Periodic Sobolev-Besov regularity in terms of Chui-Wang wavelet coefficients

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***Abstract*—This paper investigates the representation of periodic Sobolev and Besov norms in terms of wavelet coefficients. Function spaces of mixed smoothness, fundamental in functional analysis and approximation theory, are traditionally defined through weak derivatives, integrability conditions, and smoothness parameters. By studying wavelet bases, we derive equivalent norms for these spaces expressed as weighted sums of wavelet coefficients, including explicit constants. This reveals the interplay between the function spaces and wavelet properties such as smoothness, vanishing moments, and scaling. These characterizations provide computational advantages and offer a unified perspective on Sobolev and Besov spaces, emphasizing their hierarchical structure and scale-dependent behavior.**

***Index Terms*—Wavelet characterizations, Sobolev spaces, Besov spaces**

I. Introduction

Let $f \in L_2(\mathbb{T}^d)$ be a periodic function. We denote the complex scalar product in the space $L_2(\mathbb{T}^d)$ by $\langle f, g\rangle := \int_{\mathbb{T}^d} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}$. In this paper, we state a result about the connection between the wavelet coefficients and the Fourier decomposition of a periodic function, which we prove purely by Fourier analysis and determine the constants explicitly.

II. The Chui-Wang wavelets

For $m \in \mathbb{N}$, we define the **cardinal B-spline** $B_m \colon \mathbb{R} \to \mathbb{R}$ of order m recursively by

$$B_1(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} \le x \le \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases} \quad B_m(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} B_{m-1}(y)\,\mathrm{d}y,$$

which is a piecewise polynomial function. Using the cardinal B-spline $B_m(x)$ as scaling function for a wavelet system, one can construct the compactly supported **Chui-Wang wavelets of order** m, introduced in [1], which are given by

$$\psi(x) = \sum_{n\in\mathbb{Z}} q_n B_m(2x - n - \tfrac{m}{2}),$$

where

$$q_n = \frac{(-1)^n}{2^{m-1}} \sum_{k=0}^{m} \binom{m}{k} B_{2m}(n + 1 - k - m).$$

This wavelet is compactly supported, i.e. $\operatorname{supp}\psi = [0, 2m-1]$. Additionally, the wavelet has **vanishing moments** of order m, i.e.

$$\int_{-\infty}^{\infty} \psi(x) x^\beta \,\mathrm{d}x = 0, \quad \beta = 0, \dots, m-1.$$

We use the notation

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j \in \mathbb{N}_0, k \in \mathbb{Z},$$

and its periodization

$$\psi_{j,k}^{\mathrm{per}}(x) = \begin{cases} \sum_{\ell\in\mathbb{Z}} \psi_{j,k}(x+\ell) & \text{if } j \ge 0, \\ 1 & \text{if } j = -1. \end{cases}$$

The multidimensional wavelets are then tensorized by

$$\psi_{\boldsymbol{j},\boldsymbol{k}}^{\mathrm{per}}(\boldsymbol{x}) = \prod_{i=1}^{d} \psi_{j_i,k_i}^{\mathrm{per}}(x_i),$$

where $-\boldsymbol{1} \le \boldsymbol{j} \in \mathbb{Z}^d$ is the dilatation and $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{j}}$ is the translation with

$$\mathcal{I}_{\boldsymbol{j}} := \bigtimes_{i=1}^{d} \begin{cases} \{0, 1, \dots, 2^{j_i} - 1\} & \text{if } j_i \ge 0, \\ \{0\} & \text{if } j_i = -1. \end{cases}$$

The periodized wavelets $\psi_{j,k}^{\mathrm{per}}$ as well as their duals $\psi_{j,k}^{*,\mathrm{per}}$ form a **Riesz basis**, i.e., in every level j,

$$\gamma_m \sum_{k=0}^{2^j-1} |d_{j,k}|^2 \le \left\| \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}^{\mathrm{per}} \right\|_{L_2(\mathbb{T})}^2 \le \delta_m \sum_{k=0}^{2^j-1} |d_{j,k}|^2. \quad (1)$$

The Riesz constants γ_m, δ_m for the Chui-Wang wavelets are calculated in [6]. The following lemma states a useful property of the Fourier transform of the Chui-Wang wavelets.

Lemma II.1. *The term* $p(t) := |\hat{\psi}(2\pi t)|^2 (2\pi t)^{2m}$ *is bounded by*

$$p_{\max} := \max_{t\in\mathbb{R}} p(t) = 2^{4m}.$$

Proof. For the Fourier transform $\hat{\psi}(\omega) = \int_{\mathbb{R}} \psi(x)\mathrm{e}^{-\mathrm{i}x\omega}\,\mathrm{d}x$ of the wavelet function, we calculate

$$\begin{aligned} \hat{\psi}(\omega) &= \sum_{r\in\mathbb{Z}} q_r \left(B_m(2x - r - \frac{m}{2})\right)\hat{}(\omega) \\ &= \sum_{r\in\mathbb{Z}} q_r \frac{1}{2} \mathrm{e}^{-\mathrm{i}\frac{\omega}{2}(r+\frac{m}{2})} \hat{B}_m\left(\frac{\omega}{2}\right) \\ &= \frac{1}{2}\sum_{r\in\mathbb{Z}} q_r \mathrm{e}^{-\mathrm{i}\frac{\omega}{2}(r+\frac{m}{2})} \frac{\sin^m(\frac{\omega}{4})}{(\frac{\omega}{4})^m} \\ &= 2^{2m-1}\omega^{-m}\mathrm{e}^{-\mathrm{i}\frac{\omega m}{4}} \sum_{r\in\mathbb{Z}} q_r \mathrm{e}^{-\mathrm{i}\frac{\omega r}{2}} \sin^m\left(\frac{\omega}{4}\right). \end{aligned}$$

Inserting this into the term $p(t)$ gives

$$p(t) = 2^{4m-2}\left|\sum_{r\in\mathbb{Z}} q_r \mathrm{e}^{-\mathrm{i}\pi t r}\sin^m\left(\frac{\pi t}{2}\right)\right|^2,$$
$$= 2^{4m-2}\sin^{2m}\left(\frac{\pi t}{2}\right)\left|\sum_{r\in\mathbb{Z}} q_r \mathrm{e}^{-\mathrm{i}\pi t r}\right|^2.$$

The cardinal B-splines B_m are nonnegative. Since the coefficients q_r have alternating sign, the maximum $p_{\max}$ is attained at all odd integers t, such that

$$p(t) = \begin{cases} 0 & \text{for even } t\in\mathbb{Z}, \\ p_{\max} & \text{for odd } t\in\mathbb{Z}. \end{cases}$$

The term $p_{\max}$ is calculated by

$$p_{\max} = 2^{4m-2}\left(\sum_{r\in\mathbb{Z}}|q_r|\right)^2$$
$$= 2^{2m}\left(\sum_{r\in\mathbb{Z}}\sum_{k=0}^{m}\binom{m}{k}B_{2m}(r+1-k-m)\right)^2$$
$$= 2^{2m}\left(\sum_{k=0}^{m}\binom{m}{k}\right)^2 = 2^{4m}. \qquad \square$$

Especially, the following relation will be useful later on,

$$\left|\hat{\psi}(2\pi t)\right| = \frac{\sqrt{p(t)}}{(2\pi t)^m} \le \frac{2^{m-1}}{(\pi t)^m}\left|\sin^m\left(\frac{\pi t}{2}\right)\right|. \tag{2}$$

III. Function spaces

Define the Fourier coefficients by

$$c_{\boldsymbol{n}}(f) = \int_{\mathbb{T}^d} f(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\langle\boldsymbol{n},\boldsymbol{x}\rangle}\,\mathrm{d}\boldsymbol{x},$$

such that

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} c_{\boldsymbol{n}}(f)\,\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{n},\boldsymbol{x}\rangle}.$$

We introduce the dyadic blocks

$$\mathcal{Q}_{\boldsymbol{q}} := \bigtimes_{i=1}^{d}\begin{cases}\{n\in\mathbb{Z} \mid 2^{q_i-1}\le |n| < 2^{q_i}\} & \text{if } q_i\ge 1. \\ \{0\} & \text{if } q_i = 0. \end{cases} \tag{3}$$

Using these dyadic blocks, we decompose the Fourier series of the function $f\in L_2(\mathbb{T}^d)$ into

$$f = \sum_{\boldsymbol{q}\in\mathbb{N}_0^d}\Phi_{\boldsymbol{q}}(f), \quad \Phi_{\boldsymbol{q}}(f)(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathcal{Q}_{\boldsymbol{q}}} c_{\boldsymbol{k}}(f)\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{k},\boldsymbol{x}\rangle}. \tag{4}$$

Then, we define the **periodic Sobolev norm of dominating mixed smoothness** by

$$\|f\|^2_{H^s_{\mathrm{mix}}(\mathbb{T}^d)} := \sum_{\boldsymbol{q}\in\mathbb{N}_0^d} 2^{2\|\boldsymbol{q}\|_1 s}\|\Phi_{\boldsymbol{q}}(f)\|^2_{L_2(\mathbb{T}^d)} \tag{5}$$
$$\asymp \sum_{\boldsymbol{k}\in\mathbb{Z}^d}|c_{\boldsymbol{k}}|^2\prod_{i=1}^{d}\left(1+|k_i|^2\right)^s.$$

For $s > \frac{1}{2}$, we define the **periodic Besov-Nikolskij norm of mixed smoothness** by

$$\|f\|_{\boldsymbol{B}^s_{2,\infty}(\mathbb{T}^d)} := \sup_{\boldsymbol{q}\in\mathbb{N}_0^d} 2^{\|\boldsymbol{q}\|_1 s}\|\Phi_{\boldsymbol{q}}(f)\|_{L_2(\mathbb{T}^d)}, \tag{6}$$

using the decomposition (4) of f. In the one-dimensional case these spaces are denoted in the usual way as $B^s_{2,\infty}(\mathbb{T})$.

IV. Decay of wavelet coefficients

In the following theorem, we show a characterization of functions in $H^s_{\mathrm{mix}}(\mathbb{T}^d)$ for Chui-Wang wavelets in terms of the decay of the wavelet coefficients. The proof is purely in the Fourier domain. Therefore, we denote the wavelet decomposition $f(\boldsymbol{x}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} f_{\boldsymbol{j}}(\boldsymbol{x})$ with

$$f_{\boldsymbol{j}}(\boldsymbol{x}) := \begin{cases}\sum\limits_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{j}}}\langle f, \psi^{\mathrm{per},*}_{\boldsymbol{j},\boldsymbol{k}}\rangle\psi^{\mathrm{per}}_{\boldsymbol{j},\boldsymbol{k}}(\boldsymbol{x}) & \text{if } \boldsymbol{j}\ge -\boldsymbol{1}, \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

Theorem IV.1. *Let $f\in H^s_{\mathrm{mix}}(\mathbb{T}^d)$, $0 < s < m-\frac{1}{2}$ and $\psi^{\mathrm{per}}_{j,k}$ be the Chui-Wang wavelets of order m. Then,*

$$c\,\|f\|_{H^s_{\mathrm{mix}}(\mathbb{T}^d)} \le \left|\sum_{\boldsymbol{j}\ge -\boldsymbol{1}} 2^{2|\boldsymbol{j}|_1 s}\sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{j}}}|\langle f, \psi^{\mathrm{per},*}_{\boldsymbol{j},\boldsymbol{k}}\rangle|^2\right|^{1/2} \le C\,\|f\|_{H^s_{\mathrm{mix}}(\mathbb{T}^d)},$$

where $|\boldsymbol{j}|_1 = \sum_{i,j_i\ge 0} j_i$ and

$$c = \left(\frac{2^{4m}}{(1-2^{-2s})(1-2^{-2m})\pi^{2m}} + \frac{2^{2m-1}}{\pi^{2m}}\frac{1}{1-2^{2s+1-m}}\right)^{-d/2},$$
$$C = \left(\frac{1}{\gamma_m^{1/2}}\frac{2^s}{2^s-1} + \frac{(2m-1)^{1/2}}{2^{(m-s)}-1}\right)^d.$$

The right inequality holds even for all $s < m$.

Proof. We start with the one-dimensional case and the left inequality. We use the norm given in (5) and the wavelet decomposition in (7),

$$\|f\|^2_{H^s(\mathbb{T})} = \sum_{q=0}^{\infty} 2^{2qs}\|\Phi_q(f)\|^2_{L_2(\mathbb{T})}$$
$$= |c_0(f)|^2 + \sum_{q=1}^{\infty}2^{2qs}\sum_{n\in\mathcal{Q}_q}\left|\sum_{\ell=-1+q}^{\infty}c_n(f_{q+\ell})\right|^2 \tag{8}$$
$$= |c_0(f)|^2 + \sum_{q=1}^{\infty}2^{2qs}\sum_{n\in\mathcal{Q}_q}\left|\sum_{\ell=-1+q}^{\infty}\sum_{k\in\mathcal{I}_j}\langle f,\psi^{\mathrm{per}}_{j,k}\rangle c_n(\psi^{\mathrm{per}}_{j,k})\right|^2.$$

For the Fourier coefficients of $\psi^{\mathrm{per}}_{j,k}$, we calculate for $j\ne -1$

$$c_n(\psi^{\mathrm{per}}_{j,k}) = \int_{\mathbb{T}}\psi^{\mathrm{per}}_{j,k}(x)\,\mathrm{e}^{-2\pi\mathrm{i}nx}\,\mathrm{d}x$$
$$= 2^{-\frac{j}{2}}\int_{\mathbb{R}}\psi(x)\,\mathrm{e}^{-2\pi\mathrm{i}n(x+k)2^{-j}}\,\mathrm{d}x$$
$$= 2^{-\frac{j}{2}}\mathrm{e}^{-2\pi\mathrm{i}\frac{nk}{2^j}}\,\hat{\psi}\left(\frac{2\pi n}{2^j}\right).$$

Denote the vector $\boldsymbol{a}_j = \left(\langle f, \psi^{\mathrm{per},*}_{j,k}\rangle\right)_{k\in\mathcal{I}_j}$ and the matrix $(\boldsymbol{M}_{q,j})_{n,k} = 2^{-j/2}\mathrm{e}^{-2\pi\mathrm{i}\frac{nk}{2^j}}\,\hat{\psi}\left(\frac{2\pi n}{2^j}\right)$. Then, the vector containing the Fourier coefficients of f_j is

$$(c_n(f_j))_{n\in\mathcal{Q}_q} = \boldsymbol{M}_{q,j}\boldsymbol{a}_j,$$

Inserting this into (8) yields

$$\|f\|_{H^s(\mathbb{T})}^2 - |c_0(f)|^2 = \sum_{q=1}^{\infty} 2^{2qs} \left\| \sum_{\ell=-1-q}^{\infty} \boldsymbol{M}_{q,q+\ell} \boldsymbol{a}_{q+\ell} \right\|_2^2$$

$$\leq 2 \sum_{q=1}^{\infty} 2^{2qs} \left(\underbrace{\left\| \sum_{\ell=0}^{\infty} \boldsymbol{M}_{q,q+\ell} \boldsymbol{a}_{q+\ell} \right\|_2^2}_{T_q} + \underbrace{\left\| \sum_{\ell=-q}^{-1} \boldsymbol{M}_{q,q+\ell} \boldsymbol{a}_{q+\ell} \right\|_2^2}_{\tilde{T}_q} \right),$$

where $\|\cdot\|$ of a vector is the Euclidean norm. We investigate the two different terms. We start with $\ell \geq 0$,

$$T_q \leq \sum_{\ell=0}^{\infty} \left\| 2^{\ell s} \boldsymbol{M}_{q,q+\ell} \boldsymbol{a}_{\ell+q} \right\|_2^2 \frac{1}{1-2^{-2s}} \tag{9}$$

$$\leq \frac{1}{1-2^{-2s}} \sum_{\ell=0}^{\infty} 2^{2\ell s} \|\boldsymbol{M}_{q,q+\ell}\|_2^2 \|\boldsymbol{a}_{q+\ell}\|_2^2,$$

where $\|\cdot\|_2$ for a matrix is the spectral norm. For the matrices $\boldsymbol{M}_{q,j}$, which are products of a diagonal matrix and a part of a Fourier matrix, we have with $j = \ell + q$,

$$\|\boldsymbol{M}_{q,j}\|_2^2 \leq \max_{n \in \mathcal{Q}_q} \left| \hat{\psi}\left(\frac{2\pi n}{2^j}\right) \right|^2 \leq \frac{p_{\max}}{(2\pi 2^{\ell-1})^{2m}},$$

such that

$$\sum_{q=0}^{\infty} 2^{2qs} T_q \leq \frac{p_{\max}}{(1-2^{-2s})\pi^{2m}} \sum_{q=0}^{\infty} 2^{2qs} \sum_{\ell=0}^{\infty} 2^{-2m\ell} 2^{2\ell s} \|\boldsymbol{a}_{\ell+q}\|_2^2$$

$$= \frac{2^{4m}}{(1-2^{-2s})(1-2^{-2m})\pi^{2m}} \sum_{j=0}^{\infty} 2^{2js} \|\boldsymbol{a}_j\|_2^2.$$

For the remaining case where $\ell < 0$, we denote by $\mathcal{F}_j = 2^{-j/2}\left(\mathrm{e}^{-2\pi\mathrm{i}\frac{n'k}{2^j}}\right)_{n',k\in\mathcal{I}_j}$ the Fourier matrix and $\hat{\boldsymbol{a}}_j = \mathcal{F}_j \boldsymbol{a}_j$. Then,

$$\boldsymbol{M}_{q,j}\boldsymbol{a}_j = \operatorname{diag}\left(\hat{\psi}\left(\frac{2\pi n}{2^j}\right)\right)\left(\boldsymbol{1}_{2^{-\ell-1}} \otimes \boldsymbol{I}_{2^{q+\ell}}\right)\hat{\boldsymbol{a}}_j,$$

where $\boldsymbol{1}$ is the vector of ones and $\boldsymbol{I}$ is the identity matrix. We rearrange all terms to one linear system of equations, where the matrices $\boldsymbol{M}_{q,j}$ are connected horizontally and the vectors $\hat{\boldsymbol{a}}_j$ are connected vertically by

$$\tilde{T}_q = \left\| \boldsymbol{\Lambda}_q \left(2^{\ell(m-\frac{1}{2})} \hat{\boldsymbol{a}}_{q+\ell} \right)_{\ell=-q,\dots,-1} \right\|_2^2.$$

To show the structure of the matrix $\boldsymbol{\Lambda}_q$, we give an example for $q = 3$ and $m = 2$. For shortened notation, we use $[x] := \hat{\psi}(2\pi x)$ here, such that $\boldsymbol{\Lambda}_3 =$

$$\begin{pmatrix} \overbrace{2^{4.5}[\frac{4}{1}]}^{\ell=-3} & \overbrace{2^{3}[\frac{4}{2}] \quad\quad}^{\ell=-2} & \overbrace{2^{1.5}[\frac{4}{4}] \quad\quad\quad\quad\quad\quad}^{\ell=-1} \\ 2^{4.5}[\frac{5}{1}] & \quad\quad 2^{3}[\frac{5}{2}] & \quad\quad 2^{1.5}[\frac{5}{4}] \quad\quad\quad\quad \\ 2^{4.5}[\frac{6}{1}] & 2^{3}[\frac{6}{2}] \quad\quad & \quad\quad\quad\quad 2^{1.5}[\frac{6}{4}] \quad\quad \\ 2^{4.5}[\frac{7}{1}] & \quad\quad 2^{3}[\frac{7}{2}] & \quad\quad\quad\quad\quad\quad 2^{1.5}[\frac{7}{4}] \end{pmatrix}$$

Every entry of the matrix $\boldsymbol{\Lambda}_q$ is either 0 or $2^{-\ell(m-\frac{1}{2})}\hat{\psi}(\frac{2\pi n}{2^j})$. The spectral norm of this matrix is bounded by Gerschgorins circle theorem for the matrix $\boldsymbol{\Lambda}_q\boldsymbol{\Lambda}_q^*$ by

$$\|\boldsymbol{\Lambda}_q\|_2^2 \leq \max_{n\in\mathcal{Q}_q} \sum_{\substack{n'\in\mathcal{Q}_q \\ n'=2^j t+n}} \sum_{\ell=-q}^{-2} 2^{-2\ell m+\ell} \left|\hat{\psi}\left(\frac{2\pi n}{2^j}\right)\right| \left|\hat{\psi}\left(\frac{2\pi n'}{2^j}\right)\right|$$

$$\overset{(2)}{\leq} \frac{2^{2m-2}}{\pi^{2m}} \max_{n\in\mathcal{Q}_q} \frac{2^{2qm}}{n^m} \sum_{\substack{n'\in\mathcal{Q}_q \\ n'=2^j t+n}} \sum_{\ell=-q}^{-2} 2^{\ell} \frac{1}{(n')^m} \left|\sin^{2m}\left(\frac{\pi n}{2^{j+1}}\right)\right|$$

$$= \frac{2^{2m-2}}{\pi^{2m}} \max_{n\in\mathcal{Q}_q} \frac{2^{2qm}}{n^m} \sum_{\ell=-q}^{-2} 2^{\ell} \sum_{t=0}^{2^{-\ell-1}} \frac{1}{(n+2^{q+\ell}t)^m} \left|\sin^{2m}\left(\frac{\pi n}{2^{j+\ell}}\right)\right|$$

and because of the periodicity of the sin function,

$$= \frac{2^{2m-2}}{\pi^{2m}} \max_{\substack{n\in\mathcal{Q}_q \\ n=2^k p}} \frac{2^{2qm}}{n^m} \sum_{\ell=-q+k}^{-2} 2^{\ell} \left|\sin^{2m}\left(\frac{\pi 2^k p}{2^{j+\ell}}\right)\right| \sum_{t=0}^{2^{-\ell-1}} \frac{1}{(n+2^{q+\ell}t)^m},$$

the last sum is bounded by the number of summands and the maximum,

$$\leq \frac{2^{2m-2}}{\pi^{2m}} \max_{\substack{n\in\mathcal{Q}_q \\ n=2^k p}} \frac{2^{2qm}}{n^m} \sum_{\ell=-q+k}^{-2} 2^{\ell} \left|\sin^{2m}\left(\frac{\pi 2^k p}{2^{j+\ell}}\right)\right| 2^{-\ell-1}\frac{1}{n^m},$$

$$\leq \frac{2^{2m-2}}{\pi^{2m}} \max_{\substack{n\in\mathcal{Q}_q \\ n=2^k p}} \frac{2^{2qm}}{n^{2m}} \sum_{\ell=0}^{q} \left|\sin^{2m}\left(\frac{\pi}{2^{\ell}}\right)\right| \leq \frac{2^{2m-1}}{\pi^{2m}}.$$

We conclude the case $\ell < 0$ with

$$\sum_{q=0}^{\infty} 2^{2qs} \tilde{T}_q \leq \sum_{q=0}^{\infty} 2^{2qs} \|\boldsymbol{\Lambda}_q\|_2^2 \left\| \left(2^{\ell(m-\frac{1}{2})}\hat{\boldsymbol{a}}_{q+\ell}\right)_{\ell=-q,\dots,-1} \right\|_2^2$$

$$\leq \sum_{q=0}^{\infty} 2^{2qs} \|\boldsymbol{\Lambda}_q\|_2^2 \sum_{\ell=-q}^{-1} 2^{\ell(2m-1)} \|\hat{\boldsymbol{a}}_{q+\ell}\|_2^2$$

$$\leq \frac{2^{2m-1}}{\pi^{2m}} \sum_{j=0}^{\infty} 2^{2js} \|\hat{\boldsymbol{a}}_j\|_2^2 \sum_{\ell=-\infty}^{-1} 2^{\ell(2m-2s-1)}$$

$$\leq \frac{2^{2m-1}}{\pi^{2m}} \frac{1}{1-2^{2s+1-2m}} \sum_{j=0}^{\infty} 2^{2js} \|\boldsymbol{a}_j\|_2^2 \quad \text{if } s < m - \tfrac{1}{2}.$$

For the multivariate result, we have to sum over all vectors $\boldsymbol{\ell} \in \mathbb{Z}^d$. We distinguish for every ℓ_i, $i \in [d]$ separately, if it is positive or negative. Then, we apply the one-dimensional inequalities in every direction separately, which leads to the exponent d in the constants.

The right inequality follows from [4, Thm. 3.9]. □

With the same argumentation and replacing the sum over j by a supremum, the following result for the Besov spaces holds true.

Theorem IV.2. *Let $f \in \boldsymbol{B}^s_{2,\infty}(\mathbb{T}^d)$, $0 < s < m - \frac{1}{2}$ and $\psi^{\mathrm{per}}_{\boldsymbol{j},\boldsymbol{k}}$ be the Chui-Wang wavelets of order m. Then,*

$$c\,\|f\|_{\boldsymbol{B}^s_{2,\infty}(\mathbb{T}^d)} \leq \sup_{\boldsymbol{j}\geq -1} 2^{|\boldsymbol{j}|_1 s} \left| \sum_{k\in\mathcal{I}_{\boldsymbol{j}}} |\langle f, \psi^{\mathrm{per},*}_{\boldsymbol{j},\boldsymbol{k}}\rangle|^2 \right|^{1/2} \leq C\,\|f\|_{\boldsymbol{B}^s_{2,\infty}(\mathbb{T}^d)},$$

with the constants from Theorem IV.1. The right inequality holds even for all $s < m$.

V. Counterexamples for $m - \frac{1}{2} \leq s \leq m$

The left inequalities in Theorems IV.1 and IV.2 are shown only for $s < m - \frac{1}{2}$. Indeed, for bigger smoothness parameter $m - \frac{1}{2} < s \leq m$, one wavelet function $f = \psi_{j,k}^{\mathrm{per}}$ has a finite wavelet decomposition, but is neither in $H^s(\mathbb{T})$ nor in $B^s_{2,\infty}(\mathbb{T})$. For the special case $s = m - \frac{1}{2}$, the wavelet functions $f = \psi_{j,k}^{\mathrm{per}}$ are not in $H^s(\mathbb{T})$, but in $B^s_{2,\infty}(\mathbb{T})$. In [3, Thm. 1.10], a counterexample for the Haar wavelets $m = 1$ was given. However, there is also a counterexample for higher orders m, such that the left inequality in Theorem IV.2 does not hold, as the following calculation shows.

To define the function f, we choose the wavelet coefficients

$$\boldsymbol{a}_j := \left(\langle f, \psi_{j,k}^{\mathrm{per},*} \rangle \right)_{k \in \mathcal{I}_j} = 2^{-j(s+1/2)} \, \mathbf{1}_{2^j},$$

such that

$$\sup_{j \geq -1} 2^{2|j|_1 s} \sum_{k \in \mathcal{I}_j} |\langle f, \psi_{j,k}^{\mathrm{per},*} \rangle|^2 = 1.$$

Then, by using the substitution $j = q + \ell$,

$$(\boldsymbol{M}_{q,j} \boldsymbol{a}_j)_n = \begin{cases} 2^{-js} \hat{\psi}(2\pi t) & \text{if } n = 2^j t, t \in \mathbb{N}, t \text{ odd}, \\ 0 & \text{otherwise} \end{cases}$$

where in every row of the matrix $\boldsymbol{\Lambda}_q$ is only one non-zero entry, such that

$$\left\| \sum_{\ell=-q}^{-1} \boldsymbol{M}_{q,j} \boldsymbol{a}_j \right\|_2^2 = \sum_{n=2^{q-1}}^{2^q - 1} \left| \sum_{\substack{\ell=-q \\ n = 2^{\ell+q} t}}^{-1} 2^{-js} \frac{1}{(2\pi t)^m} (-1)^{\frac{t-1}{2}} \sqrt{p_{\max}} \right|^2$$

$$= \sum_{\substack{n=2^{q-1} \\ n = 2^k t}}^{2^q - 1} 2^{-2ks} \frac{p_{\max}}{(2\pi t)^{2m}}.$$

Taking then the supremum over the index q yields,

$$\|f\|_{B^s_{2,\infty}(\mathbb{T})} \geq \sup_{q \geq 0} 2^{2qs} \left\| \sum_{\ell=-q}^{-1} \boldsymbol{M}_{q,j} \boldsymbol{a}_j \right\|_2^2$$

$$= \sup_{q \geq 0} 2^{2qs} \sum_{\substack{n=2^{q-1} \\ n=2^k t}}^{2^q-1} 2^{-2ks} \frac{p_{\max}}{(2\pi t)^{2m}}$$

$$= \frac{2^{2m}}{\pi^{2m}} \sup_{q \geq 0} 2^{2qs} \sum_{k=0}^{q-1} 2^{-2ks} \sum_{\substack{t=1 \\ t \text{ odd}}}^{2^{q-k}-1} \frac{1}{(2\pi t)^{2m}}$$

$$= \frac{2^{2m}}{\pi^{2m}} \sup_{q \geq 0} 2^{2qs} \sum_{k=0}^{q-1} 2^{-2ks} \sum_{\substack{t=1 \\ t \text{ odd}}}^{2^{q-k}-1} \frac{1}{t^{2m}}$$

by estimating the sum from above by the minimal summand and the number of summands,

$$\geq \frac{2^{2m}}{\pi^{2m}} \sup_{q \geq 0} 2^{2qs} \sum_{k=0}^{q-1} 2^{-2ks} 2^{-2m(q-k)} 2^{q-k-1}$$

$$= \frac{2^{2m-1}}{\pi^{2m}} \sup_{q \geq 0} \sum_{k=0}^{q-1} 2^{(q-k)(-2m+2s+1)}.$$

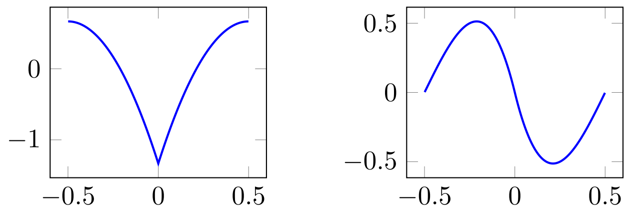

Fig. 1: Counterexample functions for $m = 2$ (left) and $m = 3$ (right), which are not in $B^s_{2,\infty}(\mathbb{T})$, but the weighted sum of the wavelet coefficients is finite.

This term is infinite in the case where $s = m - \frac{1}{2}$, such that $f \notin B^{m-1/2}_{2,\infty}(\mathbb{T})$.

The counterexample shows that in the boundary case $s = m - \frac{1}{2}$ for $f \in B^{m-1/2}_{2,\infty}(\mathbb{T})$ the Besov norm and the wavelet norm are not equivalent. It is still an open problem if for $f \in H^{m-1/2}(\mathbb{T})$ the Sobolev and the wavelet norm are equivalent, since the wavelet functions $\psi_{j,k}^{\mathrm{per}}$ are not included in this space.

VI. Conclusion

Theorems IV.1 and IV.2 offer significant insights into the characterization of functions in mixed Sobolev spaces and Besov spaces respectively, using Chui-Wang wavelets. These results establish rigorous equivalences between the norms of these function spaces and their representations in terms of wavelet coefficients for $m - \frac{1}{2} < s < m$. The decomposition of f into wavelet coefficients inherently captures both the smoothness of the function (via the scaling factor $2^{2|\boldsymbol{j}|_1 s}$) and its hierarchical structure across different scales. This makes wavelets particularly suited for adaptive methods and sparse approximations.

Here, we presented sharp characterizations for Chui-Wang wavelets, especially for estimating the sum of the wavelet coefficients from below by a Sobolev or Besov norm. For related inequalities with more complicated assumptions to the wavelets and the parameters, see [7] for higher order spline wavelets, [2] for Faber splines, [9] for orthonormal spline wavelet systems and [8] for anisotropic hyperbolic Besov and Triebel-Lizorkin spaces. Here, we even give the constants explicitly. The special case $m = 1$ for the Chui-Wang wavelets is the Haar basis, for which characterizations are for example studied in [5], [10].

We want to point out that we found a counterexample, that the left inequality in Theorem IV.2 does not hold in the case $s = m - 1/2$, such that we end up with sharp inequalities for the parameter s. For this boundary case and the Haar wavelets in [10], results are given only for the spaces $\boldsymbol{B}^s_{p,q}$ with $q < \infty$.

Acknowledgments

The author would like to thank Daniel Potts and Tino Ullrich for many valuable suggestions.

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