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# Analysis of input derivative dynamic systems and its stabilization by state feedback

**Bahram Shafai**[1] · **Fatemeh Zarei**[1]

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## Abstract

This paper considers the analysis of dynamic systems represented by state space models and general differential equations incorporating input derivatives. First, the connection between the class of systems with derivative inputs and singular systems is established. An elimination procedure for input derivatives is given, which transfers the derivative terms from the state equation to the output equation allowing stabilization by state feedback to be performed. Second, we consider the general differential equations with derivatives of inputs and outputs and derive an equivalent state space description in standard form, which incorporates the derivatives in the state variables. This is essential since a suitable feedback configuration should involve both input and output as well as their derivatives. We provide an appropriate state space model that allows stabilization by state feedback controller to be performed. Finally, a subclass of positive system involving input derivatives is considered as a special case. It is shown that stabilization of input derivative positive system can be achieved through its equivalent positive system in standard form using LMI. Numerical examples are included to support the theoretical results.


**Keywords** Dynamic systems with derivative inputs · Singular systems · State feedback control · Positive stabilization

## 1 Introduction

A dynamic system is a mathematical formalization of a physical object or process, which describes its motion by the functional relationship between time-dependent external (input and output) and/or internal (state) variables of the system. The fundamental problems associated with the dynamic systems consist of modeling, analysis, and control design for the purpose of stabilization and performance improvement. One may encounter various underlying problems depending on the class of systems and its complexity. Fundamental results are available in classical textbooks for both linear and nonlinear systems [1, 2] (see extensive references therein). However, recent mathematical and technical development made it possible to solve some of the unsolved traditional problems by proposing new solutions that are fundamentally easier to understand and computationally preferable.

The class of dynamic systems with derivative inputs appears naturally in applications such as electrical circuits when capacitors and independent voltage sources constitute fundamental loops. Similarly, all inductor cut-sets and independent current sources may lead to state space representation with input and its derivatives. This class of systems may also appear in electromechanical systems when transducers are used in setting the reference inputs for analysis and control design. Other scenarios include modeling and control of certain robotic manipulators and in active suspension systems when vibration control is required. Furthermore, there exist applications when physical variables are constrained by algebraic equations leading to the so-called descriptor or singular systems [3–6] that inherently experience derivative inputs. This is evident in the process of obtaining a state space model in standard form from a given descriptor representation by using a shuffle algorithm, SVD-based method or a combination of both [7].

Methods dealing with the class of singular systems for both continuous and discrete time representations can be found in [8–20]. In Sects. 2.1 and 2.2, we provide the major development and available techniques for analysis and design of such systems. However, we strongly believe that the proposed method of this paper, which eliminates the derivative input representation of singular systems and its transforma-

✉ Bahram Shafai
shafai@ece.neu.edu

1 Department of Electrical Engineering, Northeastern University, Boston, MA 02115, USA

tion to standard systems, allows advanced control techniques to be employed conveniently.

The theory of linear systems with derivative inputs was originally studied in [21–25] and recently found renewed interest in [26, 27]. The classical approach for analyzing such systems was based on polynomial matrix representation, strict system equivalent and realization. This approach is lengthy and inconvenient for the interpretation of results since the state variable information is lost in calculation of transfer function. To be more specific, the states of systems are made available from actual physical structures. Also, to define the state space description from general differential equation representation of dynamic systems, we need to know not only the output and its derivatives but also the input and its derivatives. Therefore, we would expect that a suitable feedback configuration should involve both output and input as well as their derivatives. Obtaining the transfer function of linear systems described by scalar or vector differential equations and subsequent state space realizations does not capture the original time domain representations involving derivatives. Consequently, one cannot apply state feedback controller to arbitrary realization when the derivatives of input and output are incorrectly diluted in constructing the state variables.

In this paper, we first introduce dynamical systems with derivative inputs and give simple examples of real systems that include derivative inputs in their state space descriptions. Second, we show that the important class of singular systems can be changed into regular systems incorporating derivative inputs. Using the input derivative dynamic systems, we provide two cases to represent equivalent state space description in a standard form. The first case considers the state space description of a linear system with input derivatives that appears from input derivative representation of a singular system. Then, we apply a direct algebraic procedure to eliminate the derivative inputs, which leads to a standard system with a modified input matrix without changing the structure of system matrix. The second case considers the general differential equation with derivatives of output and input as its starting model representation of a linear system. Then, an equivalent state space description is derived in a standard form, which incorporates the derivatives. We demonstrate that stabilization by state feedback controller can be performed for both cases.

Finally, we define the special class of positive input derivative system, which can also be regarded as an equivalent representation of positive singular systems, and provide a procedure for its positive stabilization. Positive systems are important class of dynamic systems that appear in various domains of science and engineering. There exist interesting results on robust stability and control of such systems [28–35]. We take advantage of available results and apply them to positive input derivative systems. It should be pointed out that

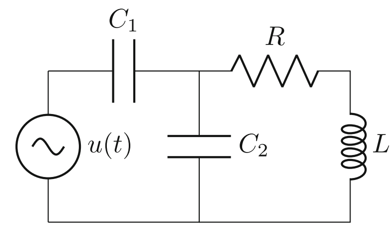


**Fig. 1** RLC circuit

an initial effort to stabilize this class of systems was made in [32]. The main contribution of this paper is to analyze the general case of input derivative dynamic systems and its stabilization. In Sect. 2, motivating application examples are provided to show that dynamic systems may involve input derivatives in its differential equation representation or by transforming the state space description of singular system to standard form. After a short analysis of singular systems, the drawback of available design approaches is outlined in Sect. 2.1. In Sects. 2.2 and 2.3, the input derivative representation of singular systems is derived. Subsequently, we also define the general scalar-valued differential equation involving both derivatives of inputs and outputs of dynamic systems, and provide its proper equivalent state space representation. The elimination of input derivatives and its equivalent standard representations are derived in Sect. 3. Design procedures for stabilization of input derivative systems by state feedback are given in Sect. 4. Finally, Sect. 5 considers the special class of positive input derivative systems and its stabilization. Numerical examples support the theoretical development of this paper.

## 2 Analysis of input derivative dynamic systems

To motivate the appearance of input derivatives in physical systems, we provide two introductory examples. Consider a linear electrical circuit model containing a loop of two capacitors and a voltage source as shown below (Fig. 1).

Defining the voltage across one capacitor and current through the inductor as state variables completely characterizes the circuit. This is evident from the KVL equation containing two capacitors and a voltage source, since the voltage across one capacitor determines the voltage across the other capacitor. Defining $v_{c1} = x_1$ and $i_L = x_2$ as state variables, it is not difficult to write the following state equation.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C_1+C_2} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u + \begin{bmatrix} \frac{c_2}{c_1+c_2} \\ 0 \end{bmatrix} \dot{u}$$

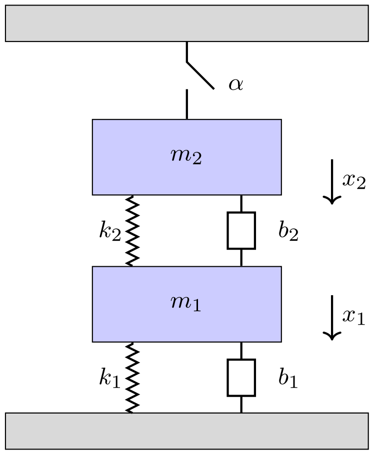


**Fig. 2** Mechanical system

The derivative of input is due to the constraint equation $\dot{v}_{c_1} + \dot{v}_{c_2} = \dot{u}$. Generally speaking, circuits containing capacitor loops with voltage sources or cut-sets with current sources lead to input derivative systems.

As a second example, consider the following mechanical system. This model of the mass-spring-damper system, which includes a rigid bar that can prevent the motion of the second mass. The exciting force is applied to mass 1 in Fig. 2. The positions of masses $m_1$ and $m_2$ are represented by $x_1$ and $x_2$, respectively. The spring coefficients are given as $k_1$ and $k_2$. The damper coefficients are given as $b_1$ and $b_2$.
Using Newton's second law, the forces acting on each mass can be written as:

$$m_1\ddot{x}_1(t) = -b_1\dot{x}_1(t) - k_1x_1(t) - b_2(\dot{x}_1(t) - \dot{x}_2(t)) - k_2(x_1(t) - x_2(t)) + u(t) \tag{1}$$

$$m_2\ddot{x}_2(t) = -b_2(\dot{x}_2(t) - \dot{x}_1(t)) - k_1(x_2(t) - x_1(t)) + \alpha\mu(t) \tag{2}$$

and the constraint equation

$$0 = \alpha(x_1(t) + x_2(t)) + (1 - \alpha)\mu(t) \tag{3}$$

where $\alpha$ represents the state of the switch $1 =$ closed and $0 =$ open, and $\mu(t)$ is the force absorbed.
Using Eqs. (1–3), the state space representation of the mechanical system can be constructed as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \dot{\mu}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -(k_1 + k_1) & k_2 & -(b_1 + b_2) & b_2 & 0 \\ k_1 & -k_1 & b_2 & -b_2 & \alpha \\ \alpha & \alpha & 0 & 0 & (1 - \alpha) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ \mu(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t) \tag{4}$$

As a third example, consider the following block diagram of a control system
Defining the state variables $x_1$, $x_2$, $x_3$ and $x_4$, one can easily derive the state equation of the system as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -b & 0 & K_2 \\ 0 & 0 & -a & 0 \\ -K_I & K_Db - K_P & K_Da^2 - K_pa + K_I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_1 \\ K_PK_1 - K_DK_1a \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ K_DK_1 \end{bmatrix} \dot{u}$$

We used the above system for the purpose of response analysis and simulations with various parameters to observe the effect of input derivative. For example, the step response of the system with or without input derivative remains the same. However, the response changes with other types of input signals (Fig. 3).

Obviously, there are many other scenarios of high-order systems with more than one input derivative. The general scalar- or vector-valued differential equations, which represents the dynamics of electromechanical systems, contains the derivatives of both inputs and outputs. A careful analysis is required to define an equivalent state space description that captures complete information of derivative in the design of control systems. This will be discussed in subsequent sections of the paper.

Based on the above introductory example, we formally define the linear system with $l$ input derivatives as:

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{l} B_i u^{(i)}(t) \tag{5}$$

$$y(t) = Cx(t) \tag{6}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$ and $u^{(i)}(t) = \frac{d^i u(t)}{dt}$. As pointed out in the introduction, such systems appear in diverse applications in several ways. It is also important to show that the above input derivative system is related to the

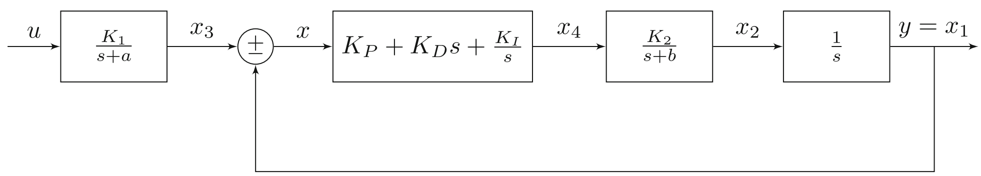


**Fig. 3** The block diagram of a feedback control system

class of singular systems

$$E\dot{x}(t) = Ax(t) + Bu(t) \quad (7)$$
$$y(t) = Cx(t) \quad (8)$$

where $E \in \mathbb{R}^{n\times n}$ is the singular matrix with $\text{rank}(E) = r < n$. We assume the same dimensions for vector and matrices as in (5) and (6).

## 2.1 Singular systems and its properties

Before we establish the connection between input derivative systems and singular systems, we provide preliminary results on singular systems. Let us define the generalized spectral abscissa for the pair $\{E, A\}$ as $\alpha(E, A) = \max \mathcal{R}(\lambda)$, where $\lambda \in \{s : \det(sE - A) = 0\}$. If $E = I$, then $\alpha(I, A) = \alpha(A)$ becomes the conventional spectral abscissa for standard systems [36].

**Definition 1** The singular system (7) is called,

1. Regular if $\det(sE - A) \neq 0$ for some $s \in \mathbb{C}$.
2. Impulse free if $\deg \det(sE - A) = \text{rank}(E) = r$.
3. Stable if all roots of $\det(sE - A)$ have negative real parts.
4. Admissible if it is regular, impulse-free, and stable.

Based on the above definition, the following results from [4, 6] can be stated.

**Lemma 1** *Consider the singular system (7),(8) and assume that it is regular. Then, there exist two nonsingular matrices L and R such that*

$$\begin{aligned}\bar{E} &= LER = block\ diag(I_{n_1}, N)\\ \bar{A} &= LAR = block\ diag(A_1, I_{n_2})\\ \bar{B} &= LB = \left[B_1^T\ B_2^T\right]^T\\ \bar{C} &= CR = \left[C_1\ C_2\right]\end{aligned} \quad (9)$$

*where $n_1 = \deg(\det(sE - A)) \le r$, $n_2 = n - n_1$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, and $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with index $\mu$ (i.e., $N^\mu = 0$, $N^{\mu-1} \neq 0$).*
*Furthermore, suppose that the pair $(E, A)$ is regular and (9) holds, then*

1. *The pair $\{E, A\}$ is impulse-free if and only if $N = 0$.*
2. *The pair $\{E, A\}$ is stable if and only if $\alpha(A_1) < 0$.*
3. *The pair $\{E, A\}$ is admissible if and only if $N = 0$ and $\alpha(A_1) < 0$.*

**Theorem 1** *The pair $\{E, A\}$ associated with the system (7) is admissible if and only if there exists a matrix P such that*

$$E^T P = P^T E \succeq 0 \quad (10)$$
$$P^T A + A^T P \prec 0 \quad (11)$$

The conditions in (10), (11) are non-strict LMIs due to the equality constraint (10), which is not desirable. Therefore, it is preferred to have a strict LMI by integrating (10) in (11). Theorem 1 and the following theorem were introduced in [37–41] to facilitate the process of stability and stabilization of singular systems.

**Theorem 2** *The pair $\{E, A\}$ associated with the system (7) is admissible if and only if there exist matrices $P \succ 0$ and Q such that*

$$(PE + SQ)^T A + A^T (PE + SQ) \prec 0 \quad (12)$$

*where $S \in \mathbb{R}^{r\times(n-r)}$ is any full rank matrix satisfying $E^T S = 0$. Furthermore, (12) can equivalently be written in terms of $\{E^T, A^T\}$ by replacing E with $E^T$ and A with $A^T$ in (12), and the side constraint $ES = 0$.*

Let the singular system (7) be unstable. Then, it is stabilizable by a state feedback control law $u(t) = Kx(t)$ if and only if $\text{rank}[sE - A \quad B] = n$ for all $s \in \mathbb{C}^+$.

The direct stabilization of singular system using an extension of theorem 2 was found to be inconvenient. Therefore, we take a different approach by transforming the singular system to input derivative system and apply a new stabilization technique.

## 2.2 Input derivative representation of singular systems

In this subsection, we show that singular systems can be represented by input derivative system using two different approaches and briefly discuss an alternative approach based on Drazin inverse.

The restricted equivalent form of singular system (7) specified in Lemma 1 can be represented by slow and fast subsystems, respectively, as:

$$\begin{aligned}\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t)\\ y_1(t) &= C_1 x_1(t)\end{aligned} \tag{13}$$

and

$$\begin{aligned}N\dot{x}_2(t) &= x_2(t) + B_2 u(t)\\ y_2(t) &= C_2 x_2(t)\end{aligned} \tag{14}$$

It is not difficult to derive the transfer function of the system $G(s) = G_1(s)+G_2(s)$, where $G_1(s) = C_1(sI_{n_1} - A_1)^{-1}B_1$ and $G_2(s) = C_2(sN - I_{n_2})^{-1}B_2$ are associated with slow and fast subsystems, respectively. Using $G_2(s)$, one can extract $X_2(s)$ and by taking the inverse Laplace transform to obtain

$$x_2(t) = -\sum_{i=0}^{\mu-1} N^i B_2 u^{(i)}(t), \quad u^{(0)} = u \tag{15}$$

assuming $x_2(0) = 0$. Differentiating (15) and combining it with slow subsystem result in the overall system:

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{i=0}^{\mu} \tilde{B}_i u^{(i)}(t) \tag{16}$$

$$y(t) = \tilde{C}x(t) \tag{17}$$

where

$$\tilde{A} = \begin{bmatrix} A_1 & 0\\ 0 & 0\end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} B_1\\ 0\end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0\\ -N^{i-1}B_2\end{bmatrix}, i > 1$$

$$\tilde{C} = [C_1 \quad C_2]$$

Thus, the singular system (5), (6) is represented by input derivative system (16), (17).

The singular system (5), (6) can also be decomposed to dynamic and static parts by applying singular value decomposition (SVD) on the matrix $E$. Subsequently, one can use the so-called Shuffle algorithm to derive an equivalent standard system with input derivatives. Performing SVD on the matrix $E$, the singular system can be rewritten as:

$$\hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \tag{18}$$

$$y(t) = \hat{C}\hat{x}(t) \tag{19}$$

where

$$\hat{A} = U^T AV = \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \quad \hat{B} = U^T B = \begin{bmatrix} B_1\\ B_2\end{bmatrix}$$

$$\hat{E} = U^T EV = \begin{bmatrix} \Sigma_r & 0\\ 0 & 0\end{bmatrix}, \quad \hat{C} = CV = \begin{bmatrix} C_1 & C_2\end{bmatrix}$$

with $\Sigma_r = \text{diag}\{\sigma_i, i = 1, \ldots, r\}$ defining the nonzero singular values and orthogonal pair of matrices $\{U, V\}$.

The matrix $\Sigma_r$ can be replaced by $I_r$ with an additional transformation step. It can be shown that the pair $\{E, A\}$ or equivalently $\{\hat{E}, \hat{A}\}$ is impulse-free if and only if $A_{22}$ is nonsingular, and in addition, the pair is admissible if and only if $\alpha(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 0$.

The transformation to SVD form (18), (19) is numerically reliable specially for large size system, and its structure prepares the system for initial step of Shuffle algorithm by defining

$$\hat{E} = \begin{bmatrix} E_1\\ 0\end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_1\\ A_2\end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1\\ B_2\end{bmatrix} \tag{20}$$

where $E_1 = [\Sigma_r \quad 0]$, $A_1 = [A_{11} \quad A_{12}]$, and $A_2 = [A_{21} \quad A_{22}]$. Therefore, we have

$$E_1\dot{x} = A_1 x + B_1 u \tag{21}$$

$$0 = A_2 x + B_2 u \tag{22}$$

Taking the derivative of (22) and combine it with (21) yield

$$\begin{bmatrix} E_1\\ A_2\end{bmatrix}\dot{x} = \begin{bmatrix} A_1\\ 0\end{bmatrix}x + \begin{bmatrix} B_1\\ 0\end{bmatrix}u + \begin{bmatrix} 0\\ -B_2\end{bmatrix}\dot{u} \tag{23}$$

If $\begin{bmatrix} E_1^T & A_2^T\end{bmatrix}^T$ is nonsingular, multiply both sides of (23) from left by its inverse and define the input derivative system. If $\begin{bmatrix} E_1^T & A_2^T\end{bmatrix}^T$ is singular, then it is required to continue the steps of Shuffle algorithm [7] until a regular pencil after $\mu - 1$ steps is obtained where $\begin{bmatrix} E_{\mu-1}^T & A_{\mu-1}^T\end{bmatrix}^T$ becomes nonsingular. Thus, the singular system (5), (6) represented by SVD form (18), (19) is transformed to the input derivative system as follows

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{i=0}^{\mu-1} \tilde{B}_i u^{(i)}(t) \tag{24}$$

$$y(t) = \tilde{C}x(t) \tag{25}$$

It is important to point out that the input derivative representation of singular system (16), (17) derived based on slow and fast subsystems differs from (24), (25), which was obtained based on dynamic and static part of singular systems. However, we used the same notations for system parameters to define the structure of input derivative systems and avoid extra symbols. One can also conclude that the difference between the number of finite modes in (13), $n_1$, and the number of dynamic modes in (21), $r$, is the number of infinite dynamic modes or impulsive modes, which is the rank $N = r - n_1$. This means that for a singular system with only finite dynamic modes we have $n_1 = r = \text{rank}(E)$. In this case, the singular system becomes impulse-free or index one.

A third approach to transform a singular system to input derivative representation is based on the Drazin inverse [42–45]. Although the Drazin-inverse approach tries to avoid input derivatives, its underline procedure and state response of the system include unavoidable time derivatives of the inputs. Here, it is sufficient to concentrate on SVD coordinate form, which is numerically preferable.

### 2.3 Scalar-valued differential equation and input derivative systems

The introduction of 2 provided several application examples that encounter time derivatives of inputs in their model descriptions. In this subsection, we show that one should represent the scalar-valued differential equations by an appropriate state space representation without loosing the derivative information of inputs and outputs.

The input derivative systems appear naturally in general differential equation representation of dynamic systems. Consider the scalar-valued differential equation described by

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) = \sum_{i=0}^{l} b_i u^{(i)}(t) \tag{26}$$

where $l \leq n$. However, we assume the properness of the system and let $l = n$. Define $a(s) = s^n + \sum_{i=0}^{n-1} a_i s^i$ and $b(s) = \sum_{i=0}^{n} b_i s^i$ and let $y(s) = b(s)z(s)$, $u(s) = a(s)z(s)$ whereby $z(t)$ is the so-called partial state. Then, by denoting $x_c = [x_1 \quad x_2 \quad \ldots \quad x_n]^T$ with $x_i = z^{(i-1)}$; $i = 1, \ldots, n$, we have the following state space description in a controllable canonical form:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u \\ y(t) &= C_c x_c(t) + D_c u \end{aligned} \tag{27}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C_c = \begin{bmatrix} b_0 - a_0 b_n & \ldots & b_{n-1} - a_{n-1} b_n \end{bmatrix}, \quad D_c = b_n$$

This realization of $G(s) = y(s)/u(s)$ looses the direct information of input and output as well as their derivatives in defining the state variables. Therefore, it is important to define a different state space description that captures all derivative terms of input and output in defining the state variables [27, 46]. As we stated in the introduction, this is essential in state feedback implementation of control systems. Let us now define the state vector as

$$x_0(t) = S_a \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - S_b \begin{bmatrix} u \\ u^{(1)} \\ \vdots \\ u^{(n-1)} \end{bmatrix} \tag{28}$$

where

$$S_a = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ a_{n-1} & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \ldots & 1 & 0 \\ a_1 & a_2 & \ldots & a_{n-1} & 1 \end{bmatrix}, \; S_b = \begin{bmatrix} b_n & 0 & \ldots & 0 & 0 \\ b_{n-1} & b_n & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_2 & b_3 & \ldots & b_n & 0 \\ b_1 & b_2 & \ldots & b_{n-1} & b_n \end{bmatrix}$$

Then, it is not difficult to show that (26) can be written as an observable canonical form

$$\begin{aligned} \dot{x}_0(t) &= A_0 x_0(t) + B_0 u(t) \\ y(t) &= C_0 x_0(t) + D_0 u(t) \end{aligned} \tag{29}$$

where

$$A_0 = \begin{bmatrix} -a_{n-1} & 1 & 0 & \ldots & 0 \\ -a_{n-2} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \ldots & 1 \\ -a_0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} b_{n-1} - a_{n-1} b_n \\ b_{n-2} - a_{n-2} b_n \\ \vdots \\ b_1 - a_1 b_n \\ b_0 - a_0 b_n \end{bmatrix}$$
$$C_0 = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad D_0 = b_n$$

Furthermore, it follows that the pair $\{A_0, B_0\}$ is controllable and $\{A_0, C_0\}$ is observable if and only if $a(S)$ and $b(s)$ is coprime. It is evident that (28) is a reliable state variable representation for state feedback implementation of input derivative systems.

Obviously, (26) can equivalently be represented by (5) and (6) as follows, if we define

$$\begin{aligned} y(t) &= x_1(t) \\ \dot{y}(t) &= \dot{x}_1(t) = x_2(t) \\ &\vdots \\ y^{(n)}(t) &= \dot{x}_n(t) = -\sum_{i=0}^{n-1} a_i x^{(i)}(t) + \sum_{i=0}^{l} b_i u^{(i)}(t) \end{aligned}$$

Then

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} x(t) + \sum_{i=0}^{l} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_i \end{bmatrix}}_{B_i} u^{(i)}(t) \quad (30)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}}_{C} x(t) \quad (31)$$

which has the same form as input derivative system (5) and (6) with special structures of system matrices.

## 3 Elimination of derivative inputs and equivalent representation

The elimination of input derivatives is required to transfer the system to an equivalent standard state space representation. This facilitate the design of state feedback controller for stabilization .

The approach of previous sections reveals that designing state feedback for singular systems may be reformulated in terms of an equivalent system with derivative inputs. Although literature reports various design techniques for singular systems, it is more convenient to use (5), (6) as its equivalent representation .

The class of systems with derivative inputs in the form of (5), (6) may appear in a natural way with only one input derivative in most applications. In this case $l = 1$ and we have

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_0u(t) + B_1\dot{u}(t) \\ y(t) &= Cx(t) \end{aligned} \quad (32)$$

Taking advantage of traditional approach of polynomial matrix representation of such system [47], we have a system matrix of the form

$$S(s) = \begin{bmatrix} sI_m - A & B_0 + B_1 s \\ -C & 0 \end{bmatrix} \quad (33)$$

which can be transformed to its strict system equivalent

$$P(s) = \begin{bmatrix} sI - A & B_0 + AB_1 \\ -C & CB_1 \end{bmatrix} \quad (34)$$

through a sequence of operations. This leads to a system representation in state space form defined by $\{A, B_0 + AB_1, C, CB_1\}$. The process becomes more involved with higher-order derivatives of inputs. Consequently, it is desirable to facilitate the process by an alternative procedure. This would particularly be beneficial for control design of singular systems.

### Elimination procedure for input derivatives:

Let us start with one derivative input case as in (32)
$l = 1$:

$$\begin{aligned} \dot{x} &= Ax + B_0u + B_1\dot{u} \\ y &= Cx \end{aligned}$$

Let $z = x - B_1u$. Then

$$\dot{z} = Az + (B_0 + AB_1)u \quad (35)$$
$$y = Cz + CB_1u \quad (36)$$

which matches with (34) with the state space matrices $\{A, B_0 + AB_1, C, CB_1\}$. Next, for two derivative inputs, we have
$l = 2$:

$$\begin{aligned} \dot{x} &= Ax + B_0u + B_1\dot{u} + B_2\ddot{u} \\ y &= Cx \end{aligned}$$

Let $v = x - B_2\dot{u}$, then

$$\dot{v} = Av + B_0u + (B_1 + AB_2)\dot{u}$$

Next, defining

$$z = v - (B_1 + AB_2)u,$$

we have

$$\dot{z} = Az + [B_0 + AB_1 + A^2B_2]u \quad (37)$$

and

$$y = Cx = C[v + B_2\dot{u}] = Cz + C(B_1 + AB_2)u + CB_2\dot{u} \quad (38)$$

One can continue the process for $l > 2$ in a similar fashion. Thus, for $l$ derivative inputs we obtain:

$$
\begin{aligned}
\dot{x} &= Ax + \sum_{i=0}^{l} B_i u^{(i)}(t) \\
v &= x - B_l u^{(l-1)}(t) \\
\dot{v} &= \dot{x} - B_l u(t) = Ax + \sum_{i=0}^{l-1} B_i u^{(i)}(t) \\
&= Av + (B_{l-1} + AB_l)u^{(l-1)}(t) + \sum_{i=0}^{l-2} B_i u^{(i)}(t) \\
&\vdots \\
\dot{z} &= Az + \left(\sum_{i=0}^{l} A^i B_i\right) u
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
y =& Cz + C\left(\sum_{i=1}^{l} A^{i-1} B_i\right) u + \\
& C\left(\sum_{i=2}^{l} A^{i-2} B_i\right)\dot{u} + C\left(\sum_{i=3}^{l} A^{i-3} B_i\right)\ddot{u} + \\
& C\left(\sum_{i=4}^{l} A^{i-4} B_i\right)\dddot{u} \cdots + C\left(\sum_{i=l}^{l} A^{i-l} B_i\right) u^{(l-1)}
\end{aligned} \tag{40}
$$

Note that the elimination of input derivatives in state equation (except for $l = 1$) shifts the derivative terms in the output equation. This should not be of concern when stabilization by state feedback controller is employed.

Defining

$$\bar{B}_j = \sum_{i=j}^{l} A^{i-j} B_i, \quad j = 0, 1, \ldots, l$$

(39) and (40) can compactly be written as

$$\dot{z} = Az + \bar{B}_0 u \tag{41}$$

$$y = Cz + C\sum_{j=1}^{l} \bar{B}_j u^{(j-1)} \tag{42}$$

The result of the above derivation can be stated as a theorem.

**Theorem 3** *The linear system (5), (6) with $l$ inputs derivatives is equivalent to system (41), (42) with output consisting of $l - 1$ input derivatives.*

Before closing this section, we state the following controllability conditions associated with regular, singular and input derivative systems [26].

**Lemma 2** *The linear system without derivatives given by (5), (6) with $B_i = 0$ for $i = 1, 2, \ldots, l$ is controllable if and only if one of the following conditions is satisfied*

*(i)* $\operatorname{rank}\left[B_0 \ AB_0 \ \ldots \ A^{n-1}B_0\right] = n$
*(ii)* $\operatorname{rank}\left[sI - A \ B_0\right] = n$ *for all* $s \in \mathbb{C}$

*Similarly, the linear singular system (7), (8) or equivalently (13), (14) is controllable if and only if*

*(i)* $\operatorname{rank}\left[B_1 \ A_1B_1 \ \ldots \ A_1^{n-1}B_1\right] = n_1$, $\operatorname{rank}\left[B_2 \ NB_2 \ \ldots \ N^{\mu-1}B_1\right] = n_2$
*(ii)* $\operatorname{rank}[sE - A|B] = n$ *for all* $s \in \mathbb{C}$ *and* $\operatorname{rank}\left[E \ B\right] = n$

*Moreover, the linear system (5), (6) with $l$ input derivatives is controllable if and only if* $\operatorname{rank}[\bar{B}_0, A\bar{B}_0, A^2\bar{B}_0, \ldots, A^{n-1}\bar{B}_0, \bar{B}_1, \bar{B}_2, \ldots, \bar{B}_l] = n$ *where* $\bar{B}_j = \sum_{i=j}^{l} A^{i-j}B_i$ *for* $j = 0, 1, \ldots, l$.

## 4 Design of state feedback for linear system with input derivatives

This section considers the design of state feedback for input derivative systems based on direct representation of scalar-valued differential equation outlined in Sect. 2.3. Then, we provide the stabilization of input derivative systems using elimination process of Sect. 3.

### 4.1 Stabilization of scalar-valued differential system

As we elaborated in the Introduction and the previous section, we need a suitable feedback configuration that involves the output and its derivatives as well as the input and its derivatives. Thus, the equivalent state space representation (29) of input derivative system (26) can be used to apply state feedback control law

$$
\begin{aligned}
u(t) = Kx_0(t) = f(&y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t); \\
&u(t), u^{(1)}(t), \ldots, u^{(n-1)}(t))
\end{aligned} \tag{43}
$$

such that the closed-loop system

$$\dot{x}_0(t) = (A_0 + B_0K)x_0(t) \tag{44}$$

has the desired set of eigenvalues, i.e., $\mathcal{R}\{\lambda_i(A_0 + B_0K)\} < 0$ for all $i = 1, \ldots, n$. In view of (28), we obtain

$$
\begin{aligned}
u(t) &= K(S_a\tilde{y}(t) - S_b\tilde{u}(t)) \\
&\triangleq K_y\tilde{y}(t) - K_u\tilde{u}(t)
\end{aligned} \tag{45}
$$

where $K_y = KS_a$ and $K_u = KS_b$.
One can obtain $K$ using available eigenvalue assignment techniques and compute $\{K_y, K_u\}$ associated with the derivatives of output and input, respectively.

## 4.2 Stabilization of state space representation of input derivative systems

In this subsection, we consider the input derivative system described by state space representation (5), (6) and provide a procedure to design state feedback controller to stabilize it using its equivalent standard form (41), (42). The natural appearance of systems with input derivatives in applications and the fact that singular systems can equivalently be represented as such, we outline the steps of the state feedback design with respect to (41), (42). To demonstrate the procedure in parallel with the elimination process of input derivatives described in Sect. 3, we start with the equation of input derivative system (5) for $l = 1$, i.e.,
$l = 1$:

$$\dot{x} = Ax + B_0 u + B_1 \dot{u}$$

Applying the elimination procedure with $z = x - B_1 u$, we have (35) as

$$\dot{z} = Az + (B_0 + AB_1)u$$

Using the state feedback control law $u = K_z z$, we get

$$\dot{z} = [A + (B_0 + AB_1)K_z]z$$

Now, one can apply any eigenvalue assignment technique to stabilize the equivalent standard system above under the controllability assumption of Sect. 4 or the usual stabilizability condition. Since $z$ is related to the original state $x$ by $x = z + B_1 u$ or

$$x = (I + B_1 K_z)z$$

one can easily obtain $u = K_x x$, where

$$K_x = K_z[I + B_1 K_z]^{-1}$$

provided that $I + B_1 K_z$ is invertible.
For $l = 2$, we have
$l = 2$:

$$\dot{x} = Ax + B_0 u + B_1 \dot{u} + B_2 \ddot{u}$$

with its equivalent standard system (37) as

$$\dot{z} = Az + (B_0 + AB_1 + A^2 B_2)u$$

and using $u = K_z z$, we get

$$\dot{z} = [A + (B_0 + AB_1 + A^2 B_2)K_z]z$$

It is not difficult to show that $z$ is related to $x$ by

$$x = [I + (B_1 + AB_2)K_z + B_2 K_z (A + B_0 + AB_1 + A^2 B_2)K_z]z$$

and $u = K_x x$ is determined by

$$K_x = K_z[I + (B_1 + AB_2)K_z + B_2 K_z (A + B_0 + AB_1 + A^2 B_2)K_z]^{-1}$$

Continuing the same process, we have the following result for systems with $l$ derivative inputs.

**Theorem 4** *Let the system (5), (6) with its equivalent form (41), (42) be controllable. Then, the state feedback control law $u = K_z z$ stabilizes (41) with arbitrarily eigenvalues and the closed-loop system becomes*

$$\dot{z} = A_z z = (A + \bar{B}_0 K_z)z, \quad \bar{B}_0 = \sum_{i=0}^{l} A^i B_i$$

*where $z$ is related to $x$ by*

$$x = \left[ I + \sum_{i=j}^{l} \bar{B}_j K_z A_z^{j-1} \right] z,$$
$$\bar{B}_j = \sum_{i=j}^{l} A^{i-j} B_i, \quad j = 1, 2, \ldots, l$$

*and $u = K_x x$ is determined by*

$$K_x = K_z \left[ I + \sum_{i=j}^{l} \bar{B}_j K_z A_z^{j-1} \right]^{-1}$$

## 5 Positive input derivative systems and its stabilization

In this section, we consider the special class of input derivative positive system and provide a solution for its positive stabilization. Subsequently, we define the positivity of singular systems and show that its stabilization can be achieved through the same procedure as in positive stabilization of input derivative positive systems.

**Definition 2** The system (5), (6) is called internally positive if for all nonnegative initial conditions $x_0 \in \mathbb{R}_+^n$ and all nonnegative $u(t) \in \mathbb{R}_+^m$ including its derivatives $u^{(i)}(t) \in \mathbb{R}_+^m$ we have $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$.

**Theorem 5** *The input derivative system (5), (6) is internally positive if and only if $A \in M_n$ is a Metzler matrix and $B_i \in \mathbb{R}_+^{n\times m}$ for all $i = 0, 1, \ldots, l$, $C \in \mathbb{R}_+^{p\times n}$ are nonnegative matrices, provided that $u^{(i)}(t) \in \mathbb{R}_+^m$.*

***Proof*** The proof of this theorem is a simple generalization of standard positive system [28] by imposing positivity of $B_i$ and $u^{(i)}(t)$ for all $i = 0, 1, \ldots, l$.

□

**Theorem 6** *The input derivative system (5), (6) is internally positive if and only if the equivalent system (41), (42) is positive with $A \in M_n$, $\bar{B}_j \in \mathbb{R}_+^{n\times m}$ and $C \in \mathbb{R}_+^{p\times n}$.*

***Remark 1*** The requirement $\bar{B}_i \in \mathbb{R}_+^{n\times m}$ can be avoided for the special subset of Metzler matrices $A \in \mathbb{R}_+^{n\times m} \subset \mathbb{M}_n$; since the Metzler matrices are defined by real matrices with nonnegative off-diagonal elements. In this case, the condition $\bar{B}_j \in \mathbb{R}_+^{n\times m}$ in the above theorem can be replaced by $B \in \mathbb{R}_+^{n\times m}$.

### 5.1 Stability and stabilization of input derivative positive systems

**Lemma 3** *Let the input derivative system (5), (6) or its equivalent form (41), (42) be positive. Then, it is asymptotically stable if and only if one of the following conditions is satisfied:*

1. *There exists a positive-definite diagonal matrix $P$ such that $A^T P + PA \prec 0$*
2. *There exists a positive vector $v \in \mathbb{R}_+^n$ such that $Av < 0$*

**Theorem 7** *Let the input derivative system (5), (6) or the equivalent form (41), (42) be positive and controllable. Then, the state feedback control law $u = K_z z$ stabilizes (41) and the closed-loop system becomes*

$$\dot{z}(t) = A_z z(t) \tag{46}$$

*where*

$$A_z = A + \bar{B}_0 K_z, \quad \bar{B}_0 = \sum_{i=0}^{l} A^i B_i, \quad i = 0, 1, \ldots, l \tag{47}$$

*and $z$ is related to $x$ by*

$$x = \left[ I + \sum_{i=j}^{l} \bar{B}_j K_z A^{j-1} \right] z \tag{48}$$

*with $u = v + K_x x$ determined by*

$$K_x = K_z \left[ I + \sum_{i=j}^{l} \bar{B}_j K_z A_z^{j-1} \right]^{-1} \tag{49}$$

*if and only if the following LMI has a feasible solution with respect to the variables $Y$ and $W$*

$$WA^T + Y^T \bar{B}_0{}^T + AW + \bar{B}_0 Y \prec 0 \tag{50}$$

$$(AW + \bar{B}_0 Y)_{ij} \geq 0 \quad for \quad i \neq j \tag{51}$$

*where $W \succ 0$ is a diagonal positive-definite matrix. Furthermore, the gain matrix $K_z$ is obtained from*

$$K_z = YW^{-1} \tag{52}$$

*and $K_x$ is determined by (49).*
*Note that (50), (51) guarantee $(AW + \bar{B}_0 Y)_{ij} < 0$*

### 5.2 Positive singular systems and stabilization

Consider [21] the singular system represented by (7), (8), and let us analyze its positivity and stability.

**Definition 3** The singular system (7), (8) is called weakly positive if and only if $E \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$, and $A \in \mathbb{M}_n$ is a Metzler matrix.

This definition is an extension of positivity as defined for standard positive systems. Unfortunately, it does not guarantee the strong positivity of the singular systems. One should apply one of the equivalent representations as discussed in the previous section to determine the positivity of the singular system through its standard form involving derivative inputs.

**Definition 4** The singular system (7), (8) is internally positive if and only if its equivalent input derivative system (24), (25) is internally positive, i.e., for every consistent initial condition $x_0 \in \mathbb{R}_+^n$ and every nonnegative input $u(t) \in \mathbb{R}_+^m$ such that $u^{(i)}(t) \in \mathbb{R}_+^m$ for $i = 0, 1, ..., \mu - 1$; $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ for $t \geq 0$ where $u^{(i)}(t) = \frac{d^i u(t)}{dt}$.

**Theorem 8** *The singular system (7), (8) is internally positive if and only if its equivalent input derivative system (24), (25) satisfies*

$$\tilde{A} \in \mathbb{M}_n, \quad \tilde{B}_i \in \mathbb{R}_+^{n\times m}, \quad \tilde{C} \in \mathbb{R}_+^{p\times n}$$

*for all* $i = 0, 1, ..., \mu - 1$. *Furthermore, it is asymptotically stable if and only if* $\tilde{A}$ *is a stable Metzler matrix satisfying one of the equivalent conditions of lemma 3.*

The development of Sect. 2 showed two different strategies of transforming the singular system to input derivative systems. This allows us to employ the algebraic transformation to transform the input derivative positive system to standard form and apply theorem 7 for stabilization of positive singular systems.

## 6 Illustrative examples

***Example 1*** Consider a system with natural input derivative with the following parameter matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2.5 & -3.125 & 5 & 10 & 4.5 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}, D = 0$$

Next, we eliminate the input derivative using the elimination procedure of lemma 2 to obtain the standard system

$$\dot{z} = Az + \underbrace{(B_0 + AB_1)}_{B} u$$

where $B = \begin{bmatrix} 0 & 0 & 0 & 2 & 4.5 \end{bmatrix}^T$.
It is not difficult to use the controllability condition of section ** to verify that $(A, B)$, or equivalently $(A, B_0, B_1)$ is controllable. Let the desired eigenvalues for the system to be $\{-1, -2, -3, -4, -5\}$. Using the eigenvalue assignment technique, we obtain

$$K_z = \begin{bmatrix} -36.41 & -89.25 & -70.18 & -11.11 & 9.269 \end{bmatrix}$$

which can be used to determine the feedback gain matrix for the original system by employing

$$\begin{aligned} K_x &= K_z[I + B_1 K_z]^{-1} \\ &= \begin{bmatrix} -3.545 & -8.691 & -6.834 & -1.081 & 0.903 \end{bmatrix} \end{aligned}$$

***Example 2*** Consider the following simple example of a singular system

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} u$$

Applying the shuffle algorithm one step, we obtain

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

Since $\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$ is nonsingular, one can easily obtain the input derivative system by (24), (25), where

$$\tilde{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It is not difficult to determine that the system is not controllable with the aid of lemma 2, however, it is stabilizable. The eigenvalues of the singular system can be obtained from $\det(sE - A) = 0$ as $\{0, -1\}$, which implies that the system is unstable. Applying the algebraic transformation and using LMI, a possible stabilizer $u = K_x X$ can be obtained for the original singular system where

$$K_x = \begin{bmatrix} -0.3860 & -1.3271 & -0.2185 \\ -0.7580 & -1.6907 & -1.0901 \end{bmatrix}$$

Thus, $A_x = A + BK_x$ with

$$\det(sE - A) = 1.386s^2 + 3.838s + 0.862$$

which has stable roots $\{-0.2402, -3.4551\}$.

***Example 3*** consider the scalar-valued differential equation as described by Eq. (26)

$$\ddot{y} + a_1\dot{y} + a_0 y = b_0 u + b_1 \dot{u} + b_2 \ddot{u}$$

where $a_0 = 2, a_1 = -3, b_0 = 1, b_1 = 2$, and $b_2 = 1$. The transfer function of the system can be obtained as $G(s) = (b_0 s^2 + b_1 s + b_2)/(s^2 + a_1 s + a_0)$.
Based on the discussion in Sect. 2, we need a state space realization that capture complete information of both input and output as well as their derivation. So, the observable

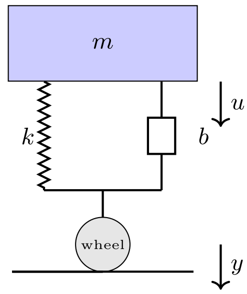


**Fig. 4** Car suspension model

canonical form (29) should be used for the above unstable system as

$$A_0 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_0 = 1$$

The state feedback gain vector $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} -2 & 6 \end{bmatrix}$ stabilizes the system and shifts the eigenvalues to $\{-6, -7\}$. Finally, $K_y$ and $K_u$ forms (45) and (28) can be obtained as:

$$K_y = K S_a = \begin{bmatrix} -2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -20 & 6 \end{bmatrix}$$

$$K_u = K S_b = \begin{bmatrix} -2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \end{bmatrix}$$

Let us continue this example by considering the car suspension model as shown below.
In this case, the differential equation above reduces to $\ddot{y} + a_1\dot{y} + a_0 y = b_0 u + b_1 \dot{u}$ (Fig. 4).
The scalar-valued differential equation of this system in the form of Eq. (26) is provided as

$$m\ddot{y} + b\dot{y} + ky = ku + b\dot{u} \tag{53}$$

Obviously, the system is stable due to the positive parameters $m$, $b$, and $k$. Therefore, we apply the design of the state feedback to improve the performance of the system. Assuming the parameters of the system are $m = 1$, $b = 3$, and $k = 2$, the observer canonical form of Sect. 2.3 can be constructed (29). It is desired to change the stiffness and spring constant to be changed from 3 to 9 and from 2 to 20, respectively. Applying the procedure of state feedback design, we obtain $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 9 & -10.5 \end{bmatrix}$, and finally, $K_y$ and $K_u$ forms (45) and (28) can be obtained as

$$K_y = K S_a = \begin{bmatrix} -22.5 & -10.5 \end{bmatrix}$$

$$K_u = K S_b = \begin{bmatrix} -31.5 & 0 \end{bmatrix}$$

***Example 4*** Consider a simple second-order singular system. In this example, we will consider an oil catalytic cracking process [14] as follows:

$$\dot{x}_1(t) = r_{11}x_1(t) + r_{12}x_2(t) + b_1 u(t)$$
$$0 = r_{21}x_1(t) + r_{22}x_2(t) + b_2 u(t)$$

where $x_1(t)$ denotes a state vector to be regulated, such as blower capacity, regenerate temperature, or valve position; $x_2(t)$ denotes a state reflecting business benefits, administration, and $u_(t)$ denotes regulation value. Let $r_{11} = 1, r_{12} = 1$, $r_{21} = 0, r_{22} = -1, b_1 = 1$, and $b_2 = 1$. Using these parameters, the above system can be expressed as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

which is unstable with eigenvalues at 1.
Applying the Shuffle Algorithm one step, we get the equivalent input derivative positive system

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\tilde{A}} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\tilde{B}_0} u + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\tilde{B}_1} \dot{u}.$$

After algebraic transformation and eliminating the input derivative, we obtain $\tilde{\tilde{B}}_0 = \tilde{B}_0 + \tilde{A}\tilde{B}_1 = \begin{bmatrix} 2 & 0 \end{bmatrix}^T$. The state feedback control law $u = K_z z$ results in stable closed-loop system $\dot{z} = A_z z$, where $A_z = \tilde{A} + \tilde{\tilde{B}}_0 K_z$ with $K_z = \begin{bmatrix} -2 & 0 \end{bmatrix}$. Finally, $K_x$ for the original singular system can be obtained as $K_x = K_z(I + \tilde{B}_1 K_z)^{-1} = \begin{bmatrix} -2 & 0 \end{bmatrix}$. To check the positivity and stability of the closed-loop singular system, we let $u = v + K_x x$ and after application of shuffle algorithm one step, we get the positive system

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{v}$$

The stability of the closed-loop singular system can be verified by $\det(sE - A_c) = 0$, where $A_c = A + BK_x$ with eigenvalues of -3. Note that other feasible solution is possible, for example if $K_z = \begin{bmatrix} -2 & -1/4 \end{bmatrix}$, then $K_x = \begin{bmatrix} -8/3 & -1/3 \end{bmatrix}$.

## 7 Conclusion

In this paper, we analyzed linear dynamic systems with derivative inputs and provided new techniques for its stabilization. Two cases of input derivative systems are considered. The first case is the state space representation involving derivative inputs, which has direct connection to singular systems. An elimination procedure for input derivatives is given, which transfers the derivative terms from state equation to the output equation allowing stabilization by state feedback to be achieved. The second case is the general differential equations with derivatives of inputs and outputs. An equivalent state space description in standard form, which incorporates the derivatives in state variables, was derived. This allows to apply state feedback to be reliably implemented to include input and output derivatives. Finally, we considered the special class of input derivative positive systems and provided a positive stabilization procedure for it using LMI. In comparison with the existing methods for stabilization of singular systems, our method is more transparent. This is due to the fact that input derivative representation of singular systems can easily be transferred to standard system by an elimination process of input derivatives. Consequently, conventional or advanced control design methods can conveniently be applied.

**Author contributions** This paper presents analysis and control of dynamic systems represented by state space models or general differential equations that incorporate input derivatives. Dynamic systems with input derivatives appear naturally in applications and they also have direct connection to the important class of singular systems through their equivalent representations. An elimination procedure for input derivatives is given, which transfers the derivative terms from the state equation to the output equation allowing stabilization by state feedback to be performed using its equivalent standard form. A separate procedure is also given for stabilization of dynamic systems represented by the general differential equations with derivatives of inputs and outputs. It is shown that a suitable feedback configuration should involve both input and output as well as their derivatives. Finally, the special class of input derivative positive systems is considered, and a positive stabilization procedure is outlined for it using LMI. The significance of the proposed techniques is validated by numerical and practical examples.

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**Data Availability** The datasets generated during the current study are available from the corresponding author on reasonable request.

## Declarations

**Conflict of interest** All opinions presented in this manuscript belong to the authors alone, and they declare that they have no competing interests.



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