Handling Uncertainty in Probabilistic Model-Based Magnetic Resonance Image Quality Transfer

Anonymity

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Abstract

Previous studies of model-based image enhancement approaches used artifact-free training data synthesised from a deterministic forward model. However, the forward model often has unknown components of a stochastic nature. In this paper, we adapt model-based image enhancement to exploit probabilistic forward models. This enables us to capture inherent model uncertainty that arises from unknown aspects of the forward model. Overall, two distinct sources of uncertainty arise: uncertainty inherently included in the stochastic forward process; and uncertainty arising from model training for the inverse process. We formulate homoscedastic and heteroscedastic models to estimate uncertainty in the mapping from input to enhanced image. The models enable us to construct uncertainty maps over the enhanced image. Here we test the new method within the task of estimating high field from low field Magnetic Resonance Images (MRI) via Image Quality Transfer (IQT).

Keywords: uncertainty estimation, super resolution, contrast enhancement, probabilistic model, model-based deep learning.

1. Introduction

Image quality transfer (IQT) has been developed to enhance images by exploiting information from a higher quality reference via image-to-image translation (Alexander et al., 2014, 2017; Tanno et al., 2017; Blumberg et al., 2018). To date, IQT has used a deterministic forward decimation model (e.g. downsampling) to generate synthetic training data, which avoids the impractical requirement to acquire perfectly aligned paired data. This model-based paradigm combines data-driven deep neural network technologies with physical-analytical models and has been shown great efficacy in reinforcement learning (Kar et al., 2019; Ruiz et al., 2018) and inverse problems (Arridge et al., 2019). Most implementations assume certainty of the forward problem; in our case, each high quality image has a single well-defined corresponding synthetic low-field image. However, in practice the mapping from high field to low field is not fully known and we would like to express the mapping probabilistically to capture the uncertainty in certain aspects of the forward problem, e.g. how relaxation constants change from high to low magnetic field. This places additional challenges on model training for the inverse problem; in particular, how we capture and utilise uncertainty in the forward problem on the learned inverse model.

In this work we substantially extend the point-estimation solution in Lin et al. (2019) to predictive uncertainty estimation. We propose probabilistic models with homoscedastic or heteroscedastic likelihood able to capture uncertainty in the inverse mapping from stochastic forward models. We compare the performance of the models in estimating uncertainty in the task of estimating high-quality high-field MRI scans from low-quality low-field images.
2. Methods and Materials

We adapt uncertainty estimation to IQT in a task of low-to-high field MR image enhancement (Bahrami et al., 2016; Kaur and Sao, 2019), more specifically here, 8× super resolution and 0.36T-to-3T contrast enhancement. The method comprises the following two steps.

**Training-set Synthesis.** In IQT, we use a forward problem to construct artifact-free synthetic paired data: pairs of perfectly aligned high-field images and corresponding synthetically generated low-field images. Here we employ the probabilistic decimation (downsampling) simulator proposed by Lin et al. (2019) for anatomical MR images. The simulator performs contrast change, down-sampling, smoothing, and addition of Gaussian noise to conduct a stochastic high-to-low field MR image translation. The high-field data were randomly sampled from 3T Human Connectome Project (HCP) dataset (Sotiropoulos et al., 2013). From each high-field datum, multiple realisations of low-field counterpart were generated by varying the parameter \( \eta \), which represents signal-to-noise ratios (SNR) in grey and white matter. We drew \( \eta \) from a distribution formed from statistics of historic low-field (0.36T) images in an anonymous academic hospital.

**Modelling Uncertainty.** We investigate homoscedastic and heteroscedastic models. First, we adopt the approach in Gal and Ghahramani (2016) as homoscedastic model. On the other hand, we specify heteroscedastic model on training and testing for IQT uncertainty estimation, as illustrated in Figure 1. The model is built on a collection of the paired 3D patches \( (x_1^\eta, y_1)_{i=1}^N \), where \( x_1^\eta \) is simulated from \( y_i \) by training-set synthesis. We first trained two networks to obtain two outputs \( \mu_{\omega_1}(x) \) and \( \sigma^2_{\omega_2}(x) \) where \( \omega_1 \) and \( \omega_2 \) are network parameters. These two networks have an identical architecture extended from 3D-SRU-Net (Heinrich et al., 2017) but each convolutional layer is followed by a dropout operation (Srivastava et al., 2014). As derived in Tanno et al. (2017), the loss function is

\[
L(\omega_1, \omega_2) = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\|y_i - \mu_{\omega_1}(x_i^\eta)\|^{2}}{\sigma_{\omega_2}(x_i^\eta)} + \log \det \sigma^2_{\omega_2}(x_i^\eta) \right] \]

when assuming that the observation noise varies with the input \( x_i^\eta \) (Kendall and Gal, 2017). As implementation details, 30 subjects comprising about 40000 patches were used for training and 10 for testing. More details about network setup can be found in Lin et al. (2019).

For testing, we predicted the mean and variance maps by employing an ensemble of models (Lakshminarayanan et al., 2017) trained by different simulated low-quality data. As derived in Appendix A, the expressions of the predictive mean and variance maps are...
Figure 2: Predictive Mean, Standard Deviation (SD) and Root Mean Squared Error (RMSE) maps of a representative test subject for both homoscedastic (Homo.) and heteroscedastic (Hetero.) models. The three low-field images simulated from the high-field one are illustrated here as references.

E_{q(y^*|x^*,θ)}(y^*) \approx \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{M} p(\eta^i) \mu(x^*; \hat{\omega}_{i,t}^{1}) \tag{1}

\text{Var}_{q(y^*|x^*,θ)}(y^*) \approx \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{M} p(\eta^i) \left[ \sigma^2(x^*; \hat{\omega}_{2,t}^{i}) + \mu^T(x^*; \hat{\omega}_{1,t}^{i}) \mu(x^*; \hat{\omega}_{1,t}^{i}) \right] - \mathbb{E}_{q(y^*|x^*,θ)} E_{q(y^*|x^*,θ)}(y^*) \tag{2}

using Monte-Carlo dropout\(^1\) with a rate of \(θ\) for \(T\) realisation at the test time. We choose \(T = 10\) and \(M = 10\) in this study.

3. Results and Conclusion

Three representative images out of ten cases are shown in Figure 2. We can see that the Standard Deviation (SD) map by the homoscedastic model has similar contrast to the Root Mean Squared Error (RMSE) map while the SD map by the heteroscedastic model appears more homogeneous in grey-white matter regions. The result also shows that the SD maps for both are sensitively correlated to the RMSE maps, with the overall correlation coefficient of 0.92 and 0.88 for homoscedastic and heteroscedastic models, respectively.

In conclusion, we exploit homoscedastic and heteroscedastic models to quantify uncertainty arising from a probabilistic decimation simulator and training process for IQT. The approach enables the use of non-deterministic forward models in model-based deep-learning applications. This allows us to accommodate incomplete information in the forward process and propagate that lack of knowledge to uncertainty in the output. The predictive uncertainty map provides a surrogate metric of reliability when the ground truth image is unavailable for producing error maps. Our results show the uncertainty maps modelled by homoscedasticity and heteroscedasticity are morphologically similar to the true error maps showing efficacy of the approach.

\(^1\) Dropout operation enables us to increase the variability of trained models at test time. It is effective when there is limited numbers of trained models, says \(M = 10\) in this study.
Acknowledgments

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References


### Appendix A. Derivation of Predictive Mean and Variance

Let $Y = \{y_i\}_{i=1}^N$ be a high-field dataset of 3D MR images of good resolution and contrast. The images in the low-field dataset $X^\eta = \{x^\eta_i\}_{i=1}^N$ are simulated from $Y$ using the probabilistic decimation model (Lin et al., 2019): $X^\eta = F(Y; \eta)$, where $\eta$ has a distribution $p(\eta)$. The distribution is also used to determine the frequency of $X^\eta$ among all simulated datasets. Conversely, the inverse process of this probabilistic decimation model is approximated by the deep neural network parameterised by $\omega$. The probability of the prediction $y^*$ given the input $x^*$ is formulated by averaging over all possible training models, weighted by both $p(\eta)$, the frequency of $X^\eta$, and the posterior $p(\omega|X^\eta, Y)$ of the network parameters given the corresponding training data. Mathematically, the formula is given by:

$$p(y^*|x^*, \{X^\eta_i\}_{i=1}^M, Y) = \mathbb{E}_{p(\eta)}[p(y^*|x^*, X^\eta_i, Y)] = \mathbb{E}_{p(\eta)}\mathbb{E}_{p(\omega|X^\eta_i, Y)}[p(y^*|x^*, \omega)]$$

(3)

However, due to the intractability of $p(\omega|X^\eta, Y)$, we utilise variational Bayesian inference to find its approximate distribution $q^\theta(\omega|\theta)$, which minimises the Kullback-Leibler (KL)
divergence (Graves, 2011), parameterised by $\theta$. Gal and Ghahramani (2016) and Kendall and Gal (2017) use Monte Carlo dropout to approximate $q^\eta(\omega|\theta)$ as a weighted sum of two Gaussian distributions. The mean of one of those two Gaussians is fixed at zero. By substituting the true posterior $p(\omega|X^0,Y)$ in Equation (3) with $q^\eta(\omega|\theta)$, we have the approximating distribution, denoted by $q(y^*|x^*, \theta)$, about $p(y^*|x^*, \{X^0\}_{i=1}^M, Y)$:

$$q(y^*|x^*, \theta) = \mathbb{E}_{p(\eta)}\mathbb{E}_{q^\eta(\omega|\theta)}[p(y^*|x^*, \omega)].$$

(4)

Now we derive the predictive mean in Equation (1) and the predictive variance in Equation (2). Following similar steps in Gal (2016), we first have the estimation of the predictive mean provided $p(y^*|x^*, \omega) = \mathcal{N}(y^*; \mu_{\omega_1}(x^*), \sigma^2_{\omega_2}(x^*))$:

$$\mathbb{E}_{q(y^*|x^*, \theta)}(y^*) = \int y^* q(y^*|x^*, \theta) dy^*
\approx \int y^* \int p(\eta) \int q^\eta(\omega|\theta)p(y^*|x^*, \omega)d\omega d\eta dy^*
= \int p(\eta) \int q^\eta(\omega|\theta)\mu_{\omega_1}(x^*)d\omega d\eta
\approx \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{M} p(\eta^i) \mu(x^*; \hat{\omega}^i_{1,t}),$$

where $\hat{\omega}^i_{1,t} \sim q^\eta(\omega|\theta)$ of $T$ realisations and $\theta$ represents the dropout rate. Subsequently, the second-order moment is given by

$$\mathbb{E}_{q(y^*|x^*, \theta)}((y^*)^T y^*) \approx \int (y^*)^T y^* \int p(\eta) \int q^\eta(\omega|\theta)p(y^*|x^*, \omega)d\omega d\eta dy^*
= \int p(\eta) \int q^\eta(\omega|\theta)\left[ (y^* - \mu_{\omega_1}(x^*))^T (y^* - \mu_{\omega_1}(x^*)) + \mu_{\omega_1}(x^*)^T \mu_{\omega_1}(x^*) \right] p(y^*|x^*, \omega) dy^* d\omega d\eta
= \int p(\eta) \int q^\eta(\omega|\theta)\left[ \sigma^2_{\omega_2}(x^*) + \mu_{\omega_1}(x^*)^T \mu_{\omega_1}(x^*) \right] d\omega d\eta
\approx \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{M} p(\eta^i) \left[ \sigma^2_{\omega_2}(x^*; \hat{\omega}^i_{2,t}) + \mu_{\omega_1}(x^*)^T \mu(x^*; \hat{\omega}^i_{1,t}) \right],$$

where $(\hat{\omega}^i_{1,t}, \hat{\omega}^i_{2,t}) \sim q^\eta(\omega|\theta)$ with respect to the $i$th training model and the $t$th realisation. Finally, the predictive variance is given by

$$\text{Var}_{q(y^*|x^*, \theta)}(y^*) = \mathbb{E}_{q(y^*|x^*, \theta)}((y^*)^T y^*) - \mathbb{E}_{q(y^*|x^*, \theta)}(y^*)^T \mathbb{E}_{q(y^*|x^*, \theta)}(y^*).$$