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Small data well-posedness for derivative nonlinear Schrödinger equations

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Abstract

We study the local and global solutions of the generalized derivative nonlinear Schrödinger equation $i\partial_t u + \Delta u = P(u, \overline{u}, \partial_x u, \partial_x \overline{u})$, where each monomial in P is of degree 3 or higher, in low-regularity Sobolev spaces without using a gauge transformation. Instead, we use a solution decomposition technique introduced in [4] during the perturbative argument to deal with the loss on derivative in nonlinearity. It turns out that when each term in P contains only one derivative, the equation is locally well-posed in $H^{\frac{1}{2}}$, otherwise we have a local well-posedness in $H^{\frac{3}{2}}$. If each monomial in P is of degree 5 or higher, the solution can be extended globally. By restricting to equations to the form $i\partial_t u + \Delta u = \partial_x P(u, \overline{u})$ with the quintic nonlinearity, we were able to obtain the global well-posedness in the critical Sobolev space.
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1. Introduction

In this paper, we study the well-posedness of the Cauchy problem for the generalized derivative nonlinear Schrödinger equation (gDNLS) on $\mathbb{R}$.

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$$\begin{cases} i\partial_t u + \Delta u = P(u, \overline{u}, \partial_x u, \partial_x \overline{u}) \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \geq s_0. \end{cases} \tag{1}$$

Here, u is a complex-valued function and $P : \mathbb{C}^4 \to \mathbb{C}$ is a polynomial of the form

$$P(z) = P(z_1, z_2, z_3, z_4) = \sum_{d \leq |\alpha| \leq l} C_\alpha z^\alpha, \tag{2}$$

and $l \geq d \geq 3$. There are several results regarding the well-posedness of this equation. In [19], Kenig, Ponce and Vega proved that the equation (1) is locally well-posed for a small initial data in $H^{\frac{7}{2}}(\mathbb{R})$. There has been some interest in the special case where $P = i\lambda|u|^k u_x$:

$$\begin{cases} i\partial_t u + \Delta u = i\lambda|u|^k u_x \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \geq s_0, \end{cases}$$

with $k \in \mathbb{R}$. Hao ([13]) proved that this equation is locally well-posed in $H^{\frac{1}{2}}(\mathbb{R})$ for $k \geq 5$, and Ambrose–Simpson ([1]) proved the result in $H^1(\mathbb{R})$ for $k \geq 2$. Recent studies show that these results can be improved. See Santos ([25]) for the local-wellposedness in $H^{\frac{1}{2}}$ when $k \geq 2$ and Hayashi–Ozawa ([14]) for the local well-posedness in H^2 when $k \geq 1$ and the global well-posedness in H^1 when $k \geq 2$.

Several studies showed that we have better results if P only consists of $\overline{u}$ and $\partial_x \overline{u}$ due to the following heuristic: if u solves the linear Schrödinger equation, then the space–time Fourier transform of $\overline{u}$ is supported away from the parabola $\{(\xi, \tau)|\tau + \xi^2 = 0\}$, leading to strong dispersive estimates. Grünrock ([12]) showed that for $P = \partial_x(\overline{u}^d)$ or $P = (\partial_x \overline{u})^d$ where $d \geq 3$, the equation (1) is locally well-posed for any $s > \frac{1}{2} - \frac{1}{d-1}$ in the former case and $s > \frac{3}{2} - \frac{1}{d-1}$ in the latter. Later, Hirayama ([16]) extended Grünrock's results for $P = \partial_x(\overline{u}^d)$ to the global well-posedness for $s \geq \frac{1}{2} - \frac{1}{d-1}$.

There are also various results for higher dimension analogues of (1)

$$\begin{cases} i\partial_t u + \Delta u = P(u, \overline{u}, \nabla u, \nabla \overline{u}) \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3}$$

The most general results in $\mathbb{R}^n$ for $n \geq 2$ are due to Kenig, Ponce and Vega in [19]. For a more specific case, we refer to [2] and [3] where Bejenaru obtained a local well-posedness result for $n = 2$ and $P(z)$ is quadratic with low regularity initial data. For results in Besov spaces, see [30] for the global well-posedness in $\dot{B}^{s_n}_{1,2}(\mathbb{R}^n)$ where $n \geq 2$ and $s_n = \frac{n}{2} - \frac{1}{d-1}$ which is the critical exponent.

For another type of derivative nonlinearities, we refer to Chihara ([10]) for nonlinearities of the form $f(u, \partial u)$, where $f : \mathbb{R}^2 \times \mathbb{R}^{2n} \to \mathbb{R}$ (identifying $\mathbb{C}$ with $\mathbb{R}^2$) is a smooth function such that $f(u, v) = O(|u|^2 + |v|^2)$ or $f(u, v) = O(|u|^3 + |v|^3)$ near $(u, v) = 0$. It turns out that the corresponding Cauchy problems are locally well-posed in $H^{\lfloor n/2 \rfloor + 4}$ for any $n \geq 1$.

Our first result is the local well-posedness of (1) in Sobolev spaces when the nonlinearity contains an arbitrary number of derivatives.

Theorem 1.1. *In the equation* (1)*, let s be any number such that*

(A) $s \geq \frac{1}{2}$ *if each term in $P(u, \overline{u}, \partial_x u, \partial_x \overline{u})$ has only one derivative,*
(B) $s \geq \frac{3}{2}$ *if a term in $P(u, \overline{u}, \partial_x u, \partial_x \overline{u})$ has more than one derivative.*

Then there exist a Banach space X^s and a constant $C = C(s, d)$ with the following properties: For any $u_0 \in H^s(\mathbb{R})$ such that $\|u_0\|_{H^s} < C$, the equation (1) *has a unique solution:*

$$u \in X := \{u \in C_t^0 H_x^s([-1, 1] \times \mathbb{R}) \cap X^s : \|u\|_{X^s} \leq 2C\}.$$

Furthermore, the map $u_0 \mapsto u$ is Lipschitz continuous from $B_C := \{u_0 \in H^s : \|u_0\|_{H^s} \leq C\}$ to X.

Remark. The definition of X^s will be made precise in Section 4 below.

This shows that, without any restriction to the number of derivatives, we are able to improve Kenig et al.'s result ([19]) from $H^{\frac{7}{2}}$ to $H^{\frac{3}{2}}$. By restricting to only one derivative per term in the nonlinearity, we can improve further to $H^{\frac{1}{2}}$. Moreover, part (A) of Theorem 1.1 extends Hao and Santos's local well-posedness result in $H^{\frac{1}{2}}$ to more general class of nonlinearities. It turns out that the global well-posedness results can be achieved if the nonlinearity is quintic or higher and the endpoint cases are excluded.

Theorem 1.2. *Suppose that $d \geq 5$ in* (2)*. Let s be any number such that*

(A) $s > \frac{1}{2}$ *if each term in $P(u, \overline{u}, \partial_x u, \partial_x \overline{u})$ has only one derivative,*
(B) $s > \frac{3}{2}$ *if a term in $P(u, \overline{u}, \partial_x u, \partial_x \overline{u})$ has more than one derivative.*

Then the equation (1) *is globally well-posed in the following sense:*

There exist a Banach space X^s and a constant $C = C(s, d)$ with the following properties: For any $u_0 \in H^s(\mathbb{R})$ such that $\|u_0\|_{H^s} < C$ and any time interval I containing 0, the equation (1) *has a unique solution:*

$$u \in X := \{u \in C_t^0 H_x^s(I \times \mathbb{R}) \cap X^s : \|u\|_{X^s} \leq 2C\}.$$

Furthermore, the map $u_0 \mapsto u$ is Lipschitz continuous from $B_C := \{u_0 \in H^s : \|u_0\|_{H^s} \leq C\}$ to X.

Remark. The definition of X^s will be made precise in Section 7 below.

Notice that when each term in $P(u, \overline{u}, \partial_x u, \partial_x \overline{u})$ has only one derivative, (1) is invariance under the scaling $u(x, t) \mapsto u_\lambda(x, t) := \lambda^{\frac{1}{d-1}} u(\lambda x, \lambda^2 t)$. Thus, the critical space is H^{s_0} where $s_0 = \frac{1}{2} - \frac{1}{d-1}$ in the sense that $\|u\|_{H^{s_0}} = \|u_\lambda\|_{H^{s_0}}$. If we follow the heuristic that a dispersive equation is expected to be locally well-posed in any subcritical Sobolev space H^s i.e. $s > s_0$, then the result in part (A) of Theorem 1.2, which requires $s > \frac{1}{2}$, is not optimal in this sense. It turns out that the global well-posedness at critical Sobolev spaces can be achieved if we assume a specific type of the gDNLS equation

$$\begin{cases} i\partial_t u + \Delta u = \partial_x P(u,\overline{u}) \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \geq s_0, \end{cases} \tag{4}$$

where $P : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial of the form

$$P(z) = P(z_1, z_2) = \sum_{d \leq |\alpha| \leq l} C_\alpha z^\alpha, \tag{5}$$

and $l \geq d \geq 5$.

The following theorem shows that for $d \geq 5$ we have the global well-posedness at the scaling critical Sobolev space.

Theorem 1.3. *Suppose that $d \geq 5$ in (5). Let $s_0 = \frac{1}{2} - \frac{1}{d-1}$. For any $s \geq s_0$, the equation (4) is globally well-posed in $H^s(\mathbb{R})$ in the following sense:*

There exist a Banach space X^s and a constant $C = C(s,d)$ with the following properties: For any $u_0 \in H^s(\mathbb{R})$ such that $\|u_0\|_{H^s} < C$ and any time interval I containing 0, the equation (4) has a unique solution:

$$u \in X := \{u \in C_t^0 H_x^s(I \times \mathbb{R}) \cap X^s : \|u\|_{X^s} \leq 2C\}.$$

Furthermore, the map $u_0 \mapsto u$ is Lipschitz continuous from $B_C := \{u_0 \in H^s : \|u_0\|_{H^s} \leq C\}$ to X.

In the case of $s = s_0$, the statement above holds true if we replace H^s by $\dot{H}^{s_0}$.

Remark. The definition of X^s will be made precise in Section 5 in the case of $d \geq 6$ and Section 6 in the case of $d = 5$ below.

This extends Grünrock and Hirayama's results to more general class of nonlinearities. The main ideas behind the proof of Theorem 1.1 and Theorem 1.3 consist of the Duhamel reformulation of the problem, followed by the contraction argument, using the local smoothing estimate (11) and the maximal function estimate (12) to deal with the loss of derivative in nonlinearity. We also use a decomposition (35) of the nonlinear Duhamel term, first introduced in [4], to deal with the truncated time integration. We then finish with the usual perturbative analysis to obtain the well-posedness results. The proof for Theorem 1.3 in the case $d = 5$ is rather delicate and needs some modulation-frequency argument, motivated by Tao's paper on the quartic generalized KdV equation ([29]), which is sensitive to the conjugates in the nonlinearity. Therefore, the proof of global well-posedness in this case will be treated separately in section 6.

One motivation of this paper came from the following specific case of (4), which has been intensively studied in the past:

$$\begin{cases} i\partial_t u + \Delta u = i\partial_x(|u|^2 u) \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \geq \frac{1}{2}. \end{cases} \tag{6}$$

We name this equation *DNLS*. It arises from studies of small-amplitude Alfv́en waves propagating parallel to a magnetic field [23] and large-amplitude magnetohydrodynamic waves in plasmas

[24]. There is also recent discovery of rogue waves as solutions for the Darboux transformation of the DNLS (see [33]). Although one expects the local well-posedness for $s \geq 0$, Biagioni and Linares ([5]) have shown that (6) is ill-posed for $s < \frac{1}{2}$ in the sense that the solution mapping $u_0 \mapsto u$ fails to be uniformly continuous. This means that our result from Theorem 1.3 when $d = 3$, which is a local well-posedness in $H^{\frac{1}{2}}$, is sharp in this sense.

We mention here a few of many results regarding this equation. The global well-posedness in the energy space $H^1(\mathbb{R})$ was proved by Hayashi and Ozawa in [15]. For data below the energy space, Takaoka has shown in [27] that DNLS is locally well-posed for $s \geq \frac{1}{2}$ using (7) with $k = -1$. In [11], Colliander, Keel, Staffilani, Takaoka and Tao used the "I-method" to show the global well-posedness of DNLS for $s > \frac{1}{2}$, assuming the smallness condition $|u_0|_{L^2} < \sqrt{2\pi}$. Later, Miao, Wu and Xu have proved the global well-posedness result for the endpoint case $s = \frac{1}{2}$ using the third generation I-method and same smallness condition in [22]. Lastly, Wu ([31] and [32]) has shown that in the energy-critical case $s = 1$, the smallness threshold is improved to $\|u_0\|_L^2 < 2\sqrt{\pi}$.

We are now shifting focus toward some qualitative aspects of the solutions. Kaup and Newell have shown that the equation is completely integrable, which implies infinitely many conservation laws. Moreover, the inverse scattering method can be applied to obtain soliton solutions which are unstable in a sense that a small perturbation could cause the soliton to disperse (see [17]). Recently, Liu, Perry and Sulem used this method to prove the global well-posedness result in $H^{2,2}(\mathbb{R})$ (see [21]). A study following Wu's above result ([9]) shows an existence of two kinds of solitons: bright solitons with mass $\sqrt{2\pi}$, and lump soliton with mass $2\sqrt{\pi}$. He showed in [31] that there is no blow-up near the $\sqrt{2\pi}$ threshold. On the other hand, the study of Cher, Simpson and Sulem ([9]) has shown some numerical evidence of a blow-up profile that closely resembles the lump soliton.

The main difficulty in studying DNLS is the spatial derivative in nonlinearity. Due to this, all of well-posedness results for DNLS so far involve the *Gauge transformation*:

$$v(x,t) := u(x,t) \exp\left\{ ik \int_{-\infty}^{x} |u(y,t)|^2 \, dy \right\} \tag{7}$$

where $k \in \mathbb{R}$. In [27], Takaoka used the transformation with $k = -1$ to turn (6) into

$$\begin{cases} i\partial_t v + \Delta v = -iv^2 \partial_x \overline{v} - \frac{1}{2}|v|^4 v \\ v(x,0) = v_0 \in H^s(\mathbb{R}), s \geq \frac{1}{2}. \end{cases} \tag{8}$$

Note that the transformation replaces the term $|u|^2 \partial_x u$ with $v^2 \partial_v \overline{u}$ which can be treated using the Fourier restriction norm method developed in [6]. In contrast to this type of proofs, we managed to get the local well-posedness of (6) (as a part of Theorem 1.3) without using a gauge transformation. The advantage is that the idea can be easily generalized to get similar result for equation (4).

The paper is organized as follows. In the next subsection, we introduce some notations that are used in this paper. In section 2, we mention several linear and smoothing estimates and prove the maximal function estimate and bilinear estimate. In section 3, we introduce the solution space X_N and nonlinear space Y_N for functions supported at frequency N and prove the main linear and bilinear estimate for functions in these spaces using a solution decomposition technique from [4]. In section 4, we prove a multilinear estimate. Having all the ingredients that we need, we finish the proof of Theorem 1.1 in the same section. For Theorem 1.3, we divide the proof into different sections by the degree d of $P(u,\overline{u})$. In section 5, we prove Theorem 1.3 in the case of $d \geq 6$. Since the case $d = 5$ requires some frequency-modulation analysis, we will introduce the notion of $X^{s,b}$ space along with several well-known estimates in section 6, and use these results to conclude the proof of Theorem 1.3 in the same section. Finally, we prove another multilinear estimate and use it to finish the proof of Theorem 1.2 in Section 7.

Notations. The following notations will be used for the rest of the paper. For $1 \leq p, q \leq \infty$, we use $\|f\|_{L^p}$ to denote the L^p norm, and we define the mixed norm

$$\|f\|_{L_x^p L_t^q} := \big\| \|f(x,t)\|_{L_t^q(I)} \big\|_{L_x^p(\mathbb{R})},$$

where $I = [-1, 1]$ if $d = 3, 4$ and $I = \mathbb{R}$ if $d \geq 5$. The norm $\|f\|_{L_t^p L_x^q}$ is defined similarly. We define the Fourier transform and the inverse Fourier transform of $f(x)$ by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx,$$

$$\check{f}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(\xi)\, d\xi.$$

To simplify the proofs, we will always drop the constant $\frac{1}{\sqrt{2\pi}}$ from these transforms. For $s \in \mathbb{R}$, we denote by $D^s = (-\Delta)^{s/2}$ the Riesz potential of order $-s$. The Sobolev space H_x^s is defined by the norm

$$\|u\|_{H_x^s} := \|(1+\xi^2)^{\frac{s}{2}} \widehat{u}(\xi)\|_{L_\xi^2}.$$

The Banach space of bounded H_x^s-valued continuous functions is denoted by

$$C_t^0 H_x^s(I \times J) := \left\{ f \in C(I; H_x^s(J)) : \sup_{t \in I} \|f(x,t)\|_{H_x^s(J)} < \infty \right\}.$$

Let $u \in L_x^2$. We define the Schrödinger propagator by

$$e^{it\Delta} u(x,t) := \int_{\mathbb{R}} e^{ix\xi - it\xi^2} \hat{u}\, d\xi.$$

The notation $a \lesssim b$ and $a \sim b$ means $a \leq Cb$ and $ca \leq b \leq CA$, respectively, for some positive constants c and C, which depend on $P(z)$ but not on the functions involved in these estimates.

We frequently split the frequency space into dyadic intervals, so whenever M and N is mentioned, we assume that $M, N \in 2^{\mathbb{Z}}$. Let $\psi(\xi)$ be a smooth cutoff function supported in $|\xi| \leq 4$ and equal 1 on $|\xi| \leq 2$. We define $\psi_N = \psi\left(\frac{\xi}{N}\right) - \psi\left(\frac{2\xi}{N}\right)$. Denote by P_N the Littlewood–Paley projection at frequency N, that is

$$\widehat{P_N f}(\xi) = \psi_N(\xi)\hat{f}(\xi).$$

Define $P_{\leq N}$ and $P_{>N}$ to be the projections of frequency less than and greater than N:

$$\widehat{P_{\leq N} f}(\xi) = \psi_{\leq N}\hat{f}(\xi) := \sum_{M\leq N} \psi_M(\xi)\hat{f}(\xi),$$
$$\widehat{P_{>N} f}(\xi) = \psi_{>N}\hat{f}(\xi) := \sum_{M>N} \psi_M(\xi)\hat{f}(\xi).$$

We will sometimes shorten the notation by $f_N := P_N f$. For $s \geq 0$, we can define the space H^s and the homogeneous Sobolev space $\dot{H}^s$ using the Littlewood–Paley projections

$$\|u\|_{\dot{H}^s} := \Big(\sum_{N_i \in 2^{\mathbb{Z}}} N_i^{2s} \|P_{N_i} u\|_{L^2}^2 \Big)^{\frac{1}{2}}$$

$$\|u\|_{H^s} := \|P_{\leq 1} u\|_{L^2} + \Big(\sum_{N_i \in 2^{\mathbb{N}}} N_i^{2s} \|P_{N_i} u\|_{L^2}^2 \Big)^{\frac{1}{2}}.$$

2. Preliminary results

2.1. Bernstein type inequality

We begin with the Bernstein inequality for the Littlewood–Paley projections. Note that this is different from the standard result in literatures which is the same estimate but for the space $L_t^q L_x^p$.

Lemma 2.1. *For any pair of* $1 \leq p, q \leq \infty$*, we have*

$$\|\partial_x P_N f\|_{L_x^p L_t^q} \lesssim N \|P_N f\|_{L_x^p L_t^q}. \tag{9}$$

Proof. Let $\tilde{P}_N := P_{N/2} + P_N + P_{2N}$ be a Littlewood–Paley projection at a wider frequency interval with corresponding multiplier $\widetilde{\psi}_N$. We can rewrite the term on the left-hand side as

$$\partial_x \tilde{P}_N P_N f = (\partial_x \check{\tilde{\psi}}_N) * P_N f(x,t).$$

For each x, we have an inequality

$$\|\partial_x P_N f\|_{L_t^q} \leq |\partial_x \check{\tilde{\psi}}_N| * \|P_N f(x,t)\|_{L_t^q}.$$

After taking the L_x^p norm and apply Young's inequality, we have

$$\|\partial_x P_N f\|_{L_x^p L_t^q} \leq \|\partial_x \check{\tilde{\psi}}_N\|_{L_x^1} \|P_N f\|_{L_x^p L_t^q} \lesssim N \|P_N f\|_{L_x^p L_t^q}. \qquad \square$$

This lemma helps us quantify derivatives of a function supported in a dyadic frequency interval, which will come in handy in the proofs of multilinear estimates in sections 4–6.

2.2. Stationary phase lemmas

We mention here stationary phase results from harmonic analysis, which will be used in the next subsection. See [26, pp. 331–334] for their proofs.

Lemma 2.2. *Suppose that ϕ and ψ are smooth functions and ψ is compactly supported in (a,b). If $\phi'(\xi) \neq 0$ for all $\xi \in [a,b]$, then*

$$\left| \int_a^b e^{i\lambda\phi(\xi)} \psi(\xi)\, d\xi \right| \leq \frac{C}{|\lambda|^k}$$

for all $k \geq 0$, where the constant C depends on ϕ, ψ and k.

Lemma 2.3. *Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is smooth, ϕ is a real-valued C^2-function in (a,b) and $\phi''(\xi) \gtrsim 1$. Then,*

$$\left| \int_a^b e^{i\lambda\phi(\xi)} \psi(\xi)\, d\xi \right| \lesssim \frac{1}{|\lambda|^{\frac{1}{2}}} \left(|\psi(b)| + \int_a^b |\psi'(\xi)|\, d\xi \right).$$

2.3. Strichartz and local smoothing estimates

In our study, the nonlinear effect of the equation (1) with small initial data u_0 plays a major role in the perturbative analysis. As we mentioned in section 1, the main difficulty is a lost of derivative in the nonlinearity. In this regard, we will need the Strichartz estimate for the Schrödinger propagator and the smoothing estimate (11) which gives a $\frac{1}{2}$-order derivative gain of the linear solution in a suitable norm. We will also prove a maximal function type estimate (12) which will be used for the analysis of the nonlinear term.

Proposition 2.4. *Let $f \in L^2$. Then, we have the following estimates*

$$\|e^{it\Delta} f\|_{L_t^q L_x^p} \lesssim \|f\|_{L_x^2}, \tag{10}$$

where $\dfrac{2}{q} + \dfrac{1}{p} = \dfrac{1}{2}$ and $2 \leq p \leq \infty$, and

$$\|D^{\frac{1}{2}} e^{it\Delta} f\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L_x^2}. \tag{11}$$

Proof. The first inequality is the well-known Strichartz estimate. The proof can be found, for example, in [8] and [28]. The proof of (11) can be found in Theorem 4.1 of [18]. □

The following maximal function type estimate tells us that for the linear equation with time-and-frequency localized initial data in $H^s(\mathbb{R})$ where $s \geq \frac{1}{2}$, the solution is well-controlled in $L_x^\gamma L_t^\infty(\mathbb{R} \times I)$, where $I = [-1, 1]$ when $\gamma = 2, 3$ and $I = \mathbb{R}$ when $\gamma \geq 4$.

Proposition 2.5. *Let $u \in L^2_x(\mathbb{R})$.*

1. *If $\gamma = 2$ or 3, assume that* $\operatorname{supp}(|\hat{u}|) \subseteq [N, 4N]$ *where $N \in 2^{\mathbb{N}}$ or* $\operatorname{supp}(|\hat{u}|) \subseteq [0, 1]$, *in which case we consider $N = 1$, then*

$$\|\chi_{[-1,1]}(t)e^{it\Delta}u(x)\|_{L^\gamma_x L^\infty_t} \lesssim N^{\frac{1}{\gamma}}\|u\|_{L^2_x}. \tag{12a}$$

2. *If $\gamma \geq 4$, assume that* $\operatorname{supp}(|\hat{u}|) \subseteq [N, 4N]$ *where $N \in 2^{\mathbb{Z}}$, we have*

$$\|e^{it\Delta}u(x)\|_{L^\gamma_x L^\infty_t} \lesssim N^{\frac{\gamma-2}{2\gamma}}\|u\|_{L^2_x}. \tag{12b}$$

Remark. We see that the estimate (12a) is local in time while (12b) is global. By setting $\gamma = d - 1$, this leads to the local and global results in Theorem 1.1 and Theorem 1.3.

Proof. We refer to Theorem 2.5 in [18] for a proof of the case $\gamma = 4$. Let $s_0 = s_0(\gamma) = \frac{1}{\gamma}$ for $\gamma = 2, 3$ and $s_0 = \frac{\gamma-2}{2\gamma}$ for $\gamma \geq 5$. We define an operator $T : L^2_x \to L^\gamma_x L^\infty_t$ by $Tu = \chi_{[-1,1]}(t)e^{it\Delta}u$, yielding $T^*F = \int_{-1}^1 e^{-it\Delta}F\,dt$. Using the TT^* argument, it follows that (12) is equivalent to either of the following estimates for $F \in L^2_x L^1_t(\mathbb{R}\times\mathbb{R})$ with the same frequency support as u in the cases of $\gamma = 2, 3$.

$$\left\|\int_{-1}^{1} e^{-it\Delta}F(x,t)\,dt\right\|_{L^2_x} \lesssim N^{s_0}\|F\|_{L_x^{\frac{\gamma}{\gamma-1}}L^1_t} \tag{13}$$

$$\left\|\chi_{[-1,1]}(t)\int_{-1}^{1} e^{i(t-s)\Delta}F(x,s)\,ds\right\|_{L^\gamma_x L^\infty_t} \lesssim N^{2s_0}\|F\|_{L_x^{\frac{\gamma}{\gamma-1}}L^1_t}. \tag{14}$$

For $\gamma \geq 5$, we have the same estimates but with integrals on $\mathbb{R}$. Thus, it suffices to prove (14). First, we assume that $F \in \mathcal{S}(\mathbb{R})$. Since $F = P_{\leq 4N}F$, the inverse Fourier transform of $e^{i(t-s)\xi^2}\widehat{F}$ is defined by

$$\begin{aligned}\mathcal{F}_x^{-1}\left(e^{i(t-s)\xi^2}\widehat{F}(\xi,s)\right) &= c\int_{\mathbb{R}} e^{i(t-s)\xi^2+ix\xi}\widehat{F}(\xi,s)\,d\xi \\ &= \mathcal{F}_x^{-1}\left(e^{-i(t-s)\xi^2}\psi\left(\frac{\xi}{4N}\right)\right) * F(x,s).\end{aligned}$$

Since $-1 \leq t, s \leq 1$ implies $-2 \leq t-s \leq 2$, the term on the right of (14) can be replaced by

$$\int_{\mathbb{R}} \mathcal{F}_x^{-1}\left(\chi_{[-2,2]}(t-s)e^{-i(t-s)\xi^2}\psi\left(\frac{\xi}{4N}\right)\right) * F(x,s)\,ds$$

$$= \mathcal{F}_x^{-1}\left(\chi_{[-2,2]}(t)e^{-it\xi^2}\psi\left(\frac{\xi}{4N}\right)\right) \star F(x,t)$$

$$= c_1 K_1 \star F$$

where $\star$ denotes the space–time convolution and

$$K_1(x,t) = \int_{\mathbb{R}} e^{-it\xi^2+ix\xi}\chi_{[-2,2]}(t)\psi\left(\frac{\xi}{4N}\right) d\xi. \tag{15}$$

Similarly, for $\gamma \geq 5$ we have

$$\int_{\mathbb{R}} e^{i(t-s)\Delta}F(x,s)\,ds = c_2 K_2 \star F$$

where

$$K_2(x,t) = \int_{\mathbb{R}} e^{-it\xi^2+ix\xi}\psi\left(\frac{\xi}{4N}\right) d\xi. \tag{16}$$

To finish the proof, we need the following lemma.

Lemma 2.6. *Let $K_1(x,t)$ and $K_2(x,t)$ be as in (15) and (16). Then, for $i = 1, 2$*

$$\|K_i\|_{L_x^{\frac{\gamma}{2}} L_t^\infty} \lesssim N^{2s_0}. \tag{17}$$

We continue the proof of Proposition 2.5. By applying Young's inequality and Lemma 2.6, we obtain

$$\|K_i \star F\|_{L_x^\gamma L_t^\infty} \leq \|K_i\|_{L_x^{\frac{\gamma}{2}} L_t^\infty} \|F\|_{L_x^{\frac{\gamma}{\gamma-1}} L_t^1}$$

as desired. We then finish the proof by the usual density argument. □

Proof of Lemma 2.6. Let $I = [-1, 1]$ when $\gamma = 2, 3$ and $I = \mathbb{R}$ when $\gamma \geq 4$. We divide $\mathbb{R} \times I$ into three regions

$$\Omega_1 := \{(x,t) \in \mathbb{R} \times I \mid |x| \leq \frac{1}{N}\}$$

$$\Omega_2 := \{(x,t) \in \mathbb{R} \times I \mid |x| \geq 64N|t|\ ,\ |x| > \frac{1}{N}\}$$

$$\Omega_3 := \{(x,t) \in \mathbb{R} \times I \mid |x| < 64N|t|\ ,\ |x| > \frac{1}{N}\},$$

and we will estimate $K_i(x,t)$ in each region. For a fixed $x \in \mathbb{R}$ and $1 \leq i \leq 3$, we define $\Omega_{x,i} := \{t \in I \mid (x,t) \in \Omega_i\}$. We consider the following two cases of values of γ.

Case 1: $\gamma = 2, 3$. Note that in this case we always assume that $N \geq 1$. By a change of variable $\eta = \frac{\xi}{4N}$, we obtain

$$K_1(x,t) = N \int_{\mathbb{R}} \chi_{[-2,2]} e^{-i16tN^2\eta^2 + i4xN\eta} \psi(\eta)\, d\eta$$

A simple estimate on Ω_1 shows that

$$\int_{|x| \leq \frac{1}{N}} |K_1(x,t)|^{\frac{\gamma}{2}}\, dx \lesssim \frac{1}{N} \cdot N^{\frac{\gamma}{2}} \Big(\int_{\mathbb{R}} \psi(\eta)\, d\eta \Big)^{\frac{\gamma}{2}} \sim N^{\frac{\gamma-2}{2}} \leq N. \tag{18}$$

Next we consider the norm on Ω_2. Note that the integrand in K_1 vanishes if $|\eta| \geq 4$. Factoring out $-i16tN^2\eta^2 + i4xN\eta = -i4xN(\eta - \frac{4tN}{x}\eta^2) := -ixN\phi_1(\eta)$ yields

$$|\phi_1'(\eta)| = |1 - 8\frac{tN}{x}\eta| \geq 1 - 32 \left| \frac{tN}{x} \right| \geq 1 - 32 \cdot \frac{1}{64} = \frac{1}{2},$$

for any $t \in \Omega_{x,2}$. Therefore, ϕ_1 has no critical point in this region. By Lemma 2.2, the integral in K_1 is bounded by $|Nx|^{-k}$ for all $k \geq 0$. In particular, by choosing $k = 2$, we obtain $|K_1(x,t)| \lesssim N(N|x|)^{-2} = N^{-1}|x|^{-2}$. We finish by computing the $L_x^{\frac{\gamma}{2}} L_t^{\infty}$ norm on Ω_2:

$$\int \sup_{t \in \Omega_{x,2}} |K_1(x,t)|^{\frac{\gamma}{2}}\, dx \lesssim N^{(\gamma-1)-\frac{\gamma}{2}} = N^{\frac{\gamma-2}{2}} \leq N. \tag{19}$$

Now we consider the norm on Ω_3. Factoring out the exponential term $-i16tN^2\eta^2 + i4xN\eta = -i4tN^2(4\eta^2 - \dfrac{x\eta}{Nt}) := i4tN^2\phi_2(\eta)$ yields $\phi_2''(\eta) \gtrsim 1$, so we can apply Lemma 2.3 to K_1.

$$\begin{aligned} |K_1(x,t)| &= N \left| \int_{\mathbb{R}} e^{-itN^2\eta^2 + ixN\eta} \psi(\eta)\, d\eta \right| \\ &\lesssim N \cdot \frac{1}{N|t|^{\frac{1}{2}}} \\ &< \frac{64N^{\frac{1}{2}}}{|x|^{\frac{1}{2}}}. \end{aligned} \tag{20}$$

Now we compute the $L_x^{\frac{\gamma}{2}} L_t^{\infty}$ norm of K_1. Observe that the finite time restriction yields $|x| \lesssim N|t| \leq 2N$ on Ω_3. Therefore,

$$\int \sup_{t \in \Omega_{x,3}} |K_1(x,t)|^{\frac{\gamma}{2}}\, dx \lesssim \int_{|x| < 64N|t|} N^{\frac{\gamma}{4}} |x|^{-\frac{\gamma}{4}}\, dx \lesssim N^{\frac{\gamma}{4} - \frac{\gamma-4}{4}} = N. \tag{21}$$

Combining (18), (19) and (21), we have that

$$\|K_1\|_{L_x^{\frac{\gamma}{2}} L_t^\infty} \lesssim N^{\frac{2}{\gamma}}.$$

Case 2: $\gamma \geq 5$. Since the estimates in (18) and (19) do not require any time restriction, we get the same results for K_2.

$$\int_{\Omega_1 \cup \Omega_2} |K_2|^{\frac{\gamma}{2}}\, dx \lesssim N^{\frac{\gamma-2}{2}}. \tag{22}$$

On Ω_3, we have the same estimate as in (20) for K_2. From the fact that $|x| > \frac{1}{N}$ in this region, we have

$$\int \sup_{t\in\Omega_{x,3}} |K_2(x,t)|^{\frac{\gamma}{2}}\, dx \lesssim \int_{|x|>\frac{1}{N}} N^{\frac{\gamma}{4}} |x|^{-\frac{\gamma}{4}}\, dx \lesssim N^{\frac{\gamma}{4}+\frac{\gamma-4}{4}} = N^{\frac{\gamma-2}{2}}. \tag{23}$$

Note that we did not use the finite time restriction in this case. Combining (22) and (23), we have that

$$\|K_2\|_{L_x^{\frac{\gamma}{2}} L_t^\infty} \lesssim N^{\frac{\gamma-2}{\gamma}}. \qquad \square$$

To estimate a product of functions as seen in the nonlinearity of DNLS, one usually employs the bilinear estimate which splits the product into estimating individual functions (see [7] where Bourgain proved the estimate in two dimensions).

Theorem 2.7 *(Bilinear Strichartz estimate). For any $u, v \in L_x^2$, we have*

$$\|P_\lambda(e^{it\Delta}u\overline{e^{it\Delta}v})\|_{L^2_{x,t}} \lesssim \lambda^{-\frac{1}{2}}\|u\|_{L^2}\|v\|_{L^2}. \tag{24}$$

In addition, if $\hat{u}$ and $\hat{v}$ have disjoint supports and $\alpha := \inf|\mathrm{supp}(\hat{u}) - \mathrm{supp}(\hat{v})|$ is strictly positive, then we have

$$\|e^{it\Delta}ue^{it\Delta}v\|_{L^2_{x,t}} \lesssim \alpha^{-\frac{1}{2}}\|u\|_{L^2}\|v\|_{L^2}. \tag{25}$$

Proof. We follow the proof in [20, Theorem 2.9]. By duality, this is equivalent to showing that for any $F \in C_c^\infty$,

$$\left|\int F(\xi-\eta, \xi^2-\eta^2)\psi_{>\lambda}(\xi-\eta)\hat{u}(\xi)\bar{\hat{v}}(\eta)\, d\xi d\eta\right| \lesssim \lambda^{-\frac{1}{2}}\|F\|_{L^2_{\xi,\tau}}\|\hat{u}\|_{L^2_\xi}\|\hat{v}\|_{L^2_\xi}.$$

For each fixed α and β, let $(\xi_{\alpha\beta}, \eta_{\alpha\beta})$ be a solution to $\alpha = \xi^2 - \eta^2$ and $\beta = \xi - \eta$. We see that the change of variables $(\xi, \eta) \mapsto (\alpha, \beta)$ gives the Jacobian $J = 2(\eta - \xi)$. This together with Cauchy–Schwarz yields

$$
\begin{aligned}
&\Big| \int F(\xi-\eta, \xi^2-\eta^2)\psi_{>\lambda}(\xi-\eta)\hat{u}(\xi)\bar{\hat{v}}(\eta)\, d\xi d\eta \Big| \\
&\quad = \Big| \int F(\alpha,\beta)\psi_{>\lambda}(\beta)\hat{u}(\xi_{\alpha\beta})\bar{\hat{v}}(\eta_{\alpha\beta})\frac{1}{J}\, d\alpha d\beta \Big| \\
&\quad \leq \|F\|_{L^2_{\xi,\tau}} \Big(\int |\psi_{>\lambda}(\beta)|^2 |\hat{u}(\xi_{\alpha\beta})|^2 |\hat{v}(\eta_{\alpha\beta})|^2 \frac{1}{J^2}\, d\alpha d\beta \Big)^{\frac{1}{2}} \\
&\quad = \|F\|_{L^2_{\xi,\tau}} \Big(\int |\psi_{>\lambda}(\xi-\eta)|^2 |\hat{u}(\xi)|^2 |\hat{v}(\eta)|^2 \frac{1}{J}\, d\xi d\eta \Big)^{\frac{1}{2}} \\
&\quad \lesssim \lambda^{-\frac{1}{2}} \|F\|_{L^2_{\xi,\tau}} \|\hat{u}\|_{L^2_\xi} \|\hat{v}\|_{L^2_\xi}.
\end{aligned}
$$

This concludes the proof of (24). The proof for (25) is essentially the same, but $\xi-\eta$ is replaced by $\xi+\eta$, $\xi^2-\eta^2$ is replaced by $\xi^2+\eta^2$ and there is no $\psi_{>\lambda}$. The conclusion follows from the observation that

$$
\frac{1}{|J|} = \frac{1}{2|\eta-\xi|} \gtrsim \frac{1}{\alpha}. \qquad \square
$$

We will need a variant of this estimate adapted to the X^s space (51) for our trilinear estimate (65). The details will be explained in the next section.

3. The main linear estimate

In this section, we consider a nonlinear Schrödinger equation

$$
\begin{aligned}
iu_t + \Delta u &= F \\
u(x,0) &= u_0.
\end{aligned} \tag{26}
$$

Let $I=[-1,1]$ if $d=3,4$ and $I=\mathbb{R}$ if $d\geq 5$. A solution $u(x,t)\in\mathbb{R}\times I$ can be represented by the Duhamel formula

$$
u(x,t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F(s)\, ds. \tag{27}
$$

In the proof of Theorem 1.1 and Theorem 1.3, the spaces that we use are based on the following norms which take a function u supported at dyadic frequency interval $\sim N$.

$$
\begin{aligned}
\|u\|_{Y_N} &= \inf\{N^{-\frac{1}{2}}\|u_1\|_{L^1_x L^2_t} + \|u_2\|_{L^1_t L^2_x} \mid u_1+u_2=u\} \\
\|u\|_{X_N} &= \|u\|_{L^\infty_t L^2_x} + N^{-s_0}\|u\|_{L^{d-1}_x L^\infty_t} + N^{\frac{1}{2}}\|u\|_{L^\infty_x L^2_t} \\
&\quad + N^{-\frac{1}{2}}\|(i\partial_t+\Delta)u\|_{Y_N},
\end{aligned} \tag{28}
$$

where $L^\infty_t L^2_x = L^\infty_t L^2_x(I\times\mathbb{R})$ and $L^p_x L^q_t = L^p_x L^q_t(\mathbb{R}\times I)$. These norms satisfy the following linear estimate, which makes them suitable for the contraction argument.

Theorem 3.1. *Let u be a solution to equation (26). Then,*

$$\|P_N u\|_{X_N} \lesssim \|u_0\|_{L_x^2} + \|P_N F\|_{Y_N}. \tag{29}$$

This immediately follows from the Duhamel formula and the following three propositions.

Proposition 3.2. *For any $u_0 \in L_x^2(\mathbb{R})$, we have*

$$\|e^{it\Delta} P_N u_0\|_{X_N} \lesssim \|u_0\|_{L_x^2}. \tag{30}$$

Proof. This follows from the Strichartz estimate (10), the smoothing estimate (11) and (12a) if $d = 3, 4$ or (12b) if $d \geq 5$. □

Proposition 3.3. *For any function $F(x,t)$ such that $P_N F \in L_x^1 L_t^2$, we have*

$$\left\| \int_0^t e^{i(t-s)\Delta} P_N F(s)\, ds \right\|_{X_N} \lesssim \|P_N F\|_{Y_N}. \tag{31}$$

Proof. It follows from Minkowski inequality and (30) that

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\Delta} P_N F(s)\, ds \right\|_{X_N} &\leq \int_{\mathbb{R}} \|e^{i(t-s)\Delta} P_N F(s)\|_{X_N}\, ds \\ &\lesssim \int_{\mathbb{R}} \|P_N F(s)\|_{L_x^2}\, ds \\ &= \|P_N F\|_{L_t^1 L_x^2}. \end{aligned}$$

Therefore, it suffices to prove that

$$\left\| \int_0^t e^{i(t-s)\Delta} P_N F(s)\, ds \right\|_{X_N} \lesssim N^{-\frac{1}{2}} \|P_N F\|_{L_x^1 L_t^2}. \tag{32}$$

Let K_0 be the fundamental solution of Schrödinger equation i.e.

$$K_0(x,t) = \mathcal{F}^{-1}(e^{-it\xi^2}) = \frac{1}{\sqrt{4\pi i t}} e^{ix^2/4t}.$$

Thus,

$$
\begin{aligned}
\int_0^t e^{i(t-s)\Delta} P_N F(x,s)\, ds &= \int_0^t \int_{\mathbb{R}} P_N\left[K_0(x-y,t-s)F(y,s)\right]\, dyds \\
&= \int_{\mathbb{R}} \int_0^t P_N\left[K_0(x-y,t-s)F(y,s)\right]\, dsdy \qquad (33)\\
&:= \int_{\mathbb{R}} w_y\, dy.
\end{aligned}
$$

In order to proceed, we will make use of the following lemma.

Lemma 3.4. *For any $N \in 2^{\mathbb{Z}}$, the function w_y defined in (33) satisfies the following estimate:*

$$
\|w_y\|_{X_N} \lesssim N^{-\frac{1}{2}} \|F(y)\|_{L_t^2}. \tag{34}
$$

Continuing the proof of Proposition 3.3, we see that the estimate (32) follows immediately from (34). □

Proof of Lemma 3.4. By translation invariance, it suffices to assume that $y = 0$. Denote $F_0(t) := F(0,t)$. To proceed, we use the following decomposition which was first introduced in [4] to deal with Schrödinger maps.

$$
w_0(x,t) = -e^{it\Delta}\mathcal{L}v_0(x) - (P_{<N/2^{50}} 1_{x>0}) e^{it\Delta} v_0(x) + h(x,t), \tag{35}
$$

where $\mathcal{L} : L_x^2(\mathbb{R}) \to L_x^2(\mathbb{R})$ is an operator and

$$
\|\mathcal{L}v_0\|_{L_x^2} + \|v_0\|_{L_x^2} + N^{-1}(\|\Delta h\|_{L_{x,t}^2} + \|h_t\|_{L_{x,t}^2}) \lesssim N^{-\frac{1}{2}} \|F_0\|_{L_t^2}. \tag{36}
$$

To prove the claim, first we rewrite the definition of w_0 as

$$
\begin{aligned}
w_0(x,t) &= \int_{\mathbb{R}} \chi_{[0,\infty)}(t-s) P_N[K_0(x,t-s)] F_0(s)\, ds \\
&\quad - e^{it\Delta} \int_{-\infty}^0 P_N[K_0(x,-s)] F_0(s)\, ds \qquad (37)\\
&= (\chi_{[0,\infty)} P_N K_0) *_t F_0 - e^{it\Delta} \int_{-\infty}^0 P_N K_0(x,-s) F_0(s)\, ds,
\end{aligned}
$$

where $*_t$ is the time convolution. The space–time Fourier transform of the first term is equal to

$$
\frac{\psi_N(\xi)}{-\tau - \xi^2 - i0} \widehat{F_0}(\tau), \tag{38}
$$

where $\widehat{F_0}$ is the time Fourier transform of F_0. We define

$$\hat{v}_0(\xi) := \psi_N(\xi)\widehat{F_0}(-\xi^2). \tag{39}$$

We see that v_0 is supported at frequency $\sim N$. By changing variables we obtain the following estimate,

$$\|v_0\|_{L^2_x} \lesssim N^{-\frac{1}{2}}\|F_0\|_{L^2_t}. \tag{40}$$

We apply the spatial Fourier transform to the second term

$$\begin{aligned}\int_{-\infty}^{0} \widehat{P_N K}_0(x,-s)F_0(s)\,ds &= \psi_N(\xi)\int_{-\infty}^{0} e^{is\xi^2}F_0(s)\,ds \\ &= \psi_N(\xi)\mathcal{F}_t(\chi_{(0,\infty]}F_0)(-\xi^2) \\ &:= \widehat{\mathcal{L}v_0}(\xi).\end{aligned} \tag{41}$$

We see that $\mathcal{L}v_0$ is supported at frequency $\sim N$. It follows from a change of variables that

$$\|\mathcal{L}v_0\|_{L^2_x} \lesssim N^{-\frac{1}{2}}\|F_0\|_{L^2_t}.$$

Applying the Fourier transform to $e^{it\Delta}v_0$,

$$\mathcal{F}(e^{it\Delta}v_0) = \psi_N(\xi)\widehat{F_0}(-\xi^2)\mathcal{F}_t(e^{-it\xi^2}) = \psi_N(\xi)\widehat{F_0}(-\xi^2)\delta_{\tau+\xi^2}.$$

Assume that $\xi > 0$ and consider the distribution $\delta_{\tau+\xi^2}$. For any $\phi \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$, by a change of variables

$$\int_0^\infty \phi(\xi,-\xi^2)\,d\xi = \int_{-\infty}^0 \frac{1}{2\sqrt{-\tau}}\phi(\sqrt{-\tau},\tau)\,d\tau.$$

Thus, $1_{\xi>0}\delta_{\tau+\xi^2} = 1_{\tau<0}\frac{1}{2\sqrt{-\tau}}\delta_{\xi-\sqrt{-\tau}}$. Therefore, the following computation holds.

$$\begin{aligned}&\mathcal{F}\big\{(P_{<N/2^{50}}1_{x>0})e^{it\Delta}v_0\big\}(\xi,\tau) \\ &\quad= (\psi_N(\xi)\widehat{F_0}(-\xi^2)\delta_{\tau+\xi^2}) * \frac{\psi_{<N/2^{50}}(\xi)}{\xi+i0} \\ &\quad= \left(\frac{\psi_N(\xi)}{2\sqrt{-\tau}}\widehat{F_0}(-\xi^2)\delta_{\xi-\sqrt{-\tau}}\right) * \frac{\psi_{<N/2^{50}}(\xi)}{\xi+i0} \\ &\quad= \frac{\psi_N(\sqrt{-\tau})\widehat{F_0}(\tau)}{2\sqrt{-\tau}}\frac{\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})}{\xi-\sqrt{-\tau}+i0} \\ &\quad= \psi_N(\sqrt{-\tau})\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})\widehat{F_0}(\tau)\frac{\xi+\sqrt{-\tau}}{2\sqrt{-\tau}}\frac{1}{\xi^2+\tau+i0}.\end{aligned}$$

With this and (38), the space–time Fourier transform of the remainder term is given by

$$\hat{h}(\xi,\tau) = \left(\psi_N(\xi) - \psi_N(\sqrt{-\tau})\psi_{<N/2^{50}}(\xi - \sqrt{-\tau})\frac{\xi+\sqrt{-\tau}}{2\sqrt{-\tau}}\right)\frac{\widehat{F_0}(\tau)}{-\xi^2-\tau-i0}$$
$$:= A(\xi,\tau)\widehat{F_0}(\tau). \tag{42}$$

The term in the bracket is bounded, supported in $\{0<\xi\sim N\}$ and vanishes when $\xi=\sqrt{-\tau}$, canceling out the singularity. Since the same result holds for $\xi<0$, this implies that

$$\|\Delta h\|_{L^2_{x,t}} + \|\partial_t h\|_{L^2_{x,t}} \sim \|(\xi^2+|\tau|)\hat{h}\|_{L^2_{\xi,\tau}} \lesssim N^{\frac{1}{2}}\|\widehat{F_0}(\tau)\|_{L^2_\tau}. \tag{43}$$

The estimate (36) then follows from (40) and (43).

Remark. It is important to note that v_0, Lv_0 and h are supported at frequency $\sim N$, since we will need this fact in any proof that employ the decomposition (35).

We are now ready to prove (34). By Bernstein's inequality and direct L^2 integration on $A(\xi,\tau)$,

$$\begin{aligned}\|h\|_{L_x^{d-1}L_t^\infty} \le \|\mathcal{F}_t h\|_{L_x^{d-1}L_\tau^1} &\lesssim \|\mathcal{F}_t h\|_{L_\tau^1 L_x^{d-1}}\\ &\lesssim N^{\frac{d-3}{2(d-1)}}\|\mathcal{F}_t h\|_{L_\tau^1 L_x^2}\\ &= N^{\frac{d-3}{2(d-1)}}\|\hat{h}\|_{L_\tau^1 L_\xi^2}\\ &\le N^{\frac{d-3}{2(d-1)}}\|A(\xi,\tau)\|_{L^2_{\tau,\xi}}\|\widehat{F_0}(\tau)\|_{L^2_\tau},\end{aligned}$$

where $A(\xi,\tau)$ is defined as in (42) when $\xi>0$. We split the integral in $\|A(\xi,\tau)\|^2_{L^2_{\tau,\xi}}$ as

$$\begin{aligned}\|A(\xi,\tau)\|^2_{L^2_{\tau,\xi}} &= \int\limits_{|\xi-\sqrt{-\tau}|<\frac{N}{2^{100}}} |A(\xi,\tau)|^2\,d\xi d\tau\\ &\quad + \int\limits_{|\xi-\sqrt{-\tau}|\ge\frac{N}{2^{100}}} |A(\xi,\tau)|^2\,d\xi d\tau\\ &:= A_1 + A_2.\end{aligned}$$

Note that $\psi_N(\xi) = \psi_N(\sqrt{-\tau}) + (\xi-\sqrt{-\tau})O(\frac{1}{N})$ as $\xi\to\sqrt{-\tau}$. If $|\xi-\sqrt{-\tau}|<\frac{N}{2^{100}}$, then $\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})=1$ and it follows that

$$\begin{aligned}&\psi_N(\xi) - \psi_N(\sqrt{-\tau})\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})\frac{\xi+\sqrt{-\tau}}{2\sqrt{-\tau}}\\ &\qquad = \frac{\psi_N(\sqrt{-\tau})(\sqrt{-\tau}-\xi)}{2\sqrt{-\tau}} + (\xi-\sqrt{-\tau})O\Big(\frac{1}{N}\Big).\end{aligned}$$

Since $A(\xi,\tau)$ is supported in the region $\xi \sim N$, we have that

$$A_1 \lesssim \int\limits_{\tau\sim -N^2}\int\limits_{\xi\sim N} \frac{1}{-2\tau(\xi+\sqrt{-\tau})^2} + \frac{1}{N^2(\xi+\sqrt{-\tau})^2}\, d\xi d\tau \lesssim \frac{1}{N}.$$

On the other hand, under the assumptions that, $\xi \sim N$ and $|\xi - \sqrt{-\tau}| \geq \frac{N}{2^{100}}$, we have $|\xi^2+\tau| = |(\xi+\sqrt{-\tau})(\xi-\sqrt{-\tau})| \gtrsim \frac{N^2}{2^{100}}$. Thus, by a change of variables $(\xi,\tau)\mapsto(\xi,\eta)$ where $\eta := \tau+\xi^2$, we have

$$\begin{aligned}
A_2 &\leq \int\limits_{-\infty}^{0}\int\limits_{|\xi-\sqrt{-\tau}|\geq \frac{N}{2^{100}}} \frac{\psi_N(\xi)}{(\xi^2+\tau)^2} + \frac{\psi_N(\sqrt{-\tau})\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})}{-4\tau(\xi+\sqrt{-\tau})^2}\, d\xi d\tau \\
&\lesssim \int\limits_{\xi\sim N}\int\limits_{|\eta|\gtrsim \frac{N^2}{2^{100}}} \frac{1}{\eta^2}\, d\eta d\xi + \int\limits_{\tau\sim -N^2}\int\limits_{\xi\sim N} \frac{1}{-4\tau(\xi+\sqrt{-\tau})^2}\, d\xi d\tau \\
&\lesssim \int\limits_{\xi\sim N} \frac{1}{N^2}\, d\xi + \frac{1}{N} \\
&\lesssim \frac{1}{N},
\end{aligned}$$

and a similar result holds when $\xi < 0$. From this, we can conclude that

$$\|h\|_{L_x^{d-1}L_t^\infty} \lesssim N^{\frac{d-3}{2(d-1)}} \|A(\xi,\tau)\|_{L^2_{\tau,\xi}} \|\widehat{F}_0(\tau)\|_{L^2_\tau} \lesssim N^{\frac{d-3}{2(d-1)}-\frac{1}{2}} \|F(0)\|_{L^2_t}. \tag{44}$$

Similarly, we have the following,

$$\|h\|_{L^\infty_{x,t}} \lesssim \|F(0)\|_{L^2_t}. \tag{45}$$

In particular, for $d = 3$ and $N \geq 1$, we have that

$$N^{-\frac{1}{2}}\|h\|_{L_x^2L_t^\infty} \leq \|h\|_{L_x^2L_t^\infty} \lesssim N^{-\frac{1}{2}}\|F(0)\|_{L^2_t}. \tag{46}$$

Similarly, by Sobolev's embedding,

$$N^{\frac{1}{2}}\|h\|_{L_x^\infty L_t^2} \lesssim N^{\frac{1}{2}}\|h\|_{L_t^2L_x^\infty} \lesssim N^{-1}\|\Delta h\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}}\|F(0)\|_{L^2_t}, \tag{47}$$

where we used (36) in the last step. Lastly, it follows from (44) that

$$\|h\|_{L_t^\infty L_x^2} \leq \|h\|_{L_x^2L_t^\infty} \lesssim N^{-\frac{1}{2}}\|F(0)\|_{L^2_t}. \tag{48}$$

Putting together (44), (47) and (48), we are done with estimating h. Similar estimate for the term $1_{x>0}e^{it\Delta}v_0$ follows easily from Strichartz-type estimates (10), (11) and (12). □

In the proof of Theorem 1.1 in the next section, we will incorporate the low frequency projection $P_{\leq 1}u$ into the spaces X^s and Y^s, which are restricted to the time interval $T=[-1,1]$, in order to obtain the local well-posedness. Therefore, we need an estimate analogous to (29) for functions supported at low frequencies, which can be obtained from the two following propositions:

Proposition 3.5. *Let $T=[-1,1]$. For any function $u_0 \in L^2(\mathbb{R})$, we have*

$$\|P_{\leq 1}e^{it\Delta}u_0\|_{X_1(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}u_0\|_{L^2_x}. \tag{49}$$

Proof. In view of Strichartz's estimate (10) with $p=2$ and $q=\infty$ and (12a), it suffices to prove that

$$\|P_{\leq 1}e^{it\Delta}u_0\|_{L^\infty_x L^2_t(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}u_0\|_{L^2_x}.$$

Using the fact that $\widehat{P_{\leq 1}u}_0(\xi,t)$ is compactly supported in ξ and Plancherel theorem, we have

$$\begin{aligned}\|P_{\leq 1}e^{it\Delta}u_0\|_{L^\infty_x L^2_t(\mathbb{R}\times T)} &\leq \|P_{\leq 1}e^{it\Delta}u_0\|_{L^2_t L^\infty_x(T\times\mathbb{R})} \leq \|\psi(\xi)\hat{u}_0\|_{L^2_t L^1_\xi(T\times\mathbb{R})}\\ &\leq \|\psi(\xi)\hat{u}_0\|_{L^\infty_t L^2_\xi(T\times\mathbb{R})} \quad = \|P_{\leq 1}u_0\|_{L^2_x}. \qquad \square\end{aligned}$$

Proposition 3.6. *Let $T=[-1,1]$. For any function $F(x,t)$ such that $P_{\leq 1}F \in Y_1$, we have*

$$\Big\|\int_0^t e^{i(t-s)\Delta}P_{\leq 1}F(x,s)\,ds\Big\|_{X_1(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}F\|_{Y_1(\mathbb{R}\times T)}. \tag{50}$$

Proof. As in the proof of Proposition 3.3, it follows from Minkowski inequality that

$$\left\|\int_0^t e^{i(t-s)\Delta}P_{\leq 1}F(s)\,ds\right\|_{X_1(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}F\|_{L^1_t L^2_x(T\times\mathbb{R})}.$$

Thus, it suffices to prove that

$$\left\|\int_0^t e^{i(t-s)\Delta}P_{\leq 1}F(s)\,ds\right\|_{X_1(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}F\|_{L^1_x L^2_t(\mathbb{R}\times T)}.$$

Note that for $t\in[0,1]$, we can rewrite

$$\begin{aligned}\int_0^t e^{i(t-s)\Delta}P_{\leq 1}F(x,s)\,ds &= \int_{\mathbb{R}} \chi_{[0,1)}(t-s)\chi_{[0,1)}(s)e^{i(t-s)\Delta}P_{\leq 1}F(x,s)\,ds\\ &:= K(x,t)\star\chi_{[0,1)}(t)P_{\leq 1}F(x,t)\end{aligned}$$

where $\star$ is the space–time convolution and

$$K(x,t) = \int_{\mathbb{R}} e^{-it\xi^2+ix\xi} \chi_{[0,1)}(t)\psi\left(\frac{\xi}{N}\right) d\xi,$$

which obeys the estimate (17) with $N = 1$. Hence, by Young's inequality

$$\left\| \chi_{[0,1]}(t) \left[K(x,t) \star \chi_{[0,1)}(t) P_{\leq 1} F(x,t)\right] \right\|_{L_x^2 L_t^\infty} \lesssim \|\chi_{[0,1]}(t) P_{\leq 1} F\|_{L_x^2 L_t^1}.$$

We use the finite time restriction and apply Bernstein's and Minkowski's inequality.

$$\begin{aligned}
\|\chi_{[0,1]}(t) P_{\leq 1} F\|_{L_x^2 L_t^1} &\lesssim \|\chi_{[0,1]}(t) P_{\leq 1} F\|_{L_{x,t}^2} \\
&\lesssim \|\chi_{[-1,1]}(t) P_{\leq 1} F\|_{L_t^2 L_x^1} \\
&\leq \|\chi_{[-1,1]}(t) P_{\leq 1} F\|_{L_x^1 L_t^2}.
\end{aligned}$$

Since similar proof applies for the time interval $[-1, 0]$, we obtain

$$\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(s)\, ds \right\|_{L_x^2 L_t^\infty(\mathbb{R}\times T)} \lesssim \|P_{\leq 1} F\|_{L_x^1 L_t^2(\mathbb{R}\times T)}.$$

This estimate has the following two consequences. Firstly, from Minkowski's inequality, we have

$$\begin{aligned}
\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(s)\, ds \right\|_{L_t^\infty L_x^2(T\times\mathbb{R})} &\leq \left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(s)\, ds \right\|_{L_x^2 L_t^\infty(\mathbb{R}\times T)} \\
&\lesssim \|P_{\leq 1} F\|_{L_x^1 L_t^2(\mathbb{R}\times T)}.
\end{aligned}$$

Secondly, it follows from Minkowski's inequality, Bernstein's inequality and the finite time restriction that

$$\begin{aligned}
\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(s)\, ds \right\|_{L_x^\infty L_t^2(\mathbb{R}\times T)} &\leq \left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(s)\, ds \right\|_{L_t^2 L_x^\infty(T\times\mathbb{R})} \\
&\lesssim \left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(s)\, ds \right\|_{L_{x,t}^2(\mathbb{R}\times T)} \\
&\lesssim \left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(s)\, ds \right\|_{L_x^2 L_t^\infty(\mathbb{R}\times T)} \\
&\lesssim \|P_{\leq 1} F\|_{L_x^1 L_t^2(\mathbb{R}\times T)}.
\end{aligned}$$

This concludes the proof of (50). □

The essential part of the contraction argument is a multilinear estimate: an estimate of the form $\|\partial_x u_1 \prod_{i=2}^d u_i\|_{Y^s} \lesssim \prod_{i=1}^d \|u_i\|_{X^s}$. One of the main tools that we will use to prove this is the following Bilinear Strichartz estimate for the X^s space.

Theorem 3.7. *Let $N \gg M$ and suppose that u and v are supported at frequency N and M, respectively. Then, we have*

$$\|uv\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \|u\|_{X_N} \|v\|_{X_M}. \tag{51}$$

Proof. Let $F_1(x,t) = (i\partial_t + \Delta)u(x,t)$ and $F_2(x,t) = (i\partial_t + \Delta)v(x,t)$. We will prove that

$$\|uv\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \left(\|u(0)\|_{L^2_x} + \|F_1\|_{L^1_t L^2_x}\right) \left(\|v(0)\|_{L^2_x} + \|F_2\|_{L^1_t L^2_x}\right) \tag{52}$$

$$\|uv\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \left(\|u(0)\|_{L^2_x} + N^{-\frac{1}{2}}\|F_1\|_{L^1_x L^2_t}\right) \left(\|v(0)\|_{L^2_x} + M^{-\frac{1}{2}}\|F_2\|_{L^1_x L^2_t}\right) \tag{53}$$

$$\|uv\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \left(\|u(0)\|_{L^2_x} + N^{-\frac{1}{2}}\|F_1\|_{L^1_x L^2_t}\right) \left(\|v(0)\|_{L^2_x} + \|F_2\|_{L^1_t L^2_x}\right) \tag{54}$$

$$\|uv\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \left(\|u(0)\|_{L^2_x} + \|F_1\|_{L^1_t L^2_x}\right) \left(\|v(0)\|_{L^2_x} + M^{-\frac{1}{2}}\|F_2\|_{L^1_x L^2_t}\right). \tag{55}$$

To achieve (52), we consider the expansion of $u\bar{v}$ after using the Duhamel formula on u and v.

$$u(x,t) = e^{it\Delta}u(0) - i\int_0^t e^{i(t-s)\Delta} F_1(s)\, ds$$

$$v(x,t) = e^{it\Delta}v(0) - i\int_0^t e^{i(t-s)\Delta} F_2(s)\, ds.$$

It follows from the bilinear estimate for free solutions (25) that

$$\|e^{it\Delta}u(0)e^{it\Delta}v(0)\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \|u(0)\|_{L^2_x} \|v(0)\|_{L^2_x}.$$

By the Minkowski inequality, we have that

$$\begin{aligned}\|\int_0^t e^{i(t-s)\Delta} F_1(s) e^{it\Delta} v(0)\, ds\|_{L^2_{x,t}} &\lesssim N^{-\frac{1}{2}} \int_{\mathbb{R}} \|F_1(s)\|_{L^2_x} \|v(0)\|_{L^2_x}\, ds \\ &= N^{-\frac{1}{2}} \|F_1\|_{L^1_t L^2_x} \|v(0)\|_{L^2_x}.\end{aligned}$$

Similarly,

$$\|\int_0^t e^{it\Delta} u(0) e^{i(t-s)\Delta} F_2(s)\, ds\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \|u(0)\|_{L^2_x} \|F_2\|_{L^1_t L^2_x}.$$

With the same proof, we can estimate the last term in the product.

$$\| \int_0^t \int_0^t e^{i(t-s)\Delta} F_1(s) e^{i(t-s)\Delta} F_2(\tilde{s})\, ds d\tilde{s} \|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \|F_1\|_{L^1_t L^2_x} \|F_2\|_{L^1_t L^2_x},$$

and (52) follows.

To prove (53), we recall (35) which allows us to decompose u and v as follows

$$u(x,t) = e^{it\Delta}u(0) - \int_{\mathbb{R}} e^{it\Delta}\mathcal{L}u_y + (P_{N/2^{50}} 1_{x>0}) e^{it\Delta} u_y - h_{1,y}(x,t)\, dy \tag{56}$$

$$v(x,t) = e^{it\Delta}v(0) - \int_{\mathbb{R}} e^{it\Delta}\mathcal{L}v_{y'} + (P_{M/2^{50}} 1_{x>0}) e^{it\Delta} v_{y'} - h_{2,y'}(x,t)\, dy', \tag{57}$$

where $\mathcal{L} : L^2_x \to L^2_x$ is a bounded operator and u_y, $\mathcal{L}u_y$ and $h_{1,y}$ are defined similarly to (42), (41) and (42), respectively. From the remark following (43), we see that these functions are supported at frequency $\sim N$. Similar $v_{y'}, \mathcal{L}v_{y'}, h_{2,y'}$ Moreover, we have

$$\|\mathcal{L}u_y\|_{L^2_x} + \|u_y\|_{L^2_x} + \frac{1}{N}(\|\Delta h_y\|_{L^2_{x,t}} + \|\partial_t h_y\|_{L^2_{x,t}}) \lesssim \frac{1}{N^{\frac{1}{2}}} \|F_1(y,t)\|_{L^2_t}. \tag{58}$$

Similar conclusions hold for $v_{y'}$, $\mathcal{L}v_{y'}$ and $h_{2,y'}$ at frequency $\sim M$ with corresponding nonlinearity $F_2(y',t)$. Consider each term in the product uv. Let $\psi_{N/2^{50}}$ be the function defined by $P_{N/2^{50}} f := \psi_{N/2^{50}} * f$. Observe that for any $G \in L^2$, we have that

$$\begin{aligned}
&\|(P_{N/2^{50}} 1_{x>0}) e^{it\Delta} u_y G(x)\|_{L^2_{x,t}} \\
&\quad = \|(\psi_{N/2^{50}} * 1_{x>0}) e^{it\Delta} u_y G(x)\|_{L^2_{x,t}} \\
&\quad \le \int \|1_{x-z>0} e^{it\Delta} u_y(x) G(x)\|_{L^2_{x,t}} |\psi_{N/2^{50}}(z)|\, dz \\
&\quad \le \int \|e^{it\Delta} u_y(x) G(x)\|_{L^2_{x,t}} |\psi_{N/2^{50}}(z)|\, dz \\
&\quad \lesssim \|e^{it\Delta} u_y G\|_{L^2_{x,t}}.
\end{aligned}$$

With this, we can take care of all the terms involving $P_{N/2^{50}} 1_{x>0}$ in the expansion of uv. For any $A, B \in L^2$, we have

$$\|(P_{N/2^{50}} 1_{x>0}) e^{it\Delta} u_y e^{it\Delta} B\|_{L^2_{x,t}} \lesssim \|e^{it\Delta} u_y e^{it\Delta} B\|_{L^2_{x,t}}$$

$$\|(P_{N/2^{50}} 1_{x>0}) e^{it\Delta} u_y h_{2,y'}\|_{L^2_{x,t}} \lesssim \|e^{it\Delta} u_y h_{2,y'}\|_{L^2_{x,t}}.$$

Similarly,

$$\|e^{it\Delta}A(P_{N/2^{50}}1_{x>0})e^{it\Delta}v_{y'}\|_{L^2_{x,t}} \lesssim \|e^{it\Delta}Ae^{it\Delta}v_{y'}\|_{L^2_{x,t}}$$
$$\|h_{1,y}(P_{N/2^{50}}1_{x>0})e^{it\Delta}v_{y'}\|_{L^2_{x,t}} \lesssim \|h_{1,y}e^{it\Delta}v_{y'}\|_{L^2_{x,t}},$$

and lastly,

$$\begin{aligned}&\left\|\left[(P_{N/2^{50}}1_{x>0})e^{it\Delta}u_y\right]\left[(P_{N/2^{50}}1_{x>0})e^{it\Delta}v_{y'}\right]\right\|_{L^2_{x,t}}\\ \lesssim& \left\|e^{it\Delta}u_y\left[(P_{N/2^{50}}1_{x>0})e^{it\Delta}v_{y'}\right]\right\|_{L^2_{x,t}}\\ \lesssim& \|e^{it\Delta}u_ye^{it\Delta}v_{y'}\|_{L^2_{x,t}}.\end{aligned}$$

Therefore, we only have to worry about the terms of the forms $e^{it\Delta}Ae^{it\Delta}B$, $e^{it\Delta}Ah_{2,y'}$, $h_{1,y}e^{it\Delta}B$ and $h_{1,y}h_{2,y'}$. Note that any choice of A, that is not $u(0)$, is an integral with respect to y. The same holds for B. By the bilinear Strichartz estimate (25), one obtains

$$\|e^{it\Delta}Ae^{it\Delta}B\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}}\|A\|_{L^2_x}\|B\|_{L^2_x}. \tag{59}$$

We get the desired bound by observing that either we have $\|A\|_{L^2_x} = \|u(0)\|_{L^2_x}$ or $\|A\|_{L^2_x} \lesssim \int_{\mathbb{R}}\|u_y\|_{L^2_x}\,dy \lesssim N^{-\frac{1}{2}}\|F_1\|_{L^1_xL^2_t}$ from (58). It remains to estimate the terms that involve $h_{1,y}$ and $h_{2,y}$. By Hölder and Bernstein inequalities, (11) and (46), we have that

$$\begin{aligned}\|e^{it\Delta}Ah_{2,y'}\|_{L^2_{x,t}} &\lesssim \|e^{it\Delta}A\|_{L^\infty_xL^2_t}\|h_{2,y'}\|_{L^2_xL^\infty_t}\\ &\lesssim N^{-\frac{1}{2}}M^{-\frac{1}{2}}\|A\|_{L^2_x}\|F_2(y')\|_{L^2_t}.\end{aligned} \tag{60}$$

By taking $\int_{\mathbb{R}}\cdot\,dy'$ when $A = u(0)$ and $\int_{\mathbb{R}}\int_{\mathbb{R}}\cdot\,dydy'$ when $A = \mathcal{L}u_y$ or $A = u_y$ on both sides of the inequality and applying (36), we get the desired bound. On the other hand, we get the estimate for $\|h_{1,y}e^{it\Delta}B\|_{L^2_{x,t}}$ by observing that from (36), we have $\|\Delta h_{1,y}\|_{L^2_{x,t}} \lesssim N^{-\frac{3}{2}}\|F_1\|_{L^2_t}$. Hence,

$$\begin{aligned}\|h_{1,y}e^{it\Delta}B\|_{L^2_{x,t}} &\lesssim \|h_{1,y}\|_{L^2_{x,t}}\|e^{it\Delta}B\|_{L^\infty_{x,t}}\\ &\lesssim N^{-\frac{3}{2}}M^{\frac{1}{2}}\|F_1\|_{L^2_x}\|B\|_{L^\infty_tL^2_x}\\ &\le N^{-\frac{1}{2}}M^{-\frac{1}{2}}\|F_1\|_{L^2_x}\|B\|_{L^\infty_tL^2_x}.\end{aligned} \tag{61}$$

Lastly, we use (46) and (47) to estimate the remaining term

$$\begin{aligned}\|h_{1,y}h_{2,y'}\|_{L^2_{x,t}} &\le \|h_{1,y}\|_{L^\infty_xL^2_t}\|h_{2,y'}\|_{L^2_xL^\infty_t}\\ &\lesssim N^{-1}M^{-\frac{1}{2}}\|F_1(y)\|_{L^2_t}\|F_2(y')\|_{L^2_t}.\end{aligned} \tag{62}$$

Taking $\int_{\mathbb{R}}\int_{\mathbb{R}} \cdot\, dydy'$, we obtain (53). We are now left to proving (54) and (55). The proof is a mix of the ideas we used to prove (52) and (53). For (54), we write u using the decomposition (56) and v using the Duhamel formula. On the product expansion of $\|uv\|_{L^2_{x,t}}$, we apply the triangle inequality and Minkowski inequality. We then apply the bilinear estimate (25) to any term of the form $\|e^{it\Delta}Ae^{it\Delta}B\|_{L^2_{x,t}}$ to get the desired bound. This leaves us with the terms of the form $\|e^{it\Delta}Ah_{2,y'}\|_{L^2_{x,t}}$, on which we can apply (60). In the same manner, we can prove (55) using the Duhamel formula for u and the decomposition (57) for v. We finish the proof by observing that the terms of the form $\|h_{1,y}e^{it\Delta}B\|_{L^2_{x,t}}$ can be bounded using (61). □

4. Proof of Theorem 1.1

Let s be the exponent which satisfies the condition in Theorem 1.1. To obtain the local well-posedness, we redefine the spaces X^s and Y^s from (28) in a way that the projections on the low frequencies are combined together. Since we assume a finite time restriction, so any spaces mentioned below are defined on the product space $\mathbb{R}\times[-1,1]$.

$$\begin{aligned}
\|u\|_{Z_N} &= \|u\|_{L^\infty_t L^2_x \cap L^4_t L^\infty_x \cap L^6_{x,t}} + N^{-\frac{1}{2}}\|u\|_{L^2_x L^\infty_t} + N^{\frac{1}{2}}\|u\|_{L^\infty_x L^2_t} \\
\|u\|_{Y_N} &= \inf\{N^{-\frac{1}{2}}\|u_1\|_{L^1_x L^2_t} + \|u_2\|_{L^1_t L^2_x} \mid u_1+u_2=u\} \\
\|u\|_{X_N} &= \|u\|_{Z_N} + \|(i\partial_t+\Delta)u\|_{Y_N} \\
\|u\|_{X^s} &= \|P_{\le 1}u\|_{X_1} + \Big(\sum_{N\in 2^{\mathbb{N}}} N^{2s}\|P_N u\|^2_{X_N}\Big)^{\frac{1}{2}} \\
\|u\|_{Y^s} &= \|P_{\le 1}u\|_{Y_1} + \Big(\sum_{N\in 2^{\mathbb{N}}} N^{2s}\|P_N u\|^2_{Y_N}\Big)^{\frac{1}{2}}.
\end{aligned} \tag{63}$$

The previous section prepares us all the estimates we need in order to obtain the linear estimate for the X^s and Y^s spaces; It follows from (29), (49) and (50) that for any $s \ge \frac{1}{2}$,

$$\|u\|_{X^s} \lesssim \|u_0\|_{H^s} + \|F\|_{Y^s}. \tag{64}$$

We are now ready to prove the multilinear estimate.

Theorem 4.1. *Let $d \ge 3$. For any $u_1, u_2, \ldots, u_d \in X^s$ where $s \ge \frac{1}{2}$, we have the following estimate.*

$$\Big\|(\partial_x u_1)\prod_{i=2}^{d} u_i\Big\|_{Y^s} \lesssim \prod_{i=1}^{d}\|u_i\|_{X^s}. \tag{65}$$

Proof. It suffices to prove that

$$\Big\|(\partial_x u_1)\prod_{i=2}^{d} u_i\Big\|_{Y^s} \lesssim \|u_1\|_{X^s}\prod_{i=2}^{d}\|u_i\|_{X^{\frac{1}{2}}}, \tag{66}$$

which implies (65) since $X^s \subset X^{\frac{1}{2}}$ due to the absence of low frequency projections. In view of (49) and (50), we can treat $P_{\leq 1}$ as P_1, so it suffices to estimate the summation over high frequencies:

$$\sum_{N,N_1,\dots,N_d} N^s \left\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i)\right\|_{Y^s}, \tag{67}$$

where $N \geq 1$ and $N_i \geq 1$ for all i in the summation. We can assume that $N_1 \geq N_2 \geq \dots \geq N_d$ and $N \lesssim N_1$. This is because u_1 is the only term in (67) that has a derivative, and so any other frequency distribution would lead to a better estimate. We define $c_{N_1,1} = N_1^s \|P_{N_1}u_1\|_{X_{N_1}}$ and $c_{N_i,i} = N_i^{\frac{1}{2}}\|P_{N_i}u_i\|_{X_{N_i}}$ for $2 \leq i \leq d$. Thus, we see that $\|c_{N_1,1}\|_{l^2(N_1)} = \|u_1\|_{X^s}$ and $\|c_{N_i,i}\|_{l^2(N_i)} = \|u_i\|_{X^{\frac{1}{2}}}$ for $2 \leq i \leq d$. In order to obtain the l^2 summation of $c_{N_i,i}$, we will repeatedly be using the following application of the Cauchy–Schwarz inequality:

$$\begin{aligned}\sum_{N_j,\dots,N_d} \frac{1}{N_j^a}\prod_{i=j}^{d} c_{N_i,i} \leq \sum_{N_j,\dots,N_d}\prod_{i=j}^{d}\frac{1}{N_i^{\frac{a}{d}}}c_{N_i,i} &\leq \prod_{i=j}^{d}\sum_{N_i\geq 1}\frac{1}{N_i^{\frac{a}{d}}}c_{N_i,i} \\ &\lesssim \prod_{i=j}^{d}\|u_i\|_{X^{\frac{1}{2}}},\end{aligned} \tag{68}$$

for any $a > 0$. To prove (66), we split the summation over three different kinds of frequency interactions.

$$\begin{aligned}&\sum_{N,N_1,\dots,N_d} N^s \left\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i)\right\|_{Y^s} \\ &= \Big(\sum_{I} + \sum_{II} + \sum_{III}\Big) N^s \left\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i)\right\|_{Y^s}.\end{aligned}$$

Each of the summations contains certain ranges of $N, N_1, \dots, N_d$ described by the following cases:

I). $N_1 \gg N_2$ and $N \sim N_1$.
By Hölder inequality, (12) with $\gamma = 2$ and (68),

$$\begin{aligned}&\sum_{N_1,\dots,N_d} \left\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i)\right\|_{L_x^1 L_t^2} \\ &\qquad \lesssim \sum_{N_i} \|P_{N_1}\partial_x u_1 P_{N_2}u_2\|_{L^2_{x,t}} \|P_{N_3}u_3\|_{L_x^2 L_t^\infty} \prod_{i=4}^{d}\|P_{N_i}u_i\|_{L^\infty_{x,t}} \\ &\qquad \lesssim \sum_{N_i} \frac{1}{N_1^{s-\frac{1}{2}} N_2^{\frac{1}{2}}} \prod_{i=1}^{d} c_{N_i,i}\end{aligned}$$

$$\lesssim \frac{1}{N^{s-\frac{1}{2}}} \sum_{N_i} \frac{1}{N_2^{\frac{1}{2}}} \prod_{i=1}^{d} c_{N_i,i}$$

$$\lesssim \frac{1}{N^{s-\frac{1}{2}}} \sum_{N_1 \sim N} c_{N_1,1} \prod_{i=2}^{d} \|u_i\|_{X^{\frac{1}{2}}}.$$

Therefore,

$$\sum_{I} N^{s-\frac{1}{2}} \Big\| P_N(P_{N_1} \partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i) \Big\|_{L_x^1 L_t^2} \lesssim \sum_{N_1 \sim N} c_{N_1,1} \prod_{i=2}^{d} \|u_i\|_{X^{\frac{1}{2}}}.$$

Taking the l^2 summation with respect to $N \geq 1$, we obtain (66).

II). $N_1 \sim N_2 \gg N_3 \geq \ldots \geq N_d$ and $N \lesssim N_1$.

In this case, we use the bilinear estimate for the product $P_{N_1} \partial_x u_1 P_{N_3} u_3$ and put $P_{N_2} u_2$ in the Strichartz space $L_t^4 L_x^\infty$:

$$\begin{aligned}
&\sum_{N_1,\ldots,N_d} \Big\| P_N(P_{N_1} \partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i) \Big\|_{L_t^1 L_x^2} \\
&\lesssim \sum_{N_1,\ldots,N_d} \Big\| P_N(P_{N_1} \partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i) \Big\|_{L_t^{\frac{4}{3}} L_x^2} \\
&\lesssim \sum_{N_i} \| P_{N_1} \partial_x u_1 P_{N_3} u_3 \|_{L_{t,x}^2} \| P_{N_2} u_2 \|_{L_t^4 L_x^\infty} \prod_{i=4}^{d} \| P_{N_i} u_i \|_{L_{t,x}^\infty} \\
&\lesssim \sum_{N_i} \frac{1}{N_1^{s-\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{1}{2}}} \prod_{i=1}^{d} c_{N_i,i} \\
&\lesssim \Big(\sum_{N_1 \sim N_2} \frac{1}{N_1^s} c_{N_1,1} c_{N_2,2} \Big) \Big(\sum_{N_3,\ldots,N_d} \frac{1}{N_3^{\frac{1}{2}}} \prod_{i=3}^{d} c_{N_i,i} \Big) \\
&\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^s} c_{N_1,1} \Big)^{\frac{1}{2}} \prod_{i=2}^{d} \|u_i\|_{\dot{X}^{\frac{1}{2}}},
\end{aligned}$$

where we used (68) in the second to last step. Therefore,

$$\sum_{II} N^{2s} \| P_N(P_{N_1} \partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i) \|_{L_t^1 L_x^2}^2 \lesssim \|u_1\|_{X^s}^2 \prod_{i=2}^{d} \|u_i\|_{X^{\frac{1}{2}}}^2.$$

III). $N_1 \sim N_2 \sim N_3 \geq \ldots \geq N_d$ and $N \lesssim N_1$.

We divide the proof into two cases depending on the degree d.

A). $d = 3$.
Even though we cannot use the bilinear estimate in this case, the fact that $N_1 \sim N_2 \sim N_3$ allows us to cancel the derivative loss in $P_{N_1}\partial_x u_1$ by the $\frac{1}{2}$ regularity from $P_{N_2}u_2$ and $P_{N_3}u_3$ via the Hölder inequality:

$$\begin{aligned}
&\sum_{N_1\sim N_2\sim N_3} \left\| P_N[(P_{N_1}\partial_x u_1)P_{N_2}u_2 P_{N_3}u_3]\right\|_{L_t^1 L_x^2} \\
&\lesssim \sum_{N_1\sim N_2\sim N_3} \left\| P_N[(P_{N_1}\partial_x u_1)P_{N_2}u_2 P_{N_3}u_3]\right\|_{L_{t,x}^2} \\
&\lesssim \sum_{N_1\sim N_2\sim N_3} \|P_{N_1}\partial_x u_1\|_{L_{t,x}^6} \|P_{N_2}u_2\|_{L_{t,x}^6} \|P_{N_3}u_3\|_{L_{t,x}^6} \\
&\lesssim \sum_{N_1\sim N_2\sim N_3} \frac{N_1^{1-s}}{N_2^{\frac{1}{2}} N_3^{\frac{1}{2}}} c_{N_1,1}c_{N_2,2}c_{N_3,3} \\
&\lesssim \Big(\sum_{N_1\gtrsim N} \frac{1}{N_1^s} c_{N_1,1}\Big)^{\frac{1}{2}} \|u_2\|_{X^{\frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}}},
\end{aligned}$$

where the last step follows from the Cauchy–Schwarz inequality on $c_{N_1,1}c_{N_2,2}c_{N_3,3}$.

B). $d \geq 4$.
We again take advantage of the finite time restriction and put $P_{N_i}u_i$ for $1 \leq i \leq 4$ in suitable Strichartz spaces, namely $L_t^\infty L_x^2$ and $L_t^4 L_x^\infty$.

$$\begin{aligned}
&\sum_{N_1,\ldots,N_d} \left\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^d P_{N_i}u_i)\right\|_{L_t^1 L_x^2} \\
&\lesssim \sum_{N_1,\ldots,N_d} \left\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^d P_{N_i}u_i)\right\|_{L_t^{\frac{4}{3}} L_x^2} \\
&\lesssim \sum_{N_i} \|P_{N_1}\partial_x u_1\|_{L_t^\infty L_x^2} \prod_{i=2}^4 \|P_{N_i}u_i\|_{L_t^4 L_x^\infty} \prod_{i=5}^d \|P_{N_i}u_i\|_{L_{t,x}^\infty} \\
&\lesssim \sum_{N_i} \frac{N_1^{1-s}}{N_2^{\frac{1}{2}} N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}} \prod_{i=1}^d c_{N_i,i} \\
&\lesssim \Big(\sum_{N_1,N_2,N_3} \frac{1}{N_1^s} c_{N_1,1}c_{N_2,2}c_{N_3,3}\Big)\Big(\sum_{N_4\ldots,N_d} \frac{1}{N_4^{\frac{1}{2}}} \prod_{i=4}^d c_{N_i,i}\Big) \\
&\lesssim \Big(\sum_{N_1\gtrsim N} \frac{1}{N_1^s} c_{N_1,1}\Big)^{\frac{1}{2}} \prod_{i=2}^d \|u_i\|_{X^{\frac{1}{2}}}.
\end{aligned}$$

In either case, it follows that

$$\sum_{III} N^{2s} \Big\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i) \Big\|^2_{L^1_t L^2_x} \lesssim \|u_1\|^2_{X^s} \prod_{i=2}^{d} \|u_i\|^2_{X^{\frac{1}{2}}},$$

and this concludes the proof. □

In view of this theorem, if every term in $P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$ has only one derivative, then we expect to close the contraction argument in a subspace of $X^{\frac{1}{2}}$. On the other hand, if we replace u_j by $\partial_x u_j$ for some $j \geq 2$, then it follows from (9) that $\|\partial_x u_i\|_{X^s} \lesssim \|u_i\|_{X^{s+1}}$ for any $s > 0$, and so (66) yields

$$\begin{aligned}
\Big\| (\partial_x u_1)(\partial_x u_j) \prod_{\substack{i=2\\ i\neq j}}^{d} u_i \Big\|_{Y^{\frac{3}{2}}} &\lesssim \|u_1\|_{X^{\frac{3}{2}}} \|\partial_x u_j\|_{X^{\frac{1}{2}}} \prod_{\substack{i=2\\ i\neq j}}^{d} \|u_i\|_{X^{\frac{1}{2}}} \\
&\lesssim \|u_1\|_{X^{\frac{3}{2}}} \prod_{i=2}^{d} \|u_i\|_{X^{\frac{3}{2}}},
\end{aligned}$$

and for any $s \geq \frac{3}{2}$, we have

$$\Big\| (\partial_x u_1)(\partial_x u_j) \prod_{\substack{i=2\\ i\neq j}}^{d} u_i \Big\|_{Y^s} \lesssim \|u_1\|_{X^s} \prod_{i=2}^{d} \|u_i\|_{X^s}.$$

Consequently, in the case that a term in $P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$ has more than one derivative, we can employ the contraction argument in $X^{\frac{3}{2}}$.

Proof of Theorem 1.1. We define $F(u) := P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$. Let u and v be functions in X^s. We use the main linear estimate (64) and simple algebra to obtain

$$\begin{aligned}
\Big\| \int_0^t e^{i(t-s)\partial_x^2} [F(u(x,s)) - F(v(x,s))]\, ds \Big\|_{X^s} \\
&\leq c_1 \|F(u) - F(v)\|_{Y^s} \\
&\leq c_1 c_2 (\|u\|^{d-1}_{X^s} + \|v\|^{d-1}_{X^s}) \|u - v\|_{X^s},
\end{aligned} \tag{69}$$

where we used the multilinear estimate (65) in the last step.

Let $C := \min\left\{ (8c_1c_2)^{-\frac{1}{d-1}}, (4c_2)^{-\frac{1}{d-1}} \right\}$ where c_1 and c_2 are constants in (69). Define a Banach space as stated in the theorem:

$$X = \{u \in C^0_t H^s_x([-1,1] \times \mathbb{R}) \cap X^s : \|u\|_{X^s} \leq 2C\}.$$

Let $u_0 \in X$ such that $\|u_0\|_{H^s} \leq C$. Then, for $u \in X$, we define an operator

$$Lu := e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F(u(x,s))\,ds.$$

Again, by the main linear estimate, we have

$$\begin{aligned}\|Lu\|_{X^s} &\leq \|u_0\|_{H^s} + \|F\|_{Y^s}\\ &\leq \|u_0\|_{H^s} + c_2\|u\|_{X^s}^d\\ &\leq \frac{3C}{2} < 2C.\end{aligned}$$

Thus, L maps X to X. Moreover, from (69),

$$\|Lu - Lv\|_{X^s} \leq c_1c_2(\|u\|_{X^s}^{d-1} + \|v\|_{X^s}^{d-1})\|u-v\|_{X^s} \leq \frac{1}{4}\|u-v\|_{X^s}.$$

Thus, L is a contraction and the local well-posedness in X immediately follows. □

5. Proof of Theorem 1.3 when $d \geq 6$

In the previous sections, we used the time restriction to avoid dealing with low frequencies at $\xi \leq 1$. However, such argument cannot be used to obtain the global well-posedness for the gDNLS with nonlinearity of order $d \geq 5$. Therefore, the function spaces that we use will take these low frequencies into account. Let $s_0(d) = \frac{1}{2} - \frac{1}{d-1} = \frac{d-3}{2(d-1)}$ for $d \geq 5$. The spaces X^s and Y^s in (28) are replaced by those defined by the quasi-norms $\dot{X}^s$ and $\dot{Y}^s$ which in turn are defined by the norms X_N and Y_N,

$$\begin{aligned}\|u\|_{X_N} &= \|u\|_{L_t^\infty L_x^2} + N^{-\frac{1}{4}}\|u\|_{L_x^4L_t^\infty} + N^{\frac{1}{2}}\|u\|_{L_x^\infty L_t^2}\\ &\quad + N^{-\frac{1}{2}}\|(i\partial_t + \Delta)u\|_{L_x^1L_t^2}\\ \|u\|_{\dot{X}^s} &= \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_Nu\|_{X_N}^2\Big)^{\frac{1}{2}}\\ \|u\|_{X^s} &= \|u\|_{\dot{X}^0} + \|u\|_{\dot{X}^s}\\ \|u\|_{Y_N} &= N^{-\frac{1}{2}}\|u\|_{L_x^1L_t^2}\\ \|u\|_{\dot{Y}^s} &= \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_Nu\|_{Y_N}^2\Big)^{\frac{1}{2}}\\ \|u\|_{Y^s} &= \|u\|_{\dot{Y}^0} + \|u\|_{\dot{Y}^s}.\end{aligned} \tag{70}$$

Thus we have embeddings $X^s \hookrightarrow H^s$ and $X^s \hookrightarrow X^{s_0} \hookrightarrow \dot{X}^{s_0}$ for any $s \geq s_0$. In view of (29), we obtain the linear estimate

$$\|u\|_{X^s} \lesssim \|u_0\|_{H^s} + \|F\|_{Y^s}. \tag{71}$$

With these choices of spaces, we can establish the multilinear estimate for $d \geq 6$. The proof for the case $d = 5$ is significantly more involved and requires some frequency-modulation analysis, so we will postpone it to the next section.

Theorem 5.1. *Let $d \geq 6$. We have the following estimates.*

1). *For any $u_1, u_2, \dots, u_d \in X^{s_0}$,*

$$\Big\| \partial_x \prod_{i=1}^{d} u_i \Big\|_{\dot{Y}^{s_0}} \lesssim \prod_{i=1}^{d} \|u_i\|_{\dot{X}^{s_0}}. \tag{72}$$

2). *Let $s \geq s_0$. For any $u_1, u_2, \dots, u_d \in X^s$,*

$$\Big\| \partial_x \prod_{i=1}^{d} u_i \Big\|_{Y^s} \lesssim \prod_{i=1}^{d} \|u_i\|_{X^s}. \tag{73}$$

Proof. Our goal is to obtain the estimate

$$\sum_N N^{2s+1} \|P_N \prod_{i=1}^{d} u_i\|^2_{L^1_x L^2_t} \lesssim \sum_{j=1}^{d} \|u_j\|^2_{\dot{X}^s} \prod_{i \neq j} \|u_i\|^2_{\dot{X}^{s_0}}, \tag{74}$$

which implies (72) by choosing $s = s_0$. We get (73) by combining two different versions of this estimate with a fixed $s \geq s_0$ and with $s = 0$. We will focus on each term on the left-hand side of (73)

$$\begin{aligned} N^{2s-1} \Big\| P_N \partial_x \prod_{i=1}^{d} u_i \Big\|^2_{L^1_x L^2_t} &= N^{2s-1} \Big\| P_N \partial_x \sum_{N_1,\dots,N_d} \prod_{i=1}^{d} P_{N_i} u_i \Big\|^2_{L^1_x L^2_t} \\ &\lesssim N^{2s+1} \sum_{N_1,\dots,N_d} \Big\| P_N \prod_{i=1}^{d} P_{N_i} u_i \Big\|^2_{L^1_x L^2_t}, \end{aligned}$$

and study different kinds of frequency interactions. As before, we assume that $N_1 \geq N_2 \geq \dots \geq N_d$. We define $c_{N_1,1} = N_1^s \|P_{N_i} u_1\|_{X_{N_1}}$ and $c_{N_i,i} = N_i^{s_0} \|P_{N_i} u_i\|_{X_{N_i}}$ for $2 \leq i \leq d$. We will use the following two estimates for a product of terms with higher and lower frequencies.

1. For $N \lesssim N_1 \sim N_2 \sim \dots \sim N_{j-1}$ where $j \geq 3$, it follows from the Cauchy–Schwarz inequality that

$$\sum_{N_i} \prod_{i=1}^{j-1} c_{N_i,i} \lesssim \Big(\sum_{N_1 \gtrsim N} c^2_{N_1,1} \Big)^{\frac{1}{2}} \prod_{i=2}^{j-1} \|u_i\|_{\dot{X}^{s_0}}. \tag{75}$$

2. For $N_j \geq N_{j+1} \geq \ldots \geq N_d$ and any $\alpha > 0$, Young's inequality and trivial estimate $c_{N_i,i} \leq \|u_i\|_{\dot{X}^{s_0}}$ imply

$$\sum_{N_j \geq \ldots \geq N_d} \left(\frac{N_d}{N_j}\right)^{\alpha} \prod_{i=j}^{d} c_{N_i,i} \leq \prod_{i=j+1}^{d-1} \|u_i\|_{\dot{X}^{s_0}} \sum_{N_j \geq \ldots \geq N_d} \left(\frac{N_d}{N_j}\right)^{\alpha} c_{N_j,j} c_{N_d,d} \lesssim_\alpha \prod_{i=j}^{d} \|u_i\|_{\dot{X}^{s_0}}. \tag{76}$$

These estimates will be used in each case after appropriate uses of Hölder inequality, Bernstein inequality and bilinear estimate (51).

By Hölder and Bernstein inequalities,

$$\begin{aligned}
&\left\| P_N \prod_{i=1}^{d} u_i \right\|_{L_x^1 L_t^2} \\
&\lesssim \sum_{N_i} \|P_{N_1} u_1\|_{L_x^\infty L_t^2} \prod_{i=2}^{5} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \prod_{i=6}^{d} \|P_{N_i} u_i\|_{L_{x,t}^\infty} \\
&\lesssim \sum_{N_i} \|P_{N_1} u_1\|_{L_x^\infty L_t^2} \prod_{i=2}^{5} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \prod_{i=6}^{d} N_i^{\frac{1}{2}} \|P_{N_i} u_i\|_{L_t^\infty L_x^2} \\
&\lesssim \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \prod_{i=2}^{5} \frac{1}{N_i^{s_0-\frac{1}{4}}} \prod_{i=6}^{d} N_i^{\frac{1}{2}-s_0} \prod_{i=1}^{d} c_{N_i,i}.
\end{aligned}$$

Since $s_0 = \frac{1}{2} - \frac{1}{d-1}$, the sums of the exponents in $\prod_{i=2}^{5} N_i^{s_0-\frac{1}{4}}$ and $\prod_{i=6}^{d} N_i^{\frac{1}{2}-s_0}$ are equal. With the assumption that $N_2 \geq N_3 \geq \ldots \geq N_d$, the right-hand side is bounded by

$$\sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_d}{N_2}\right)^{\frac{1}{4(d-1)}} \prod_{i=1}^{d} c_{N_i,i}. \tag{77}$$

To estimate this term, we consider the following two frequency interactions.

1. $N \sim N_1 \gg N_2 \geq \ldots \geq N_d$.
   Using (76) on $c_{N_2,2} c_{N_3,3} \ldots c_{N_d,d}$, we can bound (77) by

$$\sum_{N_1 \sim N} \frac{1}{N_1^{s+\frac{1}{2}}} c_{N_1,1} \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}},$$

   for each fixed N. We have that

$$\sum_N \Big(\sum_{N_1 \sim N} \frac{N^{2s+1}}{N_1^{s+\frac{1}{2}}} c_{N_1,1} \Big)^2 \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}}^2 \sim \sum_{N_1} c_{N_1,1}^2 \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}}^2$$
$$= \|u_1\|_{\dot{X}^s}^2 \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}}^2,$$

which implies (74) as desired.

2. $N \lesssim N_1 \sim N_2 \geq \ldots \geq N_d$.
Using (75) on $c_{N_1,1}c_{N_2,2}$ and (76) on $c_{N_3,3}c_{N_4,4}\ldots c_{N_d,d}$, we can bound (77) by

$$\Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+1}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}}.$$

Therefore, by switching the order of summations,

$$\sum_N \sum_{N_1 \gtrsim N} \frac{N^{2s+1}}{N_1^{2s+1}} c_{N_1,1}^2 \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}}^2 \lesssim \sum_{N_1} c_{N_1,1}^2 \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}}^2,$$

which again implies (74). This concludes the proof for $d \geq 6$. □

Using the linear estimate (71) and the multilinear estimates (72) and (73), the proof for Theorem 1.3 follows in the same manner as in Theorem 1.1. Note that we did not use any finite time restriction in any parts of the proof.

6. Proof of Theorem 1.3 when $d = 5$

The difficulty in this case arises from the fact that there is no room left to put the lowest frequency term in $L^\infty_{x,t}$. Thus, we will take this case with extra care by adding the $\dot{X}^{0,b,q}$ spaces. For each $N \in 2^{\mathbb{Z}}$, let A_N be a set defined by

$$A_M := \{(\xi, \tau) : M \leq |\tau + \xi^2| \leq 2M\}. \tag{78}$$

Recall that $\tilde{u}(\xi, \tau)$ is the space–time Fourier transform of $u(x,t)$. The $\dot{X}^{0,b,q}$ space is the closure of the test functions under the following norm:

$$\|u\|_{\dot{X}^{0,b,q}} := \Big(\sum_{M \in 2^{\mathbb{Z}}} (N^b \|\tilde{u}\|_{L^2_{\xi,\tau}(A_M)})^q \Big)^{\frac{1}{q}}.$$

Previously, the nonlinear space $\dot{Y}^s$ is based on the space Z_N defined by the following norm on each frequency N.

$$\|u\|_{Z_N} := N^{-\frac{1}{2}} \|u\|_{L^1_x L^2_t}.$$

We modify this by adding the $\dot{X}^{0,-\frac{1}{2},1}$ space.

$$Y_N := Z_N + \dot{X}^{0,-\frac{1}{2},1}.$$

The solution space is defined by

$$\begin{aligned}
\|u\|_{X_N} &= \|u(0)\|_{L^2_x} + \|(i\partial_t + \Delta)u\|_{Y_N} \\
\|u\|_{\dot{X}^s} &= \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_N u\|^2_{X_N}\Big)^{\frac{1}{2}} \\
\|u\|_{X^s} &= \|u\|_{\dot{X}^0} + \|u\|_{\dot{X}^s},
\end{aligned} \tag{79}$$

and the nonlinear space is defined by

$$\begin{aligned}
\|u\|_{\dot{Y}^s} &= \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_N u\|^2_{Y_N}\Big)^{\frac{1}{2}} \\
\|u\|_{Y^s} &= \|u\|_{\dot{Y}^0} + \|u\|_{\dot{Y}^s}.
\end{aligned} \tag{80}$$

The following proposition shows that any estimates of free solutions that we proved in Section 2 can be extended to functions in X_N using the Schrödinger equation version of Lemma 4.1 from Tao ([29]).

Proposition 6.1 *([29]). Let S be any space–time Banach space that satisfies the following inequality,*

$$\|g(t)F(x,t)\|_S \le \|g\|_{L^\infty_t}\|F(x,t)\|_S, \tag{81}$$

for any $F \in S$ and $g \in L^\infty_t(\mathbb{R})$. Let $T : L^2(\mathbb{R}) \times \ldots \times L^2(\mathbb{R}) \to S$ be a spatial multilinear operator satisfying

$$\|T(e^{it\Delta}u_{1,0}, \ldots, e^{it\Delta}u_{k,0})\|_S \lesssim \prod_{i=1}^{k} \|u_{i,0}\|_{L^2_x}$$

for any $u_{1,0}, \ldots, u_{k,0} \in L^2_x(\mathbb{R})$. Then the following estimate

$$\|T(u_1, \ldots, u_k)\|_S \lesssim \prod_{i=1}^{k} (\|u_i(0)\|_{L^2_x} + \|(i\partial_t + \Delta)u_i\|_{\dot{X}^{0,-\frac{1}{2},1}}) \tag{82}$$

holds true for any $u_1, \ldots, u_k \in \dot{X}^{0,-\frac{1}{2},1}$ provided that u_i is supported at frequency $\sim N_i$ for $1 \le i \le k$.

With this proposition, we can obtain several Strichartz-type estimates for X_N that will be useful later on.

Corollary 6.2. *For any $u \in X_N$, we have the following estimates:*

$$\|u\|_{L_t^\infty L_x^2 \cap L_{t,x}^6} \lesssim \|u\|_{X_N} \tag{83}$$

$$\|u\|_{L_x^\infty L_t^2} \lesssim N^{-\frac{1}{2}} \|u\|_{X_N} \tag{84}$$

$$\|u\|_{L_x^4 L_t^\infty} \lesssim N^{\frac{1}{4}} \|u\|_{X_N}. \tag{85}$$

Proof. We apply Proposition 6.1 to linear estimates (10), (11) and (12), and bilinear estimates (24) and (25). □

We also have the bilinear estimate adapted to the space X_N.

Proposition 6.3. *Let N, M and λ be dyadic numbers such that $M \le N$ and $\lambda \lesssim N$. For any functions u and v supported at frequency $\sim N$ and $\sim M$, respectively, we have*

$$\|P_{>\lambda}(u\bar{v})\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} \|u\|_{X_N} \|v\|_{X_M}. \tag{86}$$

In addition, if $\hat{u}$ and $\hat{v}$ have disjoint supports and $\alpha = \inf|\operatorname{supp}(\hat{u}) - \operatorname{supp}(\hat{v})|$, then we have

$$\|uv\|_{L_{t,x}^2} \lesssim \alpha^{-\frac{1}{2}} \|u\|_{X_N} \|v\|_{X_M}. \tag{87}$$

Proof. As before, the bilinear estimate for homogeneous solutions (24) and (25) is the keys to proving these estimates. It suffices to prove (86), since (87) will follow in a similar manner. Denote $F_1 := (i\partial_t + \Delta)u$ and $F_2 := (i\partial_t + \Delta)v$. Using Proposition 6.1 with $T(u_1, u_2) = u_1 u_2$ to extend the bilinear estimate (24), we obtain

$$\|P_{>\lambda}(u\bar{v})\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} (\|u(0)\|_{L_x^2} + \|F_1\|_{\dot{X}^{0,-\frac{1}{2},1}})(\|v(0)\|_{L_x^2} + \|F_2\|_{\dot{X}^{0,-\frac{1}{2},1}}).$$

Therefore, it suffices to prove that for any $u \in X_N$ and $v \in X_M$,

$$\|P_{>\lambda}(u\bar{v})\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} (\|u(0)\|_{L_x^2} + \|F_1\|_{Z_N})(\|v(0)\|_{L_x^2} + \|F_2\|_{Z_M}), \tag{88}$$

$$\|P_{>\lambda}(u\bar{v})\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} (\|u(0)\|_{L_x^2} + \|F_1\|_{Z_N})(\|v(0)\|_{L_x^2} + \|F_2\|_{\dot{X}^{0,-\frac{1}{2},1}}), \tag{89}$$

$$\|P_{>\lambda}(u\bar{v})\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} (\|u(0)\|_{L_x^2} + \|F_1\|_{\dot{X}^{0,-\frac{1}{2},1}})(\|v(0)\|_{L_x^2} + \|F_2\|_{Z_N}). \tag{90}$$

We use the decomposition from (35) for u. However, in this case, the frequency localization at $\frac{N}{2^{50}}$ is replaced by $\frac{\lambda}{2^{50}}$:

$$u(x,t) = e^{it\Delta}u(0) - \int_{\mathbb{R}} e^{it\Delta}\mathcal{L}u_y + (P_{\lambda/2^{50}} 1_{x>0})e^{it\Delta}u_y - h_y(x,t)\,dy,$$

where $\mathcal{L} : L_x^2 \to L_x^2$ is a bounded operator and $u_y, \mathcal{L}u_y$ and h_y are defined similarly to (39), (41) and (42), respectively. From the remark following (43), we see that these functions are supported at frequency $\sim N$. Moreover, the following estimate still holds even with the frequency replacement.

$$\|\mathcal{L}u_y\|_{L^2_x} + \|u_y\|_{L^2_x} + \frac{1}{N}(\|\Delta h_y\|_{L^2_{x,t}} + \|\partial_t h_y\|_{L^2_{x,t}}) \lesssim \frac{1}{N^{\frac{1}{2}}}\|F_1(y,t)\|_{L^2_t}. \tag{91}$$

We consider all the possible terms in $P_{>\lambda}(u\bar{v})$. First, we consider all the terms that involve $P_{\lambda/2^{50}}1_{x>0}$. For any $G \in L^2_x$, we have that

$$\begin{aligned} P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})e^{it\Delta}u_yG\Big] &= P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})P_{\ll\lambda}(e^{it\Delta}u_yG)\Big] \\ &\quad + P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})P_{\gtrsim\lambda}(e^{it\Delta}u_yG)\Big] \\ &= P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})P_{\gtrsim\lambda}(e^{it\Delta}u_yG)\Big]. \end{aligned}$$

Let $\psi_{N/2^{50}}$ be the function defined by $P_{N/2^{50}}f := \psi_{N/2^{50}} * f$. Consequently,

$$\begin{aligned} &\Big\|P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})e^{it\Delta}u_yG\Big]\Big\|_{L^2_{x,t}} \\ &= \Big\|P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})P_{\gtrsim\lambda}(e^{it\Delta}u_yG)\Big]\Big\|_{L^2_{x,t}} \\ &\lesssim \Big\|(P_{\lambda/2^{50}}1_{x>0})P_{\gtrsim\lambda}(e^{it\Delta}u_yG)\Big\|_{L^2_{x,t}} \\ &= \Big\|(\psi_{\lambda/2^{50}} * 1_{x>0})P_{\gtrsim\lambda}(e^{it\Delta}u_yG)\Big\|_{L^2_{x,t}} \\ &\le \int \Big\|1_{x-z>0}P_{\gtrsim\lambda}\Big[e^{it\Delta}u_y(x)G(x)\Big]\Big\|_{L^2_{x,t}}|\psi_{N/2^{50}}(z)|\,dz \\ &\lesssim \|P_{\gtrsim\lambda}(e^{it\Delta}u_yG)\|_{L^2_{x,t}}. \end{aligned}$$

In other words, to estimate such terms, we can take out the $P_{\lambda/2^{50}}1_{x>0}$ factor just like what we did in the proof of Theorem 3.7. Following the same line of proof as for (53) but using a different bilinear estimate (24) instead of (25), we obtain (88). To prove (89) and (90), we will show that for any $v_0 \in L^2_x$ supported at frequency $\sim M$,

$$\|P_{>\lambda}(u\overline{e^{it\Delta}v_0})\|_{L^2_{x,t}} \lesssim \lambda^{-\frac{1}{2}}(\|u(0)\|_{L^2_x} + \|F_1\|_{Z_N})\|v_0\|_{L^2_x}, \tag{92}$$

which, in view of Proposition 6.1 with $T(v) = P_{>\lambda}(u\bar{v})$, leads to (89). From (24) and (91), we obtain

$$\begin{aligned} \|P_{>\lambda}(e^{it\Delta}u(0)\overline{e^{it\Delta}v_0})\|_{L^2_{x,t}} &\lesssim \lambda^{-\frac{1}{2}}\|u(0)\|_{L^2_x}\|v_0\|_{L^2_x}, \\ \|P_{>\lambda}(e^{it\Delta}\mathcal{L}u_y\overline{e^{it\Delta}v_0})\|_{L^2_{x,t}} &\lesssim \lambda^{-\frac{1}{2}}\|\mathcal{L}u_y\|_{L^2_x}\|v_0\|_{L^2_x} \\ &\lesssim (\lambda N)^{-\frac{1}{2}}\|F_1(y,t)\|_{L^2_t}\|v_0\|_{L^2_x}, \end{aligned}$$

$$\begin{aligned}\|P_{\gtrsim\lambda}(e^{it\Delta}u_y\overline{e^{it\Delta}v_0})\|_{L^2_{x,t}} &\lesssim \lambda^{-\frac{1}{2}}\|u_y\|_{L^2_x}\|v_0\|_{L^2_x} \\ &\lesssim (\lambda N)^{-\frac{1}{2}}\|F_1(y,t)\|_{L^2_t}\|v_0\|_{L^2_x}.\end{aligned}$$

We use the last inequality to estimate the term in $P_{>\lambda}(u\overline{e^{it\Delta}v_0})$ that involves $P_{\lambda/2^{50}}1_{x>0}$.

$$\begin{aligned}\left\|P_{>\lambda}\left[(P_{\lambda/2^{50}}1_{x>0})e^{it\Delta}u_y\overline{e^{it\Delta}v_0}\right]\right\|_{L^2_{x,t}} &\lesssim \|P_{\gtrsim\lambda}(e^{it\Delta}u_y\overline{e^{it\Delta}v_0})\|_{L^2_{x,t}} \\ &\lesssim (\lambda N)^{-\frac{1}{2}}\|F_1(y,t)\|_{L^2_t}\|v_0\|_{L^2_x}.\end{aligned}$$

For the remaining term, we use the Hölder inequality, (91) and the fact that $\lambda \lesssim N$.

$$\begin{aligned}\|P_{>\lambda}(h_y\overline{e^{it\Delta}v_0})\|_{L^2_{x,t}} &\lesssim \|h_y\|_{L^2_{x,t}}\|e^{it\Delta}v_0\|_{L^\infty_{x,t}} \\ &\lesssim \frac{M^{\frac{1}{2}}}{N^{\frac{3}{2}}}\|F_1(y,t)\|_{L^2_t}\|v_0\|_{L^2_x} \\ &\lesssim (\lambda N)^{-\frac{1}{2}}\|F_1(y,t)\|_{L^2_t}\|v_0\|_{L^2_x}.\end{aligned} \tag{93}$$

Recalling that $\|(i\partial_t+\Delta)u\|_{Z_N} = N^{-\frac{1}{2}}\|(i\partial_t+\Delta)u\|_{L^1_xL^2_t}$, these estimates yield (92) via the Minkowski inequality. The proof for (90) is similar, except at (93) where we have the following modification:

$$\begin{aligned}\|P_{>\lambda}(e^{it\Delta}u_0\overline{h}_{y'})\|_{L^2_{x,t}} &\lesssim \|e^{it\Delta}u_0\|_{L^\infty_xL^2_t}\|h_{y'}\|_{L^2_xL^\infty_t} \\ &\lesssim (NM)^{-\frac{1}{2}}\|u_0\|_{L^2_x}\|F_2(y',t)\|_{L^2_t} \\ &\lesssim (\lambda M)^{-\frac{1}{2}}\|u_0\|_{L^2_x}\|F_2(y',t)\|_{L^2_t}.\end{aligned}$$

For the second to last inequality, we used the smoothing estimate (11) and (44) with $d=3$. This concludes the proof of (86). □

We will also use the following estimate which was taken from Tao ([29]) and modified to be suitable to our spaces.

Proposition 6.4. *Suppose that u is supported at frequency $\sim N$. Then we have*

$$\|u\|_{\dot{X}^{0,\frac{1}{2},\infty}} \lesssim \|u\|_{X_N}. \tag{94}$$

Proof. Consider the Duhamel's formula of u.

$$u(x,t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F_1(s)\,ds - i\int_0^t e^{i(t-s)\Delta}F_2(s)\,ds, \tag{95}$$

where $F_1 \in Z_N$ and $F_2 \in \dot{X}^{0,-\frac{1}{2},1}$. For $i = 1, 2$, we split the term

$$\int_0^t e^{i(t-s)\Delta} F_i(s)\, ds = \int_{-\infty}^t e^{i(t-s)\Delta} F_i(s)\, ds - e^{it\Delta} \int_{-\infty}^0 e^{-is\Delta} F_i(s)\, ds.$$

Since the $\dot{X}^{0,\frac{1}{2},\infty}$ seminorm vanishes on any free solution, it suffices to estimate the first term. For F_1, we recall the computation (33) from the proof of Lemma 3.4 that the first term is equal to

$$\int w_y\, dy \quad \text{where} \quad \widetilde{w}_y = \frac{\psi_N(\xi)}{-\tau - \xi^2 - i0} \widehat{F_1}(y, \tau).$$

With a direct integration, we see that

$$\begin{aligned}
\|\chi_{A_M} \widetilde{w}\|_{L^2_{x,\tau}} &\sim \frac{1}{N^{\frac{1}{2}}} \Big(\int \int_{\xi \sim N} \frac{|\xi|}{(\tau + \xi^2)^2} \chi_{A_M} [\widehat{F_1}(y, \tau)]^2\, d\xi d\tau \Big)^{\frac{1}{2}} \\
&\lesssim \frac{1}{N^{\frac{1}{2}} M^{\frac{1}{2}}} \|F_1(y, t)\|_{L^2_t}.
\end{aligned}$$

From the definition of $\dot{X}^{0,\frac{1}{2},\infty}$, it follows that

$$\Big\| \int_{-\infty}^t e^{i(t-s)\Delta} F_1(s)\, ds \Big\|_{\dot{X}^{0,\frac{1}{2},\infty}} \lesssim \|F_1\|_{Z_N}.$$

On the other hand, we consider the space–time Fourier transform

$$\mathcal{F} \int \chi_{(0,\infty)}(t-s) e^{i(t-s)\Delta} F_2(s)\, ds = \frac{\widetilde{F}_{2,M}(\xi, \tau)}{-\tau - \xi^2 - i0}.$$

It follows from the Plancherel's theorem that

$$\Big\| \int_{-\infty}^t e^{i(t-s)\Delta} F_2(s)\, ds \Big\|_{\dot{X}^{0,\frac{1}{2},\infty}} \lesssim \|F_2\|_{\dot{X}^{0,-\frac{1}{2},1}},$$

and the conclusion immediately follows. □

We are ready to prove the multilinear estimate. Note that the position of complex conjugates will be significant in the analysis below.

Theorem 6.5. *For $1 \le i \le 5$, let u_i represent u or $\bar{u}$. Then we have the following estimates.*

1). *For any* $u \in X^{\frac{1}{4}}$,

$$\Big\| \partial_x \prod_{i=1}^{5} u_i \Big\|_{\dot{Y}^{\frac{1}{4}}} \lesssim \|u\|^5_{\dot{X}^{\frac{1}{4}}}. \tag{96}$$

2). *Let* $s \geq \frac{1}{4}$. *For any* $u \in X^s$,

$$\Big\| \partial_x \prod_{i=1}^{5} u_i \Big\|_{Y^s} \lesssim \|u\|^5_{X^s}. \tag{97}$$

Proof. As before, our goal is to obtain the estimate

$$\sum_N N^{2s+2} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|^2_{Y_N} \lesssim \|u\|^2_{\dot{X}^s} \|u\|^8_{\dot{X}^{\frac{1}{4}}}. \tag{98}$$

First, we split each term in the left-hand side as the sum of all possible frequency interactions:

$$N^{2s+2} \Big\| P_N \partial_x \prod_{i=1}^{5} u_i \Big\|^2_{Y_N} \lesssim N^{2s+2} \sum_{N_1,\dots,N_5} \Big\| P_N \prod_{i=1}^{5} P_{N_i} u_i \Big\|^2_{Y_N}.$$

Assume that $N_1 \geq N_2 \geq \dots \geq N_5$. Define $c_{N_1,1} = N_1^s \|P_{N_1} u\|_{X_{N_1}}$ and $c_{N_i,i} = N_i^{\frac{1}{4}} \|P_{N_i} u\|_{X_{N_i}}$ for $2 \leq i \leq 5$. We make a slight abuse of notation by using $\sum_{N_i}$ for the summation over all possible $N_1, N_2, \dots, N_5$ when the restrictions on these numbers are clear. We also will be using the Cauchy–Schwarz inequality (75) and Young's inequality (76).

We split the left-hand side of (98) over four different kinds of frequency interactions:

$$\sum_{N,N_1,\dots,N_5} N^s \Big\| P_N (P_{N_1} \partial_x u_1 \prod_{i=2}^{5} P_{N_i} u_i) \Big\|_{Y_N}$$
$$= \Big(\sum_I + \sum_{II} + \sum_{III} + \sum_{IV} \Big) N^s \Big\| P_N (P_{N_1} \partial_x u_1 \prod_{i=2}^{5} P_{N_i} u_i) \Big\|_{Y_N}.$$

Each of the summations contains certain ranges of $N, N_1, \dots, N_5$ described by the following cases:

I). $N \lesssim N_1 \sim N_2 \sim N_3 \sim N_4 \sim N_5$.
By Hölder and Cauchy–Schwarz inequalities, we have

$$\Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L^1_x L^2_t} \lesssim \sum_{N_i} \|P_{N_1} u_1\|_{L^\infty_x L^2_t} \prod_{i=2}^{5} \|P_{N_i} u_i\|_{L^4_x L^\infty_t}$$

$$= \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \prod_{i=1}^{5} c_{N_i,i}$$

$$\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+1}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u\|_{\dot{X}^{\frac{1}{4}}}^4 .$$

Summing over $N \in 2^{\mathbb{Z}}$, we see that

$$\sum_{I} N^{2s+1} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2}^2 \lesssim \sum_{N_1} \sum_{N \lesssim N_1} \left(\frac{N}{N_1} \right)^{2s+1} c_{N_1,1}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8$$

$$\lesssim \|u\|_{\dot{X}^s}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8 .$$

II). $N \sim N_1 \gg N_2 \geq N_3 \geq N_4 \geq N_5$.
By the bilinear estimate (86) or (87) on $P_{N_1} u_1 P_{N_2} u_2$ (depending on the complex conjugates) and Bernstein inequality on $P_{N_5} u_5$, we have that for each fixed N,

$$\Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2}$$

$$\lesssim \sum_{N_i} \| P_{N_1} u_1 P_{N_2} u_2 \|_{L_{x,t}^2} \prod_{i=3}^{4} \| P_{N_i} u_i \|_{L_x^4 L_t^\infty} \| P_{N_5} u_5 \|_{L_{x,t}^\infty}$$

$$\lesssim \sum_{N_i} \frac{N_5^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \| P_{N_1} u_1 \|_{X_{N_1}} \| P_{N_2} u_2 \|_{X_{N_2}} \prod_{i=3}^{4} \| P_{N_i} u_i \|_{L_x^4 L_t^\infty} \| P_{N_5} u_5 \|_{L_t^\infty L_x^2}$$

$$= \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_2} \right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i} .$$

By Young's inequality (76), this term is bounded by

$$\lesssim \sum_{N_1 \sim N} \frac{1}{N_1^{s+\frac{1}{2}}} c_{N_1,1} \|u\|_{\dot{X}^{\frac{1}{4}}}^4 .$$

Therefore,

$$\sum_{II} N^{2s+1} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2}^2 \lesssim \sum_{N} \Big(\sum_{N_1 \sim N} \left(\frac{N}{N_1} \right)^{s+\frac{1}{2}} c_{N_1,1} \Big)^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8$$

$$\lesssim \Big(\sum_{N_1} \sum_{N \sim N_1} c_{N_1,1}^2 \Big) \|u\|_{\dot{X}^{\frac{1}{4}}}^8$$

$$\sim \|u\|_{\dot{X}^s}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8 .$$

III). $N \lesssim N_1 \sim N_2 \sim N_{j-1} \gg N_j \geq N_5$ where $j = 3$ or $j = 4$.
This is similar to case *II*), but instead we use the bilinear estimate on $P_{N_1}u_1 P_{N_j}u_j$.

$$\begin{aligned}
&\Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2} \\
&\quad \lesssim \sum_{N_i} \| P_{N_1}u_1 P_{N_j}u_j \|_{L^2_{x,t}} \prod_{\substack{2\le i\le 4\\ i\neq j}} \| P_{N_i}u_i \|_{L_x^4 L_t^\infty} \| P_{N_5}u_5 \|_{L^\infty_{x,t}} \\
&\quad \lesssim \sum_{N_i} \frac{N_5^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \| P_{N_1}u_1 \|_{X_{N_1}} \| P_{N_j}u_j \|_{X_{N_2}} \prod_{\substack{2\le i\le 4\\ i\neq j}} \| P_{N_i}u_i \|_{L_x^4 L_t^\infty} \| P_{N_5}u_5 \|_{L_t^\infty L_x^2} \\
&\quad \lesssim \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_j}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}.
\end{aligned}$$

Applying the Cauchy–Schwarz inequality (75) on $\prod_{i=1}^{j-1} c_{N_i,i}$ and Young's inequality (76) on $\prod_{i=j}^{5} c_{N_i,i}$, we see that

$$\sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_j}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i} \lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+1}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u\|^8_{\dot{X}^{\frac{1}{4}}}.$$

Therefore,

$$\begin{aligned}
\sum_{III} N^{2s+1} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|^2_{L_x^1 L_t^2} &\lesssim \Big(\sum_N \sum_{N_1 \gtrsim N} \left(\frac{N}{N_1}\right)^{2s+1} c_{N_1,1}^2 \Big) \|u\|^8_{\dot{X}^{\frac{1}{4}}} \\
&\sim \|u\|^2_{\dot{X}^s} \|u\|^8_{\dot{X}^{\frac{1}{4}}}.
\end{aligned}$$

IV). $N \lesssim N_1 \sim N_2 \sim N_3 \sim N_4 \gg N_5$.
In this case, we will take the number of complex conjugates in $u_1u_2u_3u_4$ into consideration. Note that the positions of conjugates does not matter here.

1). $u_1 = u_3 = u$ and $u_2 = u_4 = \bar{u}$. We divide into further subcases by comparing the sizes between N and N_5.

1.1). $N \sim N_5$.
In this case, we first use Hölder inequality and then apply the bilinear estimate (87) on $\| P_{N_1}u_1 P_{N_5}u_5 \|_{L^2_{x,t}}$

$$\begin{aligned}
&\Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2} \\
&\quad \lesssim \sum_{N_i} \| P_{N_1}u_1 P_{N_5}u_5 \|_{L^2_{x,t}} \prod_{i=2}^{3} \| P_{N_i}u_i \|_{L_x^4 L_t^\infty} \| P_{N_4}u_4 \|_{L^\infty_{x,t}}
\end{aligned}$$

$$\lesssim \sum_{N_i} \frac{N_4^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \|P_{N_1}u_1\|_{X_{N_1}} \|P_{N_5}u_5\|_{X_{N_5}} \prod_{i=2}^{3} \|P_{N_i}u_i\|_{L_x^4 L_t^\infty} \|P_{N_4}u_4\|_{L_t^\infty L_x^2}$$

$$\lesssim \sum_{N_i} \frac{N_4^{\frac{1}{4}}}{N_1^{s+\frac{1}{2}} N_5^{\frac{1}{4}}} \prod_{i=1}^{5} c_{N_i,i}$$

$$\sim \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{4}} N^{\frac{1}{4}}} \prod_{i=1}^{5} c_{N_i,i}$$

$$\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+\frac{1}{2}} N^{\frac{1}{2}}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u\|_{\dot{X}^{\frac{1}{4}}}^4,$$

where we used Cauchy–Schwarz, the fact that $N_4 \sim N_1$, $N \sim N_5$ and the trivial inequality $c_{N_5,5} \le \|u\|_{\dot{X}^{\frac{1}{4}}}$ in the last step. Consequently,

$$\sum_{\substack{IV \\ N \sim N_5}} N^{2s+1} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2}^2 \lesssim \Big(\sum_N \sum_{N_1 \gtrsim N} \Big(\frac{N}{N_1} \Big)^{2s+\frac{1}{2}} c_{N_1,1}^2 \Big) \|u\|_{\dot{X}^{\frac{1}{4}}}^8$$

$$\sim \|u\|_{\dot{X}^s}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8.$$

1.2). $N \gg N_5$.
We split $\prod_{i=1}^5 P_{N_i} u_i$ into four terms using low and high frequency projections.

$$P_{N_1}u_1 P_{N_2}u_2 = P_{\ll N}(P_{N_1}u_1 P_{N_2}u_2) + P_{\gtrsim N}(P_{N_1}u_1 P_{N_2}u_2),$$
$$P_{N_3}u_3 P_{N_4}u_4 = P_{\ll N}(P_{N_3}u_3 P_{N_4}u_4) + P_{\gtrsim N}(P_{N_3}u_3 P_{N_4}u_4).$$

Since $N \gg N_5$, so $\prod_{i=1}^4 P_{N_i} u_i$ must be at frequency $\gg N$. Thus, we can assume that each of the resulting terms after the splits contains at least one high frequency projection. Thus, it suffices to estimate the term:

$$P_{\gtrsim N}(P_{N_1}u_1 P_{N_2}u_2) \prod_{i=3}^{5} P_{N_i} u_i.$$

We start by applying the bilinear estimate (86) on $P_{\gtrsim N}(P_{N_1}u_1 P_{N_2}u_2)$,

$$\|P_{\gtrsim N}(P_{N_1}u_1 P_{N_2}u_2)\|_{L_{x,t}^2} \lesssim \frac{1}{N^{\frac{1}{2}}} \|P_{N_1}u\|_{X_{N_1}} \|P_{N_2}u\|_{X_{N_2}}. \tag{99}$$

Then, by applying the estimate (75) on $c_{N_1,1}c_{N_3,3}c_{N_4,4}$ and (76) on $c_{N_2,2}c_{N_5,5}$, we obtain

$$
\begin{aligned}
&\Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2} \\
&\lesssim \sum_{N_i} \| P_{\gtrsim N}(P_{N_1}u_1 P_{N_2}u_2)\|_{L^2_{x,t}} \prod_{i=3}^{4} \|P_{N_i}u_i\|_{L_x^4 L_t^\infty} \|P_{N_5}u_5\|_{L^\infty_{x,t}} \\
&\lesssim \sum_{N_i} \frac{N_5^{\frac{1}{2}}}{N^{\frac{1}{2}}} \|P_{N_1}u\|_{X_{N_1}} \|P_{N_2}u\|_{X_{N_2}} \prod_{i=3}^{4} \|P_{N_i}u_i\|_{L_x^4 L_t^\infty} \|P_{N_5}u_5\|_{L_t^\infty L_x^2} \\
&\lesssim \sum_{N_i N} \frac{1}{N^{\frac{1}{2}} N_1^s} \left(\frac{N_5}{N_2}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i} \\
&\sim \sum_{N_i N} \frac{1}{N^{\frac{1}{2}} N_1^s} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i} \\
&\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N N_1^{2s}} c^2_{N_1,1} \Big)^{\frac{1}{2}} \|u\|^4_{\dot{X}^{\frac{1}{4}}}, \qquad (100)
\end{aligned}
$$

where we used Cauchy–Schwarz on $\sum_{N_i} \frac{1}{N^{\frac{1}{2}} N_1^s} c_{N_1,1} c_{N_2,2}$ and Young's inequality on $\sum_{N_i} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} c_{N_3,3} c_{N_4,4} c_{N_5,5}$. Therefore,

$$
\begin{aligned}
\sum_{\substack{IV \\ N \gg N_5}} N^{2s+1} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|^2_{L_x^1 L_t^2} &\lesssim \Big(\sum_N \sum_{N_1 \gtrsim N} \left(\frac{N}{N_1}\right)^{2s} c^2_{N_1,1} \Big) \|u\|^8_{\dot{X}^{\frac{1}{4}}} \\
&\sim \|u\|^2_{\dot{X}^s} \|u\|^8_{\dot{X}^{\frac{1}{4}}}.
\end{aligned}
$$

1.3). $N \ll N_5$.
This is similar to case 1.2), but we split $\prod_{i=1}^5 P_{N_i} u_i$ at N_5 instead of N.

$$
\begin{aligned}
P_{N_1}u_1 P_{N_2}u_2 &= P_{\ll N_5}(P_{N_1}u_1 P_{N_2}u_2) + P_{\gtrsim N_5}(P_{N_1}u_1 P_{N_2}u_2), \\
P_{N_3}u_3 P_{N_4}u_4 &= P_{\ll N_5}(P_{N_3}u_3 P_{N_4}u_4) + P_{\gtrsim N_5}(P_{N_3}u_3 P_{N_4}u_4).
\end{aligned}
$$

Since the output is supported at frequency $N \ll N_5$, we can see that $\prod_{i=1}^4 P_{N_i} u_i$ must be supported at frequency $\sim N_5$. Thus, we can assume that each term in the product expansion contains at least one high frequency projection. To estimate the product, we can use (99) and (100) that we just obtained and replace $N^{-\frac{1}{2}}$ by $N_5^{-\frac{1}{2}}$.

$$
\Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2} \lesssim \sum_{N_i} \frac{1}{N_5^{\frac{1}{2}} N_1^s} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}
$$

$$\ll \sum_{N_i} \frac{1}{N^{\frac{1}{2}} N_1^s} \left(\frac{N_5}{N_3} \right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}$$

$$\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N N_1^{2s}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u\|_{\dot{X}^{\frac{1}{4}}}^4 ,$$

which leads to the same result as in the previous case.

2). $u_1 = u_2 = u_3 = u$, u_4 and u_5 can be either u or $\bar{u}$.
This is the hardest case and requires some frequency-modulation analysis. Suppose that for some $1 \le j \le 5$ the space–time Fourier transform of $P_{N_j} u$ is supported in the set

$$\{(\xi, \tau) : |\tau + N_1^2| > \frac{1}{32} N_1^2\}, \tag{101a}$$

or that of $P_{N_j} \bar{u}$ (for $4 \le j \le 5$) is supported in the set

$$\{(\xi, \tau) : |\tau - N_1^2| > \frac{1}{32} N_1^2\}. \tag{101b}$$

Then, (94) yields

$$\|P_{N_j} u_j\|_{L^2_{x,t}} \lesssim N_1^{-1} \|P_{N_j} u_j\|_{\dot{X}^{0,\frac{1}{2},\infty}} \lesssim N_1^{-1} \|P_{N_j} u_j\|_{X_{N_j}} .$$

Without loss of generality, assume that $j = 1$. Then by Hölder and Bernstein inequalities,

$$\Big\| P_N \prod_{i=1}^{5} P_{N_i} u_i \Big\|_{L^1_x L^2_t} \lesssim \|P_{N_1} u_1\|_{L^2_{x,t}} \prod_{i=2}^{3} \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \prod_{i=4}^{5} \|P_{N_i} u_i\|_{L^\infty_{x,t}}$$

$$\lesssim \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_4 N_5}{N_1^2} \right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}$$

$$\sim \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_3} \right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i} .$$

On the other hand, if the space–time Fourier transform of $P_{N_5} u_5$ is supported in the set (101a) in the case $u_5 = u$ or (101b) in the case $u_5 = \bar{u}$, then we have

$$\Big\| P_N \prod_{i=1}^{5} P_{N_i} u_i \Big\|_{L^1_x L^2_t} \lesssim \prod_{i=1}^{2} \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \|P_{N_3} u_3 P_{N_4} u_4 P_{N_5} u_5\|_{L^2_{x,t}}$$

$$\lesssim \prod_{i=1}^{2} \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \prod_{i=3}^{4} \|P_{N_i} u_i\|_{L^\infty_t L^4_x} \|P_{N_5} u_5\|_{L^2_t L^\infty_x}$$

$$\lesssim N_5^{\frac{1}{2}} \prod_{i=1}^{4} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \|P_{N_5} u_5\|_{L_{x,t}^2}$$

$$\lesssim \frac{N_5^{\frac{1}{4}}}{N_1^{s+\frac{3}{4}}} \prod_{i=1}^{5} c_{N_i,i}$$

$$\sim \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}.$$

We then get the desired result by observing that

$$\frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i} \lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+1}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u\|_{\dot{X}^{\frac{1}{4}}}^4.$$

Thus, we can assume that the space–time Fourier transform of $P_{N_j} u$ is supported in the set

$$\{\xi, \tau : |\tau + N_1^2| \le \frac{1}{32} N_1^2\}, \tag{102a}$$

and that of $P_{N_k} \bar{u}$ is supported in

$$\{\xi, \tau : |\tau - N_1^2| \le \frac{1}{32} N_1^2\}. \tag{102b}$$

Here, we introduce Riesz transforms P_+ and P_- defined by

$$\widehat{P_+ f}(\xi) = 1_{\xi \ge 0} \hat{f}, \qquad \widehat{P_- f}(\xi) = 1_{\xi < 0} \hat{f}.$$

Then, denoting $P_+ P_{N_i} := P_{N_i}^+$ and $P_- P_{N_i} := P_{N_i}^-$, for $1 \le i \le 4$, we decompose $P_{N_i} u_i$ into

$$P_{N_i} u_i = P_{N_i}^+ u_i + P_{N_i}^- u_i,$$

and consider all the terms that we get from $\prod_{i=1}^5 P_{N_i} u_i$. For any term that contains $P_{N_j}^+ u P_{N_k}^- u$, $P_{N_j}^+ u P_{N_k}^+ \bar{u}$ or $P_{N_j}^- u P_{N_k}^- \bar{u}$, where $1 \le j < k \le 4$, we can apply the bilinear estimates (86) and (87), then proceed with the Hölder's and Bernstein inequality on $L_x^1 L_t^2$ as in the previous cases. For example, if $j = 1$ and $k = 2$, then we have

$$\Big\| P_N (P_{N_1}^+ u_1 P_{N_2}^- u_2 \prod_{i=3}^{5} P_{N_i} u_i) \Big\|_{L_x^1 L_t^2}$$

$$\lesssim \frac{N_5^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \prod_{i=1}^{2} \|P_{N_i} u\|_{X_{N_i}} \prod_{i=3}^{4} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \|P_{N_5} u_5\|_{L_t^\infty L_x^2}$$

$$\lesssim \frac{1}{N_1^{s+\frac{1}{2}}}\left(\frac{N_5}{N_2}\right)^{\frac{1}{4}}\prod_{i=1}^{5} c_{N_i,i}$$
$$\sim \frac{1}{N_1^{s+\frac{1}{2}}}\left(\frac{N_5}{N_3}\right)^{\frac{1}{4}}\prod_{i=1}^{5} c_{N_i,i}.$$

Therefore, it suffices to consider the following four terms.

i. $(\prod_{i=1}^{3} P_{N_i}^{+}u)P_{N_4}^{+}u P_{N_5}u_5$
ii. $(\prod_{i=1}^{3} P_{N_i}^{-}u)P_{N_4}^{-}u P_{N_5}u_5$
iii. $(\prod_{i=1}^{3} P_{N_i}^{+}u)P_{N_4}^{-}\bar{u} P_{N_5}u_5$
iv. $(\prod_{i=1}^{3} P_{N_i}^{-}u)P_{N_4}^{+}\bar{u} P_{N_5}u_5$.

In either case, simple algebra shows that the space–time Fourier transform of the product is supported at least $\gtrsim N_1^2$ away from the parabola $\tau = -\xi^2$. The worst case is (iii) with $u_5 = u$ where the output's modulation is

$$(3N_1 - N_1 \pm N_5)^2 - 4N_1^2 + N_1^2 \sim N_1^2.$$

Thus, we can put these products in the $\dot{X}^{0,-\frac{1}{2},1}$ space and get a good bound. For example, focusing on (iii), we use Hölder inequality, Bernstein inequality and the boundedness of Riesz transforms.

$$\begin{aligned}
&\left\| P_N[(\prod_{i=1}^{3} P_{N_i}^{+}u)P_{N_4}^{-}\bar{u}P_{N_5}u_5]\right\|_{\dot{X}^{0,-\frac{1}{2},1}} \\
&\quad\lesssim \frac{1}{N_1}\left\|(\prod_{i=1}^{3} P_{N_i}^{+}u)P_{N_4}^{-}\bar{u}P_{N_5}u_5\right\|_{L^2_{t,x}} \\
&\quad\lesssim \frac{(N_4N_5)^{\frac{1}{2}}}{N_1}\prod_{i=1}^{3}\|P_{N_i}u\|_{L^6_{t,x}}\prod_{i=4}^{5}\|P_{N_i}u\|_{L^\infty_t L^2_x} \\
&\quad\lesssim \frac{1}{N_1^{s+1}}\left(\frac{N_5}{N_1}\right)^{\frac{1}{4}}\prod_{i=1}^{5} c_{N_i,i} \\
&\quad\sim \frac{1}{N_1^{s+1}}\left(\frac{N_5}{N_3}\right)^{\frac{1}{4}}\prod_{i=1}^{5} c_{N_i,i} \\
&\quad\lesssim \Big(\sum_{N_1\gtrsim N}\frac{1}{N_1^{2s+2}}c_{N_1,1}^2\Big)^{\frac{1}{2}}\|u\|^4_{\dot{X}^{\frac{1}{4}}}.
\end{aligned}$$

Hence, by summing over N and N_i's, we have

$$\sum_{IV} N^{2s+2}\Big\| P_N[(\prod_{i=1}^{3} P^{+}_{N_i}u)P^{-}_{N_4}\bar{u}P_{N_5}u_5]\Big\|^2_{\dot{X}^{0,-\frac{1}{2},1}}$$

$$\lesssim \sum_{N_1}\sum_{N\lesssim N_1}\left(\frac{N}{N_1}\right)^{2s+2} c^2_{N_1,1}\|u\|^8_{\dot{X}^{\frac{1}{4}}}$$

$$\lesssim \|u\|^2_{\dot{X}^s}\|u\|^8_{\dot{X}^{s_0}},$$

as desired.

3). $u_1 = u_2 = u_3 = \bar{u}$, u_4 and u_5 can be either u or $\bar{u}$.
The proof is the same as in the previous case. Note that we get a better result in the sense that the space–time Fourier support of $\prod_{i=1}^{5} P_{N_i}u_i$ when $\mathcal{F}_{x,t}u_i$ is supported in (102a) for all $u_i = u$ and (102b) for all $u_i = \bar{u}$ is $\gtrsim N_1^2$ away from the parabola $\tau = -\xi^2$ without relying on the Riesz transforms. This concludes the proof of the multilinear estimate. □

7. Proof of Theorem 1.2

The proof is similar to what we did in Section 5 with the same function spaces:

$$\begin{aligned}
\|u\|_{X_N} &= \|u\|_{L^\infty_t L^2_x} + N^{-\frac{1}{4}}\|u\|_{L^4_x L^\infty_t} + N^{\frac{1}{2}}\|u\|_{L^\infty_x L^2_t} \\
&\quad + N^{-\frac{1}{2}}\|(i\partial_t + \Delta)u\|_{L^1_x L^2_t} \\
\|u\|_{\dot{X}^s} &= \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_N u\|^2_{X_N}\Big)^{\frac{1}{2}} \\
\|u\|_{X^s} &= \|u\|_{\dot{X}^0} + \|u\|_{\dot{X}^s} \\
\|u\|_{Y_N} &= N^{-\frac{1}{2}}\|u\|_{L^1_x L^2_t} \\
\|u\|_{\dot{Y}^s} &= \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_N u\|^2_{Y_N}\Big)^{\frac{1}{2}} \\
\|u\|_{Y^s} &= \|u\|_{\dot{Y}^0} + \|u\|_{\dot{Y}^s}.
\end{aligned} \tag{103}$$

Now we state a multilinear estimate. The proof is shortened as it is similar to that of Theorem 5.1 for the most part.

Theorem 7.1. *Suppose that $d \geq 5$. Let $s, r > \frac{1}{2}$ and $u_i \in X^s$ for $1 \leq i \leq d$. Then we have the following estimate:*

$$\Big\|(\partial_x u_1)\prod_{i=2}^{d} u_i\Big\|_{Y^r} \lesssim \|u_1\|_{X^r}\prod_{i=2}^{d}\|u_i\|_{X^s}. \tag{104}$$

Proof. Again, we study the frequency interactions with N being the output frequency and $N_1 \geq N_2 \geq \ldots \geq N_d$ being the input frequencies. For $s > \frac{1}{2}$, we define $c_{N_1,1} = \|P_{N_1}u_1\|_{X_{N_1}}$

and $c_{N_i,i} = \|P_{N_i} u_i\|_{X_{N_i}}$ for $2 \leq i \leq d$. We consider the usual *High* $\times$ *Low* $\to$ *High* and *High* $\times$ *High* $\to$ *Low* interactions:

1. $N \sim N_1 \gg N_2 \geq \ldots \geq N_d$.
   With some abuse of notations, we define $\prod_{i=5}^{d-1} A_i = 1$ if $d = 5$. By Hölder inequality, Young's inequality and the continuous embedding of function spaces $X^s \hookrightarrow X^{s'} \hookrightarrow \dot{X}^{s'}$ for any $s' > s > \frac{1}{2}$,

$$
\begin{aligned}
&N^{r-\frac{1}{2}} \left\| P_N[(P_{N_1}\partial_x u_1) \prod_{i=2}^{d} P_{N_i} u_i] \right\|_{L_x^1 L_t^2} \\
&\lesssim N^{r-\frac{1}{2}} \sum_{N_i} \|P_{N_1}\partial_x u_1\|_{L_x^\infty L_t^2} \prod_{i=2}^{4} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \prod_{i=5}^{d-1} \|P_{N_i} u_i\|_{L_{x,t}^\infty} \|P_{N_d} u_d\|_{L_x^4 L_t^\infty} \\
&\lesssim \sum_{N_i} \left(\frac{N}{N_1}\right)^{r-\frac{1}{2}} \left(\frac{N_d}{N_2}\right)^{\frac{1}{4}} c_{N_1,1} (N_2^{\frac{1}{2}} c_{N_2,2}) c_{N_d,d} \prod_{i=3}^{4} N_i^{\frac{1}{4}} c_{N_i,i} \prod_{i=5}^{d-1} N_i^{\frac{1}{2}} c_{N_i,i} \\
&\lesssim \sum_{N_1 \sim N} \left(\frac{N}{N_1}\right)^{r-\frac{1}{2}} c_{N_1,1} \|u_2\|_{\dot{X}^{\frac{1}{2}}} \prod_{i=3}^{4} \|u_i\|_{\dot{X}^{\frac{1}{4}}} \prod_{i=5}^{d-1} \|u_i\|_{\dot{X}^{\frac{1}{2}}} \|u_d\|_{\dot{X}^0} \\
&\lesssim \sum_{N_1 \sim N} \left(\frac{N}{N_1}\right)^{r-\frac{1}{2}} c_{N_1,1} \prod_{i=2}^{d} \|u_i\|_{X^s}.
\end{aligned}
$$

   Take the l^2 summation and (104) follows.
2. $N \lesssim N_1 \sim N_2 \geq \ldots \geq N_d$.
   This is similar to the previous case, but we apply Cauchy–Schwarz to $\sum_i c_{N_1,1} c_{N_2,2}$ after applying Hölder inequality.

$$
\begin{aligned}
&N^{r-\frac{1}{2}} \left\| P_N[(P_{N_1}\partial_x u_1) \prod_{i=2}^{d} P_{N_i} u_i] \right\|_{L_x^1 L_t^2} \\
&\lesssim \sum_{N_i} \left(\frac{N}{N_1}\right)^{r-\frac{1}{2}} \left(\frac{N_d}{N_3}\right)^{\frac{1}{4}} c_{N_1,1} (N_2^{\frac{1}{4}} c_{N_2,2})(N_3^{\frac{1}{2}} c_{N_3,3})(N_4^{\frac{1}{4}} c_{N_4,4}) c_{N_d,d} \prod_{i=5}^{d-1} (N_i^{\frac{1}{2}} c_{N_i,i}) \\
&\lesssim \left(\sum_{N_1 \gtrsim N} \left(\frac{N}{N_1}\right)^{2r-1} c_{N_1,1}^2 \right)^{\frac{1}{2}} \|u_2\|_{\dot{X}^{\frac{1}{4}}} \|u_3\|_{\dot{X}^{\frac{1}{2}}} \|u_4\|_{\dot{X}^{\frac{1}{4}}} \prod_{i=5}^{d-1} \|u_i\|_{\dot{X}^{\frac{1}{2}}} \|u_d\|_{\dot{X}^0} \\
&\lesssim \left(\sum_{N_1 \gtrsim N} \left(\frac{N}{N_1}\right)^{2r-1} \|P_{N_1} u_1\|_{X_{N_1}}^2 \right)^{\frac{1}{2}} \prod_{i=2}^{d} \|u_i\|_{X^s}.
\end{aligned}
$$

   Take the l^2 summation to obtain (104). □

The proof of Theorem 1.2 part (A) now follows the same contraction argument as before. To prove part (B) of the theorem, we replace u_j by $\partial_x u_j$ for some $j \geq 2$, and it follows from (9) that $\|\partial_x u_i\|_{X^s} \lesssim \|u_i\|_{X^{s+1}}$ for any $s > \frac{1}{2}$. Hence, (104) implies that for any $s > \frac{3}{2}$,

$$\left\|(\partial_x u_1)(\partial_x u_j)\prod_{\substack{i=2\\i\neq j}}^{d} u_i\right\|_{Y^s} \lesssim \|u_1\|_{X^s}\|\partial_x u_j\|_{X^{s-1}}\prod_{\substack{i=2\\i\neq j}}^{d}\|u_i\|_{X^{s-1}}$$

$$\lesssim \prod_{i=1}^{d}\|u_i\|_{X^s}.$$

Consequently, in the case that a term in $P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$ has more than one derivative, we can employ the contraction argument in X^s.

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