Hopf Bifurcation Analysis using Zigzag Persistence

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Abstract

As bifurcations in a dynamical system are drastic behavioral changes, being able to detect when these bifurcations occur can be essential to understanding the system overall. While persistent homology has successfully been used in the field of dynamical systems, the most commonly used approaches have their limitations. Using zigzag persistence, we can simplify the methodology and capture topological changes through a collection of time series, rather that studying the topology of individual time series separately. Here we present Bifurcations using ZigZag (BuZZ), a method to detect Hopf bifurcations in dynamical systems. This work has previously been published in [38].

1 Introduction

Topological data analysis (TDA) is a field consisting of tools aimed at extracting shape in data. Persistent homology, one of the most commonly used tools from TDA, has proven useful in the field of time series analysis. Specifically, persistent homology has been shown to quantify features of a time series such as periodic and quasiperiodic behavior [27, 30, 35, 22, 40] or chaotic and periodic behavior [24, 17]. Existing applications in time series analysis include studying machining dynamics [18, 19, 41, 17, 42, 20, 16], gene expression [27, 4], financial data [12], video data [37, 36], and sleep-wake states [9, 39]. These applications typically involve summarizing the underlying topological shape of each time series in a persistence diagram then using additional methods to analyze the resulting collection of persistence diagrams. While these applications have been successful, the task of analyzing a collection of persistence diagrams can still be difficult. Many methods have been created to convert persistence diagrams into a form amenable for machine learning [5, 3, 29, 28], however, it can be difficult to choose which method is appropriate for the task, and it can add to computation time.

Our method aims to circumvent these issues using zigzag persistence, a generalization of persistent homology that is capable of summarizing information from a sequence of point clouds in a single persistence diagram. While less popular than standard persistent homology, zigzag persistence has been used in applications, including studying optical flow in computer generated videos [1, 2], analyzing stacks of neuronal images [23] and comparing different subsamples of the a dataset [34],

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however, it has not been used in the context of dynamical systems or time series analysis. In this paper, we present Bifurcations using ZigZag (BuZZ), a one-step method to analyze bifurcations in dynamical systems using zigzag persistence.

2 Methods

Here we will present tools needed to build our method, including the time delay embedding, and an overview of the necessary topological tools. Specifically, we briefly introduce homology, persistent homology and zigzag persistent homology. However, we will not go into detail and instead direct the interested reader to [14, 10, 6] for more detail on homology, persistent homology, and zigzag homology, respectively.

2.1 Homology and persistent homology

Homology is a tool from the field of algebraic topology that encodes information about shape in various dimensions. Zero-dimensional homology studies connected components, 1-dimensional homology studies loops, and 2-dimensional homology studies voids. Persistent homology is a method from TDA which studies the homology of a parameterized space.

Here we will focus on persistent homology applied to point cloud data, where we need only assume a point cloud is a collection of points with a notion of distance. Given a collection of points, we will build connections between points based on their distance. Specifically, we will build simplicial complexes, which are spaces built from different dimensional simplices. A 0-simplex is a vertex, a 1-simplex is an edge, and in general, a p-simplex is the convex hull of $p+1$ affinely independent vertices. To create a simplicial complex from a point cloud, we use the Vietoris-Rips complex (sometimes just called the Rips complex). Given a point cloud, X, and a distance value, r, the Vietoris-Rips complex, $R(X,r)$, consists of all simplices whose vertices have maximum pairwise distance at most r. Taking a range of distance values, $r_0 \leq r_1 \leq r_2 \leq \cdots r_n$ gives a set of simplicial complexes, $\{R(X,r_i)\}$. Since the distance values are strictly increasing, we have a nested sequence of simplicial complexes

$$R(X,r_0) \subseteq R(X,r_1) \subseteq \cdots \subseteq R(X,r_n) \tag{1}$$

called a filtration. Computing p-dimensional homology, $H_p(\mathcal{K})$, for each complex in the filtration gives a sequence of vector spaces and linear maps,

$$H_p(R(X,r_0)) \to H_p(R(X,r_1)) \to \cdots \to H_p(R(X,r_n)).$$

Persistent homology tracks features such as connected components and loops as you move through the filtration. Specifically, it records at what distance value a feature first appears, and when a feature disappears or connects with another feature. These distance values are called the "birth" and "death" times respectively. These birth and death times are represented as a persistence diagram, which is a multiset of the birth death pairs $\{(b,d)\}$.

2.2 Time delay embedding

One way to reconstruct the underlying dynamics given only a time series is through a time delay embedding. Given a time series, $[x_1,\ldots,x_n]$, a choice of dimension d and lag τ, the delay embedding is the point cloud $\mathbf{X} = \{\mathbf{x}_i := (x_i, x_{i+\tau}, \ldots, x_{i+(d-1)\tau})\} \subset \mathbb{R}^d$. Takens' theorem [33] shows that given most choices of parameters, the embedding retains the same topological structure as the state space of the dynamical system and that this is in fact a true embedding in the mathematical sense. In practice, not all parameter choices are optimal, so heuristics for making reasonable parameter choices have been developed [11, 15, 26, 8, 25, 21].

In existing methods combing TDA with time series analysis, most works analyze a collection of time series by embedding each one into a point cloud using the time delay embedding. Instead, we will employ a generalized version of persistent homology to avoid these additional steps.

2.3 Zigzag persistent homology

Zigzag persistence is a generalization of persistent homology that can study a collection of point clouds simultaneously. For zigzag persistence, rather than having a nested set of complexes as

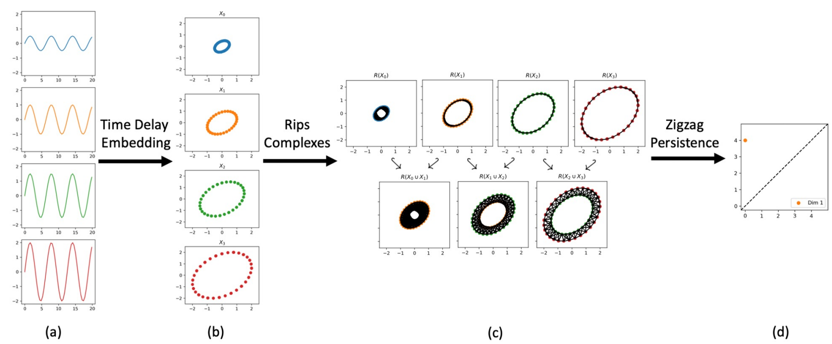

Figure 1: Outline of BuZZ method. The input time series is converted to an embedded point cloud via the time delay embedding. The Rips complexes are constructed for either a fixed r or a choice of r_i for each point cloud. Then, the zigzag persistence diagram is computed for the collection.

in Eqn. 1, you can have a collection of simplicial complexes where the inclusions go in different directions. For this paper we will focus on a specific setup for the zigzag based on a collection of point clouds.

Given an ordered collection of point clouds, $X_0, X_1, \ldots, X_n$, we can define a set of inclusions,

$$\begin{tikzcd}
X_0 \arrow[dr, hook] & & X_1 \arrow[dl, hook'] \arrow[dr, hook] & & X_2 \arrow[dl, hook'] & \cdots & X_{n-1} \arrow[dr, hook] & & X_n \arrow[dl, hook'] \\
& X_0 \cup X_1 & & X_1 \cup X_2 & & & & X_{n-1} \cup X_n.
\end{tikzcd}$$

However, these are all still point clouds, which have uninteresting homology. Thus, we can compute the Vietoris-Rips complex of each point cloud for a fixed radius, r. This results in the diagram of inclusions of simplicial complexes

$$\begin{tikzcd}
R(X_0, r) \arrow[dr, hook] & & R(X_1, r) \arrow[dl, hook'] \arrow[dr, hook] & & R(X_2, r) \arrow[dl, hook'] & \cdots & R(X_{n-1}, r) \arrow[dr, hook] & & R(X_n, r) \arrow[dl, hook'] \\
& R(X_0 \cup X_1, r) & & R(X_1 \cup X_2, r) & & & & R(X_{n-1} \cup X_n, r).
\end{tikzcd}$$

Computing the 1-dimensional homology of each complex in Eqn. 2.3 will result in a zigzag diagram of vector spaces and induced linear maps,

$$\begin{tikzcd}
H_1(R(X_0, r)) \arrow[dr] & & H_1(R(X_1, r)) \arrow[dl] & \cdots & H_1(R(X_{n-1}, r)) \arrow[dr] & & H_1(R(X_n, r)) \arrow[dl] \\
& H_1(R(X_0 \cup X_1, r)) & & & & H_1(R(X_{n-1} \cup X_n, r)).
\end{tikzcd}$$

Zigzag persistence tracks features that are homologically equivalent through this zigzag. This means it records the range of the zigzag filtration where the same feature appears. The zigzag persistence diagram records "birth" and "death" relating to location in the zigzag. If a feature appears in $R(X_i, r)$, it is assigned birth time i, and if it appears at $R(X_i \cup X_{i+1}, r)$, it is assigned birth time $i + 0.5$. Similarly, if a feature last appears in $R(X_j, r)$, it is assigned a death time $j + 0.5$, while if it last appears in $R(X_j \cup X_{j+1}, r)$, it is assigned a death time of $j + 1$.

2.4 Bifurcations using ZigZag (BuZZ)

We can now present our method, Bifurcations using ZigZag (BuZZ) for using time delay embeddings and zigzag persistence to detect changes in circular features in dynamical systems. We will focus on Hopf bifurcations [13], as these bifurcations are particularly topological in nature, and are seen when a fixed point loses stability and a limit cycle is introduced.

The necessary data for our method is a collection of time series for a varying input parameter value, in this case the amplitude of the sine wave, as shown in Fig. 1(a). Each time series is then embedded using the time delay embedding (shown in Fig. 1(b) using $d = 2$ and $\tau = 3$). Sorting the resulting point clouds based on the input parameter value, the zigzag filtration can be formed from the collection

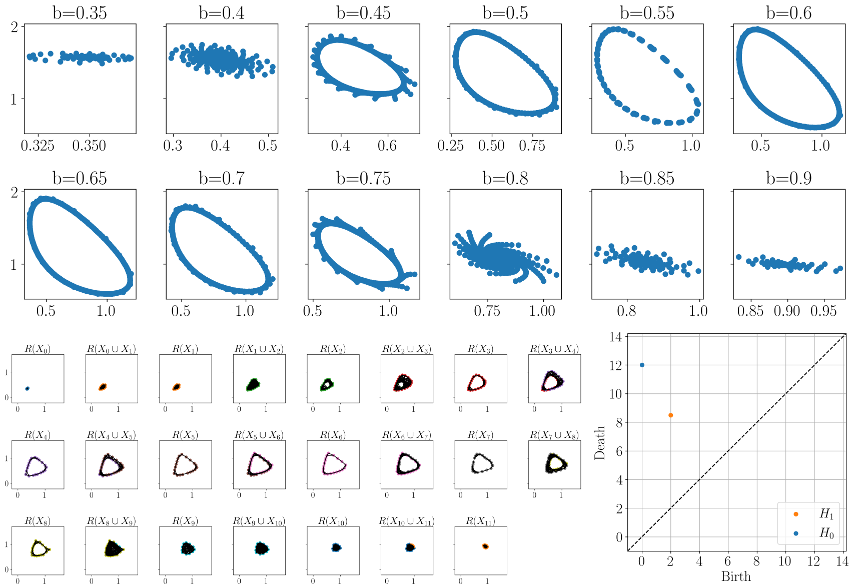

Figure 2: Top: Examples of samplings of the state space of the Sel'kov model for varying parameter value b. Bottom left: zigzag filtration using Rips complex with fixed radius of 0.25. Note that 2-simplices are not shown in the complexes. Bottom right: resulting zigzag persistence diagram.

of point clouds, as shown in Fig. 1(c). Lastly, computing zigzag persistence gives a persistence diagram, as shown in Fig. 1(d), encoding information about the structural changes moving through the zigzag.

With the right choices of parameters, the 1-dimensional persistence point with the longest lifetime in the zigzag persistence diagram will have birth and death time corresponding to the indices in the zigzag where the Hopf bifurcation appears and disappears. Lastly, mapping the birth and death times back to the parameter values used to create the corresponding point clouds will give the range of parameter values where the Hopf bifurcation occurs.

3 Results

To test our method, we study a bifurcation in the Sel'kov model [31], a model for glycolysis which is a process of breaking down sugar for energy. This model is defined by the system of differential equations,

$$\dot{x} = -x + ay + x^2y, \qquad \dot{y} = b - ay - x^2y,$$

where the overdot denotes a derivative with respect to time. This system has a Hopf bifurcation for select choices of parameters a and b.

For our experiments, we will fix $a = 0.1$ and vary the parameter b. We generate 500 time points of the data with initial conditions $(0, 0)$, and remove the first 50 points to remove transients at the beginning of the model. This data is constructed using full knowledge of the model, however, in practice, one typically only has one measurement function and then the time-delay embedding is used to reconstruct the underlying system. To mimic this setup, we will only use the time series corresponding to the x-coordinates from the model and use the delay embedding. These time series are then embedded using the time delay embedding with dimension $d = 2$ and delay $\tau = 3$.

Next, we compute zigzag persistence on subsampled versions of the point clouds to improve computation time. We subsample down to only 20 points in each point cloud using the furthest point sampling method [7], compute the Rips complex zigzag for a fixed radius value of 0.25, and then compute the zigzag persistence. Figure 2 shows the zigzag filtration of Rips complexes along with the resulting zigzag persistence diagram. In the zigzag persistence diagram, the point with the longest lifetime has coordinates $(2, 8.5)$. Again, since these coordinates correspond to the index in the zigzag sequence, this point corresponds to a feature appearing at $R(X_2)$ and disappearing at $R(X_8 \cup X_9)$. Looking back at which values of b were used to generate these point clouds, we see this corresponds to a feature appearing at $b = 0.45$ and disappearing at $b = 0.8$. For the fixed parameter value of $a = 0.1$, the Sel'kov model has a limit cycle approximately between the parameter values $0.4 \leq b \leq 0.8$ [32].

4 Discussion

Here we have introduced a method of detecting Hopf bifurcations in dynamical systems using zigzag persistent homology called BuZZ. This method was shown to work on an example using the Sel'kov model. Our method is able to detect the range of the zigzag filtration where circular features appear and disappear. Thus, this method could be applied to any application with an ordered set of point clouds and a changing topological structure.

Using existing topological time series analysis techniques, each time series would generate one persistence diagram. From there, a method of analyzing the collection of persistence diagrams would need to be chosen. This adds additional steps to the method, and could add onto computational cost. In comparison, our method is straight forward with a minimal number of steps. Additionally, the BuZZ method is implemented in an open source python package that is available on GitHub[1] for easy usability.

While the BuZZ method has shown success, it also has its limitations. The method is computationally expensive due to numerous Rips complex computations in addition to the zigzag persistence computation itself. This issue can be alleviated using subsampling, as shown with the Sel'kov model, however, future extensions of this project could include improvements of the computation time. Additionally, while the method works well in practice, it lacks theoretical guarantees. Given the method requires parameter choices for the radii of the Vietoris-Rips complexes, we would like some heuristics to be used in practice to choose these radii more easily. Because our examples in this paper are small, selecting parameters by hand is reasonable. However, in the future when applied to larger, experimental data, these sorts of heuristics will be necessary.

Lastly, the method at this point only applies in the case of one bifurcation parameter. We would like to further explore how to broaden the method to apply when changing multiple parameters.

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