BI-CRITERIA METRIC DISTORTION

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ABSTRACT

Selecting representatives based on voters' preferences is a fundamental problem in social choice theory. While cardinal utility functions offer a detailed representation of preferences, voters often cannot precisely quantify their affinity towards a given candidate. As a result, modern voting systems rely on ordinal rankings to simplistically represent preference profiles. In quantifying the suboptimality of solutions due to the loss of information when using ordinal preferences, the metric distortion framework models voters and candidates as points in a metric space, with distortion bounding the efficiency loss. Prior works within this framework use the distance between a voter and a candidate in the underlying metric as the cost of selecting the candidate for the given voter, with a goal of minimizing the sum (utilitarian) or maximum (egalitarian) of costs across voters. For deterministic election mechanisms selecting a single winning candidate, the best possible distortion is known to be 3 for any metric, as established by Gkatzelis, Halpern, and Shah (FOCS'20). In contrast, for randomized mechanisms, distortions cannot be lower than 2.112, as shown by Charikar and Ramakrishnan (SODA'22), and there exists a mechanism with a distortion guarantee of 2.753, according to Charikar, Ramakrishnan, Wang, and Wu (SODA'24 Best Paper Award). Our work asks: can one obtain a better approximation compared to an optimal candidate by selecting a committee of k candidates ($k \geq 1$), where the cost of a voter is defined to be its distance to the closest candidate in the committee? We affirmatively answer this question by introducing the concept of bi-criteria approximation within the metric distortion framework. In the line metric, it is possible to achieve optimal cost with only $O(1)$ candidates. In contrast, we also prove that in both the two-dimensional and tree metrics – which naturally generalize the line metric – achieving optimal cost is impossible unless all candidates are selected. These results apply to both utilitarian and egalitarian objectives. Our results establish a stark separation between the line metric and the 2D or tree metric in the context of the metric distortion problem.

1 INTRODUCTION

One of the fundamental challenges in the social choice theory is to elect representatives based on voters' preferences, ideally represented by cardinal utility functions that assign numerical values to each outcome. However, in most real-world scenarios, voters only provide ordinal information, such as preference orders among outcomes/candidates. This raises a natural question of how worse, if at all, a voting mechanism performs when given ordinal information rather than cardinal information. Procaccia & Rosenschein (2006) introduced the notion of *distortion* to measure such an efficiency loss – how different voting rules respond to the lack of cardinal information. Many practical voting scenarios can be formulated by considering both voters and candidates lying on a metric space (Enelow & Hinich, 1984). The distance to candidate locations determines voters' cardinal preferences for candidates – voters rank candidates based on ascending distance, with the closest candidate being the most preferable and the farthest candidate being the least preferable. The worst-case behavior of any ordinal preference order-based voting rule/mechanism is captured by the notion of *metric distortion*, introduced by Anshelevich et al. (2018). A voting mechanism, without access to the actual distances among the set of voters and candidates, seeks to minimize a specific cost function, which depends on the distances. Distortion is defined with respect to this cost: for any voting mechanism f, it is the worst-case ratio, over all instances, between the cost of f's solution and the optimal cost.

Given a single fixed candidate, the cost for a voter is defined as its distance from the given candidate. Then, the overall cost is set to be an objective that combines these values across all the voters. Depending on the specific context in the literature, the following two objectives have widely been considered: a utilitarian objective, which aims to minimize the total individual costs for all voters, and an egalitarian cost, which minimizes the maximum cost experienced by any voter. Different variants of the metric distortion problem under the above objective functions have received significant attention, e.g. Goel et al. (2017); Kempe (2020); Gkatzelis et al. (2020); Anagnostides et al. (2022); Kizilkaya & Kempe (2023a).

For the classical metric distortion problem, it is known that a distortion of 3 can be achieved for any metric. Further, it is widely known that the candidate chosen by any deterministic method cannot achieve a distortion factor of less than 3, even in a line metric. In other words, without knowing the exact distances, it is not possible to obtain a better than 3-approximation of the optimal cost. This leads to an intriguing question: Can we obtain a better approximation (distortion) by selecting more than one candidate? Specifically, assume that the algorithm is allowed to choose $k > 1$ candidates, and we set the cost for each voter to be its distance to the closest chosen candidate. Can we design an algorithm for which the overall cost (either utilitarian or egalitarian) is at most α times the cost of an optimal candidate for some $\alpha < 3$?

In this paper, we answer the above question affirmatively. We not only attain better distortion by allowing the selection of more than one candidate but, in fact, attain optimal cost with $O(1)$ candidates for the line metric. Additionally, we present several lower-bound constructions that demonstrate impossibility results for the line, tree, and 2D Euclidean metrics. Our findings establish matching upper and lower bounds for these cases. Furthermore, our results establish a separation between the line metric and the 2D Euclidean (and tree) metrics in the context of the metric distortion problem.

Our work introduces a *bi-criteria* perspective to metric distortion, which, to our knowledge, has not been considered before. The pursuit of improved *bi-criteria* approximation results for various classical optimization problems has already been well-investigated in the literature. Indeed, when the metric is known, and the goal is to pick k candidates that minimize the overall cost across voters, the election problem becomes an instance of either the k-median or the k-center clustering (for the utilitarian and egalitarian objective, respectively). For these problems, numerous (constant-factor) approximation algorithms are known which select up to $O(k)$ centers (instead of k centers), e.g. Feldman et al. (2007); Wei (2016); Alamdari & Shmoys (2018). Therefore, it is quite natural to explore a similar question in the metric distortion problem, where the underlying metric is not directly given.

Our work additionally extends the existing line of research on k-committee elections Faliszewski et al. (2017); Elkind et al. (2017); Caragiannis et al. (2022), which also select a committee of k candidates and aim to minimize some loss function across all voters. The key distinction is that we consider a single candidate as our baseline for calculating distortion, while these works consider a baseline of k candidates (see Section 1.2). Specifically, for the utilitarian cost we consider (i.e., the sum of distances of voters to their closest candidate in the committee), Caragiannis et al. (2022) show that the ratio of the cost for any deterministic method compared to the cost of an optimal committee can be unbounded in the worst case. This result fails to suggest a choice of candidates, as all possible choices are the same in the worst case. In contrast, we show that by using our baseline, one can make a meaningful distinction between different choices even though the underlying metric is unknown.

Our contribution. Our first result shows that when the underlying metric is a line, a committee of two candidates can achieve a 1-distortion of one under the *sum-cost* objective. Here, *1-distortion* refers to the worst-case ratio between the cost of the selected committee and the cost of an optimal single-winner candidate. For any voting rule f, its *distortion*, or more specifically, *1-distortion* is

$$\texttt{1-distortion}(f) := \sup_{\mathcal{E}} \sup_{d \triangleright \mathcal{E}} \frac{\texttt{cost}(f(\mathcal{E}))}{\texttt{OPT}}$$

where the cost function $\texttt{cost}$ in the above definition could be either $\texttt{cost}_s$ or $\texttt{cost}_m$ depending on the context. In other words, the 1-distortion compares the cost of the mechanism to the cost of an optimal candidate in the worst case.

In fact, we will prove a stronger result by showing that it is possible to output a list of two candidates that always contains an optimal one. Formally, we prove the following theorem.

Theorem 1. An algorithm for the 2-committee utilitarian election on the line metric exists that guarantees the choice of an optimum candidate in the elected committee. Consequently, the 1-distortion of the algorithm is 1.

We further show that when the metric is not a line, one can obtain a distortion of $1 + \frac{2}{m-1}$.

Theorem 2. There exists an algorithm for the $(m-1)$-committee utilitarian election on the general metric that obtains a 1-distortion of, at most, $1 + 2/(m-1)$. Additionally, no algorithm can obtain a 1-distortion better than $1 + 2/(m-1)$ when choosing $m-1$ candidates, even if the metric space is 2D Euclidean or tree metric.

We further study the *max-cost* objective, showing that one can obtain 1-distortions of 1, 1.5, and 2 using sets of size four, three, and two, respectively.

Theorem 3. For any $k \in \{2, 3, 4\}$, there exists an algorithm for the k-committee egalitarian election on the line metric, which obtains a 1-distortion of at most $3 - k/2$. Furthermore, no algorithm can obtain a 1-distortion better than $3 - k/2$.

For the *max-cost* objective, even though there exists an algorithm that selects one candidate with a distortion of 3, we show that no algorithm can achieve a 1-distortion better than 3, even when choosing a committee of $m-1$ candidates.

Theorem 4. There is no algorithm for the $(m-1)$-committee egalitarian election on 2D Euclidean metric such that it can obtain a 1-distortion better than 3.

It is important to note that the obtained lower bounds apply to deterministic algorithms.

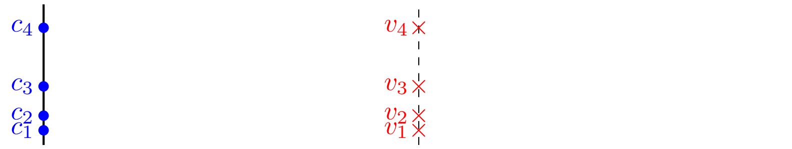
Table 1: State-of-the-Art; blue bounds are results of this paper.

Objective	Metric Space	Committee Size (Out of m)	Lower-Bound	Upper-Bound
Sum	1D	≥ 2	1	1[a]
		1	3[b]	3[c]
	2D / Tree Metric	$m-1$	$1 + \frac{2}{m-1}$	$1 + \frac{2}{m-1}$
Max	1D	≥ 4	1	1
		3	1.5	1.5
		2	2	2
		1	3	3
	2D	$m-1$	3	3

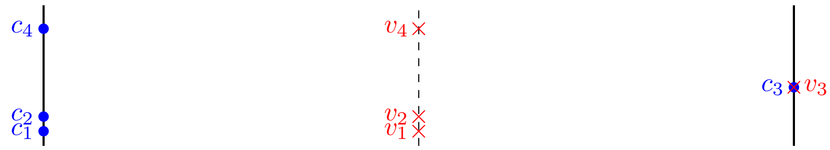
[a] Implied by Anshelevich & Postl (2017).
[b] From Anshelevich et al. (2018).
[c] From Gkatzelis et al. (2020).

1.1 Technical overview

Upper bound results. Our approach to the bi-criteria metric distortion problem builds upon the foundational result by Elkind & Faliszewski (2014), who demonstrated that the ordering of voters and candidates can be deduced from ordinal preferences in a line metric. We use this established ordering for our upper bound results, enabling us to develop algorithms that select small committees achieving optimal or near-optimal distortion relative to the single-winner benchmark. We employ a distinct method to prove this ordering, which facilitates extensions of our results to more complex



(a) The instance I_0



(b) I_3: Instance with voter v_3 and candidate c_3 moved

Figure 1: Figures of the lower bound instances. In (a), all candidates are located on the line $x = -\ell$, with voters with matching y-coordinates on the line $x = 0$. In (b), voter v_3 and candidate c_3 are moved to the line $x = \ell$, while keeping the same y-coordinate.

settings and yields improved outcomes in these generalized spaces. For the sake of completeness and to provide a clear foundation for understanding the remainder of our solution, we present a detailed exposition of this method in Section A.

Finally, to achieve the optimal solution for the sum objective with two candidates, we select candidates immediately to the left and right of any median voter. This approach follows from the result in Anshelevich & Postl (2017), with a detailed explanation provided in Section 3. In contrast, achieving a 1-distortion of 1 for the max objective requires a different strategy. We begin by identifying the leftmost and rightmost voters, whose distance is denoted by D. For the single-winner case, a lower bound of $D/2$ is known. Then, depending on whether we use 2, 3, or 4 candidates, we place them accordingly to achieve a 1-distortion of 2, 1.5, and 1, respectively.

Lower bound results. In Section 4, we provide various lower bounds for different settings. For the sum objective, we begin by presenting an example showing that any mechanism for choosing a committee of size k cannot achieve bounded distortion, even when compared to an optimal choice of $\lceil \log(k+1) \rceil + 1$ candidates. This example relies on a binary tree structure with $k+1$ leaves, where each candidate corresponds to a leaf in the tree. The set of voters is the same as the candidates. Then, for any candidate, we construct an instance where the candidates whose lowest common ancestor with this candidate in the tree is a given vertex are in a close cluster. By selecting one representative for each cluster, we obtain a set of candidates of size $\lceil \log(k+1) \rceil + 1$, for which the total cost can be made arbitrarily small. In contrast, any solution excluding this candidate will incur a constant cost. Notably, these instances lie on the real line, demonstrating that the lower bound applies even for the line metric.

Next, we present examples in the 2D Euclidean metric space and tree metric space, where achieving a 1-distortion better than $1 + \frac{2}{m-1}$ for the sum objective is not possible even when choosing $m-1$ candidates. To argue that, we consider a set of m candidates and m voters. Then, we create $m+1$ instances $I_0, I_1, \cdots, I_m$. Each of them is distinct in terms of voters' and candidates' exact locations; however, they are indistinguishable in terms of voters' preference orderings (that are available as input to any voting mechanism). See Figure 1 to have an idea about the construction of the base instance I_0 and instances I_j for each voter j. We show that not selecting any particular candidate would result in one of the above $m+1$ election instances to attain a 1-distortion of at least $1 + \frac{2}{m-1}$.

This example highlights a drastic contrast between the line metric and higher-dimensional metrics. While a 1-distortion of one can be achieved on the line with just two candidates, this is not possible in higher dimensions, even when selecting all but one candidate. We also present a similar example for the max objective, demonstrating that achieving a 1-distortion better than 3 is impossible in the $2D$ Euclidean metric, even when selecting $m-1$ candidates. Finally, we provide examples showing that our positive results for the 1-distortion of the max objective on the line are tight.

1.2 RELATED WORKS

The metric distortion framework has been central in analyzing single-winner voting. Gkatzelis et al. (2020) proved that any deterministic mechanism must have distortion at least 3, later simplified by Kizilkaya & Kempe (2023b). While it was conjectured that randomization could improve to 2, this was disproved by Charikar & Ramakrishnan (2022) and Pulyassary & Swamy (2021). Recently, Charikar et al. (2024b) achieved 2.753 with randomization. Separately, Anshelevich et al. (2024) showed that if threshold approval sets are known, distortion can be reduced to $1 + \sqrt{2}$.

In the k-committee election problem (the single-winner election being a special case with $k = 1$), the aim is to select k candidates from a pool of m candidates based on ordinal preferences provided by n voters. For any mechanism f, its distortion is defined as the worst-case ratio (across all instances) of the cost of the solution produced by f compared to the optimal cost. When the cost for a voter is considered as the sum of distances to all committee members, Goel et al. (2018) showed that the problem reduces to the single-winner election.

On the other hand, Caragiannis et al. (2022) considered a general cost function – each voter's cost is the distance to the q-th (for some integer $q \geq 1$) nearest committee member. They identified a trichotomy: For $q \leq k/3$, the distortion is unbounded; for $q \in (k/3, k/2]$, it is $\Theta(n)$; and for $q > k/2$, the problem reduces to the single-winner election. As an immediate corollary, for the 2-committee election, with each voter's cost being its distance to the nearest committee member (i.e., $q = 1$), we get a distortion of $\Theta(n)$. Further, with the same cost function, for the k-committee election when $k \geq 3$, the distortion is unbounded (even for the line metric). However, when the positions of candidates are known, for $k = m - 1$, Chen et al. (2020) demonstrated that single-vote rules achieve a distortion of 3 and provided a matching lower bound.

One of the most basic versions–where both voters and candidates are positioned on a real line–has already garnered significant attention in computational social choice theory. Strict preference profiles, with voters and candidates on a real line (also known as 1-D Euclidean), exhibit many intriguing properties, including being single-peaked and single-crossing (Black, 1948; Mirrlees, 1971; Escoffier et al., 2008). Given a preference profile and the order of voters, deciding whether it is 1-D Euclidean can be done in polynomial time (Elkind & Faliszewski, 2014). Furthermore, if the input preference order is consistent with 1-D Euclidean, Elkind & Faliszewski (2014) provides an efficient construction of a mapping realizing that. Very recently, Fotakis et al. (2024) studied the k-committee election problem on the 1-D Euclidean metric by allowing a few distance queries in addition to the voters' preference orders.

One closely related question to the problem of optimal candidate selection is the facility location problem Mahdian et al. (2006), where the goal is to place facilities at locations in a metric space to minimize the cost of serving agents. Unlike the candidate selection problem, where candidates are restricted to a fixed set, facilities in the facility location problem can be placed anywhere in the space Feldman et al. (2016). Another related concept is the Condorcet winning set: a set of candidates such that no other candidate is preferred by at least half the voters over every member of the set Elkind et al. (2015). The Condorcet dimension, defined as the minimum cardinality of a Condorcet winning set, is known to be at most logarithmic in the number of candidates, partially reaffirmed by Caragiannis et al. (2024). In the metric ranking framework, Lassota et al. (2024) demonstrated that the Condorcet dimension is at most three under the Manhattan or ℓ_∞ norms. Recently, Charikar et al. (2024a) showed that Condorcet sets of size six always exist.

2 PRELIMINARIES

Metric space. Let us consider a domain $\mathcal{X}$ and a distance function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$. We call $(\mathcal{X}, d)$ a *metric space* if the distance function d satisfies the following properties:

- **Positive definite**: For all $x, y \in \mathcal{X}$, $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.
- **Symmetry**: For all $x, y \in \mathcal{X}$, $d(x, y) = d(y, x)$.
- **Triangle inequality**: For all $x, y, z \in \mathcal{X}$, $d(x, y) \leq d(x, z) + d(z, y)$.

In this paper, we consider

- **Line metric (1-D Euclidean metric)**: The domain is $\mathcal{X} = \mathbb{R}$, and for any two points $p, q \in \mathbb{R}$, their distance is $d(p, q) = |p - q|$.
- **2D Euclidean metric**: The domain is $\mathcal{X} = \mathbb{R}^2$, and for any two points $p = (p_x, p_y), q = (q_x, q_y) \in \mathbb{R}^2$, their distance is $d(p, q) = ||p - q||_2 := \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$.
- **Tree Metric:** The domain is a set of vertices V, and there exists a weighted tree T such that for any pair of vertices $u, v \in V$, the distance $d(u, v) = d_T(u, v)$, where $d_T(u, v)$ denotes the total weight of the path between u and v in T.

Election. An *election instance* $\mathcal{E} = (V, C, \succ)$ consists of a set $V = \{v_1, \ldots, v_n\}$ of n voters and a set $C = \{c_1, \ldots, c_m\}$ of m candidates. Each voter $v_i \in V$ has a linear order $\succ_i$ over the candidates, where $c_j \succ_i c_k$ indicates that voter v_i prefers c_j over c_k. We refer to $\succ_i$ as the *ordinal preference* of voter v_i. Furthermore, $\succ = \{\succ_1, \ldots, \succ_n\}$ is called the *preference profile* of the voters. Additionally, the j-th candidate in the ordinal preference of voter v_i is denoted by $\succ_{i,j}$.

We consider the voters and candidates to lie in the same metric space $(\mathcal{X}, d)$. For ease of exposition, we extend the notion of distance d to be defined directly on the set of voters and candidates instead of the points they occupy in the underlying space $\mathcal{X}$. We say a (distance) metric d is *consistent* with an election instance $\mathcal{E} = (V, C, \succ)$, denoted as $d \rhd \mathcal{E}$, when for any voter v_i, $c_j \succ_i c_k$ if $d(v_i, c_j) \le d(v_i, c_k)$.

Social cost. Let us consider an election instance $\mathcal{E} = (V, C, \succ)$, and a distance metric $d \rhd \mathcal{E}$. Let $I = (\mathcal{E}, d)$ denote the instance $\mathcal{E}$ with d being its underlying distance metric. For any subset of candidates $S \subseteq C$, and a voter $v \in V$, we use $d(v, S)$ to denote the distance between the voter v to its nearest neighbor in S, i.e., $d(v, S) := \min_{c \in S} d(v, c)$. In this paper, we focus on the following two *social costs*:

- **Sum-cost (Utilitarian objective)**: For any subset of candidates $S \subseteq C$, its *sum-cost*, denoted by $\texttt{cost}_s(S, I)$ is defined as $\texttt{cost}_s(S, I) := \sum_{v \in V} d(v, S)$.
- **Max-cost (Egalitarian objective)**: For any subset of candidates $S \subseteq C$, its *max-cost*, denoted by $\texttt{cost}_m(S, I)$ is defined as $\texttt{cost}_m(S, I) := \max_{v \in V} d(v, S)$.

When it is clear from the context, we drop I and simply use $\texttt{cost}_s(S)$ and $\texttt{cost}_m(S)$.

Voting rule and distortion. A (deterministic) *voting rule* (also referred to as *mechanism*) f is a function that maps an election instance $\mathcal{E}$ to a subset of candidates S. We use algorithms and mechanisms interchangeably throughout this paper. In this paper, we compare the cost of a voting rule that selects a k-sized committee with the cost of an optimal single candidate. We call a single candidate c_{opt} *optimal* if

$$\texttt{cost}(c_{\text{opt}}) = \min_{c \in C} \texttt{cost}(c).$$

Throughout the paper, we use $\texttt{OPT} = \texttt{cost}(c_{\text{opt}})$ to refer to the cost of an optimal single candidate. To capture how good a voting rule is in the worst case, the notion of distortion is used. For any voting rule f, its *distortion*, or more specifically, *1-distortion* is defined as

$$\texttt{1-distortion}(f) := \sup_{\mathcal{E}} \sup_{d \rhd \mathcal{E}} \frac{\texttt{cost}(f(\mathcal{E}))}{\texttt{OPT}}$$

where the cost function $\texttt{cost}$ in the above definition could be either $\texttt{cost}_s$ or $\texttt{cost}_m$ depending on the context. In other words, the 1-distortion compares the cost of the mechanism to the cost of an optimal candidate in the worst case.

3 UPPER BOUNDS ON DISTORTION FACTOR

Upper bounds for sum-cost. We study the sum objective, where the cost of a candidate set is the sum of distances from voters to their closest selected candidate. On the line metric, using the orderings from Section A, we show that a committee of size two always contains an optimal candidate.

Theorem 5. There exists a voting rule for the 2-committee election for the sum-cost objective, on the line metric, such that the resulting committee includes an optimal candidate. Consequently, the 1-distortion of the voting rule is 1.

We proceed in two steps. First, we show that it is always possible to select three candidates such that one optimal candidate is guaranteed to be included. Second, we show that one of these three candidates can be safely discarded without ever removing that optimal candidate.

For the sum of distances, the median voter is the "balance point": moving a candidate toward the median decreases distance to the majority more than it increases it to the minority. Thus, an optimal candidate can be slid toward the median without increasing cost until it becomes one of the two candidates adjacent to the median voter (for an even number of voters, we fix either middle voter as the median). Note that the proofs of all lemmas and theorems appear in Appendix B.1.

Lemma 6. When considering the sum of distances objective and the candidates and voters are located on the real line, one of the candidates directly to the right or left of the median voter will be an optimal candidate.

Using only ordinal information, we order voters and candidates and discard candidates never closest to any voter. Let c be the candidate closest to the median voter, and take the candidates immediately to the left and right of c in the candidate ordering. By Lemma 6, the optimal candidate is the first candidate to the left or right of the median and must belong to this triple.

Lemma 7. There exists a voting rule for the 3-committee election on the line metric such that the resulting committee includes an optimal candidate.

To prove Theorem 5, let the three candidates in order on the line be c_1, c_2, c_3, and partition voters into three contiguous groups according to which of these is closest (this partition is determined by ordinal preferences). If more voters lie to the right of c_2 than to the left, replacing c_3 by c_2 cannot increase the sum of distances, so c_2 is at least as good as c_3; symmetrically, if the left side is heavier, c_2 is at least as good as c_1. Thus, one extreme is never strictly better than the middle candidate, and we can drop it while still retaining a 2-set containing an optimal candidate.

For a general metric, we show that choosing $k = m - 1$ candidates suffices to obtain 1-distortion $1 + \frac{2}{m-1}$. The rule looks only at each voter's top-ranked candidate and selects all but the least popular top choice. If the optimal single candidate is among these $m - 1$ winners, the distortion is 1; otherwise, it is the unique excluded candidate, so only voters ranking it first are affected. There are a few such voters by construction. Each can be rerouted, via a nearby voter whose favorite is selected, to an elected candidate with bounded extra distance, leading to total cost at most a factor $1 + \frac{2}{m-1}$ of the optimum.

Theorem 8. There exists a voting rule choosing $m - 1$ out of m candidates achieving a 1-distortion of $1 + \frac{2}{m-1}$ for the sum-cost objective.

Upper bounds for max-cost. We now consider the max-cost objective, where the cost of a candidate set is the maximum distance of any voter to her closest selected candidate. Our benchmark is the best single candidate. We show that two, three, and four candidates suffice to guarantee distortions 2, $3/2$, and 1, respectively.

Throughout this section, we use the leftmost and rightmost voters, denoted v_l and v_r, identified using Section A. Since we only use each voter's top candidate, ties cause no ambiguity. Proof details are in Appendix B.2.

If v_l and v_r are far apart, no single candidate can be close to both: wherever the candidate is placed, at least one of v_l or v_r is at a distance of at least $d(v_l, v_r)/2$.

Lemma 9. If the distance between v_l and v_r is D (i.e., $D = d(v_l, v_r)$), then the cost for the optimal single candidate $\texttt{OPT}$ satisfies $\texttt{OPT} \geq \frac{D}{2}$.

Thus, the "span" $d(v_l, v_r)$ forces any single candidate to incur max-cost at least $D/2$.

From the orderings of Section A, we know how voters and candidates interleave on the line. When voters form a single block with no candidate strictly between v_l and v_r, c_l is the first candidate immediately to the left of all voters and c_r the first to the right, and choosing $\{c_l, c_r\}$ is optimal.

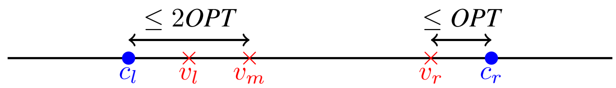


Figure 2: While the pair $\{c_l, c_r\}$ cannot achieve optimal 1-distortion, it guarantees a 1-distortion of 2.

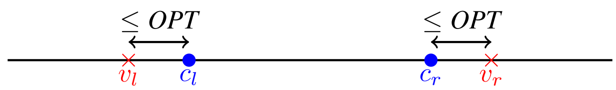


Figure 3: When c_l and c_r are between v_l and v_r; $\{c_l, c_r\}$ is optimal.

Lemma 10. If there are no candidates placed between v_l and v_r, a voting rule that selects c_l and c_r, achieves a $1 - distortion$ of 1.

The nontrivial case is when at least one candidate lies between v_l and v_r, which we assume from now on. Then any optimal single candidate c_{opt} must also lie between v_l and v_r; otherwise it is at least $d(v_l, v_r)$ away from one extreme voter.

Now v_l and v_r act as "anchors" for all voters. For any voter v between them we have $d(v_l, v) + d(v, v_r) = D = d(v_l, v_r)$. By Lemma 9, $\texttt{OPT} \geq D/2$, so every voter is within distance $\texttt{OPT}$ of at least one of v_l or v_r.

Lemma 11. If $d(v_l, v_r) = D$, then for every voter v, either $d(v, v_l) \leq \texttt{OPT}$ or $d(v, v_r) \leq \texttt{OPT}$.

This yields a 2-approximation:

Theorem 12. For any election $\mathcal{E}$ on the line metric, let v_l and v_r be the leftmost and rightmost voters, and c_l and c_r be the closest candidates to v_l and v_r, respectively. Then, a voting rule f that outputs the set $\{c_l, c_r\}$ satisfies $\texttt{1-distortion}(f) \leq 2$ with respect to the max-cost objective.

Indeed, $d(v_l, c_l) \leq \texttt{OPT}$ and $d(v_r, c_r) \leq \texttt{OPT}$ by definition, and Lemma 11 guarantees that every other voter v is within distance $\texttt{OPT}$ of at least one of v_l or v_r. Going via that extreme and then to c_l or c_r adds at most another $\texttt{OPT}$, so all voters are within $2\texttt{OPT}$ of $\{c_l, c_r\}$. The pair $\{c_l, c_r\}$ need not give distortion 1 when c_l lies left of v_l or c_r right of v_r: some voters between v_l and v_r may then be far from both c_l and c_r, even though c_{opt} sits in the middle.

Conversely, if c_l and c_r both lie between v_l and v_r (Figure 3), then $\{c_l, c_r\}$ is optimal. In this case,

$$d(v, c_l) \leq \max(d(v, v_l), \texttt{OPT}), \quad d(v, c_r) \leq \max(d(v, v_r), \texttt{OPT}),$$

and by Lemma 11 each voter is within distance $\texttt{OPT}$ of v_l or v_r, so

$$d(v, c_l) \leq \mathit{OPT} \quad \text{or} \quad d(v, c_r) \leq \mathit{OPT}.$$

To handle the general case, we "push" the selected candidates inward by adding more candidates. Adding one candidate just inside one extreme yields a $3/2$-approximation; adding one near each side gives a distortion 1.

Theorem 13. For any election $\mathcal{E}$ on the line metric, there exists a voting rule to select three candidates, which achieves a 1-distortion of $3/2$ with respect to the max-cost objective.

We start from v_l, v_r and their closest candidates c_l, c_r. Using the candidate ordering, we add c'_r immediately to the left of c_r and consider $C = \{c_l, c_r, c'_r\}$. Any voter strictly outside the interval between c_l and the rightmost candidate in $\{c_r, c'_r\}$ has her nearest candidate in C by construction. For voters in the middle region, the distance between c_l and that rightmost candidate is at most $3\texttt{OPT}$ (via v_l and c_{opt}), so any voter between them is at a distance at most $3\texttt{OPT}/2$ from one of them.

Theorem 14. For any election $\mathcal{E}$ on the line metric, there exists a voting rule to select four candidates, which achieves a 1-distortion of 1 with respect to the max-cost objective.

For the 4-candidate rule, we symmetrize. From c_l and c_r, let c'_l be the next candidate to the right of c_l and c'_r the next to the left of c_r. Let c^*_l be the one in $\{c_l, c'_l\}$ lying to the right of v_l, and c^*_r the one in $\{c_r, c'_r\}$ lying to the left of v_r. Consider $\{c_l, c'_l, c_r, c'_r\}$. Every voter to the left of c'_l or right of c'_r has her closest candidate among these four. In the central part of the line, $c^*_l, c^*_r, c_{\text{opt}}$ lie between v_l and v_r; since v_l and v_r are within distance $\texttt{OPT}$ of c_{opt}, we have $d(c^*_l, c^*_r) \leq 2\texttt{OPT}$, so any voter between them is within distance at most $\texttt{OPT}$ of one of them. Thus, every voter is within $\texttt{OPT}$ of the selected four, giving a distortion 1.

4 Lower bounds on distortion factor

Lower bounds for sum-cost. We now give lower bounds for distortion under the sum-cost objective. First, we extend the lower bound of Caragiannis et al. (2022) to show that bounded distortion is impossible for general k, even if we may select $\omega(k)$ candidates. The construction already works on the line metric. We encode each voter and candidate by an ℓ-bit binary string (with $k = 2^\ell - 1$), define preferences so that a voter prefers candidates with longer common prefixes, and embed these strings on the line via a base-3 expansion where bits agreeing with a fixed "special" candidate receive large weight and later bits are downweighted by ε.

Theorem 15. For $k \geq 3$, there exists an instance for the k-committee election on the line (with respect to the sum-cost) where no voting rule choosing k candidates can achieve a bounded distortion even when compared to an optimal choice of $\lceil \log_2(k+1) \rceil + 1$ candidates.

Any rule picking exactly $k = 2^\ell - 1$ candidates must omit some candidate c_{i^*}. We place voter v_{i^*} at the same point as c_{i^*} and arrange all other candidates so that every selected candidate is at a distance $\Omega(3^{-\ell})$ from v_{i^*}. Thus any k-sized committee omitting c_{i^*} incurs a constant cost for v_{i^*}, independent of ε. In contrast, we build a "covering" committee of size $\ell + 1$: c_{i^*} plus one candidate for each bit position where others first diverge from b_{i^*}. Since discrepancies beyond that position are downweighted by ε, every voter has a committee member at distance $O(\varepsilon)$, so by shrinking ε, the optimal cost can be made arbitrarily small. A k-sized committee excluding c_{i^*} thus pays constant cost, whereas a $\Theta(\log k)$-sized committee has cost tending to 0, making the distortion unbounded.

Next, we construct instances in the plane and on tree metrics where no rule excluding at least one out of m candidates can guarantee 1-distortion strictly less than $1 + \frac{2}{m-1}$.

Theorem 16. For any number of candidates m and any $\varepsilon > 0$, there exist instances of the $(m-1)$-committee election problem in the 2D Euclidean metric for which no voting rule can guarantee a 1-distortion factor less than $1 + \frac{2}{m-1} - \varepsilon$.

Theorem 17. For any number of candidates m and any $\varepsilon > 0$, there exist instances of the $(m-1)$-committee election problem with tree metrics for which no voting rule can guarantee a 1-distortion factor less than $1 + \frac{2}{m-1} - \varepsilon$.

The proofs of Theorems 16 and 17 share the same idea. We construct $m + 1$ metric instances $I_0, \ldots, I_m$ with identical candidate rankings but very different distances. In I_0, all candidates lie on one vertical line and all voters on a parallel line, so each voter v_i is closest to the "matching" candidate c_i. For each $j \in [m]$, instance I_j moves v_j and c_j far to the right, leaving all other points fixed; this preserves ordinal preferences, so any rule using only rankings must output the same $(m-1)$-subset on all instances.

In I_j, excluding c_j greatly increases cost: v_j is now at a distance of about ℓ from every selected candidate, whereas including only c_j keeps all voters within a distance of roughly $\ell + 1$. Summing over voters, any $(m-1)$-committee omitting c_j has cost on the order of $(m+1)\ell$, while the optimal cost is about $(m-1)(\ell+1)$. Thus

$$\frac{(m+1)\ell}{(m-1)(\ell+1)} \approx 1 + \frac{2}{m-1},$$

and by choosing ℓ large enough (as a function of ε) we make the deviation from this limit smaller than ε. The tree-metric version embeds the same "left vs. far-right" geometry into a tree (e.g., via long paths or a star) with the same effect.

Lower bounds for the max-cost We now bound the distortion for the max-cost objective. In 2D Euclidean space, even when selecting $m - 1$ out of m candidates, we cannot guarantee distortion less than 3, matching the distortion when selecting only one candidate. For the line metric, we then show that our algorithms that select $k = 2$ and $k = 3$ candidates are tight.

Theorem 18. Any deterministic algorithm for the k-committee election (with respect to the max-cost) that selects at most $k < m$ candidates out of m candidates must have a 1-distortion of at least $3 - \varepsilon$ for any $\varepsilon > 0$.

We define $m + 1$ plane instances $I_0, \ldots, I_m$ with identical ordinal preferences. In I_0, all candidates lie on one vertical line, all voters on a parallel line, and each voter v_i ranks c_i first. For I_j, we push c_j and v_j far to the right so that v_j is at a distance of about 3ℓ from every candidate except c_j, while all other distances (and rankings) are unchanged. Any deterministic, ordinal rule must choose the same committee C on all these instances. Since $k < m$, some candidate c_j is not chosen; in I_j, the max-cost of C is then driven by v_j, who sees all selected candidates at a distance of about 3ℓ, whereas the optimal solution picks c_j, keeping everyone within a distance of roughly $\ell + 1$. Hence

$$\frac{\mathtt{cost}_m(C, I_j)}{\mathtt{OPT}_m(I_j)} \approx \frac{3\ell}{\ell + 1} \approx 3,$$

and for large ℓ the distortion is at least $3 - \varepsilon$.

Theorem 19. Any deterministic algorithm for the 2-committee election (with respect to the max-cost) when voters and candidates are located on a line must have a 1-distortion of at least $2 - \varepsilon$ for any $\varepsilon > 0$.

For Theorem 19, we consider three voters and three candidates on a line with the cyclic preferences

$$v_1 : c_1 \succ c_2 \succ c_3, \quad v_2 : c_2 \succ c_1 \succ c_3, \quad v_3 : c_3 \succ c_2 \succ c_1.$$

We realize these rankings by three different metric instances on the line. In each instance, exactly one candidate is a "good center" with max distance at most 1 to every voter, and for each of the other two candidates, some voter is at distance $2 - \varepsilon$. Thus, in one instance, c_1 is uniquely optimal, in another c_2, and in the third c_3. Any deterministic 2-committee rule based only on rankings must output the same pair on all three instances. Whichever candidate it omits is uniquely optimal in one instance, where any 2-committee excluding it has max-cost at least $2 - \varepsilon$, whereas the optimal committee including it has cost 1.

Theorem 20. Any deterministic algorithm for the 3-committee election (with respect to the max-cost) when voters and candidates are located on a line must have a 1-distortion of at least $3/2 - \varepsilon$ for any $\varepsilon > 0$.

The proof of Theorem 20 uses four voters and four candidates on a line. We fix a symmetric preference profile where v_1, v_2 strongly favor c_1, c_2 and v_3, v_4 strongly favor c_3, c_4, and build four line instances $I_1, \ldots, I_4$ realizing these rankings but with different locations: in I_i, candidate c_i is well-centered, at distance at most 2 from every voter, whereas voter v_i is at distance at least $3 - 2\varepsilon$ from all other candidates. Thus, in I_i, the optimal max-cost is at most 2, achieved by including c_i, while any committee excluding c_i must incur max-cost at least $3 - 2\varepsilon$ because of v_i. Any deterministic 3-committee rule must omit at least one candidate and cannot distinguish between the four instances by rankings alone, so for some I_i it omits the uniquely good candidate. In that instance, its distortion is at least $\frac{3-2\varepsilon}{2} = \frac{3}{2} - \varepsilon$, showing that our upper bound of $3/2$ for the line metric with $k = 3$ is tight.

5 Conclusion and Future Work

In this paper, we initiate the study of the metric distortion problem under a bi-criteria approximation framework, namely whether one can match the cost of the optimal single candidate using only a fixed number of candidates. For the line metric, we show this is achievable: two candidates suffice for the utilitarian objective and four for the egalitarian objective, and for the latter we also give a smooth trade-off between committee size and approximation factor. We complement these results with matching lower bounds, establishing tightness on the line and showing that equally strong guarantees fail in richer metrics such as the 2D Euclidean plane or tree metrics.

Looking ahead, several directions remain. One is to fully characterize the metric spaces in which a constant-size committee can achieve optimal 1-distortion. Beyond this structural question, it is also natural to determine whether, in general metrics beyond the line, one can attain 1-distortion strictly better than the best achievable single-winner bound. Another direction is to investigate whether randomized mechanisms can improve guarantees in settings where deterministic mechanisms cannot attain optimal 1-distortion. One can also explore metric distortion and the k-committee election problem within this bi-criteria framework when only limited cardinal information is available.

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A LINE METRIC ELECTION: THE ORDER OF CANDIDATES AND VOTERS

In this section, we present an algorithm to determine the order of candidates and voters for any line metric election instance $\mathcal{E} = (V, C, \succ)$ based on the voters' preference profiles. This step is essential for establishing the upper bounds discussed in Section 3. Moreover, this approach may prove useful for future research, as it offers a general method for obtaining the total order of candidates and voters, as explained below.

This algorithm focuses on a specific subset of candidates with properties useful for the purposes of this paper and potential future work on line metric distortion. This subset has the following property:

Definition 21 (Core)**.** In an election $\mathcal{E} = (V, C, \succ)$, a subset $A \subseteq C$ is called a *core* if for any voter $v_i \in V$, any candidate $c_j \in A$, and any candidate $c_k \in C \setminus A$, we have $c_j \succ_i c_k$.

Also, we are able to retrieve the order of voters based on this subset of candidates. Therefore, we define the following notation:

Definition 22. For a subset of candidates $A \subseteq C$, the order of voters *with respect to* A is a sequence S of voters such that voter v_i become beforev_j in S, if the preference order of candidates in A by v_i is lexicographically no greater than the preference order of candidates in A by v_j, when A is sorted according to some fixed order S_A.

Consequently, the main theorem of this section is the following:

Theorem 23. For an election $\mathcal{E} = (V, C, \succ)$ on the line metric such that $d \rhd \mathcal{E}$, There exists an algorithm that identifies:

1. A *core* subset of candidates C^*;
2. The order of C^* candidates, denoted as S_C; and
3. The order of voters *with respect* to C^*, denoted as S_V.

Based on Theorem 23, the algorithm introduced in this section determines the order only for a subset C^*, referred to as the *determined candidates*, rather than for all candidates. It expands C^* iteratively and establishes the order of candidates within C^*. By the end of the algorithm, C^* forms a core subset of candidates.

Additionally, for some voters with similar preferences, it may be impossible to distinguish whether they are on the right or left; thus, they are considered to be at the same point. These properties of the voter and candidate order retrieved by the algorithm are accounted for in the analysis and other sections of this paper. Overall, obtaining the order of voters and candidates with this level of accuracy remains significant for the intended purposes.

The algorithm consists of three parts. The first part, referred to as `SplitLine`, involves dividing the line into two halves and determining whether each candidate occurs on the left or right side (some remain undetermined). The second part, referred to as `SortCandidates`, finds the order of candidates in each half and merges the sorted lists to obtain a total ordering. The final part, referred to as `SortVoters`, determines the ordering among the voters based on the retrieved order of candidates. Algorithm 1 demonstrates how these three components contribute to sorting the candidates and voters.

ALGORITHM 1: Sort candidates and voters

Input: Election instance $\mathcal{E}$.

Output: Sequence S_C as the order of determined candidates and sequence S_V as the order of voters.

Function `SortCandidatesAndVoters` ($\mathcal{E} = (V, C, \succ)$) **:**

    $(L, R) \leftarrow$ `SplitLine`($\mathcal{E}$)
    $S_C \leftarrow$ `SortCandidates`($\succ_1, L, R$)
    $S_V \leftarrow$ `SortVoters`($\mathcal{E}, S_C$)
    return (S_C, S_V)

SplitLine. In this part, we first find a pivot to split the line at that point. To achieve this, we introduce the following definitions. We arbitrarily choose the first voter as the pivot voter, associated with two pivot candidates as follows.

Definition 24 (Pivot Voter and Candidates)**.** The voter v_1 is called the *pivot voter*. Additionally, the two nearest candidates to the pivot voter are called the *pivot candidates*. Without loss of generality, assume that c_1 and c_2 are the two nearest candidates to the pivot voter v_1, with c_1 positioned to the left of c_2. Note that c_1 and c_2 may both be on the same side of v_1.

Finally, the point where the line is split is defined as follows:

Definition 25 (Pivot Point)**.** The midpoint of the line segment between the two pivot candidates is called the *pivot point*, denoted by p. Let L and R be the subsets of candidates on the left and right sides of p, respectively.

Finally, we aim to determine whether a candidate belongs to L or R. Therefore, we formally define determined and undetermined candidates as follows:

Definition 26 (Determined Candidates)**.** A candidate is *determined* if it is known whether the candidate belongs to L or R based on Definition 25. Otherwise, the candidate is *undetermined*. Let $C^* = L \cup R$ be the set of determined candidates

Initially, based on Definition 24, we know that $c_1 \in L$ and $c_2 \in R$. Thus, $C^* = \{c_1, c_2\}$. Then, in multiple iterations, we expand L and R by adding as many candidates as possible. Therefore, at the end of each iteration, we update C^* such that $C^* = L \cup R$ again. We also ensure that C^* always forms a consecutive subset of candidates on the line. The process for determining new candidates in each iteration is as follows:

If there exists a voter v_i such that two candidates c_k and c_j satisfy $c_k \succ_i c_j$, where $c_j \in C^*$ but $c_k \notin C^*$, then we can determine c_k's membership as follows:

- If both c_1 and c_2 are closer to v_i than c_k, then, if c_j is in L, c_k would be in R, and vice versa.
- Otherwise, c_k would be in L if c_1 is closer to v_i, and c_k would be in R if c_2 is closer to v_i.

Algorithm 2 is pseudocode for the function `Determine`, which determines a candidate if the above conditions hold and adds it to the corresponding set. Algorithm 3 determines as many candidates as possible in each iteration while maintaining the succession of the determined candidates (see A.1 for proofs). We call these candidates C_{new}. At the end of the iteration, it merges C_{new} into C^*. It is important to note that we do not add each point immediately to C^*; instead, we merge them all at the end. This approach ensures that C^* remains a consecutive list of candidates, which is crucial for our analysis.

The output of this procedure is L and R, which represent the determined candidates on the left and right sides of the pivot point p, respectively.

ALGORITHM 2: Determine a candidate with a determined candidate in a voter's ordinal preference

Input: Ordinal preference of voter v_i, denoted by $\succ_i$; candidates c_j and c_k where $c_k \succ_i c_j$, c_j is determined, but c_k is not; two sets L and R containing currently determined candidates on the left and right sides of the pivot point p, respectively; and pivot candidates c_1 and c_2.
Output: Updated sets L and R including candidate c_k.

Function `Determine`$(\succ_i, c_j, c_k, L, R, c_1, c_2)$**:**
  if $c_1 \succ_i c_k$ ***and*** $c_2 \succ_i c_k$ **then**
    Add c_k to L if $c_j \in R$; otherwise, add c_k to R
  else
    Add c_k to L if $c_1 \succ_i c_2$; otherwise, add c_k to R
  end
  return (L, R)

SortCandidates. The goal of this part is to sort the candidates in L and R. We know that L lies to the left of p, and R lies to the right of p, where p is the midpoint of the segment connecting the

ALGORITHM 3: Determine Candidates

Input: Election instance $\mathcal{E}$.

Output: Two subsets L and R of candidates, where candidates are on the left and right of the pivot point p, respectively.

Function `SplitLine` $(\mathcal{E} = (V, C, \succ))$ **:**
  $L \leftarrow \{c_1\}$
  $R \leftarrow \{c_2\}$
  $C^* \leftarrow \{c_1, c_2\}$
  repeat
    $C_{new} \leftarrow \emptyset$
    while $\exists(v_i, c_j, c_k) : c_k \succ_i c_j$ ***and*** $c_k \notin C^*$ ***and*** $c_j \in C^*$ **do**
      $(L, R) \leftarrow$ `Determine`$(\succ_i, c_k, c_j, L, R, c_1, c_2)$
      $C_{new} \leftarrow C_{new} \cup \{c_k\}$
    end
    $C^* \leftarrow C^* \cup C_{new}$
  until $C_{new} = \emptyset$
  return (L, R)

two nearest candidates of v_1 (see Definitions 24 and 25). Consequently, in the ordinal preference of v_1, candidates in L with higher preferences are positioned to the right of those with lower preferences. Similarly, candidates in R with higher preferences are positioned to the left of those with lower preferences. By combining these two observations, we can sort the candidates based on their positions along the line. Algorithm 4 presents the pseudocode for this approach.

ALGORITHM 4: Sort determined candidates

Input: Preference order of v_1, denoted as $\succ_1$, L and R, candidates on the left and the right side of pivot p.

Output: Sequence S_C, sorted candidates in L and R by their position left to right.

Function `SortCandidates` $(\succ_1, L, R)$ **:**
  S_C is an empty sequence of candidates
  for i *from* 1 *to* m **do**
    Let c_j be the i-th candidate in the preference order of v_1, i.e., $\succ_{1,i}$
    if $c_j \in L$ **then**
      Add c_j to the extreme left of S_C
    else if $c_j \in R$ **then**
      Add c_j to the extreme right of S_C
    end
  end
  return S_C

SortVoters. This part focuses on sorting voters given the sorted determined candidates. The key observation is that a voter who prefers one candidate over another tends to be closer to the candidate they prefer more. Consequently, we have a method to compare two voters. For a pair of voters v_i and v_j, assume k is the smallest index where the ordinal preferences of v_i and v_j differ. v_i is on the left side of v_j if $\succ_{i,k}$ is on the left side of $\succ_{j,k}$. Recall that the notation $\succ_{p,q}$ represents the q-th candidate in the preference order of voter v_p. If $\succ_{i,k}$ and $\succ_{j,k}$ are not determined, assuming v_i and v_j are at the same position does not affect the further analysis (see Sections A.1). Algorithms 5 and 6 illustrates how function `SortVoters` works.

A.1 Analysis

This part focuses on verifying that the order of candidates and voters has been correctly calculated.

First, recall that the algorithm returns the sequence S_C as the order of candidates (Line 3 of Algorithm 1). This sequence contains only determined candidates, denoted as $C^* = L \cup R$. Additionally, in Line 4 of Algorithm 1, we calculate the order of voters based on S_C.

It is important to note that for the remainder of this section, all lemmas consider a specific line metric election instance.

ALGORITHM 5: Compare two voters based on determined candidates

Input: Two voters v_i and v_j, preference profile $\succ$, and order of determined candidates S_C.
Output: An integer in $[-1, 1]$: -1 if v_i is left of v_j, 0 if they share the same position, and 1 if v_i is right of v_j.

Function `CompareVoters` $(v_i, v_j, \succ, S_C)$ **:**
- Let k be the min index where the preference orders of v_i and v_j differ, i.e., the smallest $k \in [m]$ such that $\succ_{i,k} \neq \succ_{j,k}$
- **if** *no such k exists **or** $\succ_{i,k}$ is not in S_C* **then**
  - **return** 0
- **else if** *$\succ_{i,k}$ appears before $\succ_{j,k}$ in S_C* **then**
  - **return** -1
- **else**
  - **return** 1
- **end**

ALGORITHM 6: Sort voters

Input: Election instance $\mathcal{E}$, and order of determined candidates S_C.
Output: Sequence of all voters from left to right S_V.

Function `SortVoters` $(\mathcal{E} = (V, C, \succ), S_C)$ **:**
- Let S_V be the sorted elements of V using comparison function `CompareVoters`
- **return** S_V

First, the following lemma demonstrates that a prefix of a voter's ordinal preference forms a consecutive subsequence of candidates.

Lemma 27. For any voter $v_i \in V$ and $1 \leq k \leq m$, the k most preferred candidates of voter v_i form a consecutive subsequence of candidates.

Proof. Assume the condition does not hold for a voter v_i and some k. Since any single candidate forms a consecutive subsequence, we must have $k \geq 2$. Let A denote the k most preferred candidates by the voter v_i. By assumption, there exist two candidates $c_l, c_r \in A$ such that a candidate $c_m \in C \setminus A$ lies between them.

Assume without loss of generality that c_l is to the left of c_m and c_r is to the right of c_m. Further, suppose v_i is located to the left of c_m. Since $c_m \notin A$, we must have $c_r \succ_{v_i} c_m$. However, since the candidates are arranged in a line, it follows that $c_m \succ_{v_i} c_r$, which is a contradiction (illustrated in Figure 4). $\square$

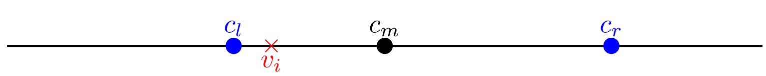


Figure 4: This figure is the illustration of the succession of nearest of any voter.

Corollary 28. c_1 and c_2 are consecutive.

Next, recalling the definitions of determined and undetermined candidates (Definition 26), the following lemma formally proves that the function `Determine` in Algorithm 2 correctly determines an undetermined candidate.

Lemma 29. Assume that C^* is a consecutive subset of determined candidates, including c_1 and c_2. If there exists a voter v_i and candidates $c_j \in C^*$ and $c_k \notin C^*$ such that $c_k \succ_i c_j$, then the function `Determine` correctly determines c_k.

Proof. Let us consider two cases regarding the positioning of c_1, c_2, and c_k in the ordinal preference of v_i.

Case 1: Both c_1 and c_2 are positioned before c_k in the ordinal preference of v_i.

Without loss of generality, we assume that c_j is in R. We use proof by contradiction to show that c_k is in L. Assume that c_j is in R. Consider the prefix of the ordinal preference of v_i ending with c_k. By Lemma 27, they must form a consecutive set of candidates. Therefore, c_j must be on the right side of c_k. However, since C^* consists of consecutive candidates, c_k would necessarily be positioned on the right side of c_j. Because of the contradiction we conclude c_k is in L (illustrated in Figure 5)

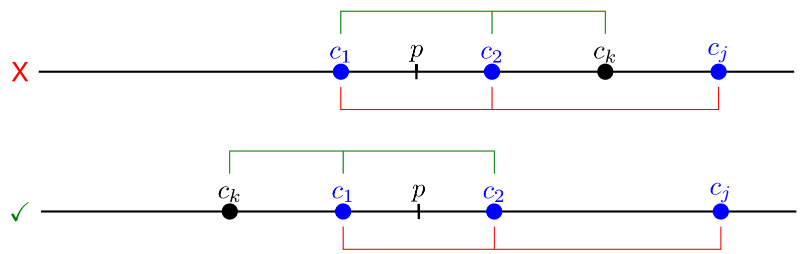


Figure 5: For a voter v_i, if we have $c_1 \succ_i c_k$, $c_2 \succ_i c_k$, and $c_k \succ_i c_k$, then c_k and c_j cannot both be in the same side of p. The figure above illustrates this contradiction, while the one below shows that they can be on opposite sides.

Case 2: At least one of c_1 and c_2 is positioned after c_k in the ordinal preference of v_i.

Without loss of generality, assume that c_1 precedes c_2 in the ordinal preference of v_i. Consider the prefix of the ordinal preference of v_i that contains c_1 and c_k but not c_2. By Lemma 27, this prefix must form a consecutive sequence of candidates, meaning c_2 cannot lie between c_1 and c_k. If c_k were in R, c_2 would lie between c_1 and c_k, as c_1 and c_2 are consecutive.

This contradiction implies that c_k is in L (illustrated in Figure 6).

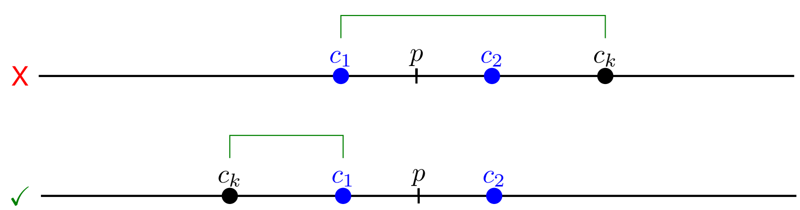


Figure 6: For a voter v_i, if we have $c_1 \succ_i c_2$ and $c_k \succ_i c_2$, then c_k is in L. The figure above illustrates that if c_k were in R, then c_2 would be in the consecutive subsequence of c_1 and c_k, which is a contradiction. On the other hand, the one below shows that c_k must be in L.

□

As explained in Lemma 29, determining a new candidate works if the current set of C^* is consecutive. The following lemma shows that at the beginning of each iteration, C^* is a consecutive subsequence of candidates.

Lemma 30. In Algorithm 3, at the beginning of each iteration (Line 5), all determined candidates, denoted as C^*, form a consecutive subsequence of all candidates.

Proof. Let C_i^* be the set of determined candidates at the beginning of the i-th iteration. We prove that for any i, C_i^* is a consecutive subsequence. $C_1^* = \{c_1, c_2\}$ satisfies the condition. (Corollary 28).

In the $(i-1)$-th iteration ($i \geq 2$), for any voter v_j, let $last_j$ be the least preferred candidate of v_j in C_{i-1}^*. Then, by Lemma 29, by the end of iteration $i-1$, any candidate c_k such that $c_k \succ_v last_j$ is determined. The set of candidates c_k for which $c_k \succ_v last_j$ is denoted by P_j; note that $C_{i-1}^* \subseteq P_j$. As P_j is a prefix of the ordinal preference of v_j, by Lemma 27, P_j is a consecutive subsequence of candidates. Also P_j contains c_1 and c_2 because $\{c_1, c_2\} \subseteq C_{i-1}^* \subseteq P_j$.

Since $C_i^* = C_{i-1}^* \cup C_{new}$ (Line 11) and $C_{new} = \bigcup_j (P_j \setminus C_{i-1}^*)$, we have $C_i^* = \bigcup_j P_j$. However, observe that a union of intersecting intervals is always an interval (this can be seen from a straightforward inductive argument). Since P_j for all j contain $\{c_1, c_2\}$, it follows that C_i^* is a consecutive subsequence as well. □

Next, we show that C^* is a core subset of candidates.

Lemma 31. By the end of the algorithm 3, C^* is core, i.e., for any voter v_i, candidate $c_j \in C^*$, $c_k \notin C^*$, we have $c_j \succ_i c_k$.

Proof. Assume otherwise that $c_k \succ_i c_i$. Then, we can determine c_k using c_j and voter v_i according to Lemma 29, and consequently, Algorithm 3 cannot have terminated at this point. Therefore, $c_j \succ_i c_i$ for any voter v_i. □

We now demonstrate that the function `SortCandidates` in Algorithm 4 returns the determined candidates in sorted order from left to right.

Lemma 32. The procedure `SortCandidates` correctly sorts determined candidates, denoted as $C^* = L \cup R$ from left to right.

Proof. Recall that sets L and R contain determined candidates (Definition 26). Now, consider the ordinal preference of v_1. For any pair of candidates $c_i, c_j \in L$, we show that $d(c_i, c_1) \leq d(c_j, c_1)$ if $c_i \succ_1 c_j$.

There are two cases:

- c_1 is on the right side of v_1 (Figure 7); we have:

$$\begin{aligned} d(c_i, c_1) - d(c_j, c_1) &= d(c_i, v_1) - d(c_1, v_1) - d(c_j, v_1) + d(c_1, v_1) \\ &= d(c_i, v_1) - d(c_j, v_1) \end{aligned}$$

- c_1 is on the left side of v_1(Figure 8); we have:

$$\begin{aligned} d(c_i, c_1) - d(c_j, c_1) &= d(c_i, v_1) + d(c_1, v_1) - d(c_j, v_1) - d(c_1, v_1) \\ &= d(c_i, v_1) - d(c_j, v_1) \end{aligned}$$

Therefore, we have:

$$d(c_i, c_1) \leq d(c_j, c_1) \Leftrightarrow d(c_i, v_1) \leq d(c_j, v_1)$$

Based on the definition of ordinal preference, we have $d(c_i, v_1) \leq d(c_j, v_1)$ if $c_i \succ_1 c_j$. Consequently $d(c_i, c_1) \leq d(c_j, c_1)$ if $c_i \succ_1 c_j$.

Similarly, for any pair of candidates $c_i, c_j \in R$, we can show that $d(c_i, c_2) \leq d(c_j, c_2)$ if $c_i \succ_1 c_j$.

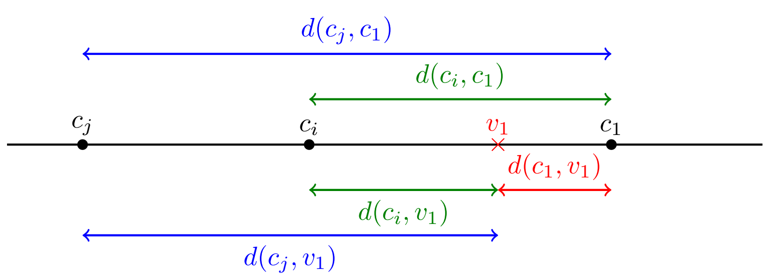


Figure 7: Illustration of the scenario where c_1 is positioned to the right of v_1.

The procedure `SortCandidates` processes candidates according to $\succ_1$, adding them to the left of S_C if they are in L and to the right otherwise. Therefore, S_C contains candidates in the correct order from left to right. □

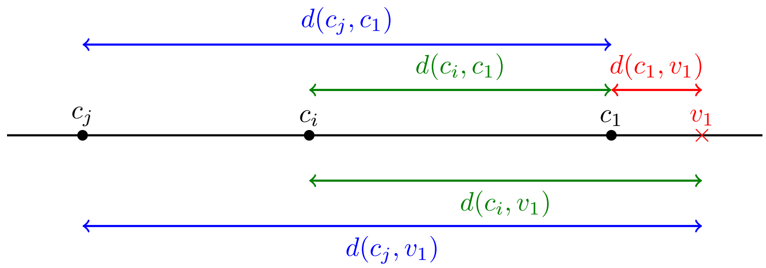


Figure 8: Illustration of the scenario where c_1 is positioned to the left of v_1.

Next, we show the correctness of the function `SortVoters`.

Lemma 33. The procedure `SortVoters` finds the order of voters with respect to the determined candidates.

Proof. By Lemma 32, we have the order among the determined candidates. Assume that for candidate c_i, order_{c_i} is the index of c_i from left to right. As `SplitLine` determines a prefix for each candidate, for a voter v_i, we let sequence $\mathit{sorted}_j = \mathit{order}_{\succ_{i,1}}, \mathit{order}_{\succ_{i,2}}, \ldots \mathit{order}_{\succ_{i,k}}$ where k is the number of determined candidates. Now, we can compare voters' *sorted* sequences lexicographically. A voter with a smaller *sorted* is on the left side of a voter with a larger *sorted*. It is important to highlight that this approach may result in ties among some voters in the ordering. □

Next, we prove Theorem 23.

Proof of Theorem 23. By Lemma 31, Algorithm 1 first determines a core subset of candidates. Then, according to Lemma 32, it finds the correct order for this subset. Lastly, Lemma 33 demonstrates that the algorithm establishes a correct order with respect to the order of determined candidates. Therefore, as Algorithm 1 correctly identifies all three, the theorem follows. □

A.2 Find the exact order of voters

In this part, we extend the order result by proving that it is possible to determine the exact order of all voters, formally:

Theorem 34. For any election $\mathcal{E}$ consistent with a line metric, it is possible to determine the ordering of the voters along the line. Furthermore, one can derive an ordering of the candidates that is consistent with this metric.

Algorithm 1 determines the order for a core subset C^* of candidates. Consequently, the first $|C^*|$ candidates in any voter's preference order are C^*. By examining the candidate at $|C^*|+1$-th position in each voter's ranking, we observe two cases.

- There exists a unique candidate c_i such that every voter ranks c_i in the $(|C^*|+1)$-th position.
- There exist two candidates, c_i and c_j, such that for each voter, the candidate in the $(|C^*|+1)$-th position is either c_i or c_j.

It is important to note that no more than two different candidates can appear in that position.

In the first case, c_i does not create any distinctions between voters, as removing this column from the preference profile has no effect.

In the second case, we can make the following assumptions:

- First, if all voters, based on our prior knowledge, are in the same position, we assume c_i is to the left of c_j. Otherwise, there are voters v_p and v_q for whom we know they are not in the same position, with one preferring c_i and the other preferring c_j. This allows us to determine the order between c_i and c_j.

- Second, when the candidates in C^* are excluded, the candidates c_i and c_j appear consecutively on the line.

Excluding candidates in C^*, these two conditions enable us to re-run `SortCandidatesAndVoters`, leading to more determined candidates and a more accurate order of voters. Furthermore, this process can be extended by recursively running the algorithm on the remaining candidates until no candidates remain. At that point, any two voters who do not share the same preference orders will be distinguished, and the result will be the exact order of the voters.

Algorithm 7 outlines this approach in pseudo-code. In this pseudo-code, the notations $\succ^{(l)}$ and $\succ_i^{(l)}$ represent the preference order of all voters and voter v_i, respectively, starting from the l-th index. Additionally, $\circ$ denotes the concatenation of sequences, and set operators can be applied to sequences as well. It is also important to note that for `SplitLine`, we input L and R as the determined candidates at the beginning. This is not explained in Algorithm 3, but it can be applied by replacing the first three lines of Algorithm 3 with the input of L and R as the determined candidates.

ALGORITHM 7: Find the exact order of voters

Input: Election instance $\mathcal{E} = (V, C, \succ)$.
Output: Sequence S_V as the exact order of voters.

Function `FindVotersOrder` $(\mathcal{E} = (V, C, \succ))$**:**
    $(S_C, S_V) \leftarrow$ `SortCandidatesAndVoters`$(\succ_1)$
    $C \leftarrow C - S_C$
    $l \leftarrow |S_C| + 1$
    while $C \neq \emptyset$ **do**
        $C_{\text{first}} \leftarrow \{c_i \in C \mid \exists v_j \text{ such that } \succ_{j,l} = c_i\}$
        if $|C_{\textit{first}}| = 1$ **then**
            $C \leftarrow C - C_{\text{first}}$
            $l \leftarrow l + 1$
            continue
        end
        $(c_i, c_j) \leftarrow C_{\text{first}}$
        if *all voters in* S_V *do **not** have equal positions* **then**
            Let v_p, v_q be two voters with different positions in S_V such that $\succ_{p,l} = c_i$ and $\succ_{q,l} = c_j$
            Swap c_i and c_j if v_p is on the right side of v_q
        end
        $L \leftarrow \{c_i\}$
        $R \leftarrow \{c_j\}$
        $(L, R) \leftarrow$ `SplitLine`$((V, C, \succ^{(l)}), L, R)$
        $S_{C'} \leftarrow$ `SortCandidates`$(\succ_1^{(l)}, L, R)$
        $S_C \leftarrow (S_{C'} \cap L) \circ S_C \circ (S_{C'} \cap R)$
        $S_V \leftarrow$ `SortVoters`$(\mathcal{E}, S_C)$
        $C \leftarrow C - S_{C'}$
        $l \leftarrow l + |S_{C'}|$
    end
    return S_V

Analysis. Now, we focus on the correctness of the approach explained earlier. Recalling the definition of core candidates (Definition 21), we define the covering range as follows:

Definition 35 (Covering Interval)**.** We define the range $[l, r]$ as the *covering interval* for an election $\mathcal{E} = (V, C, \succ)$ and a core subset of candidates C^* if it is the smallest interval that includes $C^* \cup V$.

We first show that this interval does not contain any other candidates.

Lemma 36. For an election $\mathcal{E} = (V, C, \succ)$ and a core subset of candidates C^*, let $[l, r]$ be the covering interval. Any candidate $c_i \in C \setminus C^*$ lies outside the range $[l, r]$.

Proof. Assume, for the sake of contradiction, that there exists a candidate $c_i \notin C^*$ lying within the interval $[l, r]$. We consider three cases:

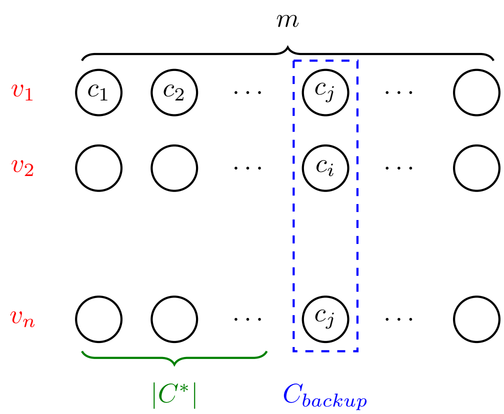


Figure 9: This figure illustrates the preference profile. Each row represents the preference order of a voter. The first $|C^*|$ cells in each row form the set C^*, while the candidates in the next column form the set C_{backup}.

- If all voters are on the left side of c_i, exist a $c_j \in C^*$ in range $(c_i, r]$. Then, any voter prefers c_j to c_i, which contradicts C^* to be a core.
- If all candidates in C^* are on the right side of c_i, then there exists a voter v_k in range $[l, c_i)$ such that v_k prefers c_i to all candidate in C^* which is a contradiction.
- Otherwise. exists candidate $c_j \in C^*$ on the left side of c_i and voter v_j on the right side of c_i. we know v_j prefers c_i to c_j which is a contradiction.

Consequently, all candidates in $C \setminus C^*$ are outside $[l, r]$. □

Next, we assume the preference profile to be a table with n rows and m columns, where each row is the preference order of each voter, ordered by priority from left to right. Figure 9 illustrates.

Then, we define *backup* candidates as those appearing in the columns immediately following a core subset of candidates (Figure 9).

Definition 37 (Backup Candidates)**.** For an election $\mathcal{E} = (V, C, \succ)$ and a core subset of candidates C^*, let the *backup* subset of candidates be defined as

$$C_{backup} = \{c_i \in C \mid \exists v_j \text{ such that } \succ_{j,|C^*|+1} = c_i\}.$$

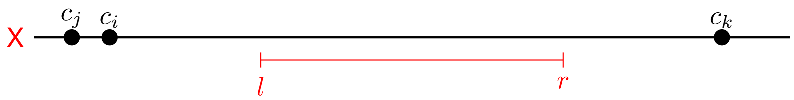
This means that a candidate c_i belongs to C_{backup} if there exists a voter v_j who ranks c_i in the $(|C^*| + 1)$-th position.

First, we show that a backup subset of candidates either has one or two candidates.

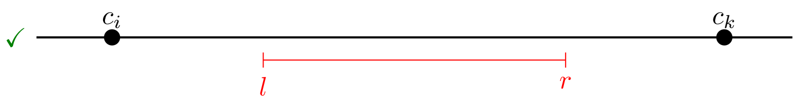
Lemma 38. For an election $\mathcal{E} = (V, C, \succ)$ and a core subset C^* that does not cover C, let C_{backup} be the backup subset of candidates. Then:

- $|C_{backup}| \in \{1, 2\}$, and
- if $|C_{backup}| = 2$, all candidates between the two elements of C_{backup} form C^*.

Proof. Let $[l, r]$ be the interval including all $V \cup C^*$. By Lemma 36, no candidates of C_{backup} is within interval $[l, r]$. Assume that on one side of the interval $[l, r]$, there exist two candidates of C_{backup}. We refer to them as c_i and c_j and assume, without loss of generality, that they are located in $(-\infty, l)$, and c_j is to the left of c_i. Then, for any voter $v \in V$, v would prefer c_i to c_j which contradicts c_j being included in C_{backup} as illustrated in Figure 10.



(a) This case cannot happen as all voters prefer c_i to c_j.



(b) This case can happen without any issues.

Figure 10: Illustration on the number of elements in C_{backup}

Consequently, each side has at most one candidate in C_{backup}. Furthermore, by the same argument, no candidate in $C \setminus (C^* \cup C_{backup})$ can be positioned between the two candidates of C_{backup}. $\square$

Corollary 39. For an election $\mathcal{E} = (V, C, \succ)$ and a core subset C^* that does not cover C, let C_{backup} be the backup subset of candidates that has two candidates. Then, C_{backup} is a consecutive subsequence of $C \setminus C^*$

Lemma 40. For an election $\mathcal{E} = (V, C, \succ)$ and C^* a core subset of candidates not covering C. Assuming C_{backup} is the backup subset of candidates for C^*, If $|C_{backup}| = 2$ and the order of voters with respect to C^* distinguishes at least two voters, we can determine the order of C_{backup}.

Proof. Based on Lemma 38, if $C_{backup} = \{c_i, c_j\}$, then each of them is on one end of the line. Let V_l and V_r be the set of leftmost and rightmost candidates with respect to C^*. As C^* distinguishes at least two voters, $V_l \neq V_r$. The goal here is find $v_l \in V_l$ and $v_r \in V_r$ such that v_l prefers one of $\{c_i, c_j\}$ while v_r prefers the other. In this case, we can conclude which of c_i and c_r is on the left side of the other. Let

$$C^*_{backup} = \{c_i \in C \mid \exists v_j \in V_l \cup V_r \text{ such that } \succ_{j,|C^*|+1} = c_i\},$$

Now, it suffices to show that $|C^*_{backup}| = 2$, as we can easily choose v_l and v_r. Suppose instead that $|C^*_{backup}| = 1$. Without loss of generality, let $C^*_{backup} = \{c_i\}$. Then, there exists a voter $v_m \notin V_l \cup V_r$ such that $\succ_{m,|C^*|+1} = c_j$. By Lemma 38, we know that c_i and c_j are different sides of V_l and V_r. Additionally, v_m is positioned between V_l and V_r, which leads to a contradiction since both V_l and V_r prefer c_i. Figure 11 illustrates this.

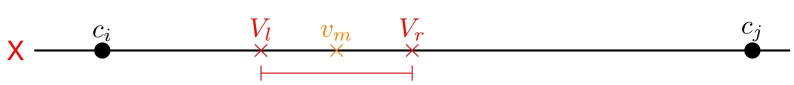


Figure 11: This figure illustrates that when V_r prefers c_i and c_j is on the right side of V_r, voter v_m cannot prefer c_j over c_i.

$\square$

Next, we focus on the order of voters in the case $|C^*| = 1$.

Lemma 41. For an election $\mathcal{E} = (V, C, \succ)$ and a core subset of candidates C^*, if C_{backup} contains only one candidate c_i, then for any subset $A \subseteq C$ that does not include c_i, the ordering of voters with respect to A is the same as the ordering of voters with respect to $A \cup \{c_i\}$.

Proof. Recalling the definition of "with respect to" (Definition 22), voters are compared based on the preference profile of the given subset in lexicographic order. Assume the order with respect to $A \cup \{c_i\}$. We know that c_i has a fixed position in all voters' preference orders. Additionally, the set of candidates before this position is disjoint from the set of candidates after it.

Thus, comparing two voters involves two steps:

- First, we compare the portion of the preference order before c_i lexicographically.
- If they are identical, we then compare the portion after c_i lexicographically.

Due to this disjointness, the order with respect to A follows the same structure. Consequently, the order of voters in both cases is identical. □

Finally, we provide the proof of Theorem 34

Proof of Theorem 34. We are showing that Algorithm 7 provides the exact order of voters. First, it runs `SortCandidatesAndVoters`, and according to Theorem 23, it determines the order of candidates for a core subset and retrieves the order of voters with respect to this order.

In each iteration of Line 5, by Lemma 38, the next position in the preference profile contains either one or two candidates.

If it contains one candidate:

- We can include this candidate in C^*, resulting in a larger core.
- Additionally, by Lemma 41, the position of this candidate does not affect the order of voters, so we can skip it in the order of this core subset.

On the other hand, if it contains two candidates:

- We determine their order using Lemma 40.
- Then, excluding C^*, Corollary 39 ensures that these two candidates are consecutive, allowing us to identify additional candidates using `SplitLine`.
- Next, we sort them based on the order of the first voter, as Lemma 36 and Lemma 38 indicate that the first voter is positioned between these two candidates.
- Furthermore, by concatenating the left and right parts of this newly ordered candidate set, we obtain a larger core with its sorted version.
- Lastly, we sort the voters based on the expanded ordering of candidates.

By the end of the algorithm, since the core is extended at each step, all effective candidates are considered in the voter ordering. Therefore, the exact order of voters is identified. □

B PROOFS OF UPPER BOUNDS ON DISTORTION FACTOR

B.1 UPPER BOUNDS FOR SUM-COST

Lemma 6. When considering the sum of distances objective and the candidates and voters are located on the real line, one of the candidates directly to the right or left of the median voter will be an optimal candidate.

Proof. Denote the median voter as v. Assume the chosen candidate is not immediately to the left or right of v. Without loss of generality, let us consider that the chosen candidate c lies to the left of v, and denote the first candidate to the left of v as c_l. We now formally show that replacing c with c_l results in a solution that is at least as effective in minimizing voter distances.

The change in the total distance is given by:

$$\Delta = \sum_u (d(u, c_l) - d(u, c)) = \sum_{u \in L_v} (d(u, c_l) - d(u, c)) + \sum_{u \in R_v} (d(u, c_l) - d(u, c)),$$

where $L_v = \{u \in V \mid u < v\}$ represents voters to the left of v, and $R_v = \{u \in V \mid u \geq v\}$ represents voters to the right of v (including v itself).

For each voter $u \in R_v$, since both c and c_l are to the left of these voters, we have:

$$\forall u \in R_v, \quad d(u, c_l) = d(u, c) - d(c, c_l).$$

For each voter $u \in L_v$, we observe that:

$$\forall u \in L_v, \quad d(u, c_l) \leq d(u, c) + d(c, c_l).$$

Substituting these relations, the change in total distance can be bounded as:

$$\Delta \leq \sum_{u \in L_v} d(c, c_l) - \sum_{u \in R_v} d(c, c_l) = |L_v| \cdot d(c, c_l) - |R_v| \cdot d(c, c_l).$$

Since v is the median voter and belongs to R_v, it follows that $|L_v| \leq |R_v|$. Thus:

$$\Delta \leq 0.$$

Therefore, replacing c with c_l does not increase the total distance, ensuring that choosing c_l is at least as good as choosing c.

Since v is the median voter, there are at least as many voters to the right of v (including v) as there are to its left (excluding v). Therefore, the total cost cannot increase when moving from c to c_l, implying that the cost of c_l is *at most* the cost of c. Similarly, if a candidate c_r directly to the right of v exists, any candidate farther to the right will have a distance that is at least as great. Hence, either c_l or c_r will be an optimal candidate that minimizes the total cost. □

Lemma 7. There exists a voting rule for the 3-committee election on the line metric such that the resulting committee includes an optimal candidate.

Proof. First, we can determine the order of a subset of candidates C^* and all voters based on the algorithms in Section A (see Theorem 23). Additionally, by Lemma 31, any candidate $c \notin C^*$ cannot be an optimal candidate, since c will be farther from all voters than any candidate in C^*. Thus, we can ignore the candidates outside C^* and proceed with the remaining candidates.

Now, we consider the median voter v in the ordering of voters. We note that there might be a tie between multiple voters for this position, but we will only utilize the voter's closest candidate, which will be the same for all tied voters.

After identifying the median voter, we consider v's closest candidate c. This candidate will be either the one immediately to the left or the right of the median voter v. Next, we use the ordering of the candidates to find the candidates c_l to the left of c and c_r to the right of c. We claim that the set $\{c, c_l, c_r\}$ is guaranteed to include an optimal candidate: if c is to the left of the median voter, then c_r will be the first candidate to the right of the median, and one of c or c_r will be an optimal candidate by Lemma 6. Similarly, if c is to the right of the median, one of c or c_l will be an optimal candidate. Therefore, the set $\{c, c_l, c_r\}$ will contain an optimal candidate (Figure 12).

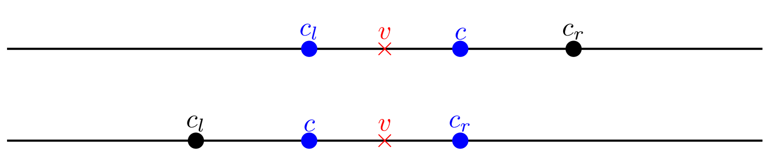


Figure 12: This figure illustrates that the set of three candidates—comprising the closest candidate to the median voter, as well as the candidates positioned to the left and right of this candidate—always includes the optimal candidate. Here, v represents the median candidate, c is the closest candidate to v, and c_l and c_r are the candidates on either side. The figure considers two cases based on the position of c relative to v.

Finally, we note that candidates c_l and c_r might not exist. In such cases, the candidates to the left and right of v are still selected if they exist. □

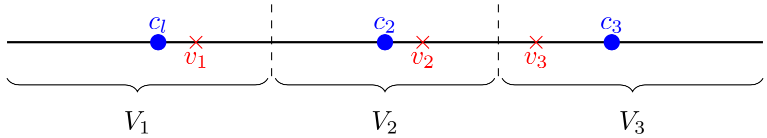


Figure 13: A figure illustrating three candidates c_1, c_2, and c_3 along with the possible locations of voters closest to each candidate, V_1, V_2, and V_3. Voters v_1, v_2, and v_3 show examples of voters in each set.

Theorem 5. There exists a voting rule for the 2-committee election for the sum-cost objective, on the line metric, such that the resulting committee includes an optimal candidate. Consequently, the 1-distortion of the voting rule is 1.

Proof. By Lemma 7, we can select three candidates that include an optimal candidate. Let c_1, c_2, and c_3 denote these candidates from left to right. We then define the sets V_1, V_2, and V_3 as the sets of voters who prefer the corresponding candidate to the other two, as illustrated in Figure 13. We note that the sets V_1, V_2, V_3 can be determined using ordinal preferences.

Thus, we can state that

$$\begin{aligned}
\texttt{cost}_s(c_2) &= \sum_{v \in V} d(v, c_2) \\
&= \sum_{v \in V_1} d(v, c_2) + \sum_{v \in V_2} d(v, c_2) + \sum_{v \in V_3} d(v, c_2) \\
&\leq \sum_{v \in V_1} d(v, c_2) + \sum_{v \in V_2} d(v, c_2) + \sum_{v \in V_3} (d(v, c_3) + d(c_2, c_3)) && \text{(Triangle Inequality)} \\
&\leq \sum_{v \in V_1} d(v, c_2) + \sum_{v \in V_2} d(v, c_3) + \sum_{v \in V_3} (d(v, c_3) + d(c_2, c_3)) \\
& && (\forall_{v \in V_2}\ d(v, c_2) \leq d(v, c_3)) \\
&= \sum_{v \in V_1} (d(v, c_3) - d(c_3, c_2)) + \sum_{v \in V_2} d(v, c_3) + \sum_{v \in V_3} (d(v, c_3) + d(c_2, c_3)) \\
& && (\forall_{v \in V_1}\ d(v, c_2) = d(v, c_3) - d(c_2, c_3)) \\
&= \texttt{cost}_s(c_3) - |V_1| \cdot d(c_2, c_3) + |V_3| \cdot d(c_2, c_3) \\
&= \texttt{cost}_s(c_3) + (|V_3| - |V_1|) \cdot d(c_2, c_3).
\end{aligned}$$

Similarly, we can show that $\texttt{cost}_s(c_2) \leq \texttt{cost}_s(c_1) + (|V_1| - |V_3|) \cdot d(c_2, c_1)$. Now, depending on whether $|V_1| < |V_3|$ or not, we can see that either $\texttt{cost}_s(c_2) \leq \texttt{cost}_s(c_1)$ or $\texttt{cost}_s(c_2) \leq \texttt{cost}_s(c_3)$. So we can disregard one of c_1 and c_3 based on this and still find the optimum, as c_2 is guaranteed to be a better candidate than the discarded one. □

Theorem 8. There exists a voting rule choosing $m - 1$ out of m candidates achieving a 1-distortion of $1 + \frac{2}{m-1}$ for the sum-cost objective.

Proof. For each voter v, let *first*$_v$ be their closest candidate. Then, we claim that the voting rule that chooses the $m - 1$ candidates appearing most frequently in *first* achieves the desired distortion. For a given instance, let C be the set of candidates selected by this algorithm and c_{opt} be an (arbitrary) optimal single candidate. If $c_{\text{opt}} \in C$, then we get a 1-distortion of 1, and we are done. Otherwise, C includes every candidate except for c_{opt}. Now, we can bound the optimal cost $\texttt{OPT}$ in this instance as

$$\begin{aligned}
\texttt{OPT} &= \sum_{i\in[n]} d(c_{\text{opt}}, v) \\
&= \sum_{\substack{v\in V \\ \mathit{first}_v = c_{\text{opt}}}} d(c_{\text{opt}}, v) + \sum_{\substack{v\in V \\ \mathit{first}_v \neq c_{\text{opt}}}} d(c_{\text{opt}}, v) \\
&\geq \sum_{\substack{v\in V \\ \mathit{first}_v = c_{\text{opt}}}} d(c_{\text{opt}}, v) + \sum_{\substack{v\in V \\ \mathit{first}_v \neq c_{\text{opt}}}} d(C, v). \qquad (1)
\end{aligned}$$

where the last inequality follows since for any voter $v \in V$, $\mathit{first}_v \in C$ if $\mathit{first}_v \neq c_{\text{opt}}$. Let v' be the voter closest to c_{opt} such that $\mathit{first}_v \neq c_{\text{opt}}$. Then, we can use this to state that

$$\begin{aligned}
\texttt{OPT} &\geq \sum_{\substack{v\in V \\ \mathit{first}_v \neq c_{\text{opt}}}} d(c_{\text{opt}}, v) \\
&\geq \sum_{\substack{v\in V \\ \mathit{first}_v \neq c_{\text{opt}}}} d(c_{\text{opt}}, v') \\
&\geq \left(n - \frac{n}{m}\right) d(c_{\text{opt}}, v') \qquad (\mathit{first}_v = c_{\text{opt}} \text{ for at most } \tfrac{n}{m} \text{ voters based on choice of } C)
\end{aligned}$$

and therefore

$$\frac{n}{m} d(c_{\text{opt}}, v') \leq \frac{1}{m-1}\texttt{OPT}. \qquad (2)$$

Now, we can bound the cost of C as follows:

$$\begin{aligned}
\texttt{cost}_m(C) &= \sum_{v\in V} d(C, v) \\
&= \sum_{\substack{v\in V \\ \mathit{first}_v = c_{\text{opt}}}} d(C, v) + \sum_{\substack{v\in V \\ \mathit{first}_v \neq c_{\text{opt}}}} d(C, v) \\
&\leq \sum_{\substack{v\in V \\ \mathit{first}_v = c_{\text{opt}}}} \left(d(c_{\text{opt}}, v) + d(c_{\text{opt}}, v') + d(C, v')\right) + \sum_{\substack{v\in V \\ \mathit{first}_v \neq c_{\text{opt}}}} d(C, v) \\
&\qquad\qquad \text{(Triangle inequality)} \\
&\leq \sum_{\substack{v\in V \\ \mathit{first}_v = c_{\text{opt}}}} \left(d(c_{\text{opt}}, v) + d(c_{\text{opt}}, v') + d(c_{\text{opt}}, v')\right) + \sum_{\substack{v\in V \\ \mathit{first}_v \neq c_{\text{opt}}}} d(C, v) \\
&\qquad\qquad (\mathit{first}_{v'} \neq c_{\text{opt}} \text{ and } \mathit{first}_{v'} \in C) \\
&\leq \texttt{OPT} + \sum_{\substack{v\in V \\ \mathit{first}_v = c_{\text{opt}}}} 2d(c_{\text{opt}}, v') \qquad \text{(By Equation 1)} \\
&= \texttt{OPT} + 2 \cdot \frac{n}{m} d(c_{\text{opt}}, v') \qquad (\mathit{first}_v = c_{\text{opt}} \text{ for at most } \tfrac{n}{m} \text{ voters}) \\
&\leq \left(1 + \frac{2}{m-1}\right)\texttt{OPT} \qquad \text{(By Equation 2)}
\end{aligned}$$

□

B.2 Upper bounds for max-cost

Lemma 9. If the distance between v_l and v_r is D (i.e., $D = d(v_l, v_r)$), then the cost for the optimal single candidate OPT satisfies $\texttt{OPT} \geq \frac{D}{2}$.

Proof. Let c_{opt} be an optimal candidate. The distance between v_l and c_{opt} is $d(v_l, c_{\text{opt}})$, and the distance between v_r and c_{opt} is $d(v_r, c_{\text{opt}})$. By the triangle inequality:

$$d(v_l, c_{\text{opt}}) + d(v_r, c_{\text{opt}}) \geq D.$$

Thus,

$$\max(d(v_l, c_{\text{opt}}), d(v_r, c_{\text{opt}})) \geq \frac{D}{2}.$$

Therefore, $\texttt{OPT} \geq \frac{D}{2}$. □

Lemma 10. If there are no candidates placed between v_l and v_r, a voting rule that selects c_l and c_r, achieves a $1 - distortion$ of 1.

Proof. In this case, the voters are next to each other in a block, with c_l being the first candidate immediately to the left of all voters and c_r being the first candidate immediately to the right of all voters. So, by selecting c_l and c_r, we ensure that the closest candidate to each voter is included. Therefore, the cost of this set is, at most, that of the single optimal candidate, and thus we conclude that $1 - distortion$ is 1. □

Lemma 11. If $d(v_l, v_r) = D$, then for every voter v, either $d(v, v_l) \leq \texttt{OPT}$ or $d(v, v_r) \leq \texttt{OPT}$.

Proof. Since v is a voter between v_l and v_r (recall v_l is the left-most and v_r is the right-most voter), we have:

$$d(v_l, v) + d(v, v_r) = D.$$

Thus, either $d(v_l, v) \leq \frac{D}{2}$ or $d(v, v_r) \leq \frac{D}{2}$.

By Lemma 9, $\frac{D}{2} \leq \texttt{OPT}$. Therefore, we conclude:

$$d(v_l, v) \leq \frac{D}{2} \leq \texttt{OPT}, \quad \text{or} \quad d(v, v_r) \leq \frac{D}{2} \leq \texttt{OPT}.$$

□

Theorem 12. For any election $\mathcal{E}$ on the line metric, let v_l and v_r be the leftmost and rightmost voters, and c_l and c_r be the closest candidates to v_l and v_r, respectively. Then, a voting rule f that outputs the set $\{c_l, c_r\}$ satisfies $\texttt{1-distortion}(f) \leq 2$ with respect to the max-cost objective.

Proof. Since c_l is closest to v_l and we know $d(v_l, c_{\text{opt}}) \leq \texttt{OPT}$, it follows that:

$$d(v_l, c_l) \leq d(v_l, c_{\text{opt}}) \leq \texttt{OPT}.$$

Similarly, $d(v_r, c_r) \leq \texttt{OPT}$.

For any voter v, by Lemma 11, either $d(v, v_l) \leq \texttt{OPT}$ or $d(v, v_r) \leq \texttt{OPT}$. Without loss of generality, assume that $d(v, v_l) \leq \texttt{OPT}$. Then:

$$d(v, c_l) \leq d(v, v_l) + d(v_l, c_l) \leq \texttt{OPT} + \texttt{OPT} = 2 \cdot \texttt{OPT}.$$

Similarly, in the other case:

$$d(v, c_r) \leq 2 \cdot \texttt{OPT}.$$

Thus, for every voter, the distance to the closest candidate in $\{c_l, c_r\}$ is at most $2 \cdot \texttt{OPT}$. Hence, this selection achieves a distortion of 2. □

Theorem 13. For any election $\mathcal{E}$ on the line metric, there exists a voting rule to select three candidates, which achieves a 1-distortion of $3/2$ with respect to the max-cost objective.

Proof. Let v_l and v_r be the leftmost and rightmost voters, and c_l and c_r their closest candidates. Using the ordering of the candidates, let c'_r be the candidate immediately to the left of c_r. We claim that the set $C = \{c_l, c_r, c'_r\}$ achieves a distortion of $3/2$.

First, we note that any voter to the left of c_l or the right of c'_r has their closest candidate included in this set. Specifically, for any voter that is either left of c_l or right of c_r, the closest candidates are c_l and c_r, respectively. For candidates between c'_r and c_r, the closest candidate is either c_r or c'_r.

Now, one of c'_r or c_r must be the rightmost candidate between v_l and v_r. Let this be candidate $c^*_r \in \{c_r, c'_r\}$. We already know that any voter outside the interval between c_l and c^*_r has their closest candidate included. In addition, we have

$$d(c_l, c^*_r) \leq d(c_l, v_l) + d(v_l, c_{\text{opt}}) + d(c_{\text{opt}}, c^*_r) \leq 3\texttt{OPT}.$$

Therefore, for any voter v in this interval,

$$\min\{d(v, c_l), d(v, c_r^*)\} \leq \frac{3}{2}\texttt{OPT}.$$

So, the maximum distance of any voter to the candidate set C is at most $3\texttt{OPT}/2$, and we achieve a $1 - distortion$ of $3/2$. □

Theorem 14. For any election $\mathcal{E}$ on the line metric, there exists a voting rule to select four candidates, which achieves a 1-distortion of 1 with respect to the max-cost objective.

Proof. Again, we consider the leftmost and rightmost voters v_l and v_r, and their closest candidates c_l and c_r. In addition, we choose candidate c_l' to the right of c_l and c_r' to the left of c_r based on the ordering of candidates. Now, one of c_l and c_l' will be the first candidate to the right of v_l: If c_l is to the right of v_l, it must be the first such candidate. Otherwise, it is the first candidate to the left of v_l, so c_l' is to the right of v_l. Let this candidate be c_l^* and define c_r^* similarly. Now, any voter v to the left of c_l' has their closest candidate included in the set, as v is either between c_l and c_l', so one of these two is v's closest candidate or v is to the left of c_l, in which case c_l must be v's closest candidate. Similarly, the closest candidate to each voter to the right of c_r' is selected (Figure 14.

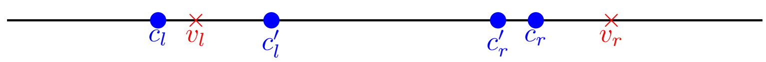


Figure 14: In this figure, the set c_l, c_l', c_r, c_r' has a 1-distortion of 1. Here, v_l and v_r represent the leftmost and rightmost voters, respectively. For v_l, the closest candidate is c_l, which is positioned to its left; therefore, c_l' is the closest candidate to its right. Conversely, for v_r, the closest candidate is c_r, which is also the closest candidate to its left, while c_r' is the next leftward candidate. However, in this case, c_r' is not necessary to achieve a 1-distortion of 1.

Next, since both c_l^* and c_r^* are between v_l and v_r and $d(v_l, v_r) \leq 2\texttt{OPT}$, we have $d(c_l^*, c_r^*) \leq 2\texttt{OPT}$. Therefore, any voter between c_l^* and c_r^* has a distance of at most $\texttt{OPT}$ to the closer candidate in $\{c_l^*, c_r^*\}$. As every voter is either between c_l^* and c_r^* or outside the interval c_l', c_r', the distance of each voter to the closest candidate in our selected set is at most $\texttt{OPT}$, and we get a distortion of 1. □

C Proofs of lower bound on distortion factor

C.1 Lower bound for sum-cost

Theorem 15. For $k \geq 3$, there exists an instance for the k-committee election (with respect to the sum-cost) where no voting rule choosing k candidates can achieve a bounded distortion even when compared to an optimal choice of $\lceil \log_2(k+1) \rceil + 1$ candidates.

Proof. Without loss of generality, we assume that k is of the form $2^\ell - 1$, and $\lceil \log_2(k+1) \rceil + 1 = \ell + 1$. Now, we consider an instance with 2^ℓ candidates and voters on a line. We further assume that voters and candidates are numbered from 0 to $2^\ell - 1$ and use b_i to denote the ℓ digit binary representation of i for $0 \leq i < 2^\ell$. For each b_i, we assume the bits are numbered left to right from 0 to $\ell - 1$ with the j-th bit shown by $b_{i,j}$.

For any two values $0 \leq i, j < 2^\ell$, let $\text{LCP}_{i,j}$ be the length of the longest common prefix between b_i and b_j. Now, for each voter v_i, the preference of v_i is determined as follows: For any two candidates c_r and c_s, if $\text{LCP}_{i,r} \neq \text{LCP}_{i,s}$, v_i prefers the candidate with the longer common prefix. Otherwise, let bit_1 be the first bit in b_i that differs from b_r and b_s. Similarly, let bit_2 be the first bit of b_r that differs from b_s. Then, if $bit_1 = bit_2$, v_i will prefer candidate c_r to candidate c_s; otherwise, c_s to c_r.

Now, any voting rule choosing $k = 2^\ell - 1$ candidates will not choose one of the candidates. Let c_{i^*} be the candidate left unselected. We show that we can arrange the voters and candidates on the line such that this selection will have unbounded distortion compared to a committee containing only $\ell + 1$ candidates.

In our construction, we will place each candidate c_i and voter v_i on the same point and describe the position for each candidate. For each candidate c_i, let $len_i = \text{LCP}_{i,i^*}$ be the length of the longest common prefix between b_i and b_{i^*}. For any candidate c_i and index $0 \le j \le \ell - 1$, let $h_{i,j}$ equal $b_{i,j}3^{-j}$ if $j \le len_i$; and $\varepsilon b_{i,j}3^{-j}$ otherwise, where $0 < \varepsilon < 1$ is a small arbitrary constant (to be set appropriately). Then, we will place candidate c_i at

$$p_i = \sum_{j=0}^{\ell-1} h_{i,j} = \sum_{j=0}^{len_i} b_{i,j}3^{-j} + \sum_{j=len_i+1}^{\ell-1} \varepsilon b_{i,j}3^{-j}.$$

Now, we show that this placement follows the voters' preference orders. First, we note that candidates (and voters) are positioned on the line in increasing order of their indices. To show this, consider candidates c_i and c_j with $i < j$. Consider the index $ind = \text{LCP}_{i,j}$ where b_i and b_j first differ. Since $i < j$, we must have $b_{i,ind} = 0$ and $b_{j,ind} = 1$. Now, $h_{i,t} = h_{j,t}$ for $t < ind$, so we have

$$p_j \ge \sum_{t=0}^{ind-1} h_{j,t} + h_{j,ind}$$

while

$$p_i \le \sum_{t=0}^{ind-1} h_{j,t} + \sum_{t=ind+1}^{\ell-1} h_{i,t}.$$

So, it suffices to show that $h_{j,ind} \ge \sum_{t=ind+1}^{\ell-1} h_{i,t}$. If $ind \le len_j$, we would have $h_{j,ind} = 3^{-ind}$ while

$$\sum_{t=ind+1}^{\ell-1} h_{i,t} \le \sum_{t=ind+1}^{\ell-1} 3^{-t} \le \frac{3^{-ind}}{2}$$

so the inequality holds. Otherwise, if $len_j \le ind$, we will also have $len_i \le ind$, as b_i and b_j match in the first ind indices. Therefore, we will have $h_{j,ind} \ge \varepsilon 3^{-ind}$ while

$$\sum_{t=ind+1}^{\ell-1} h_{i,t} \le \sum_{t=ind+1}^{\ell-1} 3^{-t} \le \varepsilon \frac{3^{-ind}}{2}.$$

In either case, we have shown that $p_i \le p_j$.

Now, take voter v_i and candidates c_r and c_s such that v_i prefers c_r to c_s. First, we consider the case where $\text{LCP}_{i,r} \ne \text{LCP}_{i,s}$. Since v_i prefers c_r to c_s, we must have $\text{LCP}_{i,r} > \text{LCP}_{i,s}$ if they are not equal. Then, we have

$$d(v_i, c_r) = \left|p_i - p_r\right| = \left|\sum_{t=0}^{\ell-1}(h_{i,t} - h_{r,t})\right| \le \sum_{t=\text{LCP}_{i,r}}^{\ell-1} \left|h_{i,t} - h_{r,t}\right|$$

while

$$d(v_i, c_s) = \left|p_i - p_s\right| = \left|\sum_{t=0}^{\ell-1}(h_{i,t} - h_{s,t})\right| = \left|\sum_{t=\text{LCP}_{i,s}}^{\ell-1}(h_{i,t} - h_{s,t})\right| \ge \left|h_{i,\text{LCP}_{i,s}} - h_{s,\text{LCP}_{i,s}}\right| - \sum_{t=\text{LCP}_{i,s}+1}^{\ell-1} h_{s,t}.$$

If $len_i \ge \text{LCP}_{i,s}$, then we will also have $len_s \ge \text{LCP}_{i,s}$, and since b_i and b_s differ at index $\text{LCP}_{i,s}$, one of $h_{i,\text{LCP}_{i,s}}$ and $h_{s,\text{LCP}_{i,s}}$ is 0 while the other is $3^{-\text{LCP}_{i,s}}$, so

$$d(v_i, c_s) \ge \left|h_{i,\text{LCP}_{i,s}} - h_{s,\text{LCP}_{i,s}}\right| - \sum_{t=\text{LCP}_{i,s}+1}^{\ell-1} h_{s,t} > 3^{-\text{LCP}_{i,s}} - 3^{-\text{LCP}_{i,s}}/2 = 3^{-\text{LCP}_{i,s}}/2.$$

Additionally, we have

$$d(v_i, c_r) \le \sum_{t=\text{LCP}_{i,r}}^{\ell-1} \left|h_{i,t} - h_{r,t}\right| < 3^{-\text{LCP}_{i,r}+1}/2 \le 3^{-\text{LCP}_{i,s}}/2 < d(v_i, c_s)$$

as $\text{LCP}_{i,r} > \text{LCP}_{i,s}$.

On the other hand, if $len_i \leq \text{LCP}_{i,s}$, we will have $len_s \leq \text{LCP}_{i,s}$ and $len_r \leq \text{LCP}_{i,s}$. Therefore, we get

$$d(v_i, c_s) \geq \left|h_{i,\text{LCP}_{i,s}} - h_{s,\text{LCP}_{i,s}}\right| - \sum_{t=\text{LCP}_{i,s}+1}^{\ell-1} h_{s,t} > \varepsilon 3^{-\text{LCP}_{i,s}} - \varepsilon 3^{-\text{LCP}_{i,s}}/2 = \varepsilon 3^{-\text{LCP}_{i,s}}/2$$

and

$$d(v_i, c_r) \leq \sum_{t=\text{LCP}_{i,r}}^{\ell-1} \left|h_{i,t} - h_{r,t}\right| < \varepsilon 3^{-\text{LCP}_{i,r}+1}/2 \leq \varepsilon 3^{-\text{LCP}_{i,s}}/2 < d(v_i, c_s).$$

Finally, if $\text{LCP}_{i,r} = \text{LCP}_{i,s}$, both c_r and c_s will fall on the same side of v_i. If $b_{i,\text{LCP}_{i,r}} = 0$, then both c_r and c_s will be to the right of v_i, and since $c_r \succ_i c_s$, we must have $b_{r,\text{LCP}_{r,s}} = 0$ while $b_{s,\text{LCP}r,s} = 1$. Therefore, $r < s$ and c_r will be to the left of c_s and closer to v_i. Similarly, if $b_{i,\text{LCP}_{i,r}} = 1$, we will have $r > s$ and c_r will again be closer to v_i than c_s. So, the preferences of each voter v_i are respected by this positioning.

Next, we show that the ratio of the cost of the selection excluding c_{i^*} to the cost of the optimal committee with $\ell + 1$ candidates is unbounded. Consider any candidate c_j other than c_{i^*}. Then, we have

$$\begin{aligned} d(v_{i^*}, c_j) &= \left|\sum_{t=0}^{\ell-1}(h_{i^*,t} - h_{j,t})\right| = \left|\sum_{t=len_j}^{\ell-1}(h_{i^*,t} - h_{j,t})\right| \\ &\geq \left|h_{i^*,len_j} - h_{j,len_j}\right| - \sum_{t=len_j+1}^{\ell-1} h_{j,t} \geq 3^{-len_j} - \varepsilon 3^{-len_j}/2 \geq 3^{-\ell}/2. \end{aligned}$$

So the cost of the selection excluding c_{i^*} is at least $3^{-\ell}/2$, a constant independent of ε. On the other hand, we consider the following selection of at most $\ell + 1$ candidates. For each $0 \leq j < \ell$, consider the candidate c_{i_j} such that $b_{i_j,t} = b_{i^*,t}$ for $t < j$, $b_{i_j,j} = 1 - b_{i^*,j}$ and $b_{i_j,t} = 0$ for all $t > j$. We show that the cost of the selection $\{c_{i_j} \mid 0 \leq j < \ell\} \cup \{c_{i^*}\}$ is a multiple of ε.

First, it is clear that v_{i^*} has cost 0 given this selection, as c_{i^*} is included. Next, take any other voter v_i and consider the candidate c_{i_j} in our selection for $j = len_i$. Then, we have

$$d(v_i, c_{i_j}) = \left|\sum_{t=0}^{\ell-1}(h_{i,t} - h_{i_j,t})\right| \leq \sum_{t=len_i+1}^{\ell-1} \left|h_{i,t} - h_{i_j,t}\right| \leq \sum_{t=len_i+1}^{\ell-1} \varepsilon 3^{-t} \leq \varepsilon 3^{-len_i}/2.$$

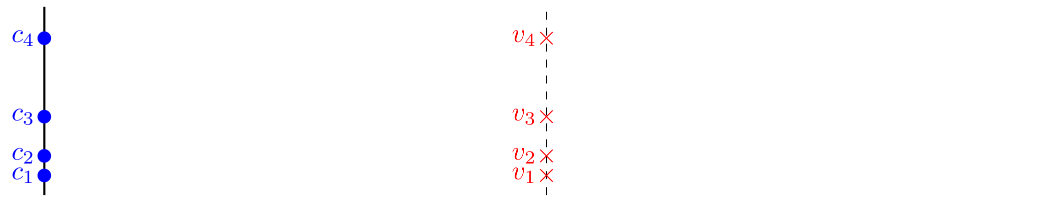
Now, since $\varepsilon > 0$ is a value we choose, we can arbitrarily decrease the cost of this selection while the cost of the selection excluding c_{i^*} remains a constant value of at least $3^{-\ell}/2$. Therefore, the distortion of any voting rule choosing $2^\ell - 1$ candidates is unbounded compared to the optimal selection of $\ell + 1$ candidates. ☐

Theorem 16. For any number of candidates m and any $\varepsilon > 0$, there exist instances of the $(m-1)$-committee election problem in the 2D Euclidean metric for which no voting rule can guarantee a 1-distortion factor less than $1 + \frac{2}{m-1} - \varepsilon$.

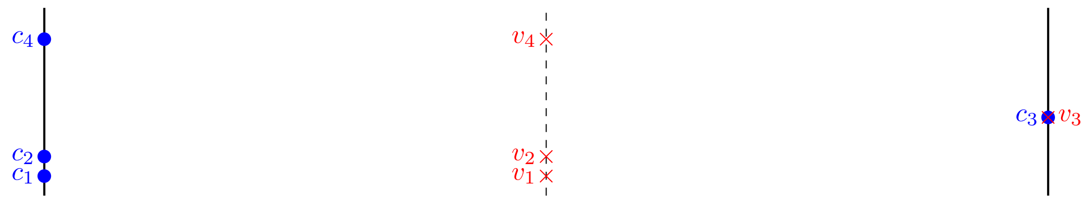
Proof. We consider a family of $m + 1$ instances $I_0, I_1, \cdots, I_m$ with m voters $\{v_1, v_2, \cdots, v_m\}$ and m candidates $\{c_1, c_2, \cdots, c_m\}$ on a two-dimensional plane. In terms of voters' and candidates' locations on the plane, each of these instances is distinct. However, in terms of voters' preference orders on candidates (i.e., ordinal information), $I_0, I_1, \cdots, I_m$ are the same. To be more specific, let us describe these instances.

Let $\ell = 3/\varepsilon - 1$. We first describe the instance I_0. For each $i \in [m]$, the candidate c_i is located at the point $\left(-\ell, 2^{i-1}/2^m\right)$. For each $i \in [m]$, the voter v_i is located at the point $\left(0, 2^{i-1}/2^m\right)$. (See Figure 15.) It is straightforward to observe that, for each $i \in [m]$, the preference order of the voter v_i is

$$c_i \succ_i c_{i-1} \succ_i \cdots \succ_i c_1 \succ_i c_{i+1} \succ_i c_{i+2} \succ_i \cdots \succ_i c_m.$$



(a) The instance I_0



(b) I_3: Instance with voter v_3 and candidate c_3 moved

Figure 15: Figures of the lower bound instances. In (a), all candidates are located on the line $x = -\ell$, with voters with matching y-coordinates on the line $x = 0$. In (b), voter v_3 and candidate c_3 are moved to the line $x = \ell$, while keeping the same y-coordinate.

Next, for each $j \in [m]$, we define the instance I_j as follows: For each $i \neq j \in [m]$, the locations of the candidate c_i and the voter v_i are the same as that in the instance I_0. Both the candidate c_j and the voter v_j are located at the point $\left(\ell, 2^{i-1}/2^m\right)$. (See Figure 15.) It is easy to observe that for any voter v_i (where $i \neq j$), the distance to any candidate c_j remains the same (as in I_0). Since the locations of other candidates remain the same, the preference order for any voter v_i (where $i \neq j$) remains the same as in I_0. Let us now consider the voter v_j. Note that its closest candidate is c_j, and the distance to any other candidate c_i remains the same as in I_0. Thus, the preference order of the voter v_j is

$$c_j \succ_j c_{j-1} \succ_j \cdots \succ_j c_1 \succ_j c_{j+1} \succ_j c_{j+2} \succ_j \cdots \succ_j c_m$$

which is the same as that in I_0.

Let us now consider any arbitrary deterministic algorithm $\texttt{ALG}$ that selects at most $m-1$ candidates. Now, suppose given the preference orders of voters as in I_0 (the same as in $I_1, \cdots, I_m$), $\texttt{ALG}$ selects a set C of candidates where $|C| \leq m - 1$. Suppose $c_k \notin C$, for some $k \in [m]$. Now, consider the instance I_k. Note, since voters' preference orders, i.e., ordinal information, are the same as in I_0, $\texttt{ALG}$ also selects the set C for the instance I_k. Then

$$\begin{aligned}\texttt{cost}_s(C, I_k) &= \sum_{i \in [m]} d(v_i, C) \\ &\geq \sum_{i \neq k} \ell + 2\ell = (m+1)\ell.\end{aligned}$$

On the other hand, selecting only the candidate c_k for the instance I_k would lead to the cost of

$$\texttt{cost}_s(c_k, I_k) = \sum_{i \in [m]} ||v_i - c_k||_2 \leq \sum_{i \neq k} (\ell + 1) = (m-1)(\ell+1)$$

and thus the optimal cost $\texttt{OPT}(I_k) \leq (m-1)(\ell+1)$.

Hence, the distortion factor of ALG on the instance I_k is

$$\begin{aligned}
\texttt{1-distortion}(\texttt{ALG}) &= \frac{\texttt{cost}_s(C, I_k)}{\texttt{OPT}(I_k)} \\
&\geq \frac{(m+1)\ell}{(m-1)(\ell+1)} \\
&= \left(1 + \frac{2}{m-1}\right)\left(1 - \frac{1}{\ell+1}\right) \\
&= 1 + \frac{2}{m-1} - \frac{1}{\ell+1}\left(1 + \frac{2}{m-1}\right) \\
&\geq 1 + \frac{2}{m-1} - \varepsilon \qquad \text{(since } \ell = 3/\varepsilon - 1\text{).}
\end{aligned}$$

□

Theorem 17. For any number of candidates m and any $\varepsilon > 0$, there exist instances of the $(m-1)$-committee election problem with tree metrics for which no voting rule can guarantee a 1-distortion factor less than $1 + \frac{2}{m-1} - \varepsilon$.

Proof. Consider a star with a central vertex and m leaves, such that the edge to leaf i has cost $1-\varepsilon_i$, where $\varepsilon \leq \varepsilon_i \leq 2\varepsilon$ are m distinct values. Then, we construct m instances on this tree, such that any voting rule has a 1-distortion of $1 + \frac{2}{m-1} - \varepsilon'$ on at least one instance. In every instance, there are m candidates, with candidate c_i located at leaf i. In addition, there are m voters with voter v_i located on the edge going toward leaf i. In the i-th instance, every voter except voter v_i is located at distance ε_i from the center of the star, while voter v_i coincides with candidate c_i on the leaf. This is illustrated in Figure 16.

In every instance, for any voter v_j, the distance to c_j is at most $1 - \varepsilon_j - \varepsilon_i \leq 1 - 2\varepsilon$, while the distance to any other candidate c_k is at least $1 - \varepsilon_k + \varepsilon_i \geq 1 - \varepsilon$. Therefore, the closest candidate to each voter v_j is candidate c_j. For other candidates, the distances are determined by $1 - \varepsilon_k$, and each voter v_j's preference order for candidates except its most preferred choice of c_j will be in descending order of ε_k.

Therefore, the voters' preference orders are identical across the m instances, and it is not possible to distinguish between the instances given the voters' ordinal preferences. Now, any deterministic voting rule choosing $m-1$ candidates must exclude one of the candidates. Let this candidate be c_{i^*}. Then, in instance i^*, the cost achieved by this voting rule is at least

$$\sum_{j \neq i^*} (1 - \varepsilon_j - \varepsilon_i) + 1 - \varepsilon_{i^*} + 1 - \max_{j \neq i^*} \varepsilon_j \geq m - 1 + 2 - (2m)(2\varepsilon) = m + 1 - 4m\varepsilon$$

while the cost of choosing only candidate i^* is

$$\sum_{j \neq i^*} (1 - \varepsilon_{i^*} + \varepsilon_{i^*}) = m - 1.$$

Therefore, the 1-distortion of any voting rule can be lower bounded by

$$\frac{m + 1 - 4m\varepsilon}{(m-1)} \geq 1 + \frac{2}{m-1} - 8\varepsilon.$$

This completes the proof with $\varepsilon' = 8\varepsilon$. □

C.2 Lower bounds for the max-cost

Theorem 18. Any deterministic algorithm for the k-committee election (with respect to the max-cost) that selects at most $k < m$ candidates out of m candidates must have a 1-distortion of at least $3 - \varepsilon$ for any $\varepsilon > 0$.

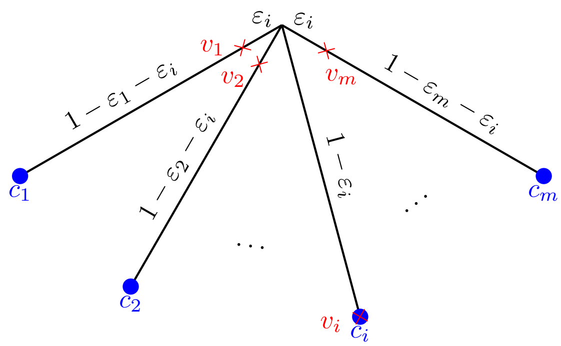


Figure 16: An illustration of the i-th instance on the star graph, where the i-th voter is located at the same spot as the i-th candidate. For each voter v_j, the closest candidate is c_j, followed by other candidates in descending order of ε_k. This ensures that all instances result in the same preference ordering for voters.

Proof. We proceed similarly to Theorem 16, using a family of instances $I_0, I_1, \ldots, I_m$ on the two-dimensional plane, such that the voters' ordinal preferences are the same in all of these instances, but the locations and distances of candidates and voters vary. In each instance, we have m voters $\{v_1, v_2, \ldots, v_m\}$ and m candidates $\{c_1, c_2, \ldots, c_m\}$.

Let $\ell = 3/\varepsilon - 1$. We use the same instance I_0 as in Theorem 16, where for each $i \in [m]$, candidate c_i is located at $(-\ell, 2^{i-1}/2^m)$ and voter v_i is located at $(0, 2^{i-1}/2^m)$. Then, for each $i \in [m]$, voter v_i will have the ordinal preference

$$c_i \succ c_{i-1} \succ \cdots \succ c_1 \succ c_{i+1} \succ c_{i+2} \succ \cdots \succ c_m.$$

in instance I_0.

Next, we define instance I_j for each $j \in [m]$. For each $i \in [m] \setminus \{j\}$, v_i and c_i will remain in the same location as in I_0. Meanwhile, candidate c_j is moved to $(\ell, 2^{j-1}/2^m)$ and voter v_j to $(2\ell, 2^{j-1}/2^m)$. It can be seen that for any $i \neq j$, the distance of voter v_i to candidate c_j will be unchanged compared to I_0, and therefore the voter's ordinal preference will remain the same. For voter j, its closest candidate will remain c_j. In addition, its preference for the other candidates will remain unchanged. Therefore, voter v_j's ordinal preference will remain the same as in I_0, too, and all voters' ordinal preferences are identical in I_0 and I_j. An example of these instances is illustrated in Figure 17.

Now, consider an arbitrary deterministic algorithm ALG that selects $k < m$ candidates. Since $k < m$, when running ALG on I_0, there exists a candidate c_j that is not in the set of selected candidates C. Now, consider the algorithm's performance on I_j. Since the voter's ordinal preferences are the same in I_0 and I_j, and ALG operates using only this information, its output on I_j must be the same as in I_0 and therefore, it does not select c_j. Now, we can lower bound the cost of ALG with respect to the max objective as

$$\begin{aligned} \texttt{cost}_m(C, I_0) &\geq d(v_j, C) \\ &\geq 3\ell. && \text{(Since } c_j \notin C\text{)} \end{aligned}$$

On the other hand, if we only select c_j, we get

$$\begin{aligned} \texttt{cost}_m(c_j, I_0) &= \max_{i\in[m]} ||v_i - c_j||_2 \\ &\leq \ell + 1. \end{aligned}$$

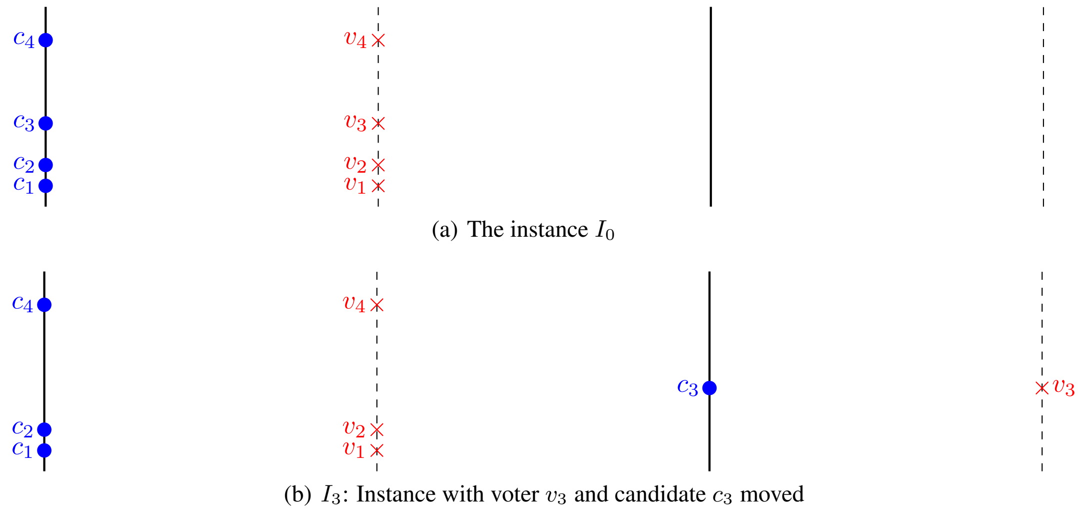


Figure 17: Figures of the lower bound instances for the max objective. In (a), all candidates are located on the line $x = -\ell$, with voters with matching y-coordinates on the line $x = 0$. In (b), candidate c_3 is moved to the line $x = \ell$, and voter v_3 is moved to the line $x = 2\ell$ while keeping the same y-coordinates.

This shows that the optimal cost $\texttt{OPT}_m(I_j) \leq \ell + 1$. Finally, combining these two, we get that the distortion of $\texttt{ALG}$ on instance I_j is at least

$$\begin{aligned}\texttt{1-distortion}(\texttt{ALG}) &\geq \frac{\texttt{cost}_m(C, I_k)}{\texttt{OPT}_m(I_k)} \\ &\geq \frac{3\ell}{\ell+1} \\ &= 3 - \frac{3}{\ell+1} \\ &= 3 - \varepsilon. \qquad (\ell = 3/\varepsilon - 1)\end{aligned}$$

□

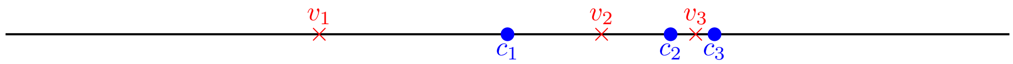
Theorem 19. Any deterministic algorithm for the 2-committee election (with respect to the max-cost) when voters and candidates are located on a line must have a 1-distortion of at least $2 - \varepsilon$ for any $\varepsilon > 0$.

Proof. We consider the following instance with three voters $\{v_1, v_2, v_3\}$ and three candidates $\{c_1, c_2, c_3\}$. Let the preferences of voters be as follows:

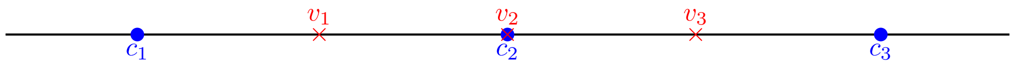
$$\begin{aligned}v_1 &: c_1 \succ c_2 \succ c_3 \\ v_2 &: c_2 \succ c_1 \succ c_3 \\ v_3 &: c_3 \succ c_2 \succ c_1.\end{aligned}$$

Next, we consider three possible placements for the voters and candidates that respect the above (desired) preference order of voters, as shown in Figure 18. In all instances, v_1 is located at point -1, and v_3 at point 1. In the first instance, v_2 is located at 0.5. Candidates c_1, c_2, c_3 are located at points $0, 1 - \varepsilon$ and $1 + \varepsilon/2$ respectively. It is easy to see that the voters' preferences in this instance match our desired orders. Now, we can see that in this instance, c_1 has a distance of at most 1 to every voter, while both c_2 and c_3 have a distance of at least $2 - \varepsilon$ from v_1. Therefore, not choosing c_1 will lead to a distortion of at least $2 - \varepsilon$.

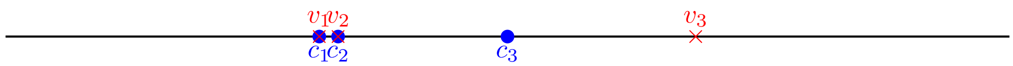
In the second instance, we place v_2 and c_2 at point 0, c_1 at point $-2 + \varepsilon$, and c_3 at point $2 - \varepsilon/2$. Again, we can see that the voters' preferences will follow the desired orders. In this instance, c_2



(a) First instance: Choosing c_1 achieves a cost of 1, while the other two candidates have a distances of at least $2 - \varepsilon$ to v_1.



(b) Second instance: Choosing c_2 achieves a cost of 1, while the other two candidates have a distance of at least $2 - \varepsilon$ to v_2.



(c) Third instance: Choosing c_3 achieves a cost of 1, while the other two candidates have a distances of at least $2 - \varepsilon$ to v_3.

Figure 18: Possible locations of the voters and candidates in the lower bound instance for $k = 2$.

has a distance of at most 1 from all voters, while v_2 is at a distance of at least $2 - \varepsilon$ from the other candidates. Therefore, not choosing c_2 will lead to a distortion of at least $2 - \varepsilon$.

Finally, in the third instance, we place c_1 at point -1, c_2 and v_2 at point $-1 + \varepsilon$ and c_3 at point 0. This placement will also respect the desired preference orders for each voter. Additionally, since c_3 has a distance of at most 1 from all voters and v_3 has a distance of at least $2 - \varepsilon$ from the other candidates, not choosing c_3 leads to a distortion of at least $2 - \varepsilon$.

Now, since these instances cannot be distinguished based on the voters' ordinal preferences, and not choosing any of the candidates leads to a distortion of at least $2 - \varepsilon$, any deterministic algorithm selecting two candidates cannot achieve a distortion better than $2 - \varepsilon$.

□

Theorem 20. Any deterministic algorithm for the 3-committee election (with respect to the max-cost) when voters and candidates are located on a line must have a 1-distortion of at least $3/2 - \varepsilon$ for any $\varepsilon > 0$.

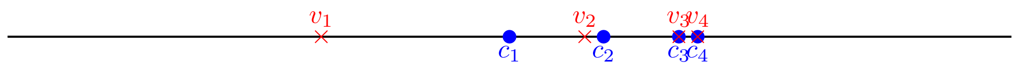
Proof. We consider instances $\{I_1, I_2, I_3, I_4\}$, each with four voters $\{v_1, v_2, v_3, v_4\}$ and four candidates $\{c_1, c_2, c_3, c_4\}$ such that the preferences of voters are as follows in every instance:

$$v_1 : c_1 \succ c_2 \succ c_3 \succ c_4$$
$$v_2 : c_2 \succ c_1 \succ c_3 \succ c_4$$
$$v_3 : c_3 \succ c_4 \succ c_2 \succ c_1$$
$$v_4 : c_4 \succ c_3 \succ c_2 \succ c_1.$$

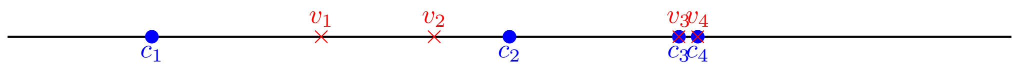
For the instance I_i, we choose the location of voters and candidates so that candidate c_i has a distance of at most 2 to each voter, while voter v_i has a distance of at least $3 - 2\varepsilon$ to every candidate except c_i. This leads to a distortion of at least $3/2 - \varepsilon$ for any deterministic algorithm ALG, as these instances cannot be distinguished based on ordinal preferences, and at least one candidate c_i is not chosen in instance I_i by ALG. These instances are shown in Figure 19.

In instance I_1, we have voters v_1, v_2, v_3, v_4 located at points $-2, 1 - \varepsilon, 2 - \varepsilon, 2$ and candidates c_1, c_2, c_3, c_4 located at points $0, 1, 2 - \varepsilon, 2$ respectively. It can be seen that the ordinal preference of each voter, in this instance, matches the desired ordering. Now, c_1 has a distance of at most 2 to every voter in this instance, while v_1 is at a distance of at least 3 from every candidate except c_1. So, if c_1 is not chosen, we get a distortion of at least $3/2$.

In instance I_2, we have voters v_1, v_2, v_3, v_4 located at points $-2, -1 - \varepsilon, 2 - \varepsilon, 2$ and candidates c_1, c_2, c_3, c_4 located at points $-4 + \varepsilon, 0, 2 - \varepsilon, 2$ respectively. Once again, we can see that the ordering



(a) I_1: Choosing c_1 achieves a cost of 2, while the other candidates have a distances of at least 3 to v_1.



(b) I_2: Choosing c_2 achieves a cost of 2, while the other candidates have a distances of at least 3 to v_2.

Figure 19: Possible locations of the voters and candidates in the lower bound instance for $k = 3$.

for voters' preferences is respected:

$$d(v_1, c_1) = 2 - \varepsilon < d(v_1, c_2) = 2 < d(v_1, c_3) = 4 - \varepsilon < d(v_1, c_4) = 4$$
$$d(v_2, c_2) = 1 + \varepsilon < d(v_2, c_1) = 3 - 2\varepsilon < d(v_2, c_3) = 3 < d(v_1, c_4) = 3 + \varepsilon$$
$$d(v_3, c_3) = 0 < d(v_3, c_4) = \varepsilon < d(v_3, c_2) = 2 - \varepsilon < d(v_3, c_1) = 6 - 2\varepsilon$$
$$d(v_4, c_4) = 0 < d(v_4, c_3) = \varepsilon < d(v_4, c_2) = 2 < d(v_4, c_1) = 6 - \varepsilon.$$

In addition, since c_2 is at a distance of at most 2 from every voter and every candidate except c_2 has a distance of at least $3 - 2\varepsilon$ from v_2, not choosing c_2 leads to a distortion of at least $3/2 - \varepsilon$.

Based on the symmetry in voters' preferences between c_1, c_2 and c_4, c_3, we create mirror versions of I_1 and I_2 as I_4 and I_3 respectively, so that not choosing c_4 or c_3 would lead to a distortion of at least $3/2 - \varepsilon$. Therefore, since any deterministic algorithm must omit one of the candidates, we cannot achieve a distortion of better than $3/2 - \varepsilon$ in this case.

□