Causal Representation Meets Stochastic Modeling under Generic Geometry

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Abstract

We investigate the identifiability problem of latent stochastic processes characterized by high dynamics that occur in continuous time with varying intensities (e.g., a multivariate Hawkes process), and we provide the corresponding identifiability theory. Building on this theoretical foundation, we implement MUTATE, a variational autoencoder framework with a time-adaptive transition module to evaluate stochastic dynamics on both synthetic stochastic processes and real-world biological signal data. This work advances causal representation learning theory by extending it to continuous-time and stochastic settings via weak topology and algebraic signature, highlighting the importance of this approach in addressing scientific questions, such as the accumulation of mutations in genomics and the mechanisms driving neuron spike triggers in response to time-varying dynamics.

1 Introduction

Inferring causal relationships among variables from observations capitalizes the potential of machine learning to advance scientific discovery, as it reveals underlying mechanisms that are not identifiable from observational distributions alone [1]. However, because of limited data sources and the challenge of interpreting high-dimensional perceptual data, causal variables with their graphical structure are often unknown and thus can be learned in a non-interpretable manner, which causes the difficulty of identifiability [2, 3]. Recently, a growing number of studies on the disentanglement of latent causal representation advance the identifiability guarantee and propose methods for latent causal variable estimation. Seminal works among them establish identifiability by leveraging sufficient variability in latent distribution due to multiple-source data [4, 5], auxiliary variable [2, 6, 7, 8], or intervention to a latent causal graph [9, 10, 11, 12, 13].

Most concurrent work aforementioned aims at recovering the latent causal variables in the time-series; A common condition they heavily hinge on is step-wise conditional independence [4, 5], termed as the number of time lags in their claims, among variables. These works mainly address cases when the mixing is assumed invertible, such that latent variables can be recovered up to component-wise indeterminacy. However, the latent dynamics driven by a stochastic process or system of stochastic differential equations are less explored. For example, in biology, fatal diseases such as cancer are principally caused by cumulative multiple mutations found in driver genes as the colonial expansion proceeds. In neuroscience, the latent event dynamics trigger visible biological signals [14, 15]. Finding cancer-associated mutational genes and tracking their behavior through their representation has been given much more paramount importance in recent few decades [16, 17, 18]. Therefore, a formal theoretical guarantee for its identifiability is missing for stochastic causal dynamics and their intervention effects.

This paper aims to establish the identifiability of latent spaces governed by stochastic dynamics driven by the intensity λ_t. Our main results demonstrate that such stochastic processes are compatible with

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the current causal representation learning framework and converge to an equivalent infinite-order INAR process, provided that an integrated fluctuation term is added. Building on this foundation, the identifiability of dynamic processes is ensured by precisely controlling the geometry of the latent space via the subtle algebraic structure of cumulants.

2 Causal disentanglement in stochastic process

2.1 Generative model for stochastic process

Let $O_t \in \mathbb{R}^n$ be observable data, and $Z_t \in \mathbb{R}^p$ be a latent process. O_t is being generated from latent point processes Z_t through an unknown mixing function f. $A^{\otimes d}$ denotes the tensor/Kronecker product. In a time process, Φ denotes the transition operator (e.g., an autoregressive coefficient matrix or continuous kernel matrix) and symbol $\star$ denotes the convolution operator. We formally set up our problem to learn dynamics of Z_t from O_t in Def. 2.1 .

Definition 2.1. Consider a series of latent processes that are stochastic and self-exciting as time increases, that is, $Z_t := N_t$, the cumulative counting process at time t. Suppose we only have access to O_t without knowledge about Z_t. The conditional intensity λ_t of N_t [19] and generative model of O_t is written as:

$$O_t = f(N_t(\Delta)),\ \lambda_t = h(u + \Phi_t \star dN_t) \tag{1}$$

Then, the objective is to recover f, λ_t as well as its causal structure Φ

$h(\cdot)$ is an unknown function reflecting how Φ_t and dN_t are mixed under a regular convolution. A simple choice $h(\cdot) = x$ immediately reduces a convolution to linear point processes. ϕ is one element of Φ_t, an integrated kernel matrix and dN_t^i the measure of a counting process N_t^j, as well as the integral measure in Itô calculus. The counting process N_t^i and the conditional intensity λ_t^t satisfy: $N^i_{t+\Delta t} - N^i_t = N^i_{\Delta t} = dN^i_t$ and $\lambda^i_t = \frac{\mathbb{E}[dN^i_t|\mathcal{F}_t]}{dt}$. For a point process to be well-defined, non-trivial constraints are needed, one of which is the stationary condition, an assumption widely adopted in most stochastic process literature to ensure a unique process. We summarize the necessary conditions to define an inhomogeneous point process in A.4.

2.2 A graph isomorphism to causal kernels

The seminal result in [20], establish the weak convergence of continuous stochastic processes under the corresponding weak topology. In particular, for any compact interval $[a, b]$, a subsequence INAR process converges to the point process N_t. This convergence is crucial for constructing a causal graph structure compatible with the term causal representation in our paper.

Lemma 1 (Bounding Point Process in Variational Approximation)**.** *Let $N_t \in \mathbb{R}^p$ be a multivariate point process whose conditional intensity function is governed by a convolution structure described in Eq. (1). Suppose the noise term ϵ_t is mean-zero and mutually independent, then the intensity model admits the following weak convergence:*

$$Z^{(\Delta)} := \mathrm{INAR}(p) \xrightarrow{w} N, \quad p \to \infty \tag{2}$$

We then show that Φ in $\mathrm{INAR}(p)$ admits an augmented DAG structure $\mathscr{G}_K$ in Lem. 2 .

Lemma 2. *Given a bipartite graph of the proposed point process with Φ, it admits a* kernel DAG*, denoted by $\mathscr{G}_K$, corresponding to a matrix $\mathcal{M}_{\mathscr{G}_K} \in \mathbb{R}^{2p\times 2p}$ such that $\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K}$ is invertible. Consequently, its inverse can be expressed as a finite order k expansion of $\mathcal{M}_{\mathscr{G}_K}$,*

$$(\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K})^{-1} = \sum_{i=0}^{k} \mathcal{M}^i_{\mathscr{G}_K}, \quad k \le p, \quad \mathcal{M}_{\mathscr{G}_K} = \begin{bmatrix} \mathcal{M}_{\mathscr{G}_K}[U] & \Phi \\ 0 & \mathcal{M}_{\mathscr{G}_K}[V] \end{bmatrix}$$

where k corresponds to the length of the longest path in the DAG and $\mathcal{M}_{\mathscr{G}_K}[U] = \mathcal{M}_{\mathscr{G}_K}[V] = \mathbf{0}_{p\times p}$

3 Recovery from algebraic signature of mixed manifolds

The endowed topological structure in $\mathcal{M}_{\mathscr{G}}$ leads to a spectrum of polynomials q_i, and an algebraic variety $V(\mathcal{I})$ is associated with the generating ideals $\mathcal{I}$ of those polynomials. We emphasize that the dimension of $V(\mathcal{I})$ determines the identifiability through algebraic quantities.

3.1 Algebraic structure for stochastic causal representation

Cumulant is an important algebraic signature of its geometric property, as a full order cumulant precisely encodes the entire distribution, including the component-wise and time-wise dependency among variables. This enables the fine-grained mathematical nature of intervention effects beyond traditional mean and variance shifts. Under generic (non-Gaussian) conditions, the d-th order cumulant of a random variable $X = As$ admits closed-form expressions as $\kappa_d(X) = \sum_{j=1}^{p} \kappa_d(s)(A_j)^{\otimes d}$, which is a secant variety $\sigma_k(X)$ that views each tensor factor $(A_j)^{\otimes d}$ as indeterminate. If the matrix A is generic in an open set, the Kruskal rank condition is satisfied and thus $\sum_{j=1}^{p} \kappa_d(s)(A_j)^{\otimes d}$ has a unique decomposition. This means the idea $\mathcal{I} : \langle \kappa_d(X) - \sum_{j=1}^{p} \kappa_d(s)(A_j)^{\otimes d} \rangle$ has dimension zero.

We connect our reasoning to this and illustrate how high-order cumulants can capture sufficient statistical variability in the system, even under temporally independent Gaussian noise. In particular, we adapt the setting in [21], modifying it to allow additive noise that is independent over time. To this end, we establish a result for full recovery of INAR processes in the asymptotic regime $p \to \infty$, by formulating identifiability conditions through algebraic-geometric constraints imposed on the latent space.

3.2 Generic identification from algebraic structure

Let J_f be the Jacobian matrix of f and $K = J_f(\mathbb{I}_p - \Phi)^{-1}$. $J_{\mathscr{G}}$ and $K_{\mathscr{G}}$ are augmented matrices by filling J_f, K in a larger matrix to match the dimension of $\mathcal{M}$. We present the following assumption.

Assumption 1.

1. *f is a generic C^d map with a full rank J_f almost surely.*
2. *There exist at least p nonzero tensors $\bar{\kappa}_d(\Delta O_t)$ for $d \in D := \{d \mid \bar{\kappa}_{d+1}(\Delta O_t) = 0\}$, where $\bar{\kappa}_d(\Delta O_t)$ is the difference cumulant truncated at the first-order Taylor expansion of $O_t := f(Z_t)$.*
3. *The ideal $\mathcal{I}^\star : \langle J_{\mathscr{G}} - K_{\mathscr{G}}^{(k)}(\mathbb{I}_{2p} - \mathcal{M}^{(k)}), k = 1, 2, \cdots, p \rangle$ has a zero-dimensional associated variety $\mathcal{V}(\mathcal{I}^\star)$*

Theorem 1. *Under Assum. 1, the latent sources with their causal structure are identifiable up to permutation and component-wise scaling.*

The C^d assumption is strictly weaker than requiring f to be a diffeomorphism, since even in the presence of directional collapse within the latent space, the cumulants may still faithfully transmit the non-redundant dependency structure to the observed domain. The core idea of our proof leverages the propagation behavior of algebraic structure under nonlinear transformations, allowing us to identify latent structure via the observed partial geometric-algebraic information on the mixed manifold O_t. That is, the cumulant hierarchy of each observed component has finite depth d, and the non-vanishing cumulants up to this order are sufficiently rich to ensure identifiability via tensor decomposition. The proposed rank condition (1) is classic and results in a generically unique decomposition of d order tensor $\kappa_d(O_t)$, which uniquely recovers the component v_i up to permutation and rescaling. Condition (2) ensures we can find such p different tensors so that $J_{\mathscr{G}}$ and $(\mathbb{I}_{2p} - \mathcal{M})^{-1}$ can be further disentangled up to the same indeterminacy. The proof strategy converts the identifiability problem into the precise geometry of latent manifolds associated with the time-dependent process. See the proof in B.1.

4 Recovery from MUTATAE

4.1 Architecture of MUTATE

Building upon our identifiability theory, we formally introduce MUTATE (**MU**lti -**T**ime **A**daptive **T**ransition **E**ncoder), a novel estimation framework for latent multivariate self-exciting point processes. MUTATE is designed as a causal representation learning architecture capable of modeling continuous-time stochastic dynamics. Importantly, the framework is modular and can be readily adapted to other types of stochastic processes with suitable modifications. Unlike prior frameworks that rely primarily on conditional independence to enforce latent structure, our approach accounts for the

nature of progressively adaptive stochastic processes. In such systems, the filtration $\mathcal{F}_T$, which captures the intrinsic history of the process, is defined as $\sigma\left(\bigcup_{0<t<T}\sigma(Z_t^{\Delta})\right)$ and grows strictly over time (Figure. 1). This dynamically expanding information structure poses unique challenges for both identifiability and representation learning, which MUTATE is explicitly designed to address.

Time adaptive transition module. We first employ an encoder $q_\phi(Z_t^{\Delta}|X_t^{\Delta})$ to learn the estimated latents $Z_t^{(\Delta)} \sim q_\phi(Z_t^{\Delta} \mid X_t^{\Delta})$ as commonly applied in representation learning frameworks. Recall that the latent process is modeled as $Z_t = \Phi \star Z_t + R_t$, where Φ denotes a global convolution kernel and $R_t = U + \epsilon_t$ is a residual process. Under our weak convergence condition, Z_t receives a well-structured representation $Z_t = (\mathbb{I} - \Phi)^{-1} \star (U + \epsilon_t)$, which is also a $(U + \epsilon_t)$-measurable process with tractable Power Spectrum Density (PSD): $S_{Z_t^{(\Delta)}}(w) = (I - \Phi)^{-1}\Sigma_{R_t}(I - \Phi)^{-H}$, where the baseline U is treated as a learnable parameter in the model and A^H is the Hamilton conjugate transpose of A. w is a continuous frequency variable. The inverse mapping f_Z^{-1} is an encoder with training parameters of neural networks. Importantly, the learned functional f maps the observation Z_t to a space of independent varying noise through the designated PSD decomposition that enforces Σ to be diagonal and recursively infers $H^\dagger = (\mathbb{I} - \Phi)^{-1}$. Then, the evaluated prior from the PSD module is sent to calculate the KL divergence. Decomposing $S(w)$, each of the transitions satisfies $\log p(Z_t^{(\Delta)}|\mathcal{F}_{t-}) = \log p[(I - \Phi) \star R_t^{\Delta}]$, which is the main part of the latent prior estimation. Our model is trained based on the Variational Auto-encoding framework. Therefore, we aim to maximize the log likelihood of observation $\log p_{data}(X)$ through the rule of the ELBO lower bound: $ELBO = -\mathcal{L}_{recon} - \alpha\mathcal{L}_{KL}$.

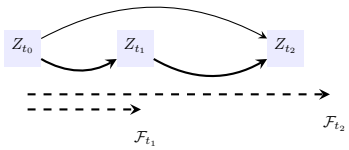

Figure 1: visualization of information loss in increasing filtration

4.2 Simulation Study

To validate our identifiability results, we evaluate against several representative baselines, including TDRL [4], BetaVAE [22], SlowVAE [23], and PCL [7]. Among them, PCL and TDRL incorporate temporal dependencies by leveraging historical information and explicitly enforcing conditional independence among latent variables to recover underlying dynamics. In contrast, BetaVAE and SlowVAE assume independent latent components and disregard any time-delayed mechanisms. A detailed simulation procedure is included in D.1.

Performance of all baselines and our model is shown in Table 1 and extended results are reported in Table A2. During training, both BetaVAE and SlowVAE tend to converge prematurely, typically reaching a local optimum within the first epoch and triggering early stopping. This behavior highlights their limitations in modeling temporal structures essential for identifying latent event-driven processes. TDRL performs reasonably when the lag module is set to a longer one (we use $L = 9$ in experiments) since it can harness shorter temporary contextual information. It is noticed that our identifiability can be readily applied to the prior framework by either adding the domain index in synthetic datasets or modulating the distribution shifts that change pairs of edges in the latent space. However, we also realize that the fully non-parametric setting is hard to interpret since our identifiability avoids such a case.

Table 1: MCC Scores with standard deviations for five kernels

Method	Ave.	Exponential	Powerlaw	Rectangular	nonlinear	nonparametric
TDRL [4]	0.599	0.593±0.028	0.609±0.043	0.618±0.056	0.556±0.016	**0.616**±0.043
BetaVAE [22]	0.141	0.153±0.863	0.128± 0.077	0.128±0.078	0.146±0.108	0.149±0.096
SlowVAE [23]	0.115	0.108±0.075	0.104±0.073	0.104±0.073	0.126±0.074	0.131±0.076
PCL [7]	0.375	0.395±0.034	0.330±0.029	0.330±0.029	0.414±0.028	0.404±0.028
MUTATE(ours)	**0.837**	**0.853**±0.218	**0.938**±0..036	**0.879**±0.102	**0.921**±0.029	**0.598**±0.013

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Supplement to

"Causal Representation Meets Stochastic Modeling"

Appendix organization:

A Useful Lemmata

A.1 Preliminary lemmas

Lemma A.1 (Weak Convergence [24]). *Let $(S, \mathcal{S})$ be a Polish space equipped with its Borel σ-algebra, and let $\{Z_n\}_{n\in\mathbb{N}}$ and Z be S-valued random elements defined on a common probability space. Then the sequence $\{Z_n\}$ converges in distribution (i.e., weakly) to Z, denoted $Z_n \Rightarrow Z$, if and only if*

$$\lim_{n\to\infty} \mathbb{E}[f(Z_n)] = \mathbb{E}[f(Z)]$$

for all bounded continuous functions $f : S \to \mathbb{R}$.

The proof and demonstration of this lemma is classic in basic probability that we omit here. The weak convergence, in most cases, corresponds to the convergence of finite dimension distribution of a process or a variable.

Lemma A.2 (Tightness of the Measure $\mathbb{P}_{Z^{(\Delta)}}$). *Let $\{Z_n^{(\Delta)}\}_{n\in\mathbb{N}}$ be a sequence of S-valued random elements (e.g., stochastic processes or path evaluations) indexed by Δ and defined on a Polish space S with Borel σ-algebra. Then the sequence of corresponding probability measures $\{\mathbb{P}_{Z_n^{(\Delta)}}\}$ is tight. In particular, any subsequence admits a further weakly convergent subsequence.*

Tightness of a sequence of probability measures ensures the existence of well-behaved subsequences: every subsequence admits a further weakly convergent subsequence. This property is particularly useful in Polish spaces, where tightness is equivalent to relative compactness (precompactness) under the weak topology. However, it is important to note that precompactness does not imply full compactness; in general, a tight sequence need not converge without an additional uniqueness or limit identification argument. Thus, tightness provides necessary control over subsequential behavior, but does not guarantee full convergence of the entire sequence.

Lemma A.3 (Higher-Order Moment Bound Implies Lower-Order Bounds). *Let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables defined on a common probability space. Fix an integer $d > 0$. Suppose there exists a constant $C > 0$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|Z_n|^d] \le C.$$

Then for any $0 < p < d$, there exists a constant $C_p > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|Z_n|^p] \le C_p.$$

A.2 Background of Hawkes Process

Assumption A.4 (Stability and stationary Increment, Proposition 1 in [25]). *The process N_t has asymptotically stationary increments, and intensity λ_t is asymptotically stationary if the kernel satisfies the assumption:*

$$\rho_{\Phi(t)} = \|\Phi(t)\| = \int_0^t |\Phi(t)|\, dt \text{ has spectral radium smaller than 1} \tag{A1}$$

Asm. A.4 gives a necessary condition so that the point process has stable, stationary increments in its intensity. In particular, it means the entire process tends to be stable with an unknown but fixed expectation of the conditional intensity $\mathbb{E}[\lambda_t^i] = \Lambda^i$. Restricted by the stationary increment assumption, the existence of the corresponding process is ensured by Lem. 3. To illustrate those conditions, we show a simpler version kernel in Example 1.

Lemma 3 (Proposition 6 in [20]). *If all conditions and results in Asm. A.4 hold almost everywhere, there exists only one determined process whose dynamics match observations with regard to Λ^i.*

Example 1. Consider a point process whose kernel functions relay causal influence with an exponential decay to other processes. The generating process thus be accordingly

$$\lambda_t^i = u^i + \sum_{j=1}^{p} \int_0^t \alpha^{ij} e^{-\beta(t-t')}\, dN_{t'}^j$$

shows the exponential kernel triggers influences that are sustaining but decaying as time proceeds. Technically, the induced causal influences, although decaying from inside the system dynamics, will not disappear unless the causal strength $\alpha = 0$ for all j.

A.2.1 Remarks on the filtration

In probability theory, the filtration $\mathcal{F}_t$ is defined as the smallest σ-algebra that renders the intensity process λ_t to be $\mathcal{F}_t$-adapted and measurable. This filtration is constructed by the minimal closure under set operations (e.g., union, intersection) over past events, ensuring that λ_t evolves consistently with the observable history [26, 27]. Therefore, for any filtration as its internal history, we have $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $s \leq t$. Note that the filtration $\mathcal{F}_t$ may theoretically differ from the intrinsic history $\mathcal{H}_t$, which introduces additional challenges in the evaluation and modeling of point processes. For a comprehensive discussion on scenarios where $\mathcal{F}_t$ and $\mathcal{H}_t$ are defined differently, we refer the interested reader to [27]. We occasionally overload the notation dN_t^i, which represents an integral element in stochastic calculus, to distinguish it from its deterministic counterpart. Despite potential similarities in notation, they are fundamentally different: while standard calculus considers infinitesimal increments over fixed mesh widths (e.g., $dg(x)$ as $\Delta t \to 0$), the increment dN_t^i is a random variable governed by the stochastic process. Specifically, its realization at each infinitesimal interval is drawn from a Bernoulli process with intensity λ_t^i, such that $\mathbb{P}(dN_t^i > 0 \mid \mathcal{F}_t) = \lambda_t^i, dt$. In contrast to deterministic differentials, dN_t^i encapsulates the uncertainty of event occurrences within each interval. The kernel matrix Φ_t consists of time-decaying kernel functions that transmit the influence of past events across processes. It captures both time-delayed and causal dependencies, and plays a central role in modeling self-exciting or mutually-exciting dynamics.

A.3 Cumulants and Tensors

Cumulant tensor notation. The d-th order cumulant tensor of a random vector $X \in \mathbb{R}^p$ is denoted $\kappa_d(X) \in \mathbb{R}^{p \times \cdots \times p}$, and is symmetric in all modes. In ICA and CRL settings, cumulants of independent components often admit a CP form:

$$\kappa_d(X) = \sum_{r=1}^{R} \lambda_r \cdot v_r^{\otimes d},$$

where $v_r \in \mathbb{R}^p$ and $\lambda_r \in \mathbb{R}$. This structure enables identifiability of latent sources from cumulant information.

Tensor notation and operations. We denote an order d tensor as $\mathcal{T} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_d}$. The outer product $u^{(1)} \otimes \cdots \otimes u^{(d)} \in \mathbb{R}^{I_1 \times \cdots \times I_d}$ produces a rank-1 tensor with entries:

$$\mathcal{T}_{i_1,\ldots,i_d} = u_{i_1}^{(1)} \cdots u_{i_d}^{(d)}.$$

Given a tensor $\mathcal{T} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ and a matrix $U \in \mathbb{R}^{J \times I_n}$, the *mode-$n$* product $\mathcal{T} \times_n U \in \mathbb{R}^{I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N}$ is defined as:

$$(\mathcal{T} \times_n U)_{i_1,\ldots,i_{n-1},j,i_{n+1},\ldots,i_N} = \sum_{i_n=1}^{I_n} \mathcal{T}_{i_1,\ldots,i_N} \cdot U_{j,i_n}.$$

B Proof of Identifiability Theory

B.1 Proof of Thm. 1

B.1.1 Useful Lemmas

To potentially identify any latent components of dynamics, we must introduce tensor algebra beyond our current setting as we present the following important results.

Corollary B.1 (CP decomposition)**.** *Let $\mathcal{T} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ be an order-N tensor. We say that $\mathcal{T}$ admits an exact rank-R* Canonical Polyadic (CP) *decomposition if there exist component vectors $a_r^{(n)} \in \mathbb{R}^{I_n}$ for each $r = 1, \ldots, R$, $n = 1, \ldots, N$, such that:*

$$\mathcal{T} = \sum_{r=1}^{R} a_r^{(1)} \otimes a_r^{(2)} \otimes \cdots \otimes a_r^{(N)} = [\![A^{(1)}, A^{(2)}, \ldots, A^{(N)}]\!],$$

Where $A^{(n)} = [a_1^{(n)}\ a_2^{(n)}\ \cdots\ a_R^{(n)}] \in \mathbb{R}^{I_n \times R}$ are the factor matrices.

Corollary B.2. *Let $X^{(1)}, X^{(2)}, \ldots, X^{(n)} \in \mathbb{R}^p$ be independent random vectors with nonzero d-th order cumulants, such that each admits the form*

$$\kappa_d(X^{(i)}) = \lambda_i \cdot v_i^{\otimes d}, \quad \text{for } i = 1, \ldots, n,$$

with $v_i \in \mathbb{R}^p$ and $\lambda_i \in \mathbb{R} \setminus \{0\}$. Let $\mathcal{T} := \kappa_d(X^{(1)} + \cdots + X^{(n)}) \in \mathbb{R}^{p \times \cdots \times p}$ be the d-th order cumulant tensor of their sum.

Assume that the matrix $V = [v_1\ v_2\ \cdots\ v_n] \in \mathbb{R}^{p \times n}$ satisfies

$$\operatorname{krank}(V) \geq \left\lceil \frac{2n + (d-1)}{d} \right\rceil.$$

Then the CP decomposition

$$\mathcal{T} = \sum_{i=1}^{n} \lambda_i \cdot v_i^{\otimes d}$$

is unique up to scaling and permutation.

B.1.2 Proof of Thm. 1

We prove Thm. 1 by showing that the tuple (f, Φ, U) is identifiable up to a component-wise transformation and permutation. Our proof is based on the dimension of the associated variety defining special hypersurfaces in a polynomial ring K^n. We study the nonlinear propagation of the cumulant structure to find the identifiability conditions for INAR processes. Given a generic nonlinear f, its exact cumulant $\kappa_d(O_t)$ follows an order d expansion of with Bell polynomial coefficients. For a smooth map $f : Z_t \to f(Z_t)$, we can construct $O_\cdot = f(Z_{t+\Delta t})$ using Taylor expansion:

$$f(Z_{t+\Delta t}) = f(Z_t) + \frac{\partial}{\partial Z_t} f(Z_t) \Delta Z_t + \frac{1}{2} \frac{\partial^2}{\partial Z_t^2} f(Z_t)(\Delta Z_t)^2 + o(\Delta Z_t^3)$$

At this time, the expansion has rather abnormal behavior, as the order can be extremely large. However, the truncated expansion at order 1 has intriguing theoretical attractions. Let higher-order components be $\mathcal{R}(1)$, and recall $Z_t = H \star \epsilon_t$, we obtain the truncated differential process $\Delta \tilde{f}(Z_t)$, denoted as:

$$\Delta f(Z_t) - \mathcal{R}(1) = J_f \sum_{k=1}^{t} H_{t-s} \epsilon_s \tag{A1}$$

We treat all quantities appearing in Eq. (A1) as indeterminates in a polynomial ring

$$R = k\big[\{\Delta f(Z_t)\}_t,\ \mathcal{R}(1),\ J_f,\ \{H_{t-s}\}_s,\ \{\epsilon_s\}_s\big],$$

where k is a base field such as $\mathbb{R}$ or $\mathbb{C}$. For each time index t, the defining polynomial is $g_t := \Delta f(Z_t) - \mathcal{R}(1) - J_f \sum_{s=1}^{t} H_{t-s}\epsilon_s \in R$. This polynomial generates the principal ideal $\mathcal{I}_t = \langle g_t \rangle \subset R$, and considering all time indices $t = 1, 2, \ldots, T$, we obtain the global ideal $\mathcal{I} = \langle g_1, g_2, \ldots, g_T \rangle \subset R$. The corresponding affine variety is then

$$V(\mathcal{I}) = \Big\{ (Z_t, \Delta f(Z_t), J_f, H_{t-s}, \epsilon_s, \mathcal{R}(1)) \in k^N \ \Big|\ g_t = 0 \text{ for all } t \Big\}.$$

It is evident that $V(\mathcal{I})$ is positive-dimensional, since the defining relations do not specify finitely many points. To obtain more structure, we consider higher-order statistics. In particular, the d-th order cumulant tensor of the transformed increments takes the form

$$\kappa_d\Big(\Delta \tilde{f}(Z_t)\Big) = \sum_{s=1}^{t} \sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \Big(J_f H_{t-s}^{(:,j)}\Big)^{\otimes d}.$$

This expression shows that the cumulant naturally defines a point in the projective tensor space

$$\mathbb{P}(V^n \otimes V^n \otimes \cdots \otimes V^n),$$

where the number of tensor factors equals d. Hence, while the affine variety $V(\mathcal{I})$ is too large to give identifiability, the cumulant tensors lift the problem into a projective geometric setting, where connections to secant varieties of the Veronese embedding provide a natural framework for studying uniqueness and decomposition. So far, the generic mixing f is preserved by its Jacobian matrix J_f; hence, identifying J_f is equivalent to the recovery of f up to a constant. Now, we are ready to prove our main theorem. Without loss of generality, we write J_f as F since they behave the same way in an algebraically closed field.

Step 1: Uniqueness of mixing kernel $F(\mathbb{I}_p - \Phi)^{-1}$ We prove this supporting result via an extension of Proposition 3.1 in [21]. In the classical linear source decomposition (LSD) setting, the d-th order cumulant of X admits the following tensor decomposition: $\kappa_d(X) = \sum_{i=1}^{q} \kappa_d(\epsilon_i) \cdot (B_i)^{\otimes d}$ under the assumption that the components of ϵ are non-Gaussian with non-vanishing d-order cumulants, and that multiple interventions are available. The sufficient order d cumulant of each Z_t for a fixed $t = t_i$ is

$$\kappa_d(Z_t) = \kappa_d[(I - \Phi)^{-1} \star \epsilon_t] = \kappa_d[\sum_{k=1}^{t} H_{t-s}\epsilon_s]$$

for each s, the linear transformation H_{t-s} results in a multi-linear transformation of their cumulants

$$\kappa_d(H_{t-s}\epsilon_k) = (H_{t-s})^{\otimes d}\mathcal{C}^d_{\epsilon_s} = \sum_{i=1}^{p} \kappa_d(\epsilon_s^i)(H_{t-s})_j^{\otimes d}$$

The full order d cumulant of the mixed manifold O_t is

$$\begin{aligned}
\kappa_d(O_t) &= \kappa_d(\underbrace{FZ_t, FZ_t, \ldots, FZ_t}_{d \text{ times}}) \\
&= F^{\otimes d} \cdot \kappa_d(\underbrace{Z_t, Z_t, \ldots, Z_t}_{d \text{ times}}) \qquad \text{(A2)} \\
&= \underbrace{F \otimes F \otimes \cdots \otimes F}_{d \text{ times}} \cdot \kappa_d \left(\underbrace{\sum_{s_1=1}^{t} H_{t-s_1}\epsilon_{s_1}, \sum_{s_2=1}^{t} H_{t-s_2}\epsilon_{s_2}, \ldots, \sum_{s_d=1}^{t} H_{t-s_d}\epsilon_{s_d}}_{d \text{ times}} \right) \\
&= \underbrace{F \otimes F \otimes \cdots \otimes F}_{d \text{ times}} \cdot \sum_{s_1=1}^{t} \cdots \sum_{s_d=1}^{t} \kappa_d \left(H_{t-s_1}\epsilon_{s_1}, H_{t-s_2}\epsilon_{s_2}, \ldots, H_{t-s_d}\epsilon_{s_d} \right) \\
&= \underbrace{F \otimes F \otimes \cdots \otimes F}_{d \text{ times}} \cdot \sum_{s_1=1}^{t} \cdots \sum_{s=1}^{t} \sum_{j=1}^{p} \kappa_d \left(H_{t-s}^{(:,j)}\epsilon_s^{(j)}, H_{t-s}^{(:,j)}\epsilon_s^{(j)}, \ldots, H_{t-s}^{(:,j)}\epsilon_s^{(j)} \right) \\
&= \underbrace{F \otimes F \otimes \cdots \otimes F}_{d \text{ times}} \cdot \left(\sum_{s=1}^{t} \sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \underbrace{H_{t-s}^{(:,j)} \otimes H_{t-s}^{(:,j)} \otimes \cdots \otimes H_{t-s}^{(:,j)}}_{d \text{ times}} \right) \\
&= \left(\sum_{s=1}^{t} \sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \underbrace{F \otimes F \otimes \cdots \otimes F}_{d \text{ times}} \cdot \left(H_{t-s}^{(:,j)} \right)^{\otimes d} \right) \\
&= \sum_{s=1}^{t} \sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \left(FH_{t-s}^{(:,j)} \right)^{\otimes d} \qquad \text{(A3)}
\end{aligned}$$

Unlike in a time-free process, the joint cumulant of a time process is of order d that is coupled with the number of time lags:

$$\begin{aligned}
\kappa_d(O_{t_1}, \ldots, O_{t_d}) &= \sum_{s=1}^{\min(t_1,\ldots,t_d)-1} \sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \bigotimes_{\ell=1}^{d} \left(FH_{t_\ell - s}^{(:,j)} \right) \\
&= \sum_{j=1}^{p} \sum_{s=1}^{\min(t_1,\ldots,t_d)-1} \kappa_d^{(j)}(\epsilon) \cdot \bigotimes_{\ell=1}^{d} \left(FH_{t_\ell - s}^{(:,j)} \right) \qquad \text{(A4)}
\end{aligned}$$

We denote the Fourier transform of $x(t)$ with respect to time t as $\mathcal{F}[x](\omega)$. Using the convolution theorem and linearity of the Fourier transform, we have:

$$\mathcal{F}[\kappa_d(O_t)](\omega) = \mathcal{F}\left[\sum_{s \geq 0}^{t} \sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \left(FH_{t-s}^{(:,j)} \right)^{\otimes d} \right]$$

$$
\begin{aligned}
&= \mathcal{F}\left[\int_0^t \sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \left(FH(t-s)^{(:,j)}\right)^{\otimes d} \, ds\right] \\
&= \left(\sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \mathcal{F}\left[\left(FH^{(:,j)}\right)^{\otimes d} \star \mathcal{U}(t-s)\right]\right) \\
&= \left(\sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \mathcal{F}\left[\left(FH^{(:,j)}\right)^{\otimes d}\right](\omega) \cdot \left(\pi\delta(\omega) + \frac{1}{i\omega}\right)\right) \qquad \text{(A5)}
\end{aligned}
$$

Since $\delta(\omega)$ vanishes everywhere except at $\omega = 0$, multiplying it by ω gives zero, Eq. (A5) yields

$$
\begin{aligned}
i\omega\mathcal{F}[\kappa_d(O_t)](\omega) &= \left(i\omega \sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \mathcal{F}\left[\left(FH^{(:,j)}\right)^{\otimes d}\right](\omega) \cdot \left(\pi\delta(\omega) + \frac{1}{i\omega}\right)\right) \\
&= \left(\sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \mathcal{F}\left[\left(FH^{(:,j)}\right)^{\otimes d}\right](\omega) \cdot i\omega \cdot \left(\pi\delta(\omega) + \frac{1}{i\omega}\right)\right).
\end{aligned}
$$

which reads

$$
\left(\sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \mathcal{F}\left[\left(FH^{(:,j)}\right)^{\otimes d}\right](\omega) \cdot i\omega \cdot \left(\pi\delta(\omega) + \frac{1}{i\omega}\right)\right) = \left(\sum_{j=1}^{p} \kappa_d^{(j)}(\epsilon) \cdot \mathcal{F}\left[\left(FH^{(:,j)}\right)^{\otimes d}\right](\omega) \cdot 1\right) \qquad \text{(A6)}
$$

By assuming non-Gaussianity in ϵ_t, for each j, Eq. (A6), hence $i\omega\mathcal{F}[\kappa_d(O_t)](\omega)$ has a unique decomposition of the summation of rank-1 tensor. Therefore, each column of the sub-linear mixing transferring matrix $\mathcal{F}\left[\left(FH^{(:,j)}\right)\right]$ is theoretically recovered up to a scaling and permutation π if all assumptions made are satisfied in Φ. This indicates that even if one needs to calculate the tensor decomposition unnecessarily, such uniqueness guarantees the possibility of further disentanglement. In the sequel, $\mathcal{F}\left[\left(FH^{(:,j)}\right)\right] DP$ is available; we thus obtain the unique indeterminacy as an immediate result of Lemma. B.3.

Lemma B.3. *Consider $\mathbb{F}$ is an algebraically closed field, the unknown indeterminacy DP is preserved in $\mathbb{F}$, that is the following relation*

$$
\begin{aligned}
FH_\tau^{(:,j)} &= \hat{F}\hat{H}_\tau^{(:,j)} D_{j,\tau} P_{j,\tau}, \\
\textit{with } (P_{j,\tau}, D_{j,\tau}) &\in \{(P_j, D_j) \mid \mathcal{F}[FH^{(:,j)}]\, D_j P_j = \mathcal{F}[\hat{F}\hat{H}^{(:,j)}]\}.
\end{aligned}
$$

Proof. Let $\mathbb{F}$ be a field and $n \geq 1$ an integer. The *general linear group* of degree n over $\mathbb{F}$ is

$$
\mathrm{GL}_n(\mathbb{F}) \;:=\; \{\, A \in M_n(\mathbb{F}) \;\mid\; \det(A) \neq 0 \,\}.
$$

Equivalently, if V is a n-dimensional vector space over $\mathbb{F}$,

$$
\mathrm{GL}(V) \;:=\; \mathrm{Aut}_{\mathbb{F}}(V) \;=\; \{\, T : V \to V \text{ linear isomorphisms} \,\},
$$

and any choice of basis identifies $\mathrm{GL}(V)$ with $\mathrm{GL}_n(\mathbb{F})$. It is obvious that $\mathcal{F}[\mathbb{F}]$ is exactly a subgroup of $\mathrm{GL}(\mathbb{F})$ as the group $\mathrm{GL}_n(\mathbb{F})$ satisfies: i) It is precisely the set of all invertible linear transformations (invertible matrices). ii) If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then $\mathrm{GL}_n(\mathbb{F})$ is an open subset of $M_n(\mathbb{F})$ since $\mathrm{GL}_n(\mathbb{F}) = \det^{-1}(\mathbb{F} \setminus \{0\})$, and it is a Lie group. By the definition of kernel matrix, one notes i) trivially holds due to the maximal spectrum being less than 1. For ii), $\det^{-1}$ denotes the preimage of the open set $\mathbb{F} \setminus \{0\}$ under $\det$. Cutting the one-dimensional line at 0 produces two open intervals (for $\mathbb{F} = \mathbb{R}$) or a punctured plane (for $\mathbb{F} = \mathbb{C}$), hence the preimage is open in $M_n(\mathbb{F})$. Therefore, the permutation and scaling must be preserved in $M \in \mathbb{R}^{p\times p}$. □

So far, the original kernel mixing matrix FH_τ is recovered up to the same permutation and scaling for any τ. In the sequel, what needs to be proved is the recovery of the causal structure as well as its full parameter space. Our proof focuses on the polynomial system and its associated ideal $\mathcal{I}$ generated by the multi-linear constrained polynomial system.

Step 2: Uniqueness of causal kernel Φ

Example 2. Consider a kernel matrix $\Phi \in \mathbb{R}^{2\times 2}$ with internal arrows allowed:

$$\Psi_U = \begin{bmatrix} 0 & \psi_{12} \\ 0 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \Psi_V = \begin{bmatrix} 0 & \psi'_{34} \\ 0 & 0 \end{bmatrix}$$

where Ψ_U, Ψ_V encodes the internal arrows of the sub-graph $\mathcal{G}_U$ and $\mathcal{G}_V$; Φ encodes time-delayed kernel effects from s to t.

The corresponding expanded kernel matrix $\mathcal{M} \in \mathbb{R}^{4\times 4}$ is

$$\mathcal{M} = \begin{bmatrix} 0 & \psi_{12} & \phi_{11} & \phi_{12} \\ 0 & 0 & \phi_{21} & \phi_{22} \\ 0 & 0 & 0 & \psi'_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. Let $\mathcal{M}_{\mathcal{G}_k} \in \mathbb{R}^{2p\times 2p}$ be the kernel matrix associated with the bipartite graph, where the variables are partitioned into two subsets U and V. Consider a topological ordering where all nodes in U precede those in V.

Since edges from V to U are forbidden by the bipartite structure, no element in the lower-left block of $\mathcal{M}$ can be nonzero. Moreover, edges within U are not allowed, so the diagonal and upper-left block corresponding to U have zeros on the diagonal. The edges within V form a DAG, and under a topological ordering of V, the corresponding block in $\mathcal{M}$ is strictly upper-triangular. Therefore, $\mathcal{M}$ as a whole is strictly upper-triangular, which implies that it represents a DAG.

Because $\mathcal{M}_{\mathcal{G}_K}$ is strictly upper-triangular, it is nilpotent. Let k denote the length of the longest directed path in the DAG. Then $\mathcal{M}_{\mathcal{G}_K}^{k+1} = 0$, and the inverse of $\mathbb{I} - \mathcal{M}_{\mathcal{G}_K}$ can be expressed as a finite sum of powers of $\mathcal{M}$:

$$(\mathbb{I} - \mathcal{M}_{\mathcal{G}_K})^{-1} = \sum_{i=0}^{k} \mathcal{M}_{\mathcal{G}_K}^i.$$

Each term $\mathcal{M}_{\mathcal{G}_K}^i$ corresponds to contributions from paths of length i in the DAG. This shows that the inverse is fully determined by path products up to the longest path length k, completing the proof. □

By Lem. 2, identifying the space of all parameters in F and Φ is equivalent to solving the following ideal $\mathcal{I} : \langle F_{\mathscr{G}} - F_{\mathscr{G}}(\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K})^{-1}(\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K})\rangle$.

In latent causal models, $F_{\mathscr{G}}, \mathcal{M}_{\mathscr{G}_K}$ are filled as indeterminates that need to be recovered, where $F_{\mathscr{G}}$ is the expanded linear mixing obtained by filling $F \in \mathbb{R}^{n\times p}$ in a larger block-diagonal matrix of size $2n \times 2p$, denoted by $F_{\mathscr{G}}$. Remember, we have $F(\mathbb{I} - \Phi(\tau))^{-1}$ to be unique due to decomposition up to a scaling and permutation. We write $H_{\mathscr{G}} = (\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K})^{-1}$, then

$$F_{\mathscr{G}}(\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K})^{-1} = \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & F \end{bmatrix} \begin{bmatrix} H_{\mathscr{G}}(U) & H_{\mathscr{G}}(\Phi) \\ \mathbf{0} & H_{\mathscr{G}}(V) \end{bmatrix} = \begin{bmatrix} FH_{\mathscr{G}}(U) & FH(\Phi) \\ \mathbf{0} & FH_{\mathscr{G}}(V) \end{bmatrix} = K_{\mathscr{G}}$$

Under INAR, the diagonal blocks of $K_{\mathscr{G}}$ are $\mathbf{0}$ matrix. Therefore, we have the ideal:$\mathcal{I} : \langle F_{\mathscr{G}} - K_{\mathscr{G}}(\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K})\rangle$ where $K_{\mathscr{G}}$ is known because FH is known, so not considered as indeterminate and thus does not contribute any degrees in the dimension of $\mathcal{I}$. Clearly, under passively observational settings, recovery of full models is never possible as the current $\mathcal{I}$ must be positive dimensional, leading to no fixed points defined in the associated variety $\mathcal{V}$. This leads to a central goal to find the number of contexts that indicate sufficient variability or interventional settings, to recover the parameter space. To this end, we need to first discuss important properties of $F_{\mathscr{G}}$.

Lemma B.4. *$F \in \mathbb{R}^{n\times p}$ is a generic full-rank matrix. Then $F_{\mathscr{G}}$ is full rank with* $\operatorname{rank}(F_{\mathscr{G}}) = 2 \cdot \operatorname{rank}(F) = 2\min(n, p)$, *and it is* not *generic in an open dense subset of $\mathbb{R}^{2n\times 2p}$ due to the additional linear constraints imposed by the block-diagonal structure. Consequently, $F_{\mathscr{G}}$ belongs to a proper linear subvariety of $\mathbb{R}^{2n\times 2p}$ defined by*

$$\mathcal{V}_{I_\star} := \left\{ B \in \mathbb{R}^{2n\times 2p} : B = \begin{pmatrix} F' & 0 \\ 0 & F' \end{pmatrix},\ F' \in \mathbb{R}^{n\times p} \right\}.$$

Therefore, for any square matrix A^{2p} with $\operatorname{rank}(A) \le r$, $\operatorname{rank}(F_{\mathscr{G}} A) \le r$

Proof. This proof is trivial by linear algebra. □

Note we have p processes; we generally assume we obtain different contextual information from at least K environments (i.e., $K = p$), which is a mild condition in causal representation learning. Each sub-ideal $\mathcal{I}_k : \langle F_{\mathscr{G}} - K_{\mathscr{G}}(\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K})\rangle$ constitutes a polynomial system, denoted by $\mathcal{S}^k_{\mathbb{P}}$ such that:

$$F_{\mathscr{G}} + K^{(k)}_{\mathscr{G}} \mathcal{M}^{(k)}_{\mathscr{G}_K} = K^{(k)}_{\mathscr{G}}. \tag{A7}$$

There are $2n \times 2p$ indeterminates for $F_{\mathscr{G}}$ and $2|e(\mathcal{G})|$ for $\mathcal{M}$ since each environment k introduces a new $\mathcal{M}_{k,j}$. Considering all K ideals $(\mathcal{I}_0, \mathcal{I}_2, \cdots, \mathcal{I}_K)$, we obtain the union of all K varieties:

$$V(\mathcal{I}) = \Big\{ (F_{\mathscr{G}}, \mathcal{M}^{(0)}_{\mathscr{G}_K}, \ldots, \mathcal{M}^{(K)}_{\mathscr{G}_K}) \;\Big|\; F_{\mathscr{G}} - K^{(k)}_{\mathscr{G}}\big(\mathbb{I}_{2p} - \mathcal{M}^{(k)}_{\mathscr{G}_K}\big) = 0,\ \forall k \in K \Big\}. \tag{A8}$$

Adding polynomial constraints by subtracting Eq.(A7) for 0 from that for k obtains:

$$V(\mathcal{I}^\star) = \Big\{ V(\mathcal{I}) \;\Big|\; K^{(k)}_{\mathscr{G}} \mathcal{M}^{(k)}_{\mathscr{G}_K} - K^{(0)}_{\mathscr{G}} \mathcal{M}^{(0)}_{\mathscr{G}_K} - \big(K^{(k)}_{\mathscr{G}} - K^{(0)}_{\mathscr{G}}\big) = 0,\ \forall k \in K \Big\}. \tag{A9}$$

that induces a coordinate ring $R/\mathcal{I}$ in a polynomial ring $R = k(F_{\mathscr{G}_{i,j}}, \mathcal{M}^k_{i,j})$. The order of the coordinate ring is the dimension of the original variety. We use $\left(\begin{array}{c|c} & \\ \hline & \end{array}\right)$ to represent the blocked system so as not to confuse with the matrix bracket. For each $m \in [n], j \in [p]$, and $i \in \mathrm{ch}_{\mathcal{G}}(j)$, we have $|\mathrm{ch}_{\mathcal{G}}(j)|$ columns for $\mathcal{M}_{\mathcal{G}}$. $F_v, \mathcal{M}^{(k)}_v$ write entries $f_{i,j}, \mathcal{M}^k_{m,i}$ as vectors, which describe the polynomial constraints as a linear system:

$$\left(\begin{array}{c|c} \mathbb{I}_{4np} & \star \\ \hline \mathbf{0} & \star \end{array}\right) \left(\begin{array}{c} F_v \\ \hline \mathcal{M}^{(0)}_v \\ \mathcal{M}^{(k)}_v \end{array}\right) = \left(\begin{array}{c} K^{(k)}_{\mathscr{G}} \\ \hline K^{(k)}_{\mathscr{G}} - K^{(0)}_{\mathscr{G}} \end{array}\right). \tag{A10}$$

In INAR (∞), each $\mathcal{M}_{t-s}$ preserves all paths $\{(\varphi(j) \to \varphi(i) | Z^{\Delta}_{\varphi(j),t-s} \to Z^{\Delta}_{\varphi(i),t}, \mathcal{M}_{i,j} \neq 0\}$ from Φ_{t-s} through an isomorphism φ. Therefore, entry $(\mathbb{I}_{2p} - \mathcal{M})^{-1}_{i,j}$ is the product of $\mathcal{M}_{n,m}$ for path $j \to m \to n \to i$. We drop the time index and graph label whenever the context is clear. Such $(\mathbb{I}_{2p} - \mathcal{M})^{-1}_{i,j}$ admits the representation

$$(\mathbb{I}_{2p} - \mathcal{M})^{-1} = \mathbb{I} + \mathcal{M}.$$

Results from [21] are applied to get $\mathrm{rank}(\mathbb{I}_{2p} - \mathcal{M}^{(k)})^{-1} - (\mathbb{I}_{2p} - \mathcal{M}^0)^{-1} \leq 1$. Using Lem. B.4, we obtain $\mathrm{rank}(F_{\mathscr{G}}(\mathbb{I}_{2p} - \mathcal{M}^{(k)})^{-1} - F_{\mathscr{G}}(\mathbb{I}_{2p} - \mathcal{M}^0)^{-1}) \leq 1$. Therefore, the left part of Eq.(A10) has columns that are multiples of each other:

$$(K^{(k)}_{\mathscr{G}} - K^{(0)}_{\mathscr{G}})_{l,j} = (K^{(0)}_{\mathscr{G}})_{l,k} \Delta_{k,i}, \quad \Delta := (I - \mathcal{M}^k)^{-1} - (I - \mathcal{M}^0)^{-1} \tag{A11}$$

We examine the non-zero sub-blocks of the lower-right block $\star$ in Eq.(A10), which has size $|\mathrm{de}(j) \setminus \mathrm{ch}(j)| \times |\mathrm{ch}(j)|$. Following the convention, we represent the sunblocks as $M[j]$ and choose the smaller blocks $[K^{(k)}_{\mathscr{G}} - K^{(0)}_{\mathscr{G}}]$ corresponding to the size of $M[j]$ and write it as $b[j]$. The dimension of variety $V(\mathcal{I}^\star)$ is the dimension of the points $(\mathcal{M}^0_{i,j})$, $i \in \mathrm{ch}(j)$ that satisfy:

$$M[j](\mathcal{M}^0_{i,j}) = b[j] \tag{A12}$$

The variety is a null set when the above constraints lead to no solutions. Therefore, we require $\mathrm{rank}(M[j]) = \mathrm{rank}(M[j]|b[j])$. $[M[j] \mid b[j]]$ is the common augmented matrix to check the stability of a polynomial equation system. We conclude our proof by making a formal statement about the dimension of $\mathcal{V}(\mathcal{I}^\star)$ in the next lemma.

Lemma B.5. *For generic F and FH arising from the cumulant decomposition, the full generating model is identifiable if and only if the variety $\mathcal{V}(\mathcal{I}^\star)$ has dimension zero, that is,*

$$\dim(\mathcal{V}(\mathcal{I}^\star)) = \sum_{j=1}^{q} \mathrm{ch}(j) - \mathrm{rank}(M[j]) = 0.$$

Proof. Recall the kernel-delayed DAG structure. For the left subset U_s, each node j has outgoing edges only to nodes i in the right subset V_t, all of which are direct children of j. By construction, no edges exist within U_s or within V_t. Consequently, for each j, we have $M[j] = \emptyset$, since $\mathrm{de}(j) = \mathrm{ch}(j)$. □

Step 3: Independent conditions for each Φ under the generic F When $\mathcal{M}_{t-s}$ is fully recovered, the full matrix $\mathcal{F}_{\mathscr{G}}$ can be obtained as

$$\mathcal{F}_{\mathscr{G}} = K_{\mathscr{G}}(\mathbb{I}_{2p} - \mathcal{M}_{\mathscr{G}_K}).$$

If F_v is injective, identification of F_v (and hence $\mathcal{F}_{\mathscr{G}}$ and F) is equivalent to identifying each $\mathcal{M}$ individually, due to the direct multiplication of F^{-1} (or the pseudo-inverse $F^{\dagger}$) with K. However, identification of the full generating model is hindered by the genericity of F. Even if $K_{\mathscr{G}}$ is unique up to the usual indeterminacies, recovering other kernel matrices $\mathcal{M}_{t-s'}$ requires analogous identifiability conditions for each individual kernel matrix.

Under $k \in 1, 2, ..., K$ distinct contexts—each introducing sufficient variability in the distribution, or ensuring that each lag k receives at least one intervention that shifts the downstream mechanism—the full latent structure is identifiable up to the same indeterminacy.

Recovery of baseline U Once the full causal structure is recovered up to a scaling and permutation matrix, lower level moments of $\mathbb{E}[F(\mathbb{I}_p - \Phi)^{-1} \star (U + \epsilon_t)]$ are can be directly computed to find U up to the same indeterminacy.

B.1.3 Identifiability under Gaussian Noise

Now we focus on the case where non-Gaussianity does not hold for the entire time process. When the noise is Gaussian, $\kappa_d(O_t) = 0$ for all $d \geq 3$, which leads to a $\mathbf{0} \in \mathbb{R}^{p^{\times d}}$ (the d-th order zero tensor). The solution of decomposition is infinite, thus FH cannot be recovered up to a column scaling and permutation, nor can the latent transition graph $\mathcal{G}$. We argue that preserving only d- order cumulant of order $d \leq 2$ is a minimal building block for identification. $\kappa_2(O_t)$ is an order 2 variance-covariance matrix. Let $M_1, M_2, \ldots, M_T \in \mathbb{R}^{p \times p}$ be a collection of order-2 tensors. We define the order-3 tensor $\mathcal{X} \in \mathbb{R}^{T \times p \times p}$ via concatenation along the first mode (tensor slices):

$$\mathcal{K}_c = \operatorname{con}(M_1, M_2, \ldots, M_T), \quad \text{where } \mathcal{X}_{t,:,:} = M_t. \tag{A13}$$

By standard tensor algebra, an order-3 tensor $\mathcal{X} \in \mathbb{R}^{T \times p \times p}$ can be reshaped or flattened into a higher-order tensor, under a specific indexing scheme. More generally, given a desired tensor order d, and assuming $T = p^{d-2}$, we define a transformation:

$$\mathcal{T} : \mathbb{R}^{T \times p \times p} \to \mathbb{R}^{p^d}, \ \kappa_d(\varepsilon_t) \in \mathbb{R}^{\underbrace{p \times \cdots \times p}_{d \text{ times}}}$$

$$\kappa_2(\varepsilon_t) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} \in \mathbb{R}^{p \times p}, \kappa_3(X_t) = \begin{bmatrix} \mathbf{0}_{p \times p} \\ \mathbf{0}_{p \times p} \\ \vdots \\ \mathbf{0}_{p \times p} \end{bmatrix} \in \mathbb{R}^{p \times p \times p}$$

that re-indexes the tensor slices M_t to fill the missing indices of an order d cumulant tensor. The replacing and re-indexing rule is illustrated in an order 3 tensor as a real plane, assuming $d-1$ is the maximal order such that $\kappa_d = 0$.

Under this transformation, each slice M_t is interpreted as contributing to a specific mode configuration of the higher-order tensor. That is, the tensor $\mathcal{X}$ is "lifted" into a d-way tensor by embedding each $p \times p$ matrix slice as filling in the cumulant entries with fixed positions in the first $d-2$ indices corresponding to $t \in \{1, \ldots, T\}$, and varying the remaining two indices over $p \times p$. This leads to the same form as Eq. (A4) where all $\kappa_d(\epsilon_t) \neq 0$. Under Corollary B.1 and B.2, the new tensor has a unique decomposition of rank-1 tensor summation. To be specific, we assume v_i has no pair of columns to be collinear. This ensures the identification of $F(I - \Phi)^{-1}$ and restricts F to be injective to only $\operatorname{span}(H_j)$.

The sequential steps are the same for non-Gaussian noise since the construction of the ideal $\mathcal{I}^{\star}$ associated with its variety is not influenced by ϵ once FH is fixed up to a permutation and scaling.

Discussion of noise. Consequently, when Gaussianity is assumed, the distribution of O_t is fully characterized by the first two cumulants. This property implies that the entire cumulant expansion, and hence any higher-order dependency, collapses at second order. In this sense, the Gaussian distribution is the unique fixed point of the cumulant hierarchy at order two. In temporal parametric transition [28], a widely known condition to ensure the component-wise identifiability of the latent process Z_t is to require that the driving noise ϵ is not a fully isotropic Gaussian. That is, the Gaussian noise distribution must shift under either intervention [13] or exhibit heterogeneity in its variance. This is because all cumulants of order $p > 2$, which encode the exact causal dependencies, vanish for Gaussian noise. As a result, sufficient variability can only arise from changes in the second-order cumulant. Our results reflect that non-isotropic Gaussian noise is a *necessary* but not a *sufficient* condition for full identifiability of the time-delayed generative model and parameters.

B.1.4 Identifying causal structure

Our proof is constructive: with the minimal hierarchy $\mathcal{H}^{\mathcal{C}}$ by soft interventions such that no fixed value of entry $(i \to j)$ in $\Phi(w)$ induces a dependence removal, the transitive closure $\bar{\mathcal{G}}$ of the ground truth process with its causal structure can be recovered up to the trivial transformation aforementioned. If a time process has no instantaneous influence it must have $TC(\mathcal{G}) = \mathcal{G}$, we can recover the original process and its causal structure up to a scaling and permutation π.

C Proof of Supportive Results

C.1 Proof of Lem. 1

This lemma significantly constitutes the reasoning chains that lead to our identifiability results. We restate the original statement to cover more details and background in point process and theory of weak topology and convergence.

Lemma C.1 (Bounding Point Process in intensity, *constructive*)**.** *For a measurable mapping $N^{\Delta} : (\Omega, \mathcal{F}) \to (M_p, \mathcal{M})$ such that $\omega \mapsto N(\omega)$ is a point process at scale Δ. Let Δ be the control operator for any subsequence of its point process. Consider $A \in \mathcal{B}$ generated by the topology $\mathcal{M}_p := \mathcal{B}(M_p)$. λ and ϵ is defined on this metric space. If λ satisfies the stationary increment condition, then we can establish the weak convergence of the constructed equivalent class:*

$$\sum_{k:k\Delta\in A} \lambda_k^{(\Delta)} + \epsilon_k^{(\Delta)} \overset{w}{\Rightarrow} N(A) \quad \textit{for } \lambda_k^{\Delta} = \lim_{\Delta\to 0} \lim_{\delta\to 0} \frac{\mathbb{E}[dN^{\Delta}|\mathcal{F}]}{\delta}$$

Proof. We start the proof with a trivial case. If $\delta = \Delta_1 = 1$, the condition always trivially holds. In this case, we only need to show $N_t^{(\Delta_1)} = \lambda_t^{(\Delta_1)} + R_t$ by simply using the tower rule. Therefore, our proof gives more attention to the non-trivial case for $\delta \neq 1$.

Case 2: $\delta \in (0, \Delta_1)$

The reasoning of this case becomes more complicated if the time step operator used for generating subsequences proportionally shrinks to a sufficiently small unit in $(0, \Delta_1)$. We rewrite the approximating sequence N to leverage the metricizability of the space. Since we work in a Polish space, the Borel δ-algebra is countably generated and the space is separable and metrizable. Given a measurable set $A \in \mathcal{B}$, and a metric ρ, define the open δ-neighborhood as:

$$\mathcal{A} = A^{\delta} := \{x \in \mathbb{R}^d : \rho(x, A) < \delta\}$$

By outer regularity of Borel probability measures on Polish spaces, for every $\epsilon > 0$, there exists a countable collection of open sets $\{A_i\}_{i\in\mathbb{N}}$ such that $\bigcup_i A_i \supset \mathcal{A}$ and $\sum_i \mu(A_i \setminus A) < \epsilon$. This allows us to approximate any compact subset from outside using open sets with arbitrarily small excess mass and ensures the approximating sequence is defined on a non-decreasing base. We paraphrase the convergence as

$$\sum_{k:k\Delta\in A} \lim_{i\to\infty} \frac{\mathbb{E}[N^{\Delta}(A_i)|\mathcal{F}]}{|A_i|} + \epsilon_k^{(\Delta)} \overset{w}{\Rightarrow} N(A) \quad \text{for } \Delta \to 0 \tag{A1}$$

The equation above is adapted from the continuous-time intensity for point processes. However, it requires us to work with two limit conditions for A_i with the $1/k$ closed ball shrinking to zero

measure and for the subsequence operator Δ approaching to 0. A common method is to ensure dominated and uniform convergence of the limit. To harness information regarding the intensity in our convergence to a more generalized process, we work with only Δ to induce the same time scale of intensity function. Therefore, we have the equivalent condition

$$\sum_{k:k\Delta\in A} \frac{\mathbb{E}[\sum_{k=1} Z_k^{\Delta} - \sum_{k=1} Z_{k-1}^{\Delta}|\mathcal{F}]}{|A_i|} + \epsilon_k^{(\Delta)} = \sum_{k:k\Delta\in A} \frac{\mathbb{E}[Z^{\Delta}(\Delta)|\mathcal{F}]}{\Delta} + \epsilon_k^{(\Delta)} \overset{w}{\Rightarrow} N(A) \quad \text{for } \Delta \to 0 \tag{A2}$$

We remove the limit condition as it is clear that $|A_i|$ is of measure zero when $\Delta = 0$, which ensures the alignment between our topological property and plausibility to analyze only subsequences in the sequel. According to Lemma.2 by [20], for any compact interval $[a, b]^{(\delta)}$ with the number of bins $[b-a]/\delta$, $\mathbb{E}[N^{(\delta)}([a, b])] < (b - a + 2)(I - G^{(\delta)}(a, b))^{-1}\Lambda$ where $G(a, b) = \int_a^b \Phi(s)ds$ is a solution of the stochastic differential equation systems

$$\mathbb{E}[\lambda([a, b])] = \mathbb{E}[u + G(a, b)\Lambda], \text{ for } \mathbb{E}[\lambda(a, b)] = \Lambda$$

Note that, by reapplying tower rule, Eq. (A1) implies:

$$\lim_{\delta\to 0} \mathbb{E}[N_A^{(\delta\in(0,1))}] \to \lim_{\delta\to 0} \mathbb{E}[\frac{\mathbb{E}[N_A^{(\delta)}|\mathcal{F}_t]}{\delta}]$$

Next, we show the necessity of tightness of the corresponding probability measure $\mathbb{P}^{\Delta}$ for the left-hand of Eq. (A2) to achieve the desired convergence. Without loss of generality, we consider a nonparametric intensity function $\lambda_t = \psi(u + \int \phi(t - s)Z^{\Delta}(s)\, ds)$. Consequently, $\mathbb{E}[\lambda_t] = \Lambda$ and $\mathbb{E}[\lambda_t] = \mathbb{E}[\psi(u + \int \phi(t - s)Z^{\Delta}(s)\, ds)]$. Assume that ψ is α-Lipschitz and $\alpha\|\phi\|_1 < 1$ [29], so the mapping $F(\Lambda) = \psi(u + \|\phi\|_1\Lambda)$ is a contraction on $\mathbb{R}_+$. By Banach's fixed-point theorem, there exists a unique solution $\Lambda^\star$ to the equation:

$$\Lambda^\star = \psi(u + \|\phi\|_1\Lambda^\star)$$

Formally, this can be rearranged as:

$$\psi^{-1}(\Lambda^\star) - \|\phi\|_1\Lambda^\star = u \quad \Longrightarrow \quad \Lambda^\star = \left(\mathrm{id} - \|\phi\|_1 \cdot \psi^{-1}\right)^{-1}(-u)$$

provided that $\mathrm{id} - \|\phi\|_1 \cdot \psi^{-1}$ is invertible on the image of ψ.

To control the tail probability, we apply Markov's inequality:

$$\mathbb{P}\left(\sum_{k:k\Delta\in A} \frac{\mathbb{E}[Z^{\Delta}(\Delta)|\mathcal{F}]}{\Delta} + \epsilon_k^{(\Delta)} > M_\varepsilon\right) \leq \frac{\mathbb{E}[\sum_k \Lambda^{\Delta}]}{M_\varepsilon} \leq \frac{(b - a + 2\delta) \cdot \Lambda^\star}{M_\varepsilon}$$

Here, we define:

$$M_\varepsilon := \frac{(b - a + 2\delta) \cdot \Lambda^\star}{\varepsilon} \quad \text{where } \Lambda^\star = \psi(u + \|\phi\|_1\Lambda^\star)$$

This choice ensures the upper bound remains within the prescribed ε-level for all $\Delta \in (0, \Delta_1)$. Since the only thing we need is the precompactness, we will not establish any tighter bound. Tightness of measure indicates we can always find a subsequence $\lambda_{k_n}^{\Delta} + \epsilon_{k_n}^{\Delta}$ in $\lambda_k^{\Delta} + \epsilon_k^{\Delta}$ converges weakly to a sequence $\lambda^\star + \epsilon^\star$. This weak convergence of subsequences, however, cannot control the limit uniqueness for each sequence. Therefore, we also should further control the limiting behavior of each sequence by uniform convergence of the characteristic functional defined by the approximating process and the target process. □

D Detailed MUTATE Configuration

D.1 Simulation Regime

We simulate multivariate point processes and their converging equivalent class Z_t extensively studied in our identifiability theory. We sample all point processes using the Poisson Superposition method

(rejection sampling from the upper bound of conditional intensity [30, 31]) in order to mimic highly dynamic changes in conditional intensity, to capture denser information contained in stochastic processes. Then we create corresponding converging classes as a proof-of-concept validation: A total of 20,000 latent trajectories are sampled for each of the five kernel functions—exponential, power-law, rectangular, simple nonlinear, and flexible mixing—under two noise regimes: heterogeneous noise and Gaussian mixture noise. To illustrate the latent events underlying the unstructured data, we also simulate stochastic dynamics for biological data using SERGIO [32], a GRN-guided gene expression simulator used in Lorch et al.'s [15] causal modeling as well. All observation O_t is obtained from latents Z_t through MLP and LeakyReLU nonlinearity mixing.

We demonstrate the generative process for INAR equivalent classes. For a fair comparison to those baselines mainly addressing step-wise conditional independence, we generate for both time-step dynamics and denser dynamics by changing the setup to very short kernel effects with $\tau \in (0.001, 0.01) = t - t'$. We generate stochastic point processes from three basic kernel response functions:

$$\phi_{\text{exponential}}(t) = \alpha e^{-\beta t'}, \alpha \sim \text{uniform}(0.1, 0.5) \text{ and } \beta \sim \text{uniform}[0.5, 2)$$

$$\phi_{\text{powerlaw}}(t) = \frac{\alpha}{(t+c)^{\beta}} \cdot \mathbf{1}, \alpha \sim \text{uniform}(0.5, 1.2), \beta \sim \text{uniform}[0.1, 0.8) \text{ and } \gamma \in \text{uniform}(1, 3, 1.8)$$

$$\phi_{\text{rectangular}}(t) = \frac{1}{T - T'} \cdot \mathbf{1}_{\{t' \leq T\}}$$

The baseline intensity u_0 is sampled from $\text{uniform}(0, 1, 0.2)$. All parameters of the basic kernel are uniformly sampled by ensuring $\alpha < \beta$ in exponential responses, $\alpha < \gamma$ in power-law response, respectively, to satisfy the stationary increment condition such that $|\phi| < 1$. In the simulation, we also consider two extreme cases for simple nonlinear intensity and nonparametric intensity. We construct the conditional intensity function by mixing latent features through a linear transformation followed by a non-linear activation. Specifically, we first compute a log-linear intensity using the expression

$$\lambda_t = \log(1 + \exp(z_\ell - r_\ell[:, \Delta, :]))$$

that ensures positivity and controls the scale of the output through a smoothed ReLU (i.e., softplus).In an alternative setting (`kernel == "np"`), we learn the intensity function using a small neural network (MLP): a two-layer perceptron with ReLU activation, ending in a Softplus to maintain positive outputs. This setup enables flexible, data-driven modeling of intensity dynamics beyond purely additive or linear forms. We define the mixing intensity function using a two-layer feedforward neural network with ReLU and Softplus activations. Formally, the architecture is given by:

$$\lambda_t = \sigma_+ \left(W_2 \cdot \text{ReLU}(W_1 \lambda_t(l) + b_1) + b_2 \right), \tag{A1}$$

where

- $\lambda_t(l) \in \mathbb{R}^d$ is the input linear basic intensity at time t,
- $W_1 \in \mathbb{R}^{64 \times d}$, $b_1 \in \mathbb{R}^{64}$ are the weights and bias of the first layer,
- $W_2 \in \mathbb{R}^{d \times 64}$, $b_2 \in \mathbb{R}^d$ are the weights and bias of the second layer,
- $\sigma_+(x) := \log(1 + e^x)$ denotes the Soft-plus activation.

This design ensures the output λ_t remains strictly positive and can model complex dependencies in the latent dynamics while maintaining numerical stability.

We model the transformation from the latent variable $Z_t \in \mathbb{R}^d$ to the observational space via a multi-layer mixing network. Specifically, for each layer $l = 1, \ldots, L-1$, the transformation is given by $Z_t^{(l)} = \mathbf{A}^{(l)} \cdot \sigma_{\text{leaky}}(Z_t^{(l-1)})$, where $\mathbf{A}^{(l)} \in \mathbb{R}^{d \times d}$ is an orthogonal mixing matrix and σ_{leaky} denotes the leaky ReLU activation with slope $\alpha = 0.2$. The initial input is $Z_t^{(0)} = Z_t$, and the final output $Z_t^{(L-1)}$ represents the observation-space signal.

D.2 Prior decomposition of time-adaptive module

Without loss of generality, we consider non-finite steps for a latent stochastic generative process, as discussed in Lem. 1, where $\Delta t \to 0$. This induces an equivalence that the intrinsic history—the filtration $\mathcal{F}_t := \sigma\left(\bigcup_{0<t<T} \sigma(Z_t^{\Delta})\right)$—ensures that the process $Z_t^{(\Delta)}$ is $\mathcal{F}_t$-adaptive and measurable.

We decompose the ELBO objective as follows:

$$
\begin{aligned}
\text{ELBO} &= \log p(O) - D_{\text{KL}}(q_\phi(Z|O)\|p(Z)) \\
&= \mathbb{E}_{z\sim q(Z_t|O_t)}[\log p(O_t|Z_t)] + \mathbb{E}_{z\sim q(Z_t|O_t)}\left[\log \frac{q(Z_t|O_t)}{p(Z_t)}\right] \\
&= \mathbb{E}_{z\sim q(Z_t|O_t)}[\log p(O_t|Z_t)] - \mathbb{E}_{z\sim q(Z_t|O_t)}\left[\log q(Z_t|O_t) - \log p(Z_t)\right] \\
&= \mathbb{E}_{z\sim q(Z_t|O_t)}\left[\log p(O_t|Z_t) - \log q(Z_t|O_t)\right] + \mathbb{E}_{z\sim q(Z_t|O_t)}\left[\log p(Z_t)\right] \\
&= \mathbb{E}_{z\sim q(Z_t|O_t)}\left[\log p(O_t|Z_t) - \log q(Z_t|O_t)\right] + \mathbb{E}_{z\sim q(Z_t|O_t)}\left[\sum_{\mathcal{F}_0^+}^{\mathcal{F}_T} \log p(Z_t^{(\Delta)}|\mathcal{F}_t)\right]
\end{aligned}
$$

The reason we can segment the increasing filtration in the last term is due to the nice property of $\mathcal{F}_t$-measurable sequence. We can show that filtration of $Z_t|Z_s, R_t$ and $Z_t|R_s$ is equal because it is well known that any p-order INAR sequence with stationary increments admits a moving average (MA) representation. Further construction of their filtration $\tilde{\mathcal{F}}_t(\text{resp.}R_{s<t})$ and $\mathcal{F}_t(\text{resp.}Z_{s<t}, R_t)$ can show

$$\tilde{\mathcal{F}}_t = \mathcal{F}_t$$

We prove the result in the sequel. For $\tilde{\mathcal{F}}_t$, Z_t is a measurable function for $s < t$. By causality of the convolution kernel $\Psi = (I-\Phi)^{-1}$ satisfying $\Psi_\tau = 0$ for $\tau < 0$, which indicates $Z_t \in \sigma(R_s : s < t)$. Then, we construct another filtration $\tilde{\mathcal{F}}_s : \sigma(R_u : u < s)$. By adaptivity $\tilde{\mathcal{F}}_s : \sigma(R_u : u < s) \subseteq \tilde{\mathcal{F}}_t : \sigma(R_u : u < t)$. Therefore, Z_s is also $\sigma(R_u : u < t)$-measurable. Since the minimal σ-algebra of the original $\mathcal{F}_t$-measurable function must be contained in its σ-algebra, we have $\sigma(Z_s) \subseteq \sigma(R_u : u < t)$ and $\sigma(\bigcup_{s\le t} \sigma(Z_s)) \subseteq \sigma(R_u : u < t)$. For $\mathcal{F}_t$, $R_t = Z_t - \Psi \star Z_t$ so R_t is $\sigma(Z_s : s \le t)$-measurable. Therefore, by a similar construction, it is evident that $\sigma(\bigcup_{s<t} \sigma(R_s)) \subseteq \sigma(Z_s : s \le t)$. Therefore, because $\tilde{\mathcal{F}}_t \subseteq \mathcal{F}_t$ and $\mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t$, there must be $\tilde{\mathcal{F}}_t = \mathcal{F}_t$.

Following this set-up, the prior becomes:

$$Z_t \mid \mathcal{F}_t \sim \mathcal{N}\left(\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{bmatrix} \sum_{t'<t} (I-\Phi)^{-1},\ \sum_{t'<t} (I-\Phi)^{-1}\Sigma_{t'}(I-\Phi)^{-T}\right)$$

The latents are generated by $Z_t = (I-\Phi) \star R_t$, where R_t is modeled as isotropic Gaussian noise with mean U and variance Σ. Note that the variance matrix Σ_{Z_t} is zero for any $t - t' \neq 0$. By the Wiener-Khinchin Theorem, we have the covariance matrix $C_{Z_t}(0) = \frac{1}{N}\sum_{k=0}^{N-1} S_z(w_k)$, we drop the sub-index Z_t whenever no confusion is caused. Now we can derive the decomposition of the convolution prior as

$$
\begin{aligned}
&\mathbb{E}_{z\sim q(Z_t|O_t)}\left\{\sum_{\mathcal{F}_0^+, Z_t}^{\mathcal{F}_T} \log p(Z_t^{(\Delta)}|\mathcal{F}_t)\right\} \\
&= \mathbb{E}_{z\sim q(Z_t|O_t)}\left\{\sum_{\mathcal{F}_0^+, Z_t}^{\mathcal{F}_T} \log p\left[(I-\Phi_t) \star \hat{R}_t^{(\Delta)}\right]\right\} = \mathbb{E}_{z\sim q(Z_t|O_t)}\left\{\sum_{\mathcal{F}_0^+, Z_t}^{\mathcal{F}_T} \log p\left[\int_0^t (I-\Phi_{t-t'})\hat{R}_{t'}^{(\Delta)}\, dt\right]\right\} \\
&= \mathbb{E}_{z\sim q(Z_t|O_t)}\left\{\sum_{\mathcal{F}_0^+, Z_t}^{\mathcal{F}_T} \log p\left[\mathcal{N}(\hat{U}_R, \sum H_t^{-1}\Sigma_{\hat{R}_t'}H_t^{-T})\right]\right\}
\end{aligned}
$$

$$= \mathbb{E}_{z\sim q(Z_t|O_t)} \left\{ \sum_{\mathcal{F}_0^+, Z_t}^{\mathcal{F}_T} \log p \left[\mathcal{N}(\hat{U}_R, \underbrace{\sum H_t^{-1}(PSD_{\hat{Z}_t})\Sigma_{\hat{R}_t'} H_t^{-T}(PSD_{\hat{Z}_t}))}_{C_{p(Z_t)}(0)} \right] \right\}$$

$$= \mathbb{E}_{z\sim q(Z_t|O_t)} \left\{ \sum_{\mathcal{F}_0^+, Z_t}^{\mathcal{F}_T} \log p \left[\mathcal{N}(\hat{U} \sum_{t'<t} \underbrace{(I - \Phi(t-t'))^{-1}}_{(1-\Phi)(w_k)^{-1}\Sigma(1-\Phi)(w_k)^{-H} = S_Z(w_k)}, \frac{1}{N}\sum_{k=0}^{N-1} S_{Z_t}(w_k)) \right] \right\} \tag{A2}$$

$$= \mathbb{E}_{z\sim q(Z_t|O_t)} \left\{ \sum_{\mathcal{F}_0^+, Z_t}^{\mathcal{F}_T} \log p \left[\mathcal{N}(\hat{U} \underbrace{\sum_{\tau>0, w=0} (I - \Phi(\tau))^{-1} e^{-jw\tau}}_{\text{the inverse Fourier at } \ w=0}, \frac{1}{N}\sum_{k=0}^{N-1} S_{Z_t}(w_k)) \right] \right\}$$

$$= \mathbb{E}_{z\sim q(Z_t|O_t)} \left\{ \sum_{\mathcal{F}_0^+, Z_t, N\in(N_0,T)}^{\mathcal{F}_T} \log p \left[\mathcal{N}(\hat{U}\text{PSD}_{Z_t}(H(0)), \frac{1}{N}\sum_{k=0}^{N-1} S_{Z_t}(w_k)) \right] \right\} \tag{A3}$$

D.3 Explicit control for convolution prior

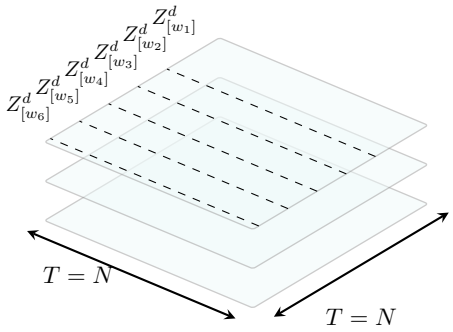

Figure A2: Visually Time-adaptive PSD Computation

Encoder-PSD flow As shown in Eq. (A3), a key component of our module is to efficiently compute the decomposition of PSD matrix. However, under milder regularity conditions, the PSD decomposition is not unique, thus only be recovered up to the minimal-phase. By the Theorem, the energy of the time domain and frequency domain is equivalent. Therefore, the encoded distribution is not sufficient to decompose the PSD matrix for which a reparameterization is needed. An encoder receives a T-length sequence O_t and returns the latent variable vector. Fast Fourier Transformation converts the latent sequence to a vector of equal length up to t:

$$[Z_{\mathcal{F}_0}, Z_{\mathcal{F}_t}, \cdots, Z_T] \Rightarrow \{[Z[f_0], Z[f_k], \cdots, Z[K]] | K = 0, 1, 2, \cdots, T\}$$

And the flow method is enforced by solving the following Wilson Factorization optimization problem for each $[Z[f_0], Z[f_k], \cdots, Z[K]]$, finding the transfer matrix

$$H^\dagger = \arg\min_{\Sigma_t = \sigma^2 I} PSD(Z_t) - H^\dagger \Sigma_t H^{\dagger H}$$

That is then sent to evaluate the true prior distribution, supporting the joint optimization of all loss components.

The summation of kernel products and integrated noise variables is guaranteed to converge to the true time-adaptive process under $\mathcal{F}_t$, provided that the time discretization is sufficiently dense. The

latent variable Z_t^{Δ} is sampled from the encoder distribution q_ϕ and passed to the PSD decomposition module to compute the frequency-domain representation of the full kernel matrix $F_w[1-\Phi_t]$ and the power spectral density $S_{\hat{R}_t}$. We further remark that the key step, spectrum decomposition, is completed for the entire encoded trajectory $\hat{Z}_{t_0:T}$, and the prior structure is ensured by segmenting filtration. This features the major difference in prior work that recursively constructs an equal-length sliding window for each latent. Filtration segmentation can work with causal masks that a more expressive encoder leverages. Note that transformer modules are not a required component for shorter sequences, i.e. $T < 100$. However, when the sequence is extremely long, as simulated in the conventional class of stochastic point processes, a transformer can be used in place of a common MLP encoder to learn much more expressive latent embeddings by utilizing the filtration attention from arbitrarily long past events.

Overall training loss. To encourage sparsity in transferring kernels, we follow the widely used penalty to jointly optimize:

$$\mathcal{L}_{Total} = \mathcal{L}_{\text{Recon}} - \beta\mathcal{L}_{\text{KLD}} - \gamma|\Phi| - \omega\mathcal{L}_{\text{PSD}} \tag{A4}$$

This training objective ensures the learned latent process is driven by a family of generalized white processes, as, in the Encoder-PSD flow, the decomposition is enforced by the prescribed isotropic noise, which omits any discriminator module as used in [28]. The coefficients in sparsity loss and PSD accuracy are registered as tunable hyperparameters.

D.4 Extended Results

Table A2: Reporting the best performance for each baseline

Method	**Metric**	Kernel Ave.	Exp	Power.	Rect.	Nonlin.	Nonpar.
TDRL	MCC	0.657	0.629	0.653	0.773	0.584	**0.644**
	$\mathcal{L}_{vae}$	0.449	0.308	0.302	0.302	0.871	**0.461**
BetaVAE	MCC	0.419	0.395	0.414	0.420	0.433	0.433
	$\mathcal{L}_{vae}$	9.480	8.538	7.533	8.424	11.683	11.220
SlowVAE	MCC	0.410	0.384	0.405	0.420	0.425	0.412
	$\mathcal{L}_{vae}$	362.890	395.107	448.105	452.472	238.520	280.247
PCL	MCC	0.440	0.469	0.379	0.430	0.474	0.449
	$\mathcal{L}_{vae}$(train)	0.693	0.693	0.694	0.693	**0.693**	0.693
MUTATE	MCC	**0.811**	**0.922**	**0.784**	**0.964**	**0.885**	0.501
	$\mathcal{L}_{vae}$	**0.670**	**0.448**	**0.508**	**0.253**	0.942	1.201

E Related Work

Causal disentanglement and learning time series. Although estimating and predicting time series is a classical problem in both traditional statistics and modern machine learning, representation learning has opened new avenues for leveraging latent information to better characterize time series data [33, 34]. Recently, learning causal representations in time series has become a foundational approach for enabling new scientific discoveries. This line of research primarily focuses on establishing identifiability of causal latent variables by exploiting nonstationary data [28, 4] and modular distribution shifts [5, 35] with sparsity constraints [36, 37] on the latent transition. Those works solve the identifiability problem of latent causal models by leveraging sufficient variability that can come from proper interventions or passive distribution shifts. Another line of research focuses on learning the underlying causal graph among latent variables

Learning Causality in Stochastic Processes. While learning causality remains a considerably more challenging task than causal discovery or representation learning, several efforts have been made to bridge these areas. Here we review existing approaches that link causal learning with stochastic modeling. Our scope is not limited to causal representation learning with stochastic processes, but extends to a broader set of problems that are closely related to either domain.

One representative direction in causal learning for dynamical systems is the study of Granger causality—a broader and looser notion compared to strictly structured causal models [38]. It is widely acknowledged that full causal recovery in such systems is impossible. Consequently, even the most recent work on stochastic processes can only determine whether a point process a is Granger-causal or non-causal with respect to another process b, typically formalized through *local independence* and the δ-separation rule [39]. Another active line of work concerns identifiability in dynamical systems [40]. However, to the best of our knowledge, none of these models provides provable guarantees for highly dynamical systems such as self-exciting or more general stochastic processes.

Connections between causal representation and dynamical systems have also been explored through ordinary differential equations (ODEs) [41]. Technically, these approaches recover only a set of parameters that are difficult to interpret as causal in the latent space, or at best allow stochastic dynamics in the observed variables. More recently, causal diffusion models have been proposed [42, 15], yet they largely treat diffusion as a standard denoising process and thus do not permit a well-structured stochastic latent causal representation.

Another important line of research investigates interventions on stochastic processes and the corresponding post-intervention distributions, which serve as the basis for causal inference [43, 44, 45, 46, 15]. The first attempt to introduce a causal interpretation into stochastic differential equations (SDEs) was made by the authors of [47], where interventions are defined as the removal of single variables in SDEs. They showed that causal principles in SDEs can be formalized as interventions, with the resulting post-interventional distribution identifiable via the infinitesimal generator. However, such interventions are too restrictive to capture more complex dynamical scenarios. Following this initial line of work, [44] further develops methods for estimating stationary causal models by minimizing the deviation of stationarity of diffusion.

Stochastic representation in biological science. Prior work on the dynamics of cancer genomics has investigated modeling the underlying stochastic processes, typically under the assumption that all mutations can be identified through observable changes in protein binding and synthesis. A seminal line of studies focuses on methods and conditions under which mutation rates are treated as fixed for each driver mutation during tumor progression. Under these assumptions, the evolutionary dynamics can be effectively modeled using a linear Moran process with fixed population size [48, 49]. One branch of this literature aims to identify driver mutations that are directly manifested in observations. However, given the limited prior knowledge, mutation interactions are unlikely to be strictly linear or fixed. As a result, due to the inherent stochasticity and dynamical nature of cancer development, most driver mutations remain latent and their patterns are not readily discernible in protein sequences [50, 51, 52]. More recently, [52] reformulated this problem by introducing a framework to distinguish between weak and strong driver mutations to better characterize cancer progression. For example, in several cell cycles, a normal cell must accumulate multiple mutations in tumor-susceptibility genes to trigger oncogenesis. This process is inherently stochastic, and many mutational events may be dependent, self-exciting, or regulated by other processes. Identifying latent processes underlying disease-specific mutations and recovering their causal relationships is therefore crucial for computational biology and the planning of sequential cancer treatment regimens.

F Conclusion and Limitation

Our paper makes extensions of causal representation learning framework to stochastic causal dynamics (i.e., multivariate Hawkes Processes), a topic not yet covered in current CRL literature. We propose a new perspective that a branch of stochastic processes can be viewed as the corresponding equivalent class through INAR representation and weak convergence. Under those conditions, we show that the latent stochastic process can be identified up to a component-wise transformation and a scaling permutation matrix. Our theoretical result bridges the gap between stochastic modeling and causal representation. We also propose a novel framework to learn the time-adaptive transition dynamics to accurately estimate the latent processes. However, our work avoids the most complicated case for a fully nonparametric kernel, which, most of the time, can be replaced with a simpler kernel. Future direction may include solving this condition and causal representation learning for stochastic differential processes that manifest in rich scientific questions.