Provable Data Subset Selection For Efficient Neural Networks Training

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Abstract

Radial basis function neural networks (*RBFNN*) are well-known for their capability to approximate any continuous function on a closed bounded set with arbitrary precision given enough hidden neurons. In this paper, we introduce the first algorithm to construct coresets for *RBFNNs*, i.e., small weighted subsets that approximate the loss of the input data on any radial basis function network and thus approximate any function defined by an *RBFNN* on the larger input data. In particular, we construct coresets for radial basis and Laplacian loss functions. We then use our coresets to obtain a provable data subset selection algorithm for training deep neural networks. Since our coresets approximate every function, they also approximate the gradient of each weight in a neural network, which is a particular function on the input. We then perform empirical evaluations on function approximation and dataset subset selection on popular network architectures and data sets, demonstrating the efficacy and accuracy of our coreset construction.

1. Introduction

Radial basis function neural networks (*RBFNNs*) are artificial neural networks that generally have three layers: an input layer, a hidden layer with a radial basis function (RBF) as an activation function, and a linear output layer. In this paper, the input layer receives a d-dimensional vector $x \in \mathbb{R}^d$ of real numbers. The hidden layer then consists of various nodes representing *RBFs*, to compute $\rho(\|x - c_i\|_2) := \exp\left(-\|x - c_i\|_2^2\right)$, where $c_i \in \mathbb{R}^d$ is the center vector for neuron i across, say, N neurons in the hidden layer. The linear output layer then computes $\sum_{i=1}^{N} \alpha_i \rho(\|x - c_i\|_2)$, where α_i is the weight of neuron i in the linear output neuron. Therefore, *RBFNNs* are feed-forward neural networks because the edges between the nodes do not form a cycle, and enjoy advantages such as simplicity of analysis, faster training time, and interpretability, compared to alternatives such as convolutional neural networks (*CNNs*) and even multi-layer perceptrons (*MLPs*) (Padmavati, 2011).

Function approximation via *RBFNNs*. *RBFNNs* are universal approximators in the sense that an *RBFNN* with a sufficient number of hidden neurons (large N) can approximate any continuous function on a closed, bounded subset of $\mathbb{R}^d$ with arbitrary precision (Park & Sandberg, 1991), i.e., given a sufficiently large input set P of n points in $\mathbb{R}^d$ and given its corresponding label function $y : P \to \mathbb{R}$, an *RBFNN*, can be trained to approximate the function y. Therefore, *RBFNNs* are commonly used across a wide range of applications, such as function approximation (Park & Sandberg, 1991; 1993; Lu et al., 1997), time series prediction (Whitehead & Choate, 1996; Leung et al., 2001; Harpham & Dawson, 2006), classification (Leonard & Kramer, 1991; Wuxing et al., 2004; Babu & Suresh, 2012), and system control (Yu et al., 2011; Liu, 2013), due to their faster learning speed.

For a given RBFNN size, i.e., the number of neurons in the hidden layer, and an input set, the aim of this paper is to compute a small weighted subset that approximates the loss of the input data on any radial basis function neural network of this size and thus approximates any function defined (approximated) by such an *RBFNN* on the big input data. This small weighted subset is called a coreset.

Coresets. Consider a prototypical machine/deep learning problem in which we are given an input set $P \subseteq \mathbb{R}^d$ of n points, its corresponding weights function $w : P \to \mathbb{R}$, a set of queries X (a set of candidate solutions for the involved optimization problem), and a loss function $f : P \times X \to [0, \infty)$. The tuple (P, w, X, f) is called the *query space*, and it defines the optimization problem at hand — where usually, the goal is to find $x^* \in \arg\min_{x \in X} \sum_{p \in P} w(p) f(p, x)$. Given a query space (P, w, X, f), a coreset is a small weighted subset of the input P that can provably approximate the cost of every query $x \in X$ on P (Feldman, 2020); see Definition 2.1. In particular, a coreset for a *RBFNN* can

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approximate the cost of an *RBFNN* on the original training data for every set of centers and weights that define the *RBFNN* (see Section 4). Hence, the coreset approximates also the centers and weights that form the optimal solution of the *RBFNN* (the solution that approximates the desired function). Thus a coreset for a *RBFNN* would facilitate training data for function approximation without reading the full training data and more generally, a strong coreset for an RBFNN with enough hidden neurons would give a strong coreset for any function that can be approximated to some precision using the RBFNN.

To this end, in this paper, we aim to provide the first coreset for *RBFNNs*, and thus provably approximating (providing a coreset to) any function that can be approximated by a given *RBFNN*.

Furthermore, we can use this small weighted subset (coreset) to suggest a provable data subset selection algorithm for training deep neural networks efficiently (on the small subset). Since our coreset approximates every function that can be approximated by an RBFNN of this size, it also approximates the gradient of each weight in a neural network (if it can be approximated by the RBFNN).

Training neural networks on data subset. Although deep learning has become widely successful with the increasing availability of data (Krizhevsky et al., 2017; Devlin et al., 2019), modern deep learning systems have correspondingly increased in their computational resources, resulting in significantly larger training times, financial costs (Sharir et al., 2020), energy costs (Strubell et al., 2019), and carbon footprints (Strubell et al., 2019; Schwartz et al., 2020). Data subset selection (coresets) allows for efficient learning at several levels (Wei et al., 2014; Kaushal et al., 2019; Coleman et al., 2019; Har-Peled & Mazumdar, 2004; Clarkson, 2010). By employing a significantly smaller subset of the big dataset, (i) we enable learning on relatively low resource computing settings without requiring a huge number of GPU and CPU servers, (ii) we may greatly optimize the end-to-end turnaround time, which frequently necessitates many training runs for hyper-parameter tweaking, and (iii) because a large number of deep learning trials must be done in practice, we allow for considerable reductions in deep learning energy usage and CO_2 emissions (Strubell et al., 2019). Multiple efforts have recently been made to improve the efficiency of machine learning models using data subset selection (Mirzasoleiman et al., 2020a; Killamsetty et al., 2021b;a; Pooladzandi et al., 2022). However, existing techniques either (i) employ proxy functions to choose data points, (ii) are specialized to specific machine learning models, (iii) use approximations of parameters such as gradient error or generalization errors, (iv) lack provable guarantees on the approximation error, or (v) require an inefficient gradient computation of the whole data. Most importantly, all of these methods are model/network dependent, and thus computing the desired subset of the data after several training epochs (for the same network) takes a lot of time and must be repeated each time the network changes.

To this end, in this paper, we introduce a provable and efficient model-independent subset selection algorithm for training neural networks. This will allow us to compute a subset of the training data, that is guaranteed to be a coreset for training multiple neural network architectures/models.

1.1. Our Contributions

In this paper, we introduce a coreset that approximates any function can be represented by an *RBFNN* architecture. Specifically:

(i) We provide a coreset for the *RBF* and Laplacian cost functions; see Algorithm 1, and Section 3.1.2.

(ii) We generate a coreset for any *RBFNN* model, in turn approximating any function that can be represented by the *RBFNN*; see Figure 1 for illustration, and Section 4 for more details.

(iii) We then exploit the properties of *RBFNNs* to approximate the gradients of any deep neural networks (*DNNs*), leading towards provable subset selection for learning/training *DNNs*. We also show the advantages of using our coreset against previous subset selection techniques; see Section 5 and Section 7.

(iv) Finally, we provide an open-source code implementation of our algorithm for reproducing our results and future research (ope, 2023).

1.2. Related Work

A long line of active work has studied efficient coreset constructions for various problems in computational geometry and machine learning, such as k-means and k-median clustering (Har-Peled & Mazumdar, 2004; Chen, 2009; Braverman et al., 2016; Huang & Vishnoi, 2020; Jubran et al., 2020; Cohen-Addad et al., 2022), regression (Dasgupta et al., 2008; Chhaya et al., 2020; Tolochinsky et al., 2022; Meyer et al., 2022; Maalouf et al., 2019; 2022b), low-rank approximation (Cohen et al., 2017; Braverman et al., 2020; Maalouf et al., 2020; 2021), volume maximization (Indyk et al., 2020; Mahabadi et al., 2020; Woodruff & Yasuda, 2022), projective clustering (Feldman et al., 2020; Tukan et al., 2022c), support vector machines (SVMs) (Clarkson, 2010; Tukan et al., 2021; Maalouf et al., 2022a), Bayesian inference (Campbell & Broderick, 2018), and sine wave fitting (Maalouf et al., 2022c). In addition, recently, coresets were leveraged for boosting sampling-based path planners

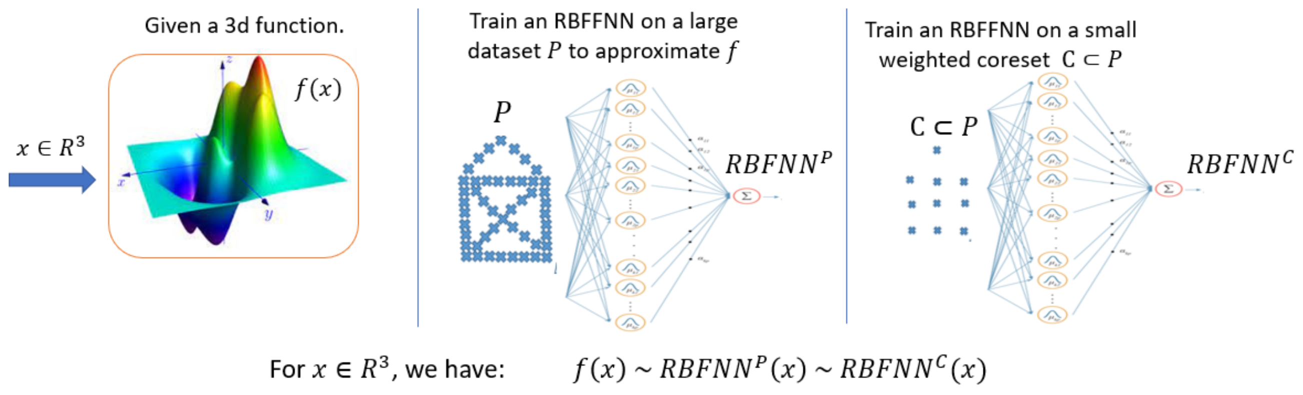

Given a 3d function.
$f(x)$
$x \in R^3$
Train an RBFFNN on a large dataset P to approximate f
P
$RBFNN^P$
Train an RBFFNN on a small weighted coreset $C \subset P$
$C \subset P$
$RBFNN^C$
For $x \in R^3$, we have: $f(x) \sim RBFNN^P(x) \sim RBFNN^C(x)$

Figure 1. Our contribution in a nutshell.

in the field of Robotics (Tukan et al., 2022a), as well as in marine applications (Tukan et al., 2023)

Baykal et al. (2022) suggested coreset-based algorithms for compressing the parameters of a trained fully-connected neural network by using sensitivity sampling on the weights of neurons after training, though without pruning full neurons. (Mussay et al., 2020; Liebenwein et al., 2019; Tukan et al., 2022b) sidestepped this issue by identifying the neurons that can be compressed regardless of their weights, due to the choice of the activation functions, thereby achieving coreset-based algorithms for neural pruning.

These approaches use coresets to achieve an orthogonal goal to data subset selection in the context of deep learning – they greatly reduce the number of neurons in the network while we greatly reduce the number of samples in the dataset that need to be read by the neural network. Correspondingly, we reduce the effective size of the data that needs to be stored or even measured prior to the training stage. Moreover, we remark that even if the number of inputs to the input layer was greatly reduced by these neural compression approaches, the union of the inputs can still consist of the entire input dataset and so these approaches generally cannot guarantee any form of data distillation.

Toward the goal of data subset selection, (Mirzasoleiman et al., 2020a;b) introduced algorithms for selecting representative subsets of the training data to accurately estimate the full gradient for tasks in both deep learning and classical machine learning models such as logistic regression and these approaches were subsequently refined by (Killamsetty et al., 2021a;b). Data distillation has also received a lot of attention in image classification (Bohdal et al., 2020; Nguyen et al., 2021; Dosovitskiy et al., 2021), natural language processing (Devlin et al., 2019; Brown et al., 2020), and federated learning (Ozkara et al., 2021; Zhu et al., 2021).

On coresets for any function. To the best of our knowledge, the only other coresets eligible for handling a wide family of functions without the need to devise a problem-dependent sensitivity are (Claici & Solomon, 2018; Claici et al., 2018). While such coresets are interesting and related, (i) both works provide coreset constructions resulting in an additive approximation, (ii) the coresets' theoretical applications seem quite a bit restrictive as they intend to handle mainly a family of functions that are either k-Lipschitz (functions with bounded gradient, usually k), having a bounded Dual-Sobolev distance, or functions satisfying the properties of reproducing kernel Hilbert space (RKHS). In addition, the running time in the worst-case scenario is not practical, i.e., exponential in the dimension of the points.

On the other hand, our coreset under mild assumptions can satisfy any function approximated by RBFNN (Wu et al., 2012), in time that is polynomial in the dimension of the points and linear in the number of nonzero entries of the points (Clarkson & Woodruff, 2017).

2. Preliminaries

For an integer $n > 0$, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. A weighted set of points is a pair (P, w), where $P \subseteq \mathbb{R}^d$ is a set of points and $w : P \to [0, \infty)$ is a weight function.

We now formally provide the notion of ε-coreset for the *RBF* loss. This will be later extended to a coreset for *RBFNN*.

Definition 2.1 (*RBF* ε-coreset)**.** Let (P, w) be a weighted of n points in $\mathbb{R}^d$, $X \subseteq \mathbb{R}^d$ be a set of queries, $\varepsilon \in (0, 1)$. For every $x \in X$ and $p \in P$ let $f(p, x) := \exp\left(-\|p - x\|_2^2\right)$ denote the *RBF* loss function between p and x. An ε-coreset for (P, w) with respect to f, is a pair (S, v) where $S \subseteq P$, $v : S \to (0, \infty)$ is a weight function, such that for every $x \in X$, $\left|1 - \frac{\sum_{q \in S} v(q) f(q,x)}{\sum_{p \in P} w(p) f(p,x)}\right| \leq \varepsilon$.

We say the *RBF* coreset is *strong* if it guarantees correctness over all $x \in X$. Otherwise, we say the coreset is *weak* if it only provides guarantees for all x only in some subset of X.

Sensitivity sampling. To compute our *RBF* ε-coreset, we utilize the sensitivity sampling framework (Braverman et al., 2016). In short, the sensitivity of a point $p \in P$ corresponds to the "importance" of this point with respect to the other points and the problem at hand. In our context (with respect to the *RBF* loss), the sensitivity is defined as $s(p) = \sup_{x \in X} \frac{w(p) f(p,x)}{\sum_{q \in P} w(q) f(q,x)}$, where the denominator is nonzero. Once we bound the sensitivities for every $p \in P$, we can sample points from P according to their corresponding sensitivity bounds, and re-weight the sampled points to obtain an RBF ε-coreset as in Definition 2.1. The size of the sample (coreset) is proportional to the sum of these bounds – the tighter (smaller) these bounds, the smaller the coreset size; we refer the reader to Section A in the appendix.

Sensitivity bounding. We now present our main tool for bounding the sensitivity of each input point with respect to the *RBF* and *Laplacian* loss functions.

Definition 2.2 (ℓ_1-SVD – Special case of Definition 4 (Tukan et al., 2020)). Let $(P, w, \mathbb{R}^d, f)$ be a query space (see Definition A.1) where for every $p \in P$ and $x \in \mathbb{R}^d$, $f(p,x) = |p^T x|$. Let $D \in [0,\infty)^{d \times d}$ be a diagonal matrix of full rank and let $V \in \mathbb{R}^{d \times d}$ be an orthogonal matrix, such that for every $x \in \mathbb{R}^d$, $\left\|DV^T x\right\|_2 \leq \sum\limits_{p \in P} w(p) \left|p^T x\right| \leq \sqrt{d} \left\|DV^T x\right\|_2$. Define $U : P \to \mathbb{R}^d$ such that $U(p) = D^{-1} V^T p$ for every $p \in P$. The tuple (U, D, V) is the $\|\cdot\|_1$-SVD of P.

Using the above tool, the sensitivity with respect to the RBF loss function can be bounded using the following.

Lemma 2.3 (Special case of Lemma 35, (Tukan et al., 2020)). *Let $(P, w, \mathbb{R}^d, f)$ be query space as in Definition A.1 where for every $p \in P$ and $x \in \mathbb{R}^d$, $f(p,x) = |p^T x|$. Let (U, D, V) be the $\|\cdot\|_1$-SVD of (P, w) with respect to $|\cdot|$ (see Definition 2.2). Then: (i) for every $p \in P$, the sensitivity of p with respect to the query space $(P, w, \mathbb{R}^d, |\cdot|)$ is bounded by $s(p) \leq \|U(p)\|_1$, and (ii) the total sensitivity is bounded by $\sum\limits_{p \in P} s(p) \leq d^{1.5}$.*

3. Method

In this section, we provide coresets for the Gaussian and Laplacian loss functions. We detail our coreset construction for the Gaussian loss function and Laplacian loss function in Section 3.1.2.

Overview of Algorithm 1. Algorithm 1 receives as input, a set P of n points in $\mathbb{R}^d$, a weight function $w : P \to [0,\infty)$, a bound R on the radius of the ball containing query space X, and a sample size $m > 0$. If the sample size m is sufficiently large, then Algorithm 1 outputs a pair (S, v) that is an ε-coreset for RBF cost function; see Theorem 3.2.

Algorithm 1 CORESET(P, w, R, m)

```
input  A set $P \subseteq \mathbb{R}^d$ of $n$ points, a weight function $w : P \to [0,\infty)$, a bound on radius
       of the query space $X$, and a sample size $m \geq 1$
output A pair $(S, v)$ that satisfies Theorem 3.2
1: Set $d'$ := the VC dimension of quadruple $(P, w, X, \rho(\cdot))$ {See Definition A.2}
2: $P' := \left\{ q_p = \left[\|p\|_2^2, -2p^T, 1\right]^T \mid \forall p \in P \right\}$
3: $(U, D, V)$ := the $f$-SVD of $(P', w, |\cdot|)$ {See Definition 2.2}
4: for every $p \in P$ do
5:     $s(p) := \eta \cdot \left( \frac{w(p)}{\sum_{q \in P} w(q)} + w(p) \|U(q_p)\|_1 \right)$ {bound on the sensitivity of $p$ as in Lemma B.1 in the appendix, where $\eta := e^{12R^2}\left(1 + 8R^2\right)$}
6: end for
7: $t := \sum_{p \in P} s(p)$
8: Set $\tilde{c} \geq 1$ to be a sufficiently large constant {Can be determined from Theorem 3.2}
9: Pick an i.i.d sample $S$ of $m$ points from $P$, where each $p \in P$ is sampled with probability $\frac{s(p)}{t}$
10: set $v : \mathbb{R}^d \to [0,\infty]$ to be a weight function such that for every $q \in S$, $v(q) = \frac{t}{s(q) \cdot m}$.
return $(S, v)$
```

First, d' is set to be the VC dimension of the quadruple $(P, w, X, \rho(\cdot))$; see Definition A.2. The heart of our algorithm lies in formalizing the RBF loss function as a variant of the regression problem, specifically, a variant of the ℓ_1-regression problem. The conversion requires manipulation of the input data as presented at Line 2. We then compute the f-SVD of the new input data with respect to the ℓ_1-regression problem followed by bounded the sensitivity of such points (Lines 3–5). Now we have all the needed ingredients to obtain an ε-coreset (see Theorem A.3), i.e., we sample i.i.d m points from P based on their sensitivity bounds (see Line 9), followed by assigning a new weight for every sampled point at Line 10.

3.1. Analysis

3.1.1. Lower bound on the coreset size for the Gaussian loss function

We first show the lower bound on the size of coresets, to emphasize the need for assumptions on the data and the query space.

Theorem 3.1. *There exists a set of n points $P \subseteq \mathbb{R}^d$ such that $\sum_{p \in P} s(p) = \Omega(n)$.*

Proof. Let $d \geq 3$ and let $P \subseteq \mathbb{R}^d$ be a set of n points distributed evenly on a 2 dimensional sphere of radius $\sqrt{\frac{\ln n}{2\cos\left(\frac{\pi}{n}\right)}}$. In other words, using the law of cosines, every $p \in P$, $\sqrt{\ln n} = \min_{q \in P \setminus \{p\}} \|p-q\|_2$; see Figure C. Observe that for every $p \in P$,

$$\begin{aligned} s(p) &:= \max_{x \in \mathbb{R}^d} \frac{e^{-\|p-x\|_2^2}}{\sum\limits_{q \in P} e^{-\|q-x\|_2^2}} \geq \frac{e^{-\|p-p\|_2^2}}{\sum\limits_{q \in P} e^{-\|p-q\|_2^2}} \\ &= \frac{1}{1 + \sum\limits_{q \in P \setminus \{p\}} e^{-\|p-q\|_2^2}} \geq \frac{1}{1 + \sum\limits_{q \in P \setminus \{p\}} \frac{1}{n}} \geq \frac{1}{2}, \end{aligned} \tag{1}$$

where the first equality holds by definition of the sensitivity, the first inequality and second equality hold trivially, the second inequality follows from the assumption that $\sqrt{\ln n} \leq \min_{q \in P \setminus \{p\}} \|p-q\|_2$, and finally the last inequality holds since $\sum_{q \in P \setminus \{p\}} \frac{1}{n} \leq 1$. □

3.1.2. Reasonable assumptions lead to existence of coresets

Unfortunately, it is not immediately straightforward to bound the sensitivities of either the Gaussian loss function or the Laplacian loss function. Therefore, we first require the following structural properties in order to relate the Gaussian and Laplacian loss functions into more manageable quantities. We shall ultimately relate the function $e^{-\left|p^T x\right|}$ to both the Gaussian and Laplacian loss functions. Thus, we first relate the function $e^{-\left|p^T x\right|}$ to the function $|p^T x| + 1$.

Let $p \in \mathbb{R}^d$ such that $\|p\|_2 \leq 1$, and let $R > 0$ be positive real number. Then for every $x \in \left\{x \in \mathbb{R}^d \middle| \|x\|_2 \leq R\right\}$, $\frac{1}{e^R(1+R)}\left(1 + \left|p^T x\right|\right) \leq e^{-\left|p^T x\right|} \leq \left|p^T x\right| + 1$.

In what follows, we provide the analysis of coreset construction for the RBF and Laplacian loss functions, considering an input set of points lying in the unit ball. We refer the reader to the supplementary material for generalization of our approaches towards general input set of points.

Theorem 3.2 (Coreset for *RBF*). *Let $R \geq 1$ be a positive real number, $X = \left\{x \in \mathbb{R}^d \middle| \|x\|_2 \leq R\right\}$, and let $\varepsilon, \delta \in (0,1)$. Let (P, w, X, f) be query space as in Definition A.1 such that every $p \in P$ satisfies $\|p\|_2 \leq 1$. For every $x \in X$ and $p \in P$, let $f(p,x) := \rho\left(\|p-x\|_2\right)$. Let (S,v) be a call to* Coreset(P, w, R, m) *where $S \subseteq P$ and $v : S \to [0,\infty)$. Then (S,v) ε-coreset of (P,w) with probability at least $1-\delta$, if $m = O\left(\frac{e^{12R^2} R^2 d^{1.5}}{\varepsilon^2}\left(R^2 + \log d + \log\frac{1}{\delta}\right)\right)$.*

Coreset for Laplacian loss function. In what follows, we provide a coreset for the Laplacian loss function. Intuitively speaking, leveraging the properties of the Laplacian loss function, we were able to construct a coreset that holds for every vector $x \in \mathbb{R}^d$ unlike the *RBF* case where the coreset holds for a ball of radius R. We emphasize that the reason for this is due to the fact that the Laplacian loss function is less sensitive than the *RBF*.

Theorem 3.3 (Coreset for the Laplacian loss function)**.** *Let $\left(P, w, \mathbb{R}^d, f\right)$ be query space as in Definition A.1 such that every $p \in P$ satisfies $\|p\|_2 \leq 1$. For $x \in \mathbb{R}^d$ and $p \in P$, let $f(p,x) := e^{-\|p-x\|_2}$. Let $\varepsilon, \delta \in (0,1)$. Then there exists an algorithm which given $P, w, \varepsilon, \delta$ return a weighted set (S,v) where $S \subseteq P$ of size $O\left(\frac{\sqrt{n} d^{1.25}}{\varepsilon^2}\left(\log n + \log d + \log\frac{1}{\delta}\right)\right)$ and a weight function $v : S \to [0,\infty)$ such that (S,v) is an ε-coreset of (P,w) with probability at least $1-\delta$.*

4. Radial Basis Function Networks

In this section, we consider coresets for *RBFNNs*. Consider an *RBFNN* with L neurons in the hidden layer and a single output neuron. First note that the hidden layer uses radial basis functions as activation functions so that the output is a scalar function of the input layer, $\phi : \mathbb{R}^d \to \mathbb{R}$ defined by $\phi(x) = \sum_{i=1}^L \alpha_i \rho(\|x - c^{(i)}\|_2)$, where $c^{(i)} \in \mathbb{R}^n$ for each $i \in [d]$.

For an input dataset P and a corresponding desired output function $y : P \to \mathbb{R}$, RBFNNs aim to optimize $\sum\limits_{p \in P}\left(y(p) - \sum_{i=1}^L \alpha_i e^{-\left\|p-c^{(i)}\right\|_2^2}\right)^2$. Expanding the cost function, we obtain that RBFNNs aim to optimize

$$\begin{aligned} &\sum_{p \in P} y(p)^2 - 2\sum_{i=1}^L \alpha_i \overbrace{\left(\sum_{p \in P} y(p) e^{-\left\|p-c^{(i)}\right\|_2^2}\right)}^{\alpha} \\ &\qquad + \underbrace{\sum_{p \in P}\left(\sum_{i=1}^L \alpha_i e^{-\left\|p-c^{(i)}\right\|_2^2}\right)^2}_{\beta}. \end{aligned} \tag{2}$$

Bounding the α term in equation 2. We first define for every $x \in \mathbb{R}^d$:

$$\begin{aligned} \phi^+(x) &= \sum_{p \in P, y(p) > 0} y(p) e^{-\|p-x\|_2^2} \\ \phi^-(x) &= \sum_{p \in P, y(p) < 0} |y(p)|\, e^{-\|p-x\|_2^2}. \end{aligned}$$

Observe that $\sum_{p \in P} y(p)\rho(\|p - c^{(i)}\|_2) = \phi^+\left(c^{(i)}\right) - \phi^-\left(c^{(i)}\right)$. Thus the α term in equation 2 can be approximated using the following.

Theorem 4.1. *There exists an algorithm that samples $O\left(\frac{e^{8R^2} R^2 d^{1.5}}{\varepsilon^2}\left(R^2 + \log d + \log\frac{2}{\delta}\right)\right)$ points to form weighted sets (S_1, w_1) and (S_2, w_2) such that with proba-*

bility at least $1-2\delta$,

$$\frac{\left|\sum_{p\in P} y(p)\phi(p) - \left(\sum_{\substack{i\in[L]\\ \alpha_i>0}} \alpha_i\gamma_{S_1} + \sum_{\substack{j\in[L]\\ \alpha_j<0}} \alpha_j\gamma_{S_2}\right)\right|}{\sum_{i\in[L]} |\alpha_i| \left(\phi^{+}\left(c^{(i)}\right) + \phi^{-}\left(c^{(i)}\right)\right)} \leq \varepsilon,$$

where $\gamma_{S_1} := \sum_{p\in S_1} w_1(p) e^{-\left\|p-c^{(i)}\right\|_2^2}$ *and* $\gamma_{S_2} := \sum_{q\in S_2} w_2(q) e^{-\left\|q-c^{(j)}\right\|_2^2}$.

Bounding the β term in equation 2. By Cauchy's inequality, it holds that

$$\sum_{p\in P}\left(\sum_{i=1}^{L}\alpha_i e^{-\left\|p-c^{(i)}\right\|_2^2}\right)^2 \leq L\sum_{p\in P}\sum_{i=1}^{L}\alpha_i^2 e^{-2\left\|p-c^{(i)}\right\|_2^2}$$
$$= L\sum_{i=1}^{L}\alpha_i^2\sum_{p\in P} e^{-2\left\|p-c^{(i)}\right\|_2^2},$$

where the equality holds by simple rearrangement.

Using Theorem 3.2, we can approximate the upper bound on β with an approximation of $L(1+\varepsilon)$. However, if for every $i \in [L]$ it holds that $\alpha_i \geq 0$, then we also have the lower bound $\sum_{i=1}^{L}\alpha_i^2 \sum_{p\in P} e^{-2\left\|p-c^{(i)}\right\|_2^2} \leq \sum_{p\in P}\left(\sum_{i=1}^{L}\alpha_i e^{-\left\|p-c^{(i)}\right\|_2^2}\right)^2$.

Since we can generate a coreset that yields a multiplicative approximation error to the full data for the left-hand side term of the above inequality, then we obtain also a coreset with multiplicative approximation error in a sense for β as well.

Thus by combining all of the above, we obtain the following corollary.

Corollary 4.2. *Let* RFBNN *be a sufficiently large (RBF) neural network,* $P \subset \mathbb{R}^d$*, and let* $f(p): P \to \mathbb{R}$ *be a function that can be approximated by a set of centers and weights q (the learned parameters that define the RBFNN) assigned to* $RFBNN$ *such that, for every* $p \in P$*:* $RFBNN(p,q) \approx f(p)$*, and* $\left\|\sum_{p\in P} RFBNN(p,q) - \sum_{p\in P} f(p)\right\| \leq \varepsilon$*, where each center of the* $RBFNN$ *lies in ball centered at the origin with radius R (to satisfy our coreset's assumption). Let* (S,v) *be the output of a call to Algorithm 1, i.e.,* (S,v) *is coreset for* P *is an RBF coreset; see Definition 2.1. Then,*

$$\left|\sum_{p\in P} f(p) - \sum_{s\in S} v(s) RBFNN(s,q)\right| \leq \varepsilon\sum_{p\in P} f(p) + \varepsilon.$$

5. Advantages of our Methods

One coreset for all networks. Our coreset is model-independent, i.e., we aim at improving the running time of multiple neural networks. Contrary to other methods that need to compute the coreset after each gradient update to support their theoretical proofs, our method gives the advantage of computing the sensitivity (or the coreset) only once, for all of the required networks. This is because our coreset can approximate any function that can be defined (approximated) using a *RBFNN* model.

Efficient coreset per epoch. Practically, our competing methods for data selection are not applied before each epoch, but every Υ epochs. This is since the competing methods require a lot of time to compute a new coreset since they compute the gradients of the network with respect to each input training data. However, our coreset can be computed before each epoch in a negligible time (~ 0 seconds), since we compute the sensitivity of each point (image) in the data once at the beginning, and then whenever we need to create a new coreset, we simply sample from the input data according to the sensitivity distribution.

6. On the Novelty of Our Approach

First, note that we have established a lower bound of $\Omega(n)$ for the coreset size when the problem of RBF fitting is handled with no assumptions; see Theorem 3.1, which consequently means that the RBF fitting problem is considered hard from the perspective of coresets. With this in mind, the assumption of boundness on the query space (R) aims to eliminate cases similar depicted by Theorem 3.1 that are deemed redundant since almost all points are needed. Our results were generalized to account for any general set of points $P \subset \mathbb{R}^d$ with modified bounds; see Section B. On the positive side of things, these bounds are usually pessimistic, and such a phenomenon can be seen in Figure 3.

In addition, from a practical point of view, the centers that the RBFNN aims to find are usually contained in (or in the vicinity of) the convex hull of the points (when setting for example the standard deviation of each neuron to 1). Combining this with the insights above, to ensure the existence of small coresets (in general), we need to lower the diameter of the data (the maximal distance between any pair of input points), which can be done using normalization/standardization. Thus, R can be chosen to be small such that the optimal RBFNN centers with respect to fitting the entire data are contained in a ball of radius R. Hence, our coreset will have a small size in such settings. We note that in machine learning, normalization, and standardization techniques are very common preprocessing used for improving the quality of the classifier/predictor. Thus, the centers will be contained in a small ball around the origin

Table 1. Data Selection Results for CIFAR10 using ResNet-18.

	Top-1 Test accuracy of the Model(%)				Model Training time(in hrs)			
Budget(%)	5%	10%	20%	30%	5%	10%	20%	30%
FULL (skyline for test accuracy)	95.09	95.09	95.09	95.09	4.34	4.34	4.34	4.34
RANDOM (skyline for training time)	71.2	80.8	86.98	87.6	0.22	0.46	0.92	1.38
GLISTER	85.5	**91.92**	92.78	93.63	0.43	0.91	1.13	1.46
CRAIG	82.74	87.49	90.79	92.53	0.81	1.08	1.45	2.399
CRAIGPB	83.56	88.77	92.24	93.58	0.4466	0.70	1.13	2.07
GRADMATCH	86.7	90.9	91.67	91.89	0.40	0.84	1.42	1.52
GRADMATCHPB	85.4	90.01	93.34	93.75	0.36	0.69	1.09	1.38
RBFNN CORESET (OURS)	**86.9**	91.4	**93.61**	**94.44**	**0.28**	**0.52**	**0.98**	**1.38**

Table 2. Data Selection Results for CIFAR10 using ResNet-18 with warm start

	Top-1 Test accuracy of the Model(%)				Model Training time(in hrs)			
Budget(%)	5%	10%	20%	30%	5%	10%	20%	30%
FULL (skyline for test accuracy)	95.09	95.09	95.09	95.09	4.34	4.34	4.34	4.34
RANDOM-WARM (skyline for training time)	83.2	87.8	90.9	92.6	0.21	0.42	0.915	1.376
GLISTER-WARM	86.57	91.56	92.98	94.09	0.42	0.88	1.08	1.40
CRAIG-WARM	84.48	89.28	92.01	92.82	0.6636	0.91	1.31	2.20
CRAIGPB-WARM	86.28	90.07	93.06	93.8	0.4143	0.647	1.07	2.06
GRADMATCH-WARM	87.2	92.15	92.11	92.01	0.38	0.73	1.24	1.41
GRADMATCHPB-WARM	86.37	**92.26**	93.59	94.17	0.32	0.62	1.05	1.36
RBFNN CORESET-WARM (OURS)	**87.82**	91.44	**93.81**	**94.6**	**0.27**	**0.51**	**0.99**	**1.36**

Table 3. Data Selection Results for CIFAR100 using ResNet-18

	Top-1 Test accuracy of the Model(%)				Model Training time(in hrs)			
Budget(%)	5%	10%	20%	30%	5%	10%	20%	30%
FULL (skyline for test accuracy)	75.37	75.37	75.37	75.37	4.871	4.871	4.871	4.871
RANDOM (skyline for training time)	19.02	31.56	49.6	58.56	0.2475	0.4699	0.92	1.453
GLISTER	29.94	44.03	61.56	70.49	0.3536	0.6456	1.11	1.5255
CRAIG	36.61	55.19	66.24	70.01	1.354	1.785	1.91	2.654
CRAIGPB	38.95	54.59	67.12	70.61	0.4489	0.6564	1.15	1.540
GRADMATCH	41.01	59.88	68.25	71.5	0.5143	0.8114	1.40	2.002
GRADMATCHPB	40.53	60.39	70.88	72.57	0.3797	0.6115	1.09	1.56
RBFNN CORESET (OURS)	**54.17**	**64.59**	**71.17**	**73.58**	**0.346**	**0.5699**	**1.01**	**1.552**

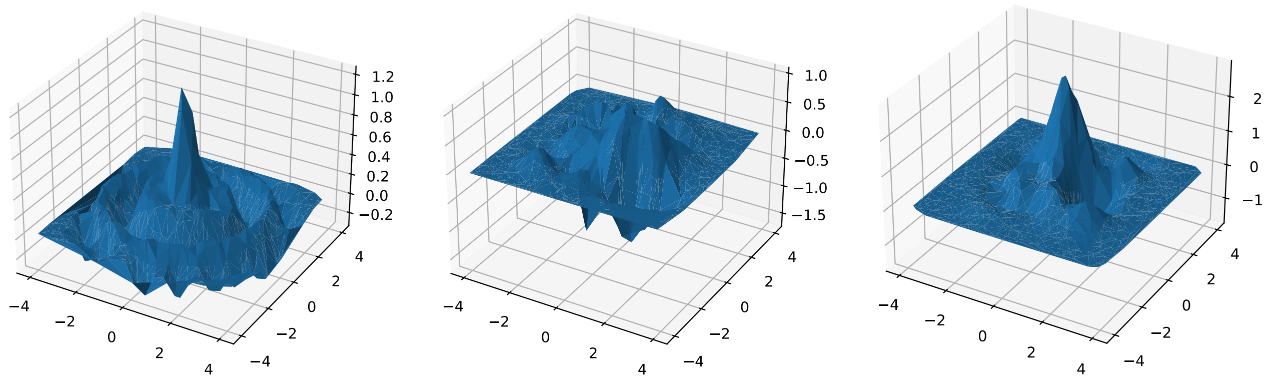
Table 4. Data Selection Results for CIFAR100 using ResNet-18 with warm start

	Top-1 Test accuracy of the Model(%)				Model Training time(in hrs)			
Budget(%)	5%	10%	20%	30%	5%	10%	20%	30%
FULL (skyline for test accuracy)	75.37	75.37	75.37	75.37	4.871	4.871	4.871	4.871
RANDOM-WARM (skyline for training time)	58.2	65.95	70.3	72.4	0.242	0.468	0.921	1.43
GLISTER-WARM	57.17	64.95	62.14	72.43	0.3185	0.6059	1.06	1.452
CRAIG-WARM	57.44	67.3	69.76	72.77	1.09	1.48	1.81	2.4112
CRAIGPB-WARM	57.66	67.8	70.84	73.79	0.394	0.6030	1.10	1.5567
GRADMATCH-WARM	57.72	68.23	71.34	74.06	0.3788	0.7165	1.30	1.985
GRADMATCHPB-WARM	58.26	**69.58**	**73.2**	74.62	**0.300**	0.5744	**1.01**	1.5683
RBFNN CORESET-WARM (OURS)	**59.22**	67.8	72.79	**75.04**	0.352	**0.5710**	1.03	**1.56**

when using these preprocessing tools.

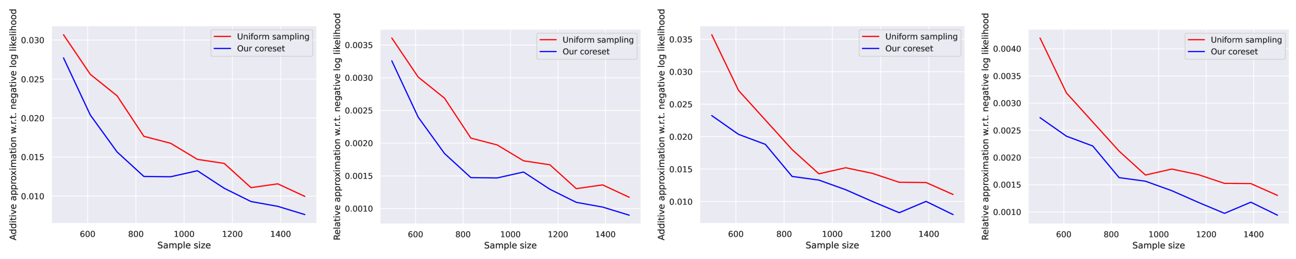
Table 5. Data Selection Results for ImageNet2012 using ResNet-18

Budget(%)	Top-1 Test accuracy of the Model(%) 5%	Model Training time(in hrs) 5%
FULL (skyline for test accuracy)	70.36	276.28
RANDOM (skyline for training time)	21.124	14.12
CRAIGPB	44.28	22.24
GRADMATCH	47.24	18.24
GRADMATCHPB	45.15	16.12
RBFNN CORESET (OURS)	**47.26**	**15.24**



(a) (b) (c)

Figure 2. **(a) is the function we wish to approximate by training an RBFNN on**: (b) a uniformly sampled subset, and (c) our coreset.



(a) (b) (c) (d)

Figure 3. **(a)** and **(b)** are depictions of the additive and relative error with respect to RBF fitting problem concerning the dataset , respectively. **(c)** and **(d)** re depictions of the additive and relative error with respect to RBF fitting problem concerning the dataset , respectively.

7. Experimental Results

In this section, we practically demonstrate the efficiency and stability of our *RBFNN* coreset approach for training deep neural networks via data subset selection. We mainly study the trade-off between accuracy and efficiency.

Competing methods. We compare our method against many variants of the proposed algorithms in (Killamsetty et al., 2021a) (denoted by, *GRAD-MATCH*), in (Mirzasoleiman et al., 2020a) (denoted by *CRAIG*), and in (Killamsetty et al., 2021b) (denoted by *GLISTER*). For each of these methods, we report the results for 4 variants: (i) the “vanilla” method, denoted by its original name, (ii) applying a warm start i.e., training on the whole data for 50% of the training time before training the other 50% on the coreset, where such methods are denoted by adding the suffix -WARM. (iii) a more efficient version of each of the competing methods denoted by adding the suffix PB (more details are given at (Killamsetty et al., 2021a)), and finally, a combination of both (ii) and (iii). In other words, the competing methods are GRAD-MATCH, GRAD-MATCHPB, GRAD-MATCH-WARM, GRAD-MATCHPB-WARM, CRAIG, CRAIGPB, CRAIG-WARM, CRAIGPB-WARM, and GLISTER-WARM. We also compare against randomly selecting points (denoted by RANDOM).

Datasets and model architecture. We performed our experiments for training CIFAR10 and CIFAR100 (Krizhevsky et al., 2009) on ResNet18 (He et al., 2016), MNIST (LeCun et al., 1998) on LeNet, and ImageNet-2012 (Deng et al., 2009) on Resnet18 (He et al., 2016).

The setting. We adapted the same setting of (Killamsetty et al., 2021a), where we used SGD optimizer for training initial learning rate equal to 0.01, a momentum of 0.9, and a weight decay of $5e-4$. We decay the learning rate using cosine annealing (Loshchilov & Hutter, 2016) for each epoch. For MNIST, we trained the LeNet model for 200 epochs. For CIFAR10 and CIFAR100, we trained the ResNet18 for 300 epochs - all on batches of size 20 for the subset selection training versions. We train the data selection methods and the entire data training with the same number of epochs; the main difference is the number of samples used for training a single epoch. All experiments were executed on V100 GPUs. The reported test accuracy in the results is after averaging across five runs.

Subset sizes and the Υ parameter. For MNIST, we use sizes of $\{1\%, 3\%, 5\%, 10\%\}$, while for CIFAR10 and CIFAR100, we use $\{5\%, 10\%, 20\%, 30\%\}$, and for ImageNet we use 5%.Since the competing methods require a lot of time to compute the gradients, we set $\Upsilon = 20$. We note that for our coreset we can test it with $\Upsilon = 1$ without adding run-time since once the sensitivity vector is defined, commuting a new coreset requires ~ 0 seconds. However, we test it with $\Upsilon = 20$, to show its robustness.

Discussion. Tables 1–4 report the results for CIFAR10 and CIFAR100. It is clear from Tables 1 and 2 that our method achieves the best accuracy, with and without warm start, for 5%, 20%, and 30% subset selection on CIFAR10. For CIFAR100, our method drastically outperforms all of the methods that do not apply a warm start. When applying a warm start, we still win in half of the cases. Note that, we outperform all of the other methods in terms of accuracy vs time. The same phenomenon is witnessed in the ImageNet experiment (Table 5) as our coreset achieves the highest accuracy. We refer the reader to the MNIST experiment (Table 6 in the appendix). We note that our sensitivity sampling vector is computed once during our experiments for each dataset. This vector can be used to sample coresets of different sizes, for different networks, at different epochs of training, in a time that is close to zero seconds. In all tables, the best results are highlighted in bold.

Function Approximations. We now compare our coreset to uniform for function approximation. Specifically, we generate around $10,000$ points in $3D$, while setting the third entry of each point to be a function of the first 2 entries, $f(x) = e^{-\|x\|_2^2} + 0.2\cos\left(4\left\|x\right\|_2\right)$. We train an RBFNN to reproduce the function using only 400 points, where we saw that our coreset (Figure 2(c)) is closer visually to the true function (Figure 2(a)) using uniform sampling for reproducing the image (Figure 2(b)).

RBF fitting problem. Finally, we show for the RBF fitting task on the HTRU and CreditCard datasets (Dua et al., 2017), where, our coreset is better than uniform sampling by at max a multiplicative factor of 1.5. In this experiment, we show the additive and multiplicative approximation associated with our method compared to uniform sampling with respect to the quality of solving the RBF fitting problem on the entire data. We refer the reader to Figure 3.

8. Conclusion and Future Work

In this paper, we have introduced a coreset that provably approximates any function that can be represented by RBFNN architectures. Our coreset construction can be used to approximate the gradients of any deep neural networks (*DNNs*), leading towards provable subset selection for learning/training *DNNs*. We also empirically demonstrate the value of our work by showing significantly better performances over various datasets and model architectures. As the first work on using coresets for data subset selection with respect to *RBFNNs*, our results lead to a number of interesting possible future directions. It is natural to ask whether there exist smaller coreset constructions that also provably give the same worst-case approximation guarantees. Furthermore, RKHS methods (Claici & Solomon, 2018; Claici et al., 2018) may be investigated in this context either by boosting their implementation or by merging ideas with this work. In addition, can our results be extended to more general classes of loss functions? Finally, we remark that although our empirical results significantly beat state-of-the-art, they nevertheless only serve as a proof-of-concept and have not been fully optimized with additional heuristics.

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Open source code for all the algorithms presented in this paper, 2023. Link for open-source code.

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A. Coreset Constructions

In what follows, we provide the necessary tools to obtain a coreset; see Definition 2.1.

Definition A.1 (Query space)**.** Let P be a set of $n \geq 1$ points in $\mathbb{R}^d$, $w : P \to [0, \infty)$ be a non-negative weight function, and let $f : P \times \mathbb{R}^d \to [0, \infty)$ denote a loss function. The tuple $(P, w, \mathbb{R}^d, f)$ is called a query space.

Definition A.2 (VC-dimension (Braverman et al., 2016))**.** For a query space $(P, w, \mathbb{R}^d, f)$ and $r \in [0, \infty)$, we define

$$\text{ranges}(x, r) = \{p \in P \mid w(p)f(p, x) \leq r\},$$

for every $x \in \mathbb{R}^d$ and $r \geq 0$. The dimension of $(P, w, \mathbb{R}^d, f)$ is the size $|S|$ of the largest subset $S \subset P$ such that

$$\left|\left\{S \cap \text{ranges}(x, r) \mid x \in \mathbb{R}^d, r \geq 0\right\}\right| = 2^{|S|},$$

where $|A|$ denotes the number of points in A for every $A \subseteq \mathbb{R}^d$.

The following theorem formally describes how to construct an ε-coreset based on the sensitivity sampling framework.

Theorem A.3 (Restatement of Theorem 5.5 in (Braverman et al., 2016))**.** *Let $\left(P, w, \mathbb{R}^d, f\right)$ be a query space as in Definition A.1. For every $p \in P$ define the* sensitivity *of p as $\sup_{x \in \mathbb{R}^d} \frac{w(p)f(p,x)}{\sum_{q \in P} w(q)f(q,x)}$, where the sup is over every $x \in \mathbb{R}^d$ such that the denominator is non-zero. Let $s : P \to [0, 1]$ be a function such that $s(p)$ is an upper bound on the sensitivity of p. Let $t = \sum_{p \in P} s(p)$ and d' be the* VC dimension *of the triplet $\left(P, w, \mathbb{R}^d, f\right)$; see Definition A.2. Let $c \geq 1$ be a sufficiently large constant, $\varepsilon, \delta \in (0, 1)$, and let S be a random sample of*

$$|S| \geq \frac{ct}{\varepsilon^2}\left(d' \log t + \log \frac{1}{\delta}\right)$$

i.i.d points from P, such that every $p \in P$ is sampled with probability $s(p)/t$. Let $v(p) = \frac{tw(p)}{s(p)|S|}$ for every $p \in S$. Then, with probability at least $1 - \delta$, (S, v) is an ε-coreset for P with respect to f.

Remark A.4 (VC dimension w.r.t. RBF/Laplacian fitting problems). In the context of RBF and Laplacian fitting problems, the corresponding VC dimension is $O(d)$ where d denotes the dimensionality (number of features) of the queries.

B. Proofs for our Main Theorems

B.1. Proof of Claim 3.1.2

Let $p \in \mathbb{R}^d$ such that $\|p\|_2 \leq 1$, and let $R > 0$ be positive real number. Then for every $x \in \left\{x \in \mathbb{R}^d \middle| \|x\|_2 \leq R\right\}$, $\frac{1}{e^R(1+R)}\left(1 + \left|p^T x\right|\right) \leq e^{-\left|p^T x\right|} \leq \left|p^T x\right| + 1$.

Proof. Put $x \in \left\{x \in \mathbb{R}^d \middle| \|x\|_2 \leq R\right\}$ and note that if $p^T x = 0$ then the claim is trivial. Otherwise, we observe that

$$e^{-\left|p^T x\right|} \geq \frac{1}{e^R} \geq \frac{1 + \left|p^T x\right|}{(1 + R)\, e^R}.$$

□

B.2. Proof of Theorem 3.2

Let $a, b \geq 0$ be pair of nonnegative real numbers and let $c, d > 0$ be a pair of positive real numbers. Then

$$\frac{a + b}{c + d} \leq \frac{a}{c} + \frac{b}{d}.$$

Proof. Observe that

$$\frac{a + b}{c + d} = \frac{a}{c + d} + \frac{b}{c + d} \leq \frac{a}{c} + \frac{b}{d}$$

where the inequality holds since $c, d > 0$. □

Lemma B.1 (Sensitivity bound w.r.t. the *RBF* loss function)**.** *Let $R \geq 1$ be a positive real number, and let $X = \left\{x \in \mathbb{R}^d \middle| \|x\|_2 \leq R\right\}$. Let $\left(P, w, \mathbb{R}^d, f\right)$ be query space as in Definition A.1 where for every $p \in P$ and $x \in \mathbb{R}^d$, $f(p,x) = e^{-\|p-x\|_2^2}$. Let $P' := \left\{q_p = \left[\|p\|_2^2, -2p^T, 1\right]^T \mid \forall p \in P\right\}$ and let $q^* \in \arg\sup_{q \in P'} e^{3R^2\|q\|_2}\left(1 + 3R^2\|q\|_2\right)$. Let $u(p) := \frac{w(p)}{e^{3R^2\|q_p\|_2}\left(1+3R^2\|q_p\|_2\right)}$ and let (U, D, V) be the $\|\cdot\|_1$-SVD of $(P', u(\cdot))$. Then for every $p \in P$,*

$$s(p) \leq e^{3R^2\|q_p\|_2}\left(1 + 3R^2\|q_p\|_2\right)\left(\frac{u(p)}{\sum\limits_{p' \in P} u(p')} + u(p)\left\|U(q_p)\right\|_1\right),$$

and

$$\sum_{q \in P} s(q) \leq e^{3R^2\|q^*\|_2}\left(1 + 3R^2\|q^*\|_2\right)\left(1 + (d+2)^{1.5}\right).$$

Proof. Let $X = \left\{x \in \mathbb{R}^d \middle| \|x\|_2 \leq 1\right\}$, and observe that for every $p \in P$ and $x \in \mathbb{R}^d$, it holds that

$$\|p - x\|_2^2 = \left|q_p^T y\right|, \tag{3}$$

where $q_p = \left[\|p\|_2^2, -2p^T, 1\right]^T$ and $y = \left[1, x, \|x\|_2^2\right]^T$.

Let $Y = \left\{\left[1, x, \|x\|_2^2\right]^T \middle| x \in X\right\}$. Following the definition of Y, for every $y \in Y$, we obtain that $\|y\|_2 \leq 3R^2$. Hence, by plugging $p := \frac{q_p}{\|q_p\|}$ and $R := 3R^2\|q_p\|$ for every $p \in P$ into Claim 3.1.2, we obtain that the for every $y \in Y$ and $p \in P$,

$$\frac{1}{e^{3R^2\|q_p\|_2}\left(1 + 3R^2\|q_p\|_2\right)}\left(1 + \left|q_p^T y\right|\right) \leq e^{-\left|q_p^T y\right|} \leq 1 + \left|q_p^T y\right|. \tag{4}$$

Note that for every $p \in P$, $u(p) := \frac{w(p)}{e^{3\|q_p\|_2}\left(1+3\|q_p\|_2\right)}$. Thus

$$\begin{aligned}
&\sup_{x \in X} \frac{w(p)f(p,x)}{\sum\limits_{q \in P} w(q)f(q,x)} \\
&= \sup_{y \in Y} \frac{w(p)e^{-\left|q_p^T y\right|}}{\sum\limits_{p' \in P'} w(q)\left|q_{p'}^T y\right|} \\
&\leq e^{3R^2\|q_p\|_2}\left(1 + 3R^2\|q_p\|_2\right)\sup_{y \in Y} \frac{u(p)\left|q_p^T y\right| + u(p)}{\sum\limits_{p' \in P} u(p')\left|q_{p'}^T y\right| + \sum\limits_{p' \in P} u(p')} \\
&\leq e^{3R^2\|q_p\|_2}\left(1 + 3R^2\|q_p\|_2\right)\left(\frac{u(p)}{\sum\limits_{p' \in P} u(p')} + \sup_{y \in Y} \frac{u(p)\left|q_p^T y\right|}{\sum\limits_{p' \in P} u(p')\left|q_{p'}^T y\right|}\right),
\end{aligned} \tag{5}$$

where the first inequality holds by Claim 3.1.2 and the second inequality is by Claim B.2.

Let $f : P' \times Y \to [0, \infty)$ be a function such that for every $q \in P'$ and $y \in Y$, $f(q,y) = \left|q^T y\right|$. Plugging in $P := P'$, $w := u$, $d := d + 2$, and $f := f$ into Lemma 2.3 yields for every $p \in P$

$$\sup_{x \in X} \frac{w(p)f(p,x)}{\sum\limits_{q \in P} w(q)f(q,x)} \leq e^{3R^2\|q_p\|_2}\left(1 + 3R^2\|q_p\|_2\right)\left(\frac{u(p)}{\sum\limits_{p' \in P} u(p')} + u(p)\left\|U(q_p)\right\|_1\right). \tag{6}$$

Note that by definition, $q^* \in \arg\sup_{q \in P'} e^{3R^2\|q\|_2}\left(1+3R^2\|q\|_2\right)$. Then the total sensitivity is bounded by

$$\sum_{q\in P} \sup_{x\in X} \frac{w(p)f(p,x)}{\sum\limits_{q\in P} w(q)f(q,x)} \leq e^{3R^2\|q^*\|_2}\left(1+3R^2\|q^*\|_2\right)\left(1+(d+2)^{1.5}\right). \tag{7}$$

□

Theorem 3.2 (Coreset for *RBF*)**.** *Let $R \geq 1$ be a positive real number, $X = \left\{x \in \mathbb{R}^d \middle| \|x\|_2 \leq R\right\}$, and let $\varepsilon, \delta \in (0,1)$. Let (P, w, X, f) be query space as in Definition A.1 such that every $p \in P$ satisfies $\|p\|_2 \leq 1$. For every $x \in X$ and $p \in P$, let $f(p,x) := \rho\left(\|p-x\|_2\right)$. Let (S,v) be a call to* Coreset(P,w,R,m) *where $S \subseteq P$ and $v : S \to [0,\infty)$. Then (S,v) ε-coreset of (P,w) with probability at least $1-\delta$, if $m = O\left(\frac{e^{12R^2}R^2d^{1.5}}{\varepsilon^2}\left(R^2+\log d+\log\frac{1}{\delta}\right)\right)$.*

Proof. First , by plugging in the query space $(P, w, \mathbb{R}^d, f)$ into Lemma B.1, we obtain that a bound on the sensitivities $s(p)$ for every $p \in P$ and a bound on the total sensitivities $t := e^{12R^2}\left(1+12R^2\right)\left(1+(d+2)^{1.5}\right)$, since the $\max_{q\in P}\|q\|_2 \leq 1$. Notice that the analysis done in Lemma B.1 is analogues to the steps done in Algorithm 1.

By plugging the bounds on the sensitivities, the bound on the total sensitivity t, probability of failure $\delta \in (0,1)$, and approximation error $\varepsilon \in (0,1)$ into Theorem A.3, we obtain a subset $S' \subseteq Q$ and $v' : S' \to [0,\infty)$ such that the tuple (S', v') is an ε-coreset for (P,w) with probability at least $1-\delta$. □

B.3. Proof of Theorem 3.3

Lemma B.2 (Sensitivity bound w.r.t. the *Laplacian* loss function)**.** *Let $\left(P, w, \mathbb{R}^d, f\right)$ be query space as in Definition A.1 where for every $p \in P$ and $x \in \mathbb{R}^d$, $f(p,x) = e^{-\|p-x\|_2}$. Let $P' := \left\{q_p = \left[\|p\|_2^2, -2p^T, 1\right]^T \mid \forall p \in P\right\}$ and let $q^* \in \arg\sup_{q\in P'} e^{3\sqrt{\|q\|_2}}\left(1+3\sqrt{\|q\|_2}\right)$. Let $u(p) := \frac{w(p)}{e^{3\sqrt{\|q_p\|_2}}\left(1+3\sqrt{\|q_p\|_2}\right)}$ and let (U,D,V) be the $\|\cdot\|_1$-SVD of $\left(P', u^2(\cdot)\right)$. Then for every $p \in P$,*

$$s(p) \leq e^{3\sqrt{\|q_p\|_2}}\left(1+3\sqrt{\|q_p\|_2}\right)\left(\frac{u(p)}{\sum\limits_{p'\in P} u(p')} + u(p)\sqrt{\|U(q_p)\|_1}\right) + \frac{e^{\|p\|_2+\sqrt{\|q^*\|_2}}w(p)}{\sum\limits_{q\in P} w(q)},$$

and

$$\sum_{q\in P} s(q) \leq 2e^{3\sqrt{\|q^*\|_2}} + e^{3\sqrt{\|q^*\|_2}}\left(1+3\sqrt{\|q^*\|_2}\right)\left(1+\sqrt{n}\,(d+2)^{1.25}\right).$$

Proof. Let $X = \left\{x \in \mathbb{R}^d \middle| \|x\|_2 \leq 1\right\}$, and observe that for every $p \in P$ and $x \in \mathbb{R}^d$, it holds that

$$\|p-x\|_2 = \sqrt{\left|q_p^T y\right|}, \tag{8}$$

where $q_p = \left[\|p\|_2^2, -2p^T, 1\right]^T$ and $y = \left[1, x, \|x\|_2^2\right]^T$.

Let $Y = \left\{\left[1, x, \|x\|_2^2\right]^T \middle| x \in X\right\}$. Hence, following Theorem A.3, the sensitivity of each point $p \in P$, can be rewritten as

$$\sup_{x\in\mathbb{R}^d} \frac{w(p)f(p,x)}{\sum\limits_{q\in P} w(q)f(q,x)} \leq \sup_{x\in X} \frac{w(p)f(p,x)}{\sum\limits_{q\in P} w(q)f(q,x)} + \sup_{x\in\mathbb{R}^d\setminus X} \frac{w(p)f(p,x)}{\sum\limits_{q\in P} w(q)f(q,x)}. \tag{9}$$

From here, we bound the sensitivity with respect to subspaces of $\mathbb{R}^d$.

Handling queries from X. Following the definition of Y, for every $y \in Y$, we obtain that $\|y\|_2 \leq 3$. Hence, by plugging $p := q_p$ for every $p \in P$ and $R := 3\|q_p\|_2$ into Claim 3.1.2, we obtain that the for every $y \in Y$ and $p \in P$,

$$\frac{1}{e^{3\sqrt{\|q_p\|_2}}\left(1+3\sqrt{\|q_p\|_2}\right)}\left(1+\sqrt{\left|q_p^T y\right|}\right) \leq e^{-\sqrt{\left|q_p^T y\right|}} \leq 1+\sqrt{\left|q_p^T y\right|}. \tag{10}$$

Note that for every $p \in P$, $u(p) := \frac{w(p)}{e^{3\sqrt{\|q_p\|_2}}\left(1+3\sqrt{\|q_p\|_2}\right)}$. Combining equation 9 and equation 10, yields that

$$\begin{aligned}\sup_{x\in X}\frac{w(p)f(p,x)}{\sum\limits_{q\in P}w(q)f(q,x)} &\leq e^{3\sqrt{\|q_p\|_2}}\left(1+3\sqrt{\|q_p\|_2}\right)\sup_{y\in Y}\frac{u(p)\sqrt{\left|q_p^T y\right|}+u(p)}{\sum\limits_{p'\in P}u(p')\sqrt{\left|q_{p'}^T y\right|}+\sum\limits_{p'\in P}w(p')}\\ &\leq e^{3\sqrt{\|q_p\|_2}}\left(1+3\sqrt{\|q_p\|_2}\right)\left(\frac{u(p)}{\sum\limits_{p'\in P}u(p')}+\sup_{y\in Y}\frac{u(p)\sqrt{\left|q_p^T y\right|}}{\sum\limits_{p'\in P}u(p')\sqrt{\left|q_{p'}^T y\right|}}\right),\end{aligned} \tag{11}$$

where the first inequality holds by Claim 3.1.2 and the second inequality is by Claim B.2.

By Cauchy-Schwartz inequality,

$$\begin{aligned}\sup_{y\in Y}\frac{u(p)\sqrt{\left|q_p^T y\right|}}{\sum\limits_{p'\in P}u(p')\sqrt{\left|q_{p'}^T y\right|}} &= \sup_{y\in Y}\frac{\sqrt{u(p)^2\left|q_p^T y\right|}}{\sum\limits_{p'\in P}\sqrt{u(p')^2\left|q_{p'}^T y\right|}}\\ &\leq \sup_{y\in Y}\frac{\sqrt{u(p)^2\left|q_p^T y\right|}}{\sqrt{\sum\limits_{p'\in P}u(q)^2\left|q_{p'}^T y\right|}}\\ &\leq \sup_{y\in\mathbb{R}^{d+2}}\frac{\sqrt{u(p)^2\left|q_p^T y\right|}}{\sqrt{\sum\limits_{p'\in P}u(q)^2\left|q_{p'}^T y\right|}},\end{aligned} \tag{12}$$

where the last inequality follows from the properties associated with the supremum operation.

Let $u' : P' \to [0,\infty)$ be a weight function such that for every $p \in P$, $u'(q_p) = u(p)^2$, $f : P' \times Y \to [0,\infty)$ be a function such that for every $q \in P'$ and $y \in Y$, $f(q,y) = \left|q^T y\right|$. Plugging in $P := P'$, $w := u$, $d := d+2$, and $f := f$ into Lemma 2.3 yields for every $p \in P$

$$\sup_{x\in X}\frac{w(p)f(p,x)}{\sum\limits_{q\in P}w(q)f(q,x)} \leq e^{3\sqrt{\|q_p\|_2}}\left(1+3\sqrt{\|q_p\|_2}\right)\left(\frac{u(p)}{\sum\limits_{p'\in P}u(p')}+u(p)\sqrt{\|U(q_p)\|_1}\right). \tag{13}$$

Note that by definition, $q^* \in \arg\sup\limits_{q\in P'} e^{3\sqrt{\|q_p\|_2}}\left(1+3\sqrt{\|q_p\|_2}\right)$. Then the total sensitivity is bounded by

$$\sum_{q\in P}\sup_{x\in X}\frac{w(p)f(p,x)}{\sum\limits_{q\in P}w(q)f(q,x)} \leq e^{3\sqrt{\|q^*\|_2}}\left(1+3\sqrt{\|q^*\|_2}\right)\left(1+\sqrt{n}\,(d+2)^{1.25}\right), \tag{14}$$

where the $\sqrt{n}$ follows from $\sum_{p'\in P}\sqrt{\|U(q_{p'})\|_1} \leq \sqrt{n}\sqrt{\sum_{p'\in P}\|U(q_{p'})\|_1}$, which is used when using Lemma 2.3. This inequality is a result of Cauchy-Schwartz's inequality.

Handling queries from $\mathbb{R}^d \setminus X$. First, we observe that for any integer $m \geq 1$ and $x, y \in \mathbb{R}^m$,

$$-\|x\|_2 - \|y\|_2 \leq -\|x-y\|_2 \leq \|x\|_2 - \|y\|_2 \,, \tag{15}$$

where the first inequality holds by the triangle inequality, and the second inequality follows from the reverse triangle inequality.

Thus, by letting $x_p \in \underset{x \in \mathbb{R}^d \setminus X}{\arg\sup} \frac{w(p)f(p,x)}{\sum\limits_{q\in P} w(q)f(q,x)}$ for every $p \in P$, we obtain that

$$\frac{w(p)f\left(p,x_p\right)}{\sum\limits_{q\in P} w(q)f\left(q,x_p\right)} \leq \frac{w(p)e^{\|p\|_2 - \|x_p\|_2}}{\sum\limits_{q\in P} w(q)e^{-\|q\|_2-\|x_p\|_2}} \leq \frac{w(p)e^{\|p\|_2 - \|x_p\|_2}}{\sum\limits_{q\in P} w(q)e^{-\sqrt{\|q^*\|_2}-\|x_p\|_2}} = \frac{e^{\|p\|_2 + \sqrt{\|q^*\|_2}}w(p)}{\sum\limits_{q\in P} w(q)}, \tag{16}$$

where the first inequality holds by equation 15, and the second inequality holds since $\sqrt{\|q^*\|_2} \geq \|p\|_2$ for every $p \in P$.

Combining equation 9, equation 13, equation 14, and equation 16, yields that for every $p \in P$

$$s(p) \leq e^{3\sqrt{\|q_p\|_2}}\left(1 + 3\sqrt{\|q_p\|_2}\right)\left(\frac{u(p)}{\sum\limits_{p'\in P} u\left(p'\right)} + u(p)\sqrt{\left\|U\left(q_p\right)\right\|_1}\right) + \frac{e^{\|p\|_2+\sqrt{\|q^*\|_2}}w(p)}{\sum\limits_{q\in P} w(q)}$$

and

$$\sum_{p\in P} s(p) \leq 2e^{3\sqrt{\|q^*\|_2}} + e^{3\sqrt{\|q^*\|_2}}\left(1+3\sqrt{\|q^*\|_2}\right)\left(1+\sqrt{n}\left(d+2\right)^{1.25}\right).$$

□

Theorem 3.3 (Coreset for the Laplacian loss function)**.** *Let $\left(P, w, \mathbb{R}^d, f\right)$ be query space as in Definition A.1 such that every $p \in P$ satisfies $\|p\|_2 \leq 1$. For $x \in \mathbb{R}^d$ and $p \in P$, let $f(p,x) := e^{-\|p-x\|_2}$. Let $\varepsilon, \delta \in (0,1)$. Then there exists an algorithm which given $P, w, \varepsilon, \delta$ return a weighted set (S,v) where $S \subseteq P$ of size $O\left(\frac{\sqrt{n}d^{1.25}}{\varepsilon^2}\left(\log n + \log d + \log\frac{1}{\delta}\right)\right)$ and a weight function $v : S \to [0,\infty)$ such that (S,v) is an ε-coreset of (P,w) with probability at least $1-\delta$.*

Proof. Plugging in the query space $(P, w, \mathbb{R}^d, f)$ into Lemma B.2, we obtain that a bound on the sensitivities $s(p)$ for every $p \in P$ and a bound on the total sensitivities $t := 2e^5 + 3e^5\left(1+5\right)\left(1+\sqrt{n}\left(d+2\right)^{1.5}\right)$, since the $\max_{q\in P}\|q\|_2 \leq 1$. By plugging the bounds on the sensitivities, the bound on the total sensitivity t, probability of failure $\delta \in (0,1)$, and approximation error $\varepsilon \in (0,1)$ into Theorem A.3, we obtain a subset $S' \subseteq Q$ and $v' : S' \to [0,\infty)$ such that the tuple (S', v') is an ε-coreset for (P,w) with probability at least $1-\delta$. □

B.4. Proof of Theorem 4.1

To prove Theorem 4.1, we first prove the following theorem.

Theorem B.3. *There exists an algorithm that samples $O\left(\frac{e^{8R^2}R^2d^{1.5}}{\varepsilon^2}\left(R^2 + \log d + \log\frac{2}{\delta}\right)\right)$ points to form weighted sets (S_1, w_1) and (S_2, w_2) such that with probability at least $1-2\delta$,*

$$\frac{\left|\sum\limits_{p\in P} y(p)\rho(\|p-c^{(i)}\|_2) - \left(\sum\limits_{p\in S_1} w_1(p)e^{-\left\|p-c^{(i)}\right\|_2^2} + \sum\limits_{q\in S_2} w_2(q)e^{-\left\|q-c^{(j)}\right\|_2^2}\right)\right|}{\left|\phi^+\left(c^{(i)}\right) + \phi^-\left(c^{(i)}\right)\right|} \leq \varepsilon.$$

Proof. We first construct strong coresets for both ϕ^+ and ϕ^-. If $\|\mathbf{c}_i\|_2 \leq R$ for all $i \in [n]$, then by Theorem 3.2, it suffices to sample a weighted weight S of size $O\left(\frac{e^{12R^2}R^2d^{1.5}}{\varepsilon^2}\left(R^2 + \log d + \log\frac{2}{\delta}\right)\right)$ to achieve a strong coreset S_1 with corresponding weighting function w_1 for ϕ^+ with probability at least $1-\frac{\delta}{2}$. Similarly, we obtain a strong coreset for ϕ^- with probability at least $1-\frac{\delta}{2}$ by sampling a set S_2 with weights w_2 of size $O\left(\frac{e^{8R^2}R^2d^{1.5}}{\varepsilon^2}\left(R^2 + \log d + \log\frac{2}{\delta}\right)\right)$.

Hence by the definition of a strong coreset, we have that

$$\begin{aligned}\left|\sum_{\substack{p\in P\\y(p)>0}} y(p)\rho(\|p-c^{(i)}\|_2) - \sum_{p\in S_1} w_1(p)\rho(\|p-c^{(i)}\|_2)\right| &\leq \varepsilon \sum_{\substack{p\in P\\y(p)>0}} y(p)\rho(\|p-c^{(i)}\|_2)\\ &= \varepsilon\phi^+(c^{(i)})\end{aligned}$$

and

$$\begin{aligned}\left|\sum_{\substack{p\in P\\y(p)<0}} y(p)\rho(\|p-c^{(i)}\|_2) - \sum_{q\in S_2} w_2(q)\rho(\|q-c^{(i)}\|_2)\right| &\leq \varepsilon \sum_{\substack{p\in P\\y(p)<0}} |y(p)|\rho(\|p-c^{(i)}\|_2)\\ &= \varepsilon\phi^-(c^{(i)})\end{aligned}$$

for any input $\mathbf{x}\in\mathbb{R}^n$.

Thus by triangle inequality and a slight rearrangement of the inequality, we have that with probability at least $1-\delta$,

$$\frac{\left|\sum_{p\in P} y(p)\rho(\|p-c^{(i)}\|_2) - \left(\sum_{p\in S_1} w_1(p)e^{-\left\|p-c^{(i)}\right\|_2^2} + \sum_{q\in S_2} w_2(q)e^{-\left\|q-c^{(j)}\right\|_2^2}\right)\right|}{\left|\phi^+\left(c^{(i)}\right)+\phi^-\left(c^{(i)}\right)\right|} \leq \varepsilon.$$

□

To prove Theorem 4.1, we split α_i into the sets $\{i|\alpha_i>0\}$ and $\{i|\alpha_i<0\}$ and apply Theorem B.3 to each of the sets. We emphasize that this argument is purely for the purposes of analysis so that the algorithm itself does not need to partition the α_i quantities (and so the algorithm does not need to recompute the coreset when the values of the weights α_i change over time). We thus get the following guarantee:

Theorem 4.1. *There exists an algorithm that samples $O\left(\frac{e^{8R^2}R^2d^{1.5}}{\varepsilon^2}\left(R^2+\log d+\log\frac{2}{\delta}\right)\right)$ points to form weighted sets (S_1,w_1) and (S_2,w_2) such that with probability at least $1-2\delta$,*

$$\frac{\left|\sum_{p\in P} y(p)\phi(p) - \left(\sum_{\substack{i\in[L]\\\alpha_i>0}} \alpha_i\gamma_{S_1} + \sum_{\substack{j\in[L]\\\alpha_j<0}} \alpha_j\gamma_{S_2}\right)\right|}{\sum_{i\in[L]} |\alpha_i|\left(\phi^+\left(c^{(i)}\right)+\phi^-\left(c^{(i)}\right)\right)} \leq \varepsilon,$$

where $\gamma_{S_1} := \sum_{p\in S_1} w_1(p)e^{-\left\|p-c^{(i)}\right\|_2^2}$ and $\gamma_{S_2} := \sum_{q\in S_2} w_2(q)e^{-\left\|q-c^{(j)}\right\|_2^2}$.

B.5. Proof sketch of Corollary 4.2

Corollary B.4. *Let* RFBNN *be a sufficiently large (RBF) neural network, $P\subset\mathbb{R}^d$, and let $f(p):P\to\mathbb{R}$ be a function that can be approximated by a set of centers and weights q (the learned parameters that define the RBFNN) assigned to $RFBNN$ such that, for every $p\in P$: $RFBNN(p,q)\approx f(p)$, and $\left\|\sum_{p\in P} RFBNN(p,q) - \sum_{p\in P} f(p)\right| \leq \varepsilon$, where each center of the $RBFNN$ lies in ball centered at the origin with radius R (to satisfy our coreset's assumption). Let (S,v) be the output of a call to Algorithm 1, i.e., (S,v) is coreset for P is an RBF coreset; see Definition 2.1. Then,*

$$\left|\sum_{p\in P} f(p) - \sum_{s\in S} v(s)RBFNN(s,q)\right| \leq \varepsilon\sum_{p\in P} f(p)+\varepsilon.$$

Proof. First using our RBF coreset properties and the definition of $RBFNN$ which is a weighted sum of RBFs, we easily show that $\left|\sum_{s\in S} v(s)RBFNN(s,q) - \sum_{p\in P} RBFNN(p,q)\right| \leq \varepsilon \sum_{p\in P} RBFNN(p,q)$.

Combing the above with the fact that $\sum_{p\in P} RBFNN(p,q)$ approximates $\sum_{p\in P} f(p)$ yields our result. □

C. Lower bound on the coreset size for the Gaussian loss function - illustration

Here, we illustrate one dataset such that for any approximation ε, the ε-coreset must contain at least half the points to ensure the desired approximation from a theoretical point of view.

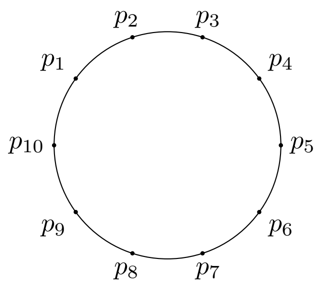

Figure 4. Evenly distributed points on some circle where the minimal distance between each point and any other point is at least $\sqrt{\ln n}$, where in this example $n = 10$.

D. Coresets and Function Approximation

The weight function w in the query space refers to a function that assigns weights to each input in the input set. These weights can be used to indicate the relative importance of each input in the analysis or experiment.

In traditional supervised learning, the weight function is not typically used, as the focus is on learning a function that maps inputs x to outputs y. Usually, all input examples are equally important. Notably, in some cases, such as when dealing with imbalanced datasets or when certain inputs are more informative than others, the use of weights can improve the performance of the learning algorithm. In general, the role of the weight function in analysis and experiments is to provide a mechanism for incorporating prior knowledge into the learning process. By assigning weights to the input set, we can guide the learning algorithm to focus on certain inputs of the input space (or certain features of the inputs in other cases), which can lead to improved performance.

In the context of function approximation via coresets, the weight function is used to give greater weight to some chosen coreset points that represent a larger quantity of the input data for the task at hand. For example, if the function being approximated has many points (denoted by Q) with a similar cost on the given queries, then, a point from this set Q will hopefully be chosen in the coreset as a representative for the whole set Q, and the weight function may assign a greater weight to this point to ensure that the approximation captures the other points not included in the coreset, that are close to this coreset point. In the deep learning experiments, once we sample coresets and compute corresponding weights, we can define the loss of the trained neural network, as the **weighted mean** of the losses of the computed coreset, instead of using uniform weights when updating each batch.

E. Gradient Approximation via Our Coreset for RBFNNs

Radial basis function (RBF) neural networks are a type of neural network that use radial basis functions as activation functions. Given a "large enough" RBF network, it can approximate any continuous function to arbitrary precision. In other words, let $RFBNN$ be a sufficiently large (RBF) neural network, and let $f(p) : P' \to \mathbb{R}$ where $P' \subset \mathbb{R}^d$, then, there exists a set of centers and weights (the learned parameters that define the RBFNN) q to assign *RFBNN* such that for every $x \in X$, RFBNN$(p,q) \approx f(p)$.

Focusing on the derivative of single parameters/weight in a given DNN in a given step of the training process, the derivative of this weight can be defined as function $g(p)$, where p here is a vector representation of an input example for the deep neural network. Thus, if $RBFNN$ is large enough, then there exists a set of centers and weights q to assign $RFBNN$ such that:

- For every $p \in P'$: $RFBNN(p, q) \approx g(p)$ —P' is the space where the input may come from. This leads to the following.
- For every $\sum_{p\in P} RFBNN(p, q) \approx \sum_{p\in P} g(p)$ — P is the input training data of size n.

Now, according to the definition of coreset for RBFNN, given a set of inputs P and an RBFNN (structure), a coreset for P with respect to RBFNN, is a pair (S, v), where $S \subset P$, and $v : S \to R$ is a weight function such that for any set of centers and weights q to assign $RFBNN$, we have

$$\sum_{p\in P} RFBNN(p, q) \approx \sum_{s\in S} v(s)RFBNN(s, q).$$

Finally, put q to be the set of centers and weights (of RBFNN) that approximates some desired derivative $g(x)$ of a weight in the neural network. Combining all of the above yields

$$\sum_{p\in P} g(p) \approx \sum_{p\in P} RFBNN(p, q) \approx \sum_{s\in S} v(s)RFBNN(s, q).$$

F. Experimental Results - Extended

F.1. Intuition behind re-sampling the coreset

Theoretically speaking, our coreset is independent of the model and can be computed once to all epochs. However, practically speaking, to improve the practical results we construct a new coreset after Υ epochs to expose the model to new samples and thus ensure a model with better generalization capabilities. We note the following very important comments:

- Once we compute the sensitivity, the sampling process takes ~ 0 time. Thus, it is free to sample a new coreset, even after each epoch.
- Since we rely on sensitivity sampling, then our samples across different epochs should be similar, as high sensitivity points, will be sampled with a high probability. However, the sampled points which have low sensitivity may be changed across different samples/epochs, thus, making sure we still capture the real important samples, while we get a wider diversity in the less important samples.

Finally, we note that we could have conducted experiments for sampling after each epoch, as it takes zero time to do that, and we would have achieved higher test accuracy results. However, we did not report these results to ensure a fair comparison with our competitors that cannot afford to compute a new coreset every epoch due to time constraints.

F.2. MNIST results

In what follows, we present our subset selection results on the MNIST dataset at Table 6.

F.3. Standard deviation and statistical significance results

Tables 7–9 show the standard deviation results over five training runs on CIFAR10, CIFAR100, and MNIST datasets, respectively.

Table 6. Data Selection Results for MNIST using LeNet

	Top-1 Test accuracy of the Model(%)				Model Training time(in hrs)			
Budget(%)	1%	3%	5%	10%	1%	3%	5%	10%
FULL (skyline for test accuracy)	99.35	99.35	99.35	99.35	0.82	0.82	0.82	0.82
RANDOM (skyline for training time)	94.55	97.14	97.7	98.38	0.0084	0.03	0.04	0.084
RANDOM-WARM (skyline for training time)	98.8	99.1	99.1	99.13	0.0085	0.03	0.04	0.085
GLISTER	93.11	98.062	99.02	99.134	0.045	0.0625	0.082	0.132
GLISTER-WARM	97.63	98.9	99.1	99.15	0.04	0.058	0.078	0.127
CRAIG	96.18	96.93	97.81	98.7	0.3758	0.4173	0.434	0.497
CRAIG-WARM	98.48	98.96	99.12	99.14	0.2239	0.258	0.2582	0.3416
CRAIGPB	97.72	98.47	98.79	99.05	0.08352	0.106	0.1175	0.185
CRAIGPB-WARM	98.47	99.08	99.01	99.16	0.055	0.077	0.0902	0.1523
GRADMATCH	98.954	99.174	99.214	99.24	0.05	0.0607	0.097	0.138
GRADMATCH-WARM	98.86	99.22	99.28	99.29	0.046	0.057	0.089	0.132
GRADMATCHPB	98.7	99.1	99.25	99.27	0.04	0.051	0.07	0.11
GRADMATCHPB-WARM	**99.0**	**99.23**	99.3	99.31	0.038	0.05	0.065	0.10
RBFNN CORESET (OURS)	98.98	99.2	**99.31**	**99.32**	0.028	0.051	0.062	0.098

Table 7. Data Selection Results for CIFAR10 using ResNet-18: Standard deviation of the Model (for 5 runs)

	Standard deviation of the Model(for 5 runs)			
Budget(%)	5%	10%	20%	30%
FULL (skyline for test accuracy)	0.032	0.032	0.032	0.032
RANDOM (skyline for training time)	0.483	0.518	0.524	0.538
RANDOM-WARM (skyline for training time)	0.461	0.348	0.24	0.1538
GLISTER	0.453	0.107	0.046	0.345
GLISTER-WARM	0.325	0.086	0.135	0.129
CRAIG	0.289	0.2657	0.1894	0.1647
CRAIG-WARM	0.123	0.1185	0.1058	0.1051
CRAIGPB	0.152	0.1021	0.086	0.064
CRAIGPB-WARM	0.0681	0.061	0.0623	0.0676
GRADMATCH	0.192	0.123	0.112	0.1023
GRADMATCH-WARM	0.1013	0.1032	0.091	0.1034
GRADMATCHPB	0.0581	0.0571	0.0542	0.0584
GRADMATCHPB-WARM	0.0542	0.0512	0.0671	0.0581
RBFNN CORESET (OURS)	0.25	0.2	0.17	0.13
RBFNN CORESET-WARM (OURS)	0.21	0.16	0.13	0.12

Table 8. Data Selection Results for CIFAR100 using ResNet-18: Standard deviation of the Model (for 5 runs)

	Standard deviation of the Model(for 5 runs)			
Budget(%)	5%	10%	20%	30%
FULL (skyline for test accuracy)	0.051	0.051	0.051	0.051
RANDOM (skyline for training time)	0.659	0.584	0.671	0.635
RANDOM-WARM (skyline for training time)	0.359	0.242	0.187	0.175
GLISTER	0.463	0.15	0.061	0.541
GLISTER-WARM	0.375	0.083	0.121	0.294
CRAIG	0.3214	0.214	0.195	0.187
CRAIG-WARM	0.18	0.132	0.125	0.115
CRAIGPB	0.12	0.134	0.123	0.115
CRAIGPB-WARM	0.1176	0.1152	0.1128	0.111
GRADMATCH	0.285	0.176	0.165	0.156
GRADMATCH-WARM	0.140	0.134	0.142	0.156
GRADMATCHPB	0.104	0.111	0.105	0.097
GRADMATCHPB-WARM	0.093	0.101	0.100	0.098
RBFNN CORESET (OURS)	0.3	0.19	0.18	0.16
RBFNN CORESET-WARM (OURS)	0.19	0.14	0.11	0.1

Table 9. Data Selection Results for MNIST using LeNet: Standard deviation of the Model (for 5 runs)

	Standard deviation of the Model(for 5 runs			
Budget(%)	1%	3%	5%	10%
FULL (skyline for test accuracy)	0.012	0.012	0.012	0.012
RANDOM (skyline for training time)	0.215	0.265	0.224	0.213
RANDOM-WARM (skyline for training time)	0.15	0.121	0.110	0.103
GLISTER	0.256	0.218	0.145	0.128
GLISTER-WARM	0.128	0.134	0.119	0.124
CRAIG	0.186	0.178	0.162	0.125
CRAIG-WARM	0.0213	0.0223	0.0196	0.0198
CRAIGPB	0.021	0.0209	0.0216	0.0204
CRAIGPB-WARM	0.023	0.0192	0.0212	0.0184
GRADMATCH	0.156	0.128	0.135	0.12
GRADMATCH-WARM	0.087	0.084	0.0896	0.0815
GRADMATCHPB	0.0181	0.0163	0.0147	0.0129
GRADMATCHPB-WARM	0.0098	0.012	0.0096	0.0092
RBFNN CORESET (OURS)	0.18	0.13	0.11	0.1
RBFNN CORESET-WARM (OURS)	0.09	0.08	0.05	0.01