME as the name of the concepts in provided concept list!!!

<<< concept1; concept2; ... >>>

```

```

If there is no concept to filter out (KEEPING ALL CONCEPTS), please output an empty list: <<< >>>

Task Context:
Current DSL Documentation:

{dsl_doc}

Concepts for Filtering:

{concept_list}

```

D.1.3 STEP 3: CDG CONSTRUCTION

You are an expert in mathematics, logic, programming languages, and formal verification with deep knowledge of all fields of mathematics and formalization of mathematics. You are given a list of mathematical concepts in English, a list of previously analyzed concepts, and the documentation of a Domain-Specific Language (DSL) for formal mathematics. {additional_specs}

Your task is to analyze the dependency structure of each concept and determine how they can be formalized in the DSL.

Dependency Analysis
For each concept in the list, you need to determine:

- 1. **Direct Expressibility**:** Can this concept be directly expressed using existing formal types, relations, and functions in the current DSL? If yes, list the specific DSL elements that can express this concept.
- 2. **Concept Dependencies**:** If not directly expressible, what other mathematical concepts (with no direct correspondents in the current DSL) does this concept depend on? List all the prerequisite concepts needed to define this concept. These dependencies should be abstract mathematical concepts, not specific instances.
- 3. **Formalization Status**:** Based on the analysis, classify the concept as:
 - 'directly_expressible': Can be directly expressed using existing DSL elements
 - 'needs_dependencies': Requires other concepts to be defined first, but ultimately formalizable
 - 'impossible': Cannot be expressed in the current DSL and would require fundamental DSL extensions

Mathematical Concepts
There are 3 types of mathematical concepts you can add as additional dependencies:

- 1. **Definitions of Mathematical Objects**:** Point, Line, Integer, Real Number, Series, Group, Ring, etc.
- 2. **Relations between Mathematical Objects**:** 'a point being on a line', 'an integer being even', 'a group being a subgroup of another group', etc.
- 3. **Functions mapping Mathematical Objects to Mathematical Objects**:** 'the Euclidean distance function mapping two points in Euclidean space to a real number', 'the Determinant function mapping a square matrix to a real number', etc.

Guidelines

- 1. **Conservative Classification**:** When in doubt between 'needs_dependencies' and 'impossible', choose 'needs_dependencies'. Only mark as 'impossible' if the concept fundamentally cannot be expressed in the mathematical framework of the DSL. When in doubt between 'directly_expressible' and 'needs_dependencies', choose 'directly_expressible' since the DSL is designed to be expressive enough for these concepts and you should ****TRY YOUR BEST**** to find directly expressible formal correspondents for these concepts.
- 2. **Direct Correspondents Analysis**:** Be precise about what DSL elements (types, relations, functions) can express each concept. Quote the exact DSL definitions and explain how they relate to the concept.
- 3. **Concept Dependencies Analysis**:** First of all, ****TRY YOUR BEST NOT TO**** introduce new concepts unless absolutely necessary!!! If you must introduce new concepts, note that the list of concepts you are analyzing might have dependencies on each other. It's great if we can define one concept on top of another, but be careful about circular dependencies! **NO TWO CONCEPTS CAN DEPEND ON EACH OTHER!!!**
- 4. **Concept Dependencies Criteria**:** When adding new mathematical concepts as dependencies, you **MUST** follow these criteria:
 - a. **Precision**:** When adding new mathematical concepts as dependencies, you **MUST** be as precise and detailed as possible. For example 'congruent' is not a precise mathematical relation! You **MUST** specify how many arguments and what type of arguments the relation or function takes like 'two integers being congruent modulo 3', 'two triangles being congruent', 'a list of line segments being congruent to each other', etc.
 - b. **Well-Definedness**:** Please ****MAKE SURE**** that each mathematical object, relation, and function you add as a dependency is well-defined, or has a conventionally well-accepted definition. For example, 'two lines being on opposing sides of a point' is not well-defined because a point can't partition a plane into sides.
 - c. **Abstractness**:** The definitions, relations and functions you add as a dependency **MUST** be abstract i.e. not involving any particular objects, variables, names in the current statements. For example, 'line AB being parallel to line CD' is not abstract because it involves the names 'AB' and 'CD'.
- 4. **Reasoning**:** Please carefully examine concept by concept, recall the definition of each concept, analyze which other concepts are necessary to define the current concept, which of those have direct correspondents in the current DSL, and provide your detailed step-by-step reasoning ****BEFORE**** generating the JSON output surrounded by triple angle brackets.

Output Format
You ****MUST**** provide your final analysis in JSON format within triple angle brackets. The JSON should have the following structure:

```
```json
{
 "concept_name_1": {
 "analysis": "detailed step-by-step reasoning for dependency analysis"
 "formal_correspondents": ["list of DSL elements that can express this concept"],
 "concept_dependencies": ["list of prerequisite concepts"],
 "status": "directly_expressible|needs_dependencies|impossible",
 },
 "concept_name_2": {
 ...
 }
}
```
```

<<< JSON_OUTPUT_HERE >>>

Task Context:
**Current DSL Documentation**

```
{dsl_doc}
```

```

1944
1945   ### **Previously Analyzed Concepts:**
1946   {previous_analysis}
1947
1948   ### **Concepts for Analysis:**
1949   {concept_list}
1950

```

1951 D.1.4 STEP 4: CONCEPT FORMALIZATION

1952 You are an expert in mathematics, logic, programming languages, and formal verification with deep knowledge of
 1953 all fields of mathematics, proof assistants like Lean, and Domain-Specific Language (DSL) design. You
 1954 are given a concept dependency graph (CDG) of mathematical concepts, previous extension file(s)
 1955 containing the concepts you have already implemented, and the documentation of a Domain-Specific
 Language (DSL) for formal mathematics.

1956 Your task is to extend the current DSL by implementing / formalizing the provided concepts in the CDG based on
 1957 the current DSL.

```

1958 {additional_specs}
1959

```

```

1959 ## Guidelines

```

- 1960 1. ****CDG Interpretation****: In the CDG, for each concept, the field 'status' indicates if we can directly
 1961 formalize this concept as a relation, function, abbreviation, alias, etc. in the current DSL. If it is '
 1962 directly_expressible', then the field 'formal_correspondents' lists the specific DSL elements that can
 1963 express this concept. You ****MUST**** first try your best to implement the concepts using these '
 1964 formal_correspondents'. However, if they contradicts any specifications given above, You should
 1965 carefully use your own knowledge and judgement to determine the most appropriate way to implement /
 1966 formalize each concept based on the current DSL. If the field 'status' is 'needs_dependencies', then the
 1967 field 'concept_dependencies' lists the prerequisite concepts that we need to formalize first before we
 1968 can formalize this concept.
- 1969 2. ****Faithfulness to Original Concept****: Please try your best to faithfully implement / formalize the original
 1970 concept, nothing more and nothing less. For example, if the original concept is a 'a quadrilateral in
 1971 2-dimensional Euclidean space', then you should only involve the four points and four lines that form
 1972 the quadrilateral, ****no other elements such as the diagonal lines****!!! Please ****PAY EXTRA ATTENTION**** to
 1973 this!!! Such subtlety will result in entirely different formalizations.
- 1974 3. ****Code Reuse****: Please try your best to reuse the helper functions or formalized concepts from the previous
 1975 extension file(s) to implement / formalize the current new concepts. Please don't reinvent the wheel!!!
- 1976 4. ****Consistency with Current DSL****: Please ensure your extension matches the current DSL in spirit and style.
 1977 For example, naming, API convention, formalization choices, etc..
- 1978 5. ****Documentation****: Please carefully document every new type, relation, function, axiom, abbreviation, alias,
 1979 notation, etc. in your implementation. Please make sure that the style and level of detail are
 1980 consistent with the current DSL documentation.
- 1981 6. ****Reasoning****: Please carefully examine concept by concept, analyze the dependency structure of each
 1982 concept, determine the best way to implement / formalize each concept based on the current DSL, and
 1983 provide your detailed step-by-step reasoning ****BEFORE**** outputting your final implementation.

```

1976 ## Output Format

```

1977 You don't need to re-generate the previous extension file(s), ****ONLY**** output your implementation of the new
 1978 concepts, which will be appended to the previous extension file(s). You ****MUST**** provide your final
 1979 implementation within triple angle brackets.
 1980 <<< your entire implementation goes here >>>

```

1979 ## Task Context:

```

```

1980 ### **Current DSL Documentation:**

```

```

1981 {dsl_doc}
1982

```

```

1982 ### **Previous Extension File(s):**

```

```

1983 {previous_extension_files}
1984

```

```

1984 ### **Concept Dependency Graph (CDG):**

```

```

1985 {cdg_list}
1986

```

1988 D.1.5 STEP 5: FORMALIZATION REFACTORING

1989 You are an expert in mathematics, logic, programming languages, and formal verification with deep knowledge of
 1990 all fields of mathematics, proof assistants like Lean, and Domain-Specific Language (DSL) design. You
 1991 are given the documentation of the current Domain-Specific Language (DSL) for formal mathematics and
 1992 file(s) containing extensions (new definitions, relations, functions, axioms, theorems, notations, etc.)
 to this DSL.

1993 Your task is to refactor the extension file to improve its quality, maintainability, and consistency with the
 1994 DSL design principles.

```

1995 {additional_specs}
1996

```

```

1996 ## Guidelines

```

- 1997 1. ****Aggressive Refactoring****: You should be bold in refactoring the extension file:
 - ****Eliminate Redundancy****: If multiple definitions express the exact same mathematical concept with
 different levels of generality, keep only the most general and well-abstracted version.

- ****Consolidate Similar Patterns****: When you find multiple similar definitions that can be unified, remove the individual ones and create a single, parameterized abstraction

- ****Quality Over Quantity****: It's better to have fewer, well-designed abstractions than many specific, poorly abstracted ones. You are not required to preserve every single definition from the original file. Focus on creating a clean, maintainable, and mathematically sound extension.

- ****Pick Best One Only****: If there are many formal relations that are similar to each other, or can substitute each other, you should either carefully choose one of them, or merge them into a single, more general and canonical definition. Multiple similar APIs can be confusing to the user, ****BE CONCISE, BE CONCISE, BE CONCISE****!!!

2. ****Preserve The Necessary****: Even though you are required to refactor aggressively, you should preserve the formal concepts are not truly not in the current DSL. It's ok if they might be easy to implement using the current DSL primitives, but this will enhance the readability and further extendability of the DSL.

3. ****Documentation****: Please carefully update the documentation of every relation, function, axiom, abbreviation, alias, notation, etc. you refactored. Please make sure that the style and content are consistent with the current DSL documentation, NOT the comments in the extension file(s). If there are comments in the extension file(s) that are not consistent with the DSL documentation, please refactor them as well.

4. ****Reasoning****: Please carefully analyze the extension file, identify areas for improvement according to the refactoring criteria, and provide your detailed step-by-step reasoning ****BEFORE**** outputting your refactored implementation.

**** Output Format**

You ****MUST**** provide your refactored Lean file within triple angle brackets. The output should be a complete, valid Lean file that can be appended to the DSL. Do not include any import statements - they will be handled separately.

<<< your refactored implementation goes here >>>

**** Task Context:**

***** **Current DSL Documentation:****

{dsl_doc}

***** **Extension File to Refactor:****

{extension_file}

D.1.6 STEP 6: DOCUMENTATION UPDATE

You are an expert in mathematics, logic, programming languages, and formal verification with deep knowledge of all fields of mathematics, proof assistants like Lean, and Domain-Specific Language (DSL) design. You are given the documentation of the current Domain-Specific Language (DSL) for formal mathematics and extension file(s) containing new definitions, relations, functions, axioms, theorems, notations, etc. that have been added to this DSL. Your task is to generate a new DSL documentation to incorporate all the new elements from the extension file(s), while preserving the previous documentation intact.

{additional_specs}

**** Guidelines**

1. ****Preserve Existing Content****: Keep all existing documentation content unchanged. This is a ****MUST****.
2. ****Consistency with Current Doc****: Keep the style and content consistent with the current DSL documentation. For example, if the current documentation explains the syntax or gives usage examples, then you should do the same.
3. ****Reasoning****: Please carefully analyze the extension file(s), identify all new elements that need to be documented, determine the best way to integrate them into the existing documentation structure, and provide your detailed step-by-step reasoning ****BEFORE**** outputting the new updated DSL documentation.

**** Output Format**

You ****MUST**** provide the new DSL documentation within triple angle brackets. The output should be a complete, new version of the DSL documentation that includes everything: both elements from the current DSL documentation and new documentation for the extension file(s).

<<< your entire new DSL documentation goes here >>>

**** Task Context:**

***** **Current DSL Documentation:****

{dsl_doc}

***** **Extension File(s) to Document:****

{extension_files}

D.2 HIERARCHICAL DECOMPOSITION

D.2.1 LEANEUCLIDPLUS INSTRUCTIONS WITH LEARNED ABSTRACTION

You are given an English Statement of a theorem from Euclidean Geometry. Note that all points and lines mentioned in the statement are distinct, unless otherwise implied by some premises.

2052 Your task is to first semi-formalize the English Statement into a json-style structure (see Guidelines #2),
 2053 then formalize each clause in the Semi-Formalized Structure resulting in a Formalized Structure, and
 2054 finally convert the Formalized Structure into a formal statement in Lean 4 strictly adhering to the
 2055 following formal definitions and guidelines.

```

2056 -- Basic Geometric Sorts --
2057 axiom Point : Type
2058 axiom Line : Type
2059 axiom Circle : Type
2060
2061 -- Inductive Types for Geometric Entities --
2062 inductive Angle | right | ofPoints (A B C : Point)
2063 inductive Segment | endpoints (a b : Point)
2064 inductive Triangle | ofPoints (a b c : Point)
2065
2066 -- Notations and Macros for Geometric Entities --
2067 "|(a-b)|" means the length of the line segment (type  $\mathbb{R}$ , not type Segment) between point a and point
2068 b.
2069 "\angle a:b:c" means the degree of the angle (type  $\mathbb{R}$ , not type Angle) formed by points a, b, and c
2070 , where b is the vertex of the angle and a and c are points respectively on the two sides of the angle.
2071 Add parentheses around the angle notation like "(\angle a:b:c)" to avoid ambiguity.
2072
2073 "\perp " means the degree of the right angle (type  $\mathbb{R}$ , not type Angle).
2074
2075 "\triangle a:b:c" means the triangle (type Triangle) formed from points a, b and c.
2076 "Triangle.area \triangle a:b:c" means the area of the triangle formed by points a, b and c. Add parentheses
2077 around the triangle notation like "(\triangle a:b:c)" to avoid ambiguity.
2078
2079 -- Relations and Axioms for Geometric Sorts --
2080 namespace Point
2081 def onLine (a : Point) (L : Line) -- point a is on line L. The syntax is `a.onLine L`.
2082
2083 def sameSide (a b : Point) (L : Line) -- point a and b are on the same side of line L. They both are not on
2084 line L, but can be the same point. The syntax is `a.sameSide b L`.
2085
2086 def opposingSides (a b : Point) (L : Line) -- distinct point a and b are on opposite sides of line L. They
2087 both are not on line L. The syntax is `a.opposingSides b L`.
2088
2089 def onCircle (a : Point) (C : Circle) -- point a is on circle C. The syntax is `a.onCircle C`.
2090
2091 def insideCircle (a : Point) (C : Circle) -- point a is inside circle C. It can't be on the circle. The syntax
2092 is `a.insideCircle C`.
2093
2094 def outsideCircle (a : Point) (C : Circle) -- point a is outside circle C. It can't be on the circle. The syntax
2095 is `a.outsideCircle C`.
2096
2097 def isCentre (a : Point) (C : Circle) -- point a is on the unique center circle C. The syntax is `a.isCentre C`.
2098
2099 def isMidpointOf (m a b : Point) -- point m is the midpoint of segment AB iff m is between a and b (hence all
2100 three are distinct and collinear), and  $|AM| = |MB|$ . The syntax is `m.isMidpointOf a b`.
2101
2102 def isMidpointOfSegmentEndpoints (m a b : Point) (SP : Segment) -- same as `m.isMidpointOf a b` with an
2103 explicit segment witness. Requires `SP = Segment.endpoints a b`. The syntax is `m.
2104 isMidpointOfSegmentEndpoints a b SP`.
2105
2106 def onExtensionBeyondB (p a b : Point) -- point p lies on the extension of segment AB beyond endpoint B iff `
2107 between a b p`. The syntax is `p.onExtensionBeyondB a b`.
2108
2109 def onExtensionBeyondA (p a b : Point) -- point p lies on the extension of segment AB beyond endpoint A iff `
2110 between b a p`. The syntax is `p.onExtensionBeyondA a b`.
2111 end Point
2112
2113 namespace Line
2114 def intersectsLine (L M : Line) -- two lines L and M intersect at some point. They can be the same line. The
2115 syntax is `L.intersectsLine M`.
2116
2117 def intersectsCircle (L : Line) (C : Circle) -- line L and circle C intersect. The syntax is `L.
2118 intersectsCircle C`.
2119 end Line
2120
2121 namespace Circle
2122 def intersectsCircle (C1 C2 : Circle) -- circle C1 and C2 intersect. The syntax is `C1.intersectsCircle C2`.
2123 end Circle
2124
2125 namespace Triangle
2126 def sideAB (a b : Point) (_c : Point) : Segment -- the side AB of triangle ABC as a segment, i.e. `Segment.
2127 endpoints a b`. The syntax is `Triangle.sideAB a b c`.
2128
2129 def sideBC (_a : Point) (b c : Point) : Segment -- the side BC of triangle ABC as a segment, i.e. `Segment.
2130 endpoints b c`. The syntax is `Triangle.sideBC a b c`.

```

```

2106 def sideCA (a : Point) (_b : Point) (c : Point) : Segment -- the side CA of triangle ABC as a segment, i.e. `
2107     Segment.endpoints c a`. The syntax is `Triangle.sideCA a b c`.
2108 end Triangle
2109
2110 -- Geometric Relations --
2111 def distinctPointsOnLine (a b : Point) (L : Line) -- points a and b are distinct and on line L. The syntax is
2112     `distinctPointsOnLine a b L`.
2113
2114 def between (a b c : Point) -- mutually distinct points a, b and c are collinear and ordered. Point b is
2115     between point a and c. The syntax is `between a b c`.
2116
2117 def formTriangle (a b c : Point) (AB BC CA : Line) -- mutually distinct points a, b and c form a triangle,
2118     where point a and b are on line AB, point b and c are on line BC, point a and c are on line CA. The
2119     lines AB, BC, and CA must be distinct. Note that the order and correspondence of arguments a, b, c, AB,
2120     BC, CA is strictly required i.e. a and b must be on AB, b and c must be on BC, a and c must be on CA,
2121     and they must be passed in the exact order! The syntax is `formTriangle a b c AB BC CA`.
2122
2123 def formRectilinearAngle (a b c : Point) (AB BC : Line) -- points a, b and c form a rectilinear angle, where
2124     b is the vertex of the angle and a and c are points respectively on the sides AB and BC. The sides AB
2125     and BC can be the same line, the points a and c can be the same point, but the vertex b must be distinct
2126     from a and c. The syntax is `formRectilinearAngle a b c AB BC`.
2127
2128 def formParallelogram (a b c d : Point) (AB CD AC BD : Line) -- mutually distinct points a, b, d, and c (in
2129     clockwise/counterclockwise order i.e. ad is a diagonal) form a parallelogram, where points a and b are
2130     on line AB, points c and d are on line CD, points a and c are on line AC, and points b and d are on line
2131     BD. The lines AB, CD, AC, and BD must be distinct. Note that the order and correspondence of arguments
2132     a, b, c, d, AB, CD, AC, BD is strictly required i.e. a and b must be on AB, c and d must be on CD, a and
2133     c must be on AC, b and d must be on BD, and they must be passed in the exact order! The syntax is `
2134     formParallelogram a b c d AB CD AC BD`.
2135
2136 def twoDistinctLinesIntersectAtPoint (L M : Line) (i : Point) -- two distinct lines L and M intersect at point
2137     i, i.e.  $L \neq M$ , i lies on both L and M, and `L.intersectsLine M`. The syntax is `
2138     twoDistinctLinesIntersectAtPoint L M i`.
2139
2140 def sequentiallyAligned (pts : List Point) -- a list of points is sequentially aligned (ordered collinear
2141     chain) iff every consecutive triple  $(p_i, p_{i+1}, p_{i+2})$  satisfies `between  $p_i p_{i+1} p_{i+2}$ '.
2142     No extra line or global distinctness is required beyond betweenness. The syntax is `sequentiallyAligned
2143     [p_0, p_1, p_2, ...]`.
2144
2145 def sequentiallyAlignedThreeOrMore (pts : List Point) -- like `sequentiallyAligned`, additionally requiring
2146     the list to contain at least three points. The syntax is `sequentiallyAlignedThreeOrMore [p_0, p_1, p_
2147     _2, ...]`.
2148
2149 def twoSetsOpposingSidesOnLine (xs ys : List Point) (L : Line) -- two lists of points lie on opposite sides of
2150     line L: (1) no point from either list is on L; (2) every cross-pair  $(x \text{ in } xs, y \text{ in } ys)$  is on
2151     opposing sides of L. The syntax is `twoSetsOpposingSidesOnLine [x_1, x_2, ...] [y_1, y_2, ...] L`.
2152
2153 def supplementaryAngles (a b c d e f : Point) -- angles  $\angle a:b:c$  and  $\angle d:e:f$  are supplementary
2154     iff  $(\angle a:b:c) + (\angle d:e:f) = \perp$ . The syntax is `supplementaryAngles a b c d
2155     e f`.
2156
2157 def parallel (L M : Line) -- two lines L and M are parallel iff they are distinct and do not intersect. The
2158     syntax is `parallel L M`.
2159
2160 def nonCollinearPoints (a b c : Point) -- points a, b, c are non-collinear iff  $\text{Triangle.area } (a:b:
2161     c) \neq 0$ . The syntax is `nonCollinearPoints a b c`.
2162
2163 def segmentFromVertexToOppositeSide (v s1 s2 p : Point) (SP : Segment) -- a segment SP connects vertex v to a
2164     point p on the opposite side s1s2, witnessed by `between s1 p s2`. Requires `SP = Segment.endpoints v p
2165    `. The syntax is `segmentFromVertexToOppositeSide v s1 s2 p SP`.
2166
2167 def congruentSegments (a b c d : Point) -- segments AB and CD are congruent iff  $|AB| = |CD|$ . The syntax is `
2168     congruentSegments a b c d`.
2169
2170 def perpendicularAt (L M : Line) (i a c : Point) -- lines L and M are perpendicular with right angle at
2171     witness point i using points a on L and c on M iff: (1) `L.intersectsLine M`; (2) `formRectilinearAngle
2172     a i c L M` (so a, i lie on L and i, c lie on M, with i as the vertex); (3)  $\angle a:i:c = \perp$ . The
2173     syntax is `perpendicularAt L M i a c`.
2174
2175 def equalLengthRatios (a b c d e f g h : Point) -- equality of two segment-length ratios:  $(|AB| / |CD|) = (|EF|
2176     / |GH|)$ . This is a real-number equation; no non-zero-length guard is enforced. The syntax is `
2177     equalLengthRatios a b c d e f g h`.
2178
2179 def triangleAngleSum (a b c : Point) -- for non-degenerate triangle ABC (i.e. `nonCollinearPoints a b c`), the
2180     interior angle sum is  $\perp + \perp$ :  $(\angle a:b:c) + (\angle a:b:c) + (\angle a:c:b) = \perp + \perp$ . The syntax is `triangleAngleSum a b c`.
2181
2182 def trianglesSimilar (a b c d e f : Point) -- triangles ABC and DEF are similar iff: (1) corresponding angles
2183     are equal:  $\angle BAC = \angle EDF$ ,  $\angle ABC = \angle DEF$ ,  $\angle ACB = \angle DFE$ ; (2)
2184     corresponding sides are proportional:  $|AB|/|DE| = |BC|/|EF|$  and  $|BC|/|EF| = |CA|/|FD|$ . The
2185     correspondence is  $A \leftrightarrow D$ ,  $B \leftrightarrow E$ ,  $C \leftrightarrow F$ . No non-degeneracy is assumed. The syntax is `trianglesSimilar a b
2186     c d e f`.
2187
2188 def trianglesCongruent (a b c d e f : Point) -- triangles ABC and DEF are congruent iff: (1) corresponding
2189     angles are equal; (2) corresponding sides are equal:  $|AB|=|DE|$ ,  $|BC|=|EF|$ ,  $|CA|=|FD|$ . The correspondence
2190     is  $A \leftrightarrow D$ ,  $B \leftrightarrow E$ ,  $C \leftrightarrow F$ . The syntax is `trianglesCongruent a b c d e f`.
2191
2192 def formConvexQuadrilateral (a b c d : Point) (AB BC CD DA : Line) -- ordered points a, b, c, d with side-
2193     lines AB, BC, CD, DA form a convex quadrilateral iff: (1) consecutive vertices lie on their side lines
2194     with distinct endpoints: `distinctPointsOnLine a b AB`, `distinctPointsOnLine b c BC`, `
2195     distinctPointsOnLine c d CD`, `distinctPointsOnLine d a DA`; (2) for each side line, the two nonincident

```



```

2160         vertices lie on the same side of that line: c.sameSide d AB, d.sameSide a BC, a.sameSide b CD, b.
2161         sameSide c DA. No further distinctness, intersection, or non-collinearity is imposed. The order and
2162         correspondence are strict: AB is the side ab, BC is the side bc, CD is the side cd, DA is the side da,
2163         and they must be passed in this exact order. The syntax is `formConvexQuadrilateral a b c d AB BC CD DA`.
2164
2165     def diagonalACofQuadrilateral (a b c d : Point) (SP : Segment) -- SP is the diagonal AC of the ordered
2166         quadruple (a, b, c, d), i.e. `SP = Segment.endpoints a c`. The syntax is `diagonalACofQuadrilateral a b
2167         c d SP`.
2168
2169     def diagonalBDOFQuadrilateral (a b c d : Point) (SP : Segment) -- SP is the diagonal BD of the ordered
2170         quadruple (a, b, c, d), i.e. `SP = Segment.endpoints b d`. The syntax is `diagonalBDOFQuadrilateral a b
2171         c d SP`.
2172
2173     def segmentBisectsAngleAtVertex (a b c p : Point) (SP : Segment) (AB BC BP : Line) -- segment BP bisects angle
2174         ABC at vertex b, with: a,b on AB; b,c on BC; b,p on BP; interior witnessed by same-side constraints a.
2175         sameSide p BC and c.sameSide p AB; sub-angles equal ( $\angle a:b:p = \angle p:b:c$ ); and `SP =
2176         Segment.endpoints b p`. The order is strict: AB is the side ab, BC is the side bc, BP is the carrier
2177         line of the bisector bp, and they must be passed in this exact order. The syntax is `
2178         segmentBisectsAngleAtVertex a b c p SP AB BC BP`.
2179
2180     def trianglesShareSideBy
2181         (sel_1 : Point -> Point -> Point -> Segment)
2182         (sel_2 : Point -> Point -> Point -> Segment)
2183         (a b c d e f : Point) -- triangles ABC and DEF share a side witnessed by segment selectors `sel_1` and `sel
2184         _2` iff `sel_1 a b c = sel_2 d e f`. Typical selectors are `Triangle.sideAB`, `Triangle.sideBC`, or
2185         `Triangle.sideCA`. The syntax is `trianglesShareSideBy Triangle.sideAB Triangle.sideBC a b c d e f`.
2186
2187     def equilateralTriangle (a b c : Point) -- triangle ABC is equilateral iff  $|AB| = |BC|$  and  $|BC| = |CA|$ . The
2188         syntax is `equilateralTriangle a b c`.
2189
2190     def isoscelesAtA (a b c : Point) -- triangle ABC is isosceles at vertex A iff  $|AB| = |AC|$ . The syntax is `
2191         isoscelesAtA a b c`.
2192
2193     def isoscelesAtB (a b c : Point) -- triangle ABC is isosceles at vertex B iff  $|BA| = |BC|$ . The syntax is `
2194         isoscelesAtB a b c`.
2195
2196     def isoscelesAtC (a b c : Point) -- triangle ABC is isosceles at vertex C iff  $|CA| = |CB|$ . The syntax is `
2197         isoscelesAtC a b c`.
2198
2199     def lineCutsTriangleOnABandACat (a b c p q : Point) (AB BC CA L : Line) -- line L intersects sides AB and AC
2200         of triangle ABC at points p and q respectively, with: `formTriangle a b c AB BC CA`; betweenness `
2201         between a p b` and `between c q a`; and both p and q on L. The order and correspondence are strict: AB
2202         is the side ab, BC is the side bc, CA is the side ca, and L is the cutting line; they must be passed in
2203         this exact order. The syntax is `lineCutsTriangleOnABandACat a b c p q AB BC CA L`.
2204
2205     def lineCutsTriangleOnABandBCat (a b c p q : Point) (AB BC CA L : Line) -- line L intersects sides AB and BC
2206         at points p and q respectively, with: `formTriangle a b c AB BC CA`; betweenness `between a p b` and `
2207         between b q c`; and both p and q on L. The order and correspondence are strict as above. The syntax is `
2208         lineCutsTriangleOnABandBCat a b c p q AB BC CA L`.
2209
2210     def lineCutsTriangleOnACandBCat (a b c p q : Point) (AB BC CA L : Line) -- line L intersects sides AC and BC
2211         at points p and q respectively, with: `formTriangle a b c AB BC CA`; betweenness `between c p a` and `
2212         between b q c`; and both p and q on L. The order and correspondence are strict as above. The syntax is `
2213         lineCutsTriangleOnACandBCat a b c p q AB BC CA L`.
2214
2215     def lineCutsABandACParallelBC (a b c p q : Point) (AB BC CA L : Line) -- line L cuts sides AB and AC at p and
2216         q and is parallel to side BC; requires `lineCutsTriangleOnABandACat a b c p q AB BC CA L` and `parallel
2217         L BC`. The syntax is `lineCutsABandACParallelBC a b c p q AB BC CA L`.
2218
2219     def lineCutsABandBCParallelCA (a b c p q : Point) (AB BC CA L : Line) -- line L cuts sides AB and BC at p and
2220         q and is parallel to side CA; requires `lineCutsTriangleOnABandBCat a b c p q AB BC CA L` and `parallel
2221         L CA`. The syntax is `lineCutsABandBCParallelCA a b c p q AB BC CA L`.
2222
2223     def lineCutsACandBCParallelAB (a b c p q : Point) (AB BC CA L : Line) -- line L cuts sides AC and BC at p and
2224         q and is parallel to side AB; requires `lineCutsTriangleOnACandBCat a b c p q AB BC CA L` and `parallel
2225         L AB`. The syntax is `lineCutsACandBCParallelAB a b c p q AB BC CA L`.
2226
2227 -- Guidelines --
2228 1. Formalized Statement Format: Your formalized statement must be of the form  $\llcorner \forall \text{forall } (...), P_1 \wedge \text{and } P_2 \dots \wedge \text{and } P_n \rightarrow Q_1 \wedge \text{and } Q_2 \dots \wedge \text{and } Q_m \ggcorner$  where each  $P_i$  and  $Q_i$  is built from
2229     the above building blocks using conjunction ( $\wedge$ ) disjunction ( $\vee$ ) and negation ( $\neg$ ). All
2230     variable declarations must be placed in parentheses after the universal quantifier. You shouldn't
2231     declare variables or use quantifiers in any other places! For different types of variables, you should
2232     put them in different parentheses. For example, if the English statement contains "The points A, B are
2233     on line AB", then you should declare (A B : Point) (AB : Line) in the formalized statement.
2234
2235 2. Semi-Formalized Structure: For the Semi-Formalized Structure, you must output a json5-style structure (you
2236     can interleave comments in between data) with 2 fields "premises" and "conclusions", and the value of
2237     each field is a list of clauses.
2238
2239 {
2240     "premises": [
2241         English clause 1,
2242         English clause 2,
2243         ...
2244     ],
2245     "conclusions": [
2246         English clause 1,
2247         English clause 2,
2248         ...
2249     ]
2250 }

```

Each clause is an English sentence representing a proposition. The relationship between premises and conclusions is implication. The premises are the antecedent and the conclusions are the consequent. The relationship between each clause is conjunction. You should try to make each clause as atomic as possible i.e. within a clause, there shouldn't be any logical operators. If there has to be some, you should use as few logical operators within a clause as possible, and use multiple clauses instead to express the same meaning.

3. Formalized Structure: The structure is exactly the same as the Semi-Formalized Structure, but you will formalize each English clauses in premises and conclusions into Lean expressions, to get the Formalized Structure.

```

{
  "premises": [
    Formalized clause 1,
    Formalized clause 2,
    ...
  ],
  "conclusions": [
    Formalized clause 1,
    Formalized clause 2,
    ...
  ]
}

```

For the formalization of each clause, you should first try to find if there is a direct corresponding formal relation provided. For example, the English sentence "distinct points a is between b and c" has a potential directly corresponding formal relation 'between'. You should then double check the detailed description of the formal relation to see if the details match the actual English clause, and pass the arguments in the correct order specified in the description: 'between b a c'. If no direct formalization is available, or the relation description does not match, you should formalize it equivalently using available operators, constants, and other formal relations. For example, "the points A, B, C form an inscribed triangle in circle α " doesn't have any direct formalization, but it's equivalent to "the points A, B, C form a triangle, and they are all on circle α ", which can be formalized as `formTriangle A B C AB BC CA α \wedge A.onCircle α \wedge B.onCircle α \wedge C.onCircle α` .

4. Implication: There can be only a single implication in the formula; either side of the implication must be a conjunction of formulae.

5. Variable Naming: You should always use the EXACT SAME variable name in the formalized statement as the one in the English statement. For example, if the English statement contains "Point A, B is on line AB", then you should use the variable names (A B : Point) (AB : Line) in the formalized statement. **You must not add any prefix or suffix to the variable names like "point_A" or "AB_line"!!!** Sometimes the English statement might first refer to a line as "AB", but later use "BA" for the same line, in which case you should use the first occurring name "AB" for that same line consistently in your formalized statement.

6. Numeric Values Restrictions: Denote 90-degree angle by `\perp` , 180-degree angle by `\perp + \perp` , etc. Also, when referring to segments, we always mean its length (i.e. `|a-b|`).

7. Quantified Variables: Your quantified variables must be limited to primitive geometric types: points, lines, and circles. ALL bound variables that you declared must be mentioned in some clauses later.

8. Intermediate Variables: You should never define an intermediate variable inside the proposition. For example, `"let α := (something);"` is not allowed.

9. Numeric Operators: You should only uses addition (+), subtraction (-), multiplication (*), and division (/). Avoid using other mathematical operators such as exponentiation.

10. Equality Relation: You can use the equality relation (=) and non-equality relation (!=) to compare points, lines, circles, angles, and line segments. However, please **avoid** using the equality relation to compare expressions and numeric values as it's not supported by the current formal language implementation. For example, you should write `"|a-b| = |c-d|"` instead of `"|(a-b)| / |(c-d)| = 1"`.

11. Angle Notation: When you see a short-hand angle notation like `" \angle X"`, this means that X is the vertex of the angle. You should always expand it to the full angle notation `" \angle A X B"` (formally, `\angle A:X:B`) where A and B are points respectively on the two sides of the angle.

12. Syntax Tip 1: You should only use the provided relations and axioms. Please examine the provided axioms and relations carefully to find the best way to express your proposition. Don't create or use new ones that are not provided above! When you see an error message that contains "unknown identifier", it means that you might have used some relations that are not provided in the guidelines. Please double-check this, or your formalized statement won't compile!

13. Syntax Tip 2: You must declare all Points, Lines, and Circles that you will use in the formal statement. You shouldn't declare any extra Points, Lines, or Circles that are not used in the formalized statement. When you see an error message that contains "unknown identifier", it means that you might have forgotten to declare some Points, Lines, or Circles that are used in the formalized statement. Please double-check this, or your formalized statement won't compile!

14. Syntax Tip 3: You shouldn't declare any extra Points, Lines, or Circles that are not used in the formalized statement. When you see an error message that contains "Unexpected expression", it means that you might have declared some extra variables that are not used in the formalized statement. Please double-check this, or your formalized statement won't compile!

15. Syntax Tip 4: A formalized angle `\angle A:B:C` always expects three point identifiers separated by colons! Even though you will see the English expressions like `" \angle Y"`, but `" \angle Y"` is not a valid formalization. You should formalize it into `" \angle X:Y:Z"` where X, Y, and Z are the three points that form the angle. When you see an error message like "unexpected token '>'; expected ':'", it means that you have not provided the correct number of identifiers for the angle, or you need to surround the angle

with parentheses like ($\angle X:Y:Z$) to avoid ambiguity. Please double-check this, or your formalized statement won't compile!

16. Reasoning: Please think carefully step-by-step and state your intermediate reasoning steps before you semi-formalize each clause when converting the English Statement to Semi-Formalized Structure, formalize each clause when converting the Semi-Formalized Structure into Formalized Structure, and write down the final Formalized Statement. **This is crucial!!! Please do not skip this!!! If there are examples provided by the user, you must strictly follow the format and reasoning in the examples!!!**

17. Response Format: Present your formalized Lean expression within triple angle brackets (\llcorner Lean expression here \gg). Do not add any annotations/explanations, or markdown syntax.

D.2.2 LEANEUCLIDPLUS ONE-SHOT EXAMPLE

Here is an example:

English Statement:

There are two triangles $\triangle GHJ$ and $\triangle IFJ$ with a shared vertex J . The lines GI and HF intersect at point J . The point J lies between G and I , and also lies between H and F .

Given $G J / I J = H J / F J$. Complete the proof that $\triangle GHJ \sim \triangle IFJ$.

Semi-Formalized Structure:

```
/*
  To semi-formalize the English Statement into a structure of premises and conclusions, we will examine the
  English Statement sentence by sentence, and break down each sentence into a natural language clause
  that is as atomic as possible.
*/
{
  "premises": [
    /*
      The sentence "There are two triangles  $\triangle GHJ$  and  $\triangle IFJ$  with a shared vertex  $J$ ." first
      declares 2 triangles  $\triangle GHJ$  and  $\triangle IFJ$ .

      From the instructions, we know that all points and lines are distinct, so it also declares 5 distinct
      points:  $H, J, G$  that form triangle  $\triangle GHJ$ , and  $I, F, J$  that form triangle  $\triangle IFJ$ .

      We can break down the sentence into the following 2 premises:
    */
    "The distinct points  $G, H$ , and  $J$  form a triangle  $\triangle GHJ$ ",
    "The distinct points  $I, F$ , and  $J$  form a triangle  $\triangle IFJ$ ",
    /*
      For now, the premises above only partially express the fact that points  $G, H, J, I, F$  are all distinct
      , and some lines (sides of the triangles) are distinct.

      However, we might be able to incorporate and express this distinctness relation in later premises, so
      we don't create additional premises for them right now.

      Instead, we will first examine all sentences in the English statement and eventually create premises
      for the remaining unexpressed distinctness relations if necessary.
    */

    /*
      The sentence "The lines  $GI$  and  $HF$  intersect at point  $J$ ." first declares 2 lines:  $GI, HF$ .

      From the instructions, we know that all points and lines are distinct, so it also declares 4 distinct
      points and 2 distinct lines:  $H, F$  on line  $HF$ ,  $G, I$  on line  $GI$ .

      It then states an intersection relation: the lines  $GI$  and  $HF$  intersect at point  $J$ .

      If we don't explicitly state the distinctness of the points and lines, it might be interpreted that
      either  $G$  and  $I$  can be the same point,  $H$  and  $F$  can be the same point, or  $GI$  and  $HF$  are the same
      line, which is not what we want.

      Therefore, we can break down the sentence into the following 3 premises:
    */
    "H and F are distinct points on line HF",
    "G and I are distinct points on line GI",
    "The distinct lines  $GI$  and  $HF$  intersect at point J",
    /*
      Since we had explicitly expressed that the points  $G, H, J$  are mutually distinct, and that the points  $I, F, J$  are mutually distinct, we have also expressed all distinctness relations between the points  $G, H, J, I, F$  except for the distinctness relation between  $G$  and  $F, H$  and  $I$ .

      If  $G$  were the same as  $F$ , then  $GI$  ( $FI$ ) would intersects  $HF$  at point  $F$  as well as point  $J$ , which means that either  $J = F$  or  $GI = HF$ , which contradicts the fact that  $J$  and  $F$  are distinct points or  $GI$  and  $HF$  are distinct lines. Therefore,  $G$  and  $F$  must be distinct points.

      If  $H$  were the same as  $I$ , following the exact same reasoning, we will get a contradiction. Therefore,  $H$  and  $I$  must be distinct points.

      Therefore, we checked that all distinctness relations among the points  $G, H, I, F$  are expressed. There is only one more distinctness relation left to check: the lines  $GH$  and  $FI$  are distinct.
    */
  ]
}
```

2322 If GH were the same as FI, we would have the intersection of GI and HF, point J also lying on GH (FI),
 2323 which contradicts the fact that G, H, and J (I, F, and J) form a triangle. Therefore, GH and FI
 2324 must be distinct lines.

2325 Therefore, we checked that all distinctness relations among the points and lines are expressed.
 2326 */

2327 /*
 2328 The sentence "The point J lies between G and I, and also lies between H and F." states order relations
 2329 : the point J is between G and I, and also between H and F.

2329 If we don't explicitly state the distinctness of the points, it might be interpreted that J can be the
 2330 same as G or I, or J can be which is not what we want.

2331 Therefore, we have the following 2 premises:
 2332 */
 2333 "Among the distinct points G, J, I, the point J is between G and I",
 2334 "Among the distinct points H, J, F, the point J is between H and F",

2335 /*
 2336 The sentence "Given $GJ / IJ = HJ / FJ$." states a numeric equation between the ratios of line
 2337 segments: $GJ / IJ = HJ / FJ$.

2336 Therefore, we have the following premise:
 2337 */
 2338 "The ratio of segment GJ to segment IJ equals the ratio of segment HJ to segment FJ, denoted as $GJ /$
 2339 $IJ = HJ / FJ$ ",

2339],
 2340 "conclusions": [
 2341 /*
 2342 The sentence "Complete the proof that $\triangle GHJ \sim \triangle IFJ$." states the goal of the
 2343 proof: to show that the triangles $\triangle GHJ$ and $\triangle IFJ$ are similar.

2342 Therefore, we have the following conclusion:
 2343 */
 2344 "The triangles $\triangle GHJ$ and $\triangle IFJ$ are similar, denoted as $\triangle GHJ \sim \triangle$
 2345 IFJ "
 2346]
 2347 }

2346 Formalized Structure:

2347 /*
 2348 To convert the Semi-Formalized Structure into a Formalized Structure, we will examine the natural language
 2349 clauses in the Semi-Formalized Structure one by one, and formalize each into a formal clause that
 2350 is well-formed and well-defined according to the guidelines.
 2351 */

2351 {
 2352 "premises": [
 2353 /*
 2354 The clause "The distinct points G, H, and J form a triangle $\triangle GHJ$ " has a potential direct
 2355 formalization in the guidelines: `formTriangle`.

2355 We double check the description of the `formTriangle` relation: "mutually distinct points a, b and c
 2356 form a triangle, where point a and b are on line AB, point b and c are on line BC, point a and c
 2357 are on line CA. The lines AB, BC, and CA must be distinct".

2357 The mutual distinctness of points in our target clause matches the description of "mutually distinct
 2358 points a, b and c". By definition, the 3 points that form a triangle must be non-collinear,
 2359 which matches the description "The lines AB, BC, and CA must be distinct". Therefore, this
 2360 relation is indeed equivalent to our target clause.

2360 `formTriangle` takes (a b c : Point) (AB BC CA : Line) as arguments in order, so we can formalize the
 2361 clause as: `formTriangle G H J GH HJ JG`.

2362 The following clause "The distinct points I, F, and J form a triangle $\triangle IFJ$ " can be
 2363 formalized in the exact same way.

2364 Therefore, we can formalize the first 2 clauses as follows:
 2365 */
 2366 "formTriangle G H J GH HJ JG",
 2367 "formTriangle I F J IF FJ JI",

2366 /*
 2367 The clause "H and F are distinct points on line HF" has a potential direct formalization in the
 2368 guidelines: `distinctPointsOnLine`.

2368 We double check the description of the `distinctPointsOnLine` relation: "points a and b are distinct
 2369 and on line L". It is indeed equivalent to our target clause.

2370 It takes (a b : Point) (L : Line) as arguments in order, so we can formalize the clause as: `
 2371 distinctPointsOnLine H F HF`

2372 The following clause "G and I are distinct points on line GI" can be formalized in the exact same way.

2373 Therefore, we can formalize the next 2 clauses as follows:
 2374 */
 2375 "distinctPointsOnLine H F HF",
 2376 "distinctPointsOnLine G I GI",

```

2376
2377      /*
2378      The clause "The distinct lines GI and HF intersect at point J" has a potential direct formalization in
2379      the guidelines: `twoDistinctLinesIntersectAtPoint`.
2380
2381      We double check the description of the `twoDistinctLinesIntersectAtPoint` relation: "two distinct
2382      lines L and M intersect at point i, i.e. L != M, i lies on both L and M, and `L.intersectsLine M
2383      `. It is indeed equivalent to our target clause since it guarantees that the lines are distinct
2384      .
2385
2386      It takes (L M : Line) (i : Point) as arguments in order, so we can formalize the clause as follows:
2387      */
2388      "twoDistinctLinesIntersectAtPoint GI HF J",
2389
2390      /*
2391      The clause "Among the distinct points G, J, I, the point J is between G and I" has a potential direct
2392      formalization in the guidelines: `between`.
2393
2394      We double check the description of the `between` relation: "mutually distinct points a, b and c are
2395      collinear and ordered. Point b is between point a and c". It is indeed equivalent to our target
2396      clause since it guarantees that the points are mutually distinct.
2397
2398      It takes (a b c : Point) as arguments in order, with Point b being the one that is between Point a and
2399      Point c, so we can formalize the clause as: `between G J I`.
2400
2401      The following clause "Among the distinct points H, J, F, the point J is between H and F" can be
2402      formalized in the exact same way.
2403
2404      Therefore, we can formalize the next 2 clauses as follows:
2405      */
2406      "between G J I",
2407      "between H J F",
2408
2409      /*
2410      The clause "The ratio of segment GJ to segment IJ equals the ratio of segment HJ to segment FJ,
2411      denoted as  $GJ / IJ = HJ / FJ$ " has a potential direct formalization in the guidelines: `
2412      equalLengthRatios`.
2413
2414      We double check the description of the `equalLengthRatios` relation: "equality of two segment-length
2415      ratios:  $(|AB| / |CD|) = (|EF| / |GH|)$ . This is a real-number equation; no non-zero-length guard
2416      is enforced". It is indeed equivalent to our target clause.
2417
2418      It takes (a b c d e f g h : Point) as arguments in order. Therefore, the parameters a, b corresponds
2419      our concrete arguments G and J, the parameters c, d corresponds to I and J, the parameters e, f
2420      corresponds to H and J, and the parameters g, h corresponds to F and J, so we can formalize the
2421      clause as follows:
2422      */
2423      "equalLengthRatios G J I J H J F J"
2424    ],
2425    "conclusions": [
2426      /*
2427      The clause "The triangles  $\triangle GHJ$  and  $\triangle IFJ$  are similar, denoted as  $\triangle GHJ \sim \triangle IFJ$ " has a potential direct formalization in the guidelines: `
2428      trianglesSimilar`.
2429
2430      We double check the description of the `similar` relation: "triangles ABC and DEF are similar iff: (1)
2431      corresponding angles are equal:  $\angle BAC = \angle EDF$ ,  $\angle ABC = \angle DEF$ ,  $\angle ACB = \angle DFE$ ; (2) corresponding sides are proportional:  $|AB|/|DE| = |BC|/|EF|$  and  $|BC|/|EF| = |CA|/|FD|$ . The correspondence is  $A \leftrightarrow D$ ,  $B \leftrightarrow E$ ,  $C \leftrightarrow F$ . No non-degeneracy is assumed".
2432      It is indeed equivalent to our target clause.
2433
2434      We check that the point G corresponds to the point I, the point H corresponds to the point F, and the
2435      point J corresponds to the point J.
2436
2437      The `trianglesSimilar` relation takes (a b c d e f : Point) as arguments in order. Therefore, the
2438      parameters a, b, c corresponds our concrete arguments G, H, J, the parameters d, e, f
2439      corresponds to I, F, J, so we can formalize the clause as follows:
2440      */
2441      "trianglesSimilar G H J I F J"
2442    ]
2443  }
2444
2445  Formalized Statement:
2446
2447  Before converting the Formalized Structure into a Formalized statement, we first need to declare all the
2448  geometric objects that are mentioned in the clauses of the Formalized Structure as variables.
2449
2450  According to the guidelines, since all our declared variables will be bounded by a universal quantifier  $\forall$ 
2451  forall , we shouldn't declare any extra variables that are not mentioned in any clauses of Formalized
2452  Structure.
2453
2454  We will examine the clauses in the Formalized Structure one by one, and declare variables for the mentioned
2455  geometric objects in order.
2456
2457  The first 2 clauses, `formTriangle G H J GH HJ JG`, and `formTriangle I F J IF FJ JI`, require the declaration
2458  of Points: G, H, J, I, F, and Lines: GH, HJ, JG, IF, FJ, JI.
2459
2460  The next 2 clauses, `distinctPointsOnLine H F HF`, and `distinctPointsOnLine G I GI`, requires the additional
2461  declaration of Lines: HF, GI.
2462
2463  The remaining clauses require no additional declarations.

```

Therefore, we need to declare all and only the following geometric objects as variables:

```
(G H J I F : Point) (GH HJ JG IF FJ JI HF GI : Line)
```

Now we can combine the declarations, premises, and conclusions to convert the Formalized Structure to a Formalized Statement. According to the guidelines, we will do the following:

1. Quantify the declared variables correctly.
2. Use the conjunction of all formal clauses in the premises as the antecedent.
3. Use the conjunction of all formal clauses in the conclusions as the consequent.
4. Connect the antecedent and consequent with an implication (\rightarrow).
5. Wrap the entire Formalized Statement with triple angle brackets (\llcorner Lean expression here \rceil) for parsing.

Finally, we have the Formalized Statement:

```
<<< \forall (G H J I F : Point) (GH HJ JG IF FJ JI HF GI : Line), formTriangle G H J GH HJ JG \land
formTriangle I F J IF FJ JI \land distinctPointsOnLine H F HF \land distinctPointsOnLine G I GI \land
twoDistinctLinesIntersectAtPoint GI HF J \land between G J I \land between H J F \land
equalLengthRatios G J I J H J F J \rightarrow trianglesSimilar G H J I F J >>>
```

D.2.3 PROOFNET-HARD INSTRUCTIONS WITH LEARNED ABSTRACTION

You are given an English Statement of a mathematical theorem. Target environment: Lean 4.7.0-rc2 with Mathlib4. Do NOT use Lean 3 or deprecated identifiers. Use only current Lean 4/Mathlib4 names and notations.

Your task is to formalize the English Statement into a formal theorem in Lean 4 using the Mathlib library and no other libraries, strictly adhering to the following formal definitions and guidelines.

Here are an extra set of helpers you can ****DIRECTLY USE**** in addition to Mathlib:

```
import Mathlib
```

```
/-!
```

This file extends Mathlib with a small, self-contained collection of reusable definitions used in undergraduate-level formalization tasks. The design emphasizes generality, concise naming, and consistency with Mathlib conventions.

Contents:

1. Iterated square-root map on $\mathbb{R}_{\geq 0}$ and a recursive sequence:
 - `nnrealSqrt2PlusSqrt` : $x \mapsto \sqrt{2 + \sqrt{x}}$ on $\mathbb{R}_{\geq 0}$
 - `nnrealSqrt2` : the nonnegative real $\sqrt{2}$
 - `nnrealIterSqrtSeq` : the sequence defined by $s_0 = \sqrt{2}$, $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$
2. Simple sequence predicates and examples:
 - `StrictlyBoundedAboveBy` : strict upper bound predicate for sequences
 - `succDiff` : the real sequence $n \mapsto \sqrt{n+1} - \sqrt{n}$
3. Comparability of topologies:
 - `TopologiesComparable` : $t \leq u \vee u \leq t$
4. Subbasis and generated topologies on \mathbb{R} :
 - `IsSubbasisFor` : a family is a subbasis for a topology if the topology is generated from it
 - `LowerLimitSubbasis` and `sorgenfreyTopology` (lower limit / Sorgenfrey line)
 - `KNatRecip`, `KTopologySubbasis`, and `KTopology` (K-topology on \mathbb{R})
 - `RationalIooSubbasis` and `realTopologyFromRationalIoo` (topology from rational intervals)
5. Pointwise convergence on a set and limit points:
 - `UnitInterval` : the type $\text{Icc } 0 \ 1$
 - `powSeqOnUnitInterval` : the sequence of functions $x \mapsto x^n$ on the unit interval
 - `powLimitOnUnitInterval` : its pointwise limit
 - `PointwiseConvergesOn` : pointwise convergence on a set
 - `IsLimitPoint` and `LimitPointCompact` : limit points and limit point compactness
6. Nested closed nonempty families:
 - `NestedClosedNonempty` : antitone closed nonempty families
 - `NestedClosedNonemptyInterNonempty` : their intersection is nonempty

All definitions are marked with `@[simp]` as requested.

```
-/
```

noncomputable section

```
open Real Filter Set TopologicalSpace
open scoped NNReal
```

```
/-- The function on  $\mathbb{R}_{\geq 0}$  given by  $x \mapsto \sqrt{2 + \sqrt{x}}$ . -/
@[simp]
def nnrealSqrt2PlusSqrt (x :  $\mathbb{R}_{\geq 0}$ ) :  $\mathbb{R}_{\geq 0}$  :=
  \langle Real.sqrt (2 + Real.sqrt x), Real.sqrt_nonneg _ \rangle
```

```
/-- The nonnegative real number  $\sqrt{2}$ . -/
```

```
@[simp]
def nnrealSqrt2 :  $\mathbb{R}_{\geq 0}$  :=
  \langle Real.sqrt 2, Real.sqrt_nonneg 2 \rangle
```

```
/-- The recursively defined sequence on  $\mathbb{R}_{\geq 0}$ :
```

```

2484 `s 0 = sqrt(2)` and `s (n+1) = sqrt(2 + sqrt(s n))`. -/
2485 @[simp]
2486 def nnrealIterSqrtSeq :  $\mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ 
2487   | 0 => nnrealSqrt2
2488   | n+1 => nnrealSqrt2PlusSqrt (nnrealIterSqrtSeq n)
2489
2488 /-- A predicate expressing that a sequence `s :  $\mathbb{N} \rightarrow \alpha$ ` is strictly bounded above by `c`,
2489 i.e.  $\forall n, s n < c$ . -/
2489 @[simp]
2490 def StrictlyBoundedAboveBy { $\alpha$  : Type*} [LT  $\alpha$ ] (s :  $\mathbb{N} \rightarrow \alpha$ ) (c :  $\alpha$ ) :
2491   Prop :=
2492      $\forall n, s n < c$ 
2493
2492 /-- The real sequence `n |> sqrt(n + 1) - sqrt(n)`. -/
2492 @[simp]
2493 def sqrtSuccDiff (n :  $\mathbb{N}$ ) :  $\mathbb{R}$  :=
2494   Real.sqrt ((n :  $\mathbb{R}$ ) + 1) - Real.sqrt (n :  $\mathbb{R}$ )
2495
2495 /-- Two topologies on the same type are comparable if one is included in the other
2496 (i.e.  $t \leq u \vee u \leq t$ ). -/
2496 @[simp]
2497 def TopologiesComparable { $\alpha$  : Type*} (t u : TopologicalSpace  $\alpha$ ) : Prop :=
2498   t <= u  $\vee$  u <= t
2499
2499 /-- A family `S : Set (Set X)` is a subbasis for a topology `t` on `X` if
2500 `t` is generated from `S`. -/
2500 @[simp]
2501 def IsSubbasisFor {X : Type*} (S : Set (Set X)) (t : TopologicalSpace X) : Prop :=
2502   t = TopologicalSpace.generateFrom S
2503
2502 /-- The generating family for the lower limit (Sorgenfrey) topology on  $\mathbb{R}$ :
2503 all half-open intervals  $[a, b)$  with  $a < b$ . -/
2503 @[simp]
2504 def LowerLimitSubbasis : Set (Set  $\mathbb{R}$ ) :=
2505   {U |  $\exists a b : \mathbb{R}, a < b \wedge U = \text{Set.Ico } a b$ }
2506
2506 /-- The lower limit (Sorgenfrey) topology on  $\mathbb{R}$ ,
2507 generated by  $\{ [a, b) \mid a < b \}$ . -/
2507 @[simp]
2508 def sorgenfreyTopology : TopologicalSpace  $\mathbb{R}$  :=
2509   TopologicalSpace.generateFrom LowerLimitSubbasis
2510
2509 /-- The set  $K := \{ 1/n \mid n \in \mathbb{N}^+ \} \subseteq \mathbb{R}$ , used in the definition of the K-
2510 topology. -/
2510 @[simp]
2511 def KNatRecip : Set  $\mathbb{R}$  :=
2512   {x |  $\exists n : \mathbb{N}, x = (1 : \mathbb{R}) / (n : \mathbb{N})$ }
2513
2513 /-- The generating family for the K-topology on  $\mathbb{R}$ :
2514 all open intervals  $(a, b)$  and the punctured intervals  $(a, b) \setminus K$ . -/
2514 @[simp]
2515 def KTopologySubbasis : Set (Set  $\mathbb{R}$ ) :=
2516   {U |  $\exists a b : \mathbb{R}, a < b \wedge (U = \text{Set.Ioo } a b \vee U = \text{Set.Ioo } a b \setminus \text{KNatRecip})$ }
2517
2516 /-- The K-topology on  $\mathbb{R}$ , generated by the subbasis consisting of all sets
2517 of the form  $(a, b)$  and  $(a, b) \setminus K$ , where  $K = \{ 1/n \mid n \in \mathbb{N}^+ \}$ . -/
2517 @[simp]
2518 def kTopology : TopologicalSpace  $\mathbb{R}$  :=
2519   TopologicalSpace.generateFrom KTopologySubbasis
2520
2519 /-- The generating family for the topology on  $\mathbb{R}$  with a basis of intervals having
2520 rational endpoints:  $\text{Ioo } a b$  with  $a, b \in \mathbb{Q}$  and  $a < b$ . -/
2520 @[simp]
2521 def RationalIooSubbasis : Set (Set  $\mathbb{R}$ ) :=
2522   {U |  $\exists a b : \mathbb{Q}, a < b \wedge U = \text{Set.Ioo } (a : \mathbb{R}) (b : \mathbb{R})$ }
2523
2523 /-- The topology on  $\mathbb{R}$  generated by the subbasis of open intervals with rational endpoints. -/
2523 @[simp]
2524 def realTopologyFromRationalIoo : TopologicalSpace  $\mathbb{R}$  :=
2525   TopologicalSpace.generateFrom RationalIooSubbasis
2526
2526 /-- The unit interval  $[0, 1]$  seen as a subtype of  $\mathbb{R}$ . -/
2526 @[simp] abbrev UnitInterval : Type := Set.Icc (0 :  $\mathbb{R}$ ) 1
2527
2527 /-- The sequence of functions on the unit interval  $[0, 1]$  given by
2528  $(\text{powSeqOnUnitInterval } n) x = x^n$ . -/
2528 @[simp]
2529 def powSeqOnUnitInterval :  $\mathbb{N} \rightarrow \text{UnitInterval} \rightarrow \mathbb{R}$  :=
2530   fun n x => (x :  $\mathbb{R}$ ) ^ n
2531
2531 /-- The pointwise limit function on  $[0, 1]$  associated with the sequence  $x \mapsto x^n$ :
2532 it is 0 for  $x < 1$  and 1 otherwise (in particular at  $x = 1$ ). -/
2532 @[simp]
2533 def powLimitOnUnitInterval : UnitInterval  $\rightarrow \mathbb{R}$  :=
2534   fun x => if (x :  $\mathbb{R}$ ) < 1 then 0 else 1
2535
2535 /-- Pointwise convergence on a set: a sequence of functions  $F : \mathbb{N} \rightarrow X \rightarrow \beta$  converges
2536 pointwise on `S` to `g : X  $\rightarrow \beta$ ` if  $\forall x \in S, \text{Tendsto } (\text{fun } n => F n x) \text{ atTop } (\text{nhds } (g x))$ .
2536 -/
2536 @[simp]
2537 def PointwiseConvergesOn {X  $\beta$  : Type*} [TopologicalSpace  $\beta$ ]
2538   (F :  $\mathbb{N} \rightarrow X \rightarrow \beta$ ) (g : X  $\rightarrow \beta$ ) (S : Set X) : Prop :=

```

```

2538   \forall x \in S, \text{Tendsto} (fun n => F n x) atTop (nhds (g x))
2539
2540 /-- A point `x` is a limit point (accumulation point) of a subset `A` of a topological space `X`
2541 if every neighborhood of `x` meets `A \ {x}`. -/
2542 @[simp]
2543 def IsLimitPoint {X : Type*} [TopologicalSpace X] (A : Set X) (x : X) : Prop :=
2544   \forall U \in nhds x, (U \cap (A \ {x})).Nonempty
2545
2546 /-- A topological space is limit point compact if every infinite subset has a limit point. -/
2547 @[simp]
2548 def LimitPointCompact {X : Type*} [TopologicalSpace X] : Prop :=
2549   \forall A : Set X, A.Infinite -> \exists x : X, IsLimitPoint A x
2550
2551 /-- A nested (decreasing) sequence `C : \mathbb{N} -> Set X` of closed, nonempty sets in a
2552 topological space:
2553 `Antitone C \land (\forall n, IsClosed (C n)) \land (\forall n, (C n).Nonempty)` -/
2554 @[simp]
2555 def NestedClosedNonempty {X : Type*} [TopologicalSpace X] (C : \mathbb{N} -> Set X) : Prop :=
2556   Antitone C \land (\forall n : \mathbb{N}, IsClosed (C n)) \land (\forall n : \mathbb{N}, (C n).Nonempty)
2557
2558 /-- The property that every nested (decreasing) sequence of closed nonempty subsets of `X`
2559 has nonempty intersection `bigcap n, C n`. -/
2560 @[simp]
2561 def NestedClosedNonemptyInterNonempty {X : Type*} [TopologicalSpace X] : Prop :=
2562   \forall (C : \mathbb{N} -> Set X),
2563     Antitone C ->
2564       (\forall n : \mathbb{N}, IsClosed (C n)) ->
2565         (\forall n : \mathbb{N}, (C n).Nonempty) ->
2566           (\bigcap n, C n).Nonempty
2567
2568 1. Target & Environment
2569   - Formalize an English statement into Lean 4 using ONLY Mathlib
2570   - NO other libraries or import should be used.
2571   - Environment: Lean 4.7.0-rc2 with Mathlib4. Use only current Lean 4/Mathlib4 identifiers. Do NOT use Lean
2572     3-era names.
2573
2574 2. Forbidden Lean 3-Era Identifiers & Notations
2575   - Identifiers: `Convergent`, `Function.IsFieldHom`, `QuotientGroup.quotient`, `Metric.bounded`, `IsPerfect`
2576     , `Real.cbrt`.
2577   - Notations:  $\mathbb{R}^m$ ,  $\mathbb{Z}[i]$ , raw  $\int x \in \mathbb{R}, \ldots$ , binder shorthand  $\forall x y$ 
2578      $\in S, \ldots$ , negative exponents with  $^{\wedge}$  (e.g.,  $^{\wedge}(-p)$ ).
2579   - Use `RingHom`/`AlgHom` (not `Function.IsFieldHom`).
2580   - Use `Subgroup.Quotient` (not `QuotientGroup.quotient`).
2581   - Use `Bornology.IsBounded E` (not `Metric.bounded E`).
2582   - Avoid `Real.cbrt`; use `Real.rpow` or rational exponents.
2583
2584 3. Header Format:
2585   - You need to generate BOTH the header and the theorem in your final response
2586   - The ONLY import statement you should generate is `import Mathlib`.
2587   - After that, write `open ...` and `open scoped ...` for the namespaces you need to use in the theorem.
2588   - MAKE SURE that the names and notations you are using are opened in the header!!! Please check this if
2589     you receive syntax errors!!!
2590
2591 4. Theorem Format: ALWAYS include theorem name, and it MUST be exactly `thm_Q`.
2592   - Required format: `theorem thm_Q (params) : conclusion := by sorry`
2593   - You MUST use `by sorry` as the placeholder. Do NOT generate a proof!!!
2594   - Note that some formalizations of a statement might not have parameters/binders, only the type/goal. For
2595     example, "Prove that  $\sqrt{3} + \sqrt{3}$  is irrational." can be formalized as `theorem thm_Q :
2596     Irrational (2^(1:\mathbb{R})/3) + 3^(1:\mathbb{R})/3) := by sorry`
2597   - You should try to use the binder/parameter form as much as possible. For example, try to convert
2598     existential statements to universal statements so that you can use the binder/parameter form.
2599
2600 5. Parameter Format: Use explicit parentheses for ALL parameters/binders  $(\Omega : \text{Set } \mathbb{C}) (f : \mathbb{C} \rightarrow \mathbb{C}) (h : \text{IsOpen } \Omega)$ . NEVER use implicit  $(\Omega)$ .
2601
2602 6. Parameter/Binder Naming:
2603   - For object names, use the EXACT SAME names as the in problem text. For example, if the problem
2604     mentions "complex function f" then you should name the corresponding parameter `f` like  $(f : \mathbb{C} \rightarrow \mathbb{C})$ .
2605   - For hypothesis names, use the object name with a prefix `h` and index them from 1. For example, if the
2606     problem mentions "complex function f that is holomorphic on X" then you should name the corresponding
2607     hypothesis `hf_1` like  $(hf_1 : \text{DifferentiableOn } \mathbb{C} \text{ } f \text{ } X)$ .
2608
2609 7. Reasoning
2610   - Please think carefully step-by-step and state your intermediate reasoning steps before write down the
2611     final Formalized Statement.
2612   - This is EXTREMELY CRUCIAL!!! Please do not skip this!!! PAY EXTRA ATTENTION TO THIS!!!
2613   - If there are examples provided by the user, you must strictly follow the format and reasoning in the
2614     examples!!! THINK EXTRA HARD!!!
2615
2616 8. Response Format: Return the header and the theorem together, surrounded by triple angle brackets <<< import
2617   Mathlib
2618   open ...
2619   open scoped ...
2620
2621   theorem thm_Q ... := by sorry >>>.
2622
2623 Do NOT include ANY comments or proof tactics in your final response!!! PAY EXTRA ATTENTION TO THIS
2624   !!!

```



```

Example of CORRECT output (declare all variables, name must be thm_Q):
<<< import Mathlib
open ...
open scoped ...

theorem thm_Q : \forall N : \mathbb{N}, \exists n \geq N, (3*n+1).Prime \land (3*n+1) \geq N := by sorry
>>>

Example of INCORRECT output (wrong theorem name, missing variable N declaration, includes comments, missing `
  by sorry`):
<<< import Mathlib
open ...
open scoped ...

-- some random comment
theorem some_random_name : \exists n \geq N, (3*n+1).Prime \land (3*n+1) \geq N := >>>

```

D.2.4 PROOFNET-HARD ONE-SHOT EXAMPLE

Here is an example:

English Statement:

Let $U \subset \mathbb{C}$ be a (non-empty) connected open set and let f_n be a sequence of holomorphic functions defined on U . Suppose that f_n converges uniformly to a function f on every compact subset of U . Show that f is holomorphic in U .

Semi-Formalized Structure:

```

/*
  To semi-formalize the English Statement into a nested structure of quantifications, premises and
  conclusions, we will examine the English Statement sentence by sentence, and break down each
  sentence into natural language clauses that are as atomic as possible.
*/
{
  /*
    The sentence "Let  $U \subset \mathbb{C}$  be a (non-empty) connected open set and let  $f_n$  be a sequence
    of holomorphic functions defined on  $U$ ." indicates that the choice of  $U$  and  $f_n$  are arbitrary,
    so we should choose the universal quantifier for the entire statement that quantifies over  $U$ ,
     $f_n$ , and possibly other variables.

    Since  $f_n$  is a sequence of functions with certain properties, we will further break it into a nested sub
    -structure. Now, we will first universally quantify over  $U$  as follows:
  */
  "quantification": "for all  $U$ ,  $f_n$ , possibly other variables",
  "premises": [
    /*
      The sentence first declares that  $U$  is a subset of the complex plane  $\mathbb{C}$ , that it is a non-
      empty, a connected, and an open, so we have the following 4 premises:
    */
    " $U$  is a subset of the complex plane  $\mathbb{C}$ ",
    " $U$  is non-empty",
    " $U$  is connected",
    " $U$  is open",
    /*
      The sentence also declares that  $f_n$  is a sequence of complex functions, so we have the following
      premise:
    */
    " $f_n$  is a sequence of functions from  $\mathbb{C}$  to  $\mathbb{C}$ ",
    /*
      Now we can break down the specification on  $f_n$ , a sequence of holomorphic functions defined on  $U$ .
      Recall our mathematical knowledge, a sequence is countably indexed, which means that for all  $i$ 
       $\in \mathbb{N}$ ,  $f_i$  is holomorphic on  $U$ .

      By the definition of holomorphic function, each  $f_i$  is a function from  $U$  to  $\mathbb{C}$  and is
      complex-differentiable at every point of  $U$ .

      Therefore, we break it into a nested sub-structure by universally quantifying over  $n$  as follows:
    */
    {
      "quantification": "for all  $i$ ",
      "premises": [
        " $i$  is a natural number i.e.  $i \in \mathbb{N}$ ",
      ],
      "conclusions": [
        " $f_i$  in the sequence  $f_n$  is holomorphic on  $U$ , which means that  $f_i$  is complex-
        differentiable at every point of  $U$ "
      ]
    }
  ],
  /*
    The next sentence "Suppose that  $f_n$  converges uniformly to a function  $f$  on every compact subset of
     $U$ " indicates that there exists a function  $f$  with a certain property, which indicates a
    nested existential quantifier over  $f$ .

    However, our conclusion "Show that  $f$  is holomorphic in  $U$ " requires  $f$  to be declared beforehand.
    Therefore, the correct conclusion should really be "Show that if there exists a functions  $f$ 

```

such that f_n converges uniformly to f on every compact subset of U , then f is holomorphic in U .

According to the guidelines, we should convert statement to universally quantified as much as possible so that later we can easily formalize them in the binder form. Recall our mathematical knowledge, if a limit exists, it is unique. Then we can rewrite the existential quantified conclusion as "for all f , such that f_n converges uniformly to f on every compact subset of U , f is holomorphic in U ".

In addition, since no other premises before depends on f , we can move the quantification over f to our top-level quantifications i.e. f can be quantified by the outmost universal quantifier.

Therefore, we declare f as follows:

```

/*
 * $f$ is a function from  $\mathbb{C}$  to  $\mathbb{C}$ ,
 */
/*
Now we can break down the specification on $f$, a function that for all compact subset $X$ of $U$, the
sequence $f_n$ converges uniformly to $f$ on $X$, which indicates a nested universal quantifier
over $X$.

Therefore, we break it into a nested sub-structure by universally quantifying over $X$ as follows:
*/
{
  "quantification": "for all $X$",
  "premises": [
    "$X$ is a subset of $U$",
    "$X$ is compact"
  ],
  "conclusions": [
    "the sequence $f_n$ converges uniformly to $f$ on $X$"
  ]
}
],
"conclusions": [
  /*
  The final sentence "Show that $f$ is holomorphic in $U$" means that $f$ is complex-differentiable at
  every point of $U$. Since we have declared $f$ beforehand in the premises, we can directly refer
  to the name $f$ in the conclusion.

  Therefore, we have the following conclusion:
  */
  "$f$ is holomorphic on $U$, which means that $f$ is complex-differentiable at every point of $U$"
]
}

```

Formalized Structure:

```

/*
To convert the Semi-Formalized Structure into a Formalized Structure, we will examine the natural language
clauses in the Semi-Formalized Structure one by one, and formalize each into a formal clause that
is well-formed and well-defined according to the guidelines.
*/
{
  /*
  First of all, we notice that the premises and conclusions in the Semi-Formalized Structure are quantified
  by a universal quantifier over $U$. According to the guidelines, for outmost universally quantified
  statements, we should use the binder form as much as possible.

  Therefore, we put no formal quantifier over $U$ in the Formalized Structure, but instead formalize the
  premises "$U$ is subset of the complex plane  $\mathbb{C}$ ", "$U$ is non-empty", "$U$ is connected",
  etc. as binders like `(U : some_type)`, `(hU_1 : some_property_of_U)`, etc.
  */
  "quantification": "omitted since we will use the binder form",
  "premises": [
    /*
    Now we should formalize the premises one by one. The first premise is "$U$ is a subset of the complex
    plane  $\mathbb{C}$ ". Check the provided helpers and recall our knowledge of the required
    versions of Lean and Mathlib in the guidelines, there is no direct correspondents of "the set of
    all complex numbers", so we need to come up with an equivalent way to formalize this premise
    alternatively.

    We notice that this premise is equivalent to just saying that $U$ is a set of complex numbers.
    Therefore, we can use the type former `Set` and the complex number type ` $\mathbb{C}$ ` to say
    that $U$ is of type `Set  $\mathbb{C}$ `.

    Therefore, we can formalize this premise as follows:
    */
    "(U : Set  $\mathbb{C}$ )",
    /*
    The next premise is "$U$ is non-empty". Check the provided helpers and recall our knowledge of the
    required versions of Lean and Mathlib in the guidelines, there is a direct corresponding formal
    relation `Nonempty`. Therefore, we can formalize this property of $U$ as follows:
    */
    "(hU_1 : Nonempty U)",
    /*
    The next premise is "$U$ is connected". Check the provided helpers and recall our knowledge of the
    required versions of Lean and Mathlib in the guidelines, there is a direct corresponding formal
    relation `IsConnected`. Therefore, we can formalize this property of $U$ as follows:
    */
    "(hU_2 : IsConnected U)",
  ]
}

```

```

2700
2701      /*
2702      The next premise is "$U$ is open". Check the provided helpers and recall our knowledge of the required
2703      versions of Lean and Mathlib in the guidelines, there is a direct corresponding formal relation
2704      `IsOpen`. Therefore, we can formalize this property of $U$ as follows:
2705      */
2706      "(hU_3 : IsOpen U)",
2707      /*
2708      The next premise is "$f_n$ is a sequence of complex functions". Check the provided helpers and recall
2709      our knowledge of the required versions of Lean and Mathlib in the guidelines, since a sequence
2710      is countably indexed, we can say that $f_n$ is of the curried function type  $\mathbb{N} \rightarrow \mathbb{C}$ 
2711       $\rightarrow \mathbb{C}$ . When we want to talk about the $i$-th function in the sequence, we
2712      can use the function application notation `f_n i`.
2713
2714      Therefore, we can formalize this premise as follows:
2715      */
2716      "(f_n :  $\mathbb{N} \rightarrow \mathbb{C} \rightarrow \mathbb{C}$ )",
2717      /*
2718      The next premise is a universally quantified statement (a sub-structure in our semi-formalized
2719      structure) about each $f_i$ in the sequence $f_n$: "for all $i$ in  $\mathbb{N}$ " we have "$f_i$
2720      in the sequence $f_n$ is holomorphic on $U$, which means that $f_i$ is complex-differentiable at
2721      every point of $U$".
2722
2723      For "$i$ being a natural number", check the provided helpers and recall our knowledge of the required
2724      versions of Lean and Mathlib in the guidelines, we can say that $i$ is of type  $\mathbb{N}$ , so
2725      we can formalize the quantification over $i$ as  $\forall i : \mathbb{N}$ .
2726
2727      For "$f_i$ is complex-differentiable at every point of $U$", check the provided helpers and recall our
2728      knowledge of the required versions of Lean and Mathlib in the guidelines, there is a direct
2729      corresponding formal relation `DifferentiableOn` to talk about complex-differentiability by
2730      passing  $\mathbb{C}$  as the first argument.
2731
2732      Therefore, we can formalize this property of $f_i$ as follows:
2733      */
2734      "(hf_n_1 :  $\forall i : \mathbb{N}$ , DifferentiableOn  $\mathbb{C}$  (f_n i) U)",
2735      /*
2736      The next premise is "$f$ is a function from  $\mathbb{C}$  to  $\mathbb{C}$ ". Check the provided helpers
2737      and recall our knowledge of the required versions of Lean and Mathlib in the guidelines, we can
2738      say that $f$ is of the function type  $\mathbb{C} \rightarrow \mathbb{C}$ .
2739
2740      Therefore, we can formalize this premise as follows:
2741      */
2742      "(f :  $\mathbb{C} \rightarrow \mathbb{C}$ )",
2743      /*
2744      The next premise is a universally quantified statement (a sub-structure in our semi-formalized
2745      structure) about each subset $X$ of $U$: "for all subsets $X$ of $U$, if $X$ is compact, then
2746      the sequence $f_n$ converges uniformly to $f$ on $X$".
2747
2748      For "$X$ being a subset of $U$", check the provided helpers and recall our knowledge of the required
2749      versions of Lean and Mathlib in the guidelines, there is a direct corresponding formal notation
2750       $\subseteq$ . However, according to the guidelines, we should always declare the type of any
2751      variable first. Therefore, we can formalize the quantification over $X$ as  $\forall X : \text{Set } \mathbb{C}$ 
2752      and the subset relation as  $X \subseteq U$ .
2753
2754      For "$X$ being compact", check the provided helpers and recall our knowledge of the required versions
2755      of Lean and Mathlib in the guidelines, there is a direct corresponding formal relation `
2756      IsCompact`. Therefore, we can formalize this property of $X$ as follows as `IsCompact X`.
2757
2758      For "the sequence $f_n$ converges uniformly to $f$ on $X$", check the provided helpers and recall our
2759      knowledge of the required versions of Lean and Mathlib in the guidelines, there is a direct
2760      corresponding relation `TendstoUniformlyOn F f l X` for `F :  $\iota \rightarrow \alpha \rightarrow \beta$ ` and
2761      `f :  $\alpha \rightarrow \beta$ ` along a filter `l`. Here `F` is our sequence of functions  $\lambda n x, f_n n x$ ,
2762      the limit is `f`, the index filter is `atTop` (we can directly use this name
2763      since we opened the namespace `Filter`), and the set is `X`. We can formalize the conclusion of
2764      this quantified statement as `TendstoUniformlyOn ( $\lambda n x \Rightarrow f_n n x$ ) f atTop X`.
2765
2766      Therefore, assembling every piece together, we can formalize this universally quantified statement as
2767      follows:
2768      */
2769      "(hf_1 :  $\forall X : \text{Set } \mathbb{C}$ ,  $X \subseteq U$   $\wedge$  IsCompact X  $\rightarrow$  TendstoUniformlyOn ( $\lambda$ 
2770      lambda n x => f_n n x) f atTop X)",
2771      ],
2772      "conclusions": [
2773        /*
2774        The conclusion is "$f$ is holomorphic on $U$, which means that $f$ is complex-differentiable at every
2775        point of $U$". Check the provided helpers and recall our knowledge of the required versions of
2776        Lean and Mathlib in the guidelines, there is a direct corresponding formal relation `
2777        DifferentiableOn` to talk about complex-differentiability by passing  $\mathbb{C}$  as the first
2778        argument.
2779
2780        Therefore, we can formalize this property of $f$ as follows:
2781        */
2782        "DifferentiableOn  $\mathbb{C}$  f U",
2783      ]
2784    ]
2785  }
2786
2787  Formalized Statement:
2788
2789  Now we can convert the Formalized Structure to a Formalized Statement. According to the guidelines, we will do
2790  the following:

```

1. Assemble the quantification, premises, and conclusions properly into a Formalized Statement. Name the theorem following the guidelines as `thm_Q`. Use the correct syntax if we are using the binder form.
2. Wrap the entire Formalized Statement with triple angle brackets (<<< Lean expression here >>>) for parsing.

Finally, we have the Formalized Statement:

```
<<< theorem thm_Q (U : Set \mathbb{C}) (hU_1 : Nonempty U) (hU_2 : IsConnected U) (hU_3 : IsOpen U) (f_n : \mathbb{N} -> \mathbb{C} -> \mathbb{C}) (hf_n_1 : \forall n : \mathbb{N}, DifferentiableOn \mathbb{C} (f_n n) U) (f : \mathbb{C} -> \mathbb{C}) (hf_1 : \forall X : Set \mathbb{C}, X \subseteq U \wedge IsCompact X -> TendstoUniformlyOn (\lambda n x => f_n n x) f atTop X) : DifferentiableOn \mathbb{C} f U := by sorry >>>
```