How large are the gaps in phase space?

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***Abstract*—Given a sampling measure for the wavelet transform (resp. the short-time Fourier transform) with the wavelet (resp. window) being chosen from the family of Laguerre (resp. Hermite) functions, we provide quantitative upper bounds on the radius of any ball that does not intersect the support of the measure. The estimates depend on the condition number, i.e., the ratio of the sampling constants, but are independent of the structure of the measure. Our proofs are completely elementary and rely on explicit formulas for the respective transforms.**

***Index Terms*—sampling measures, wavelet transform, short-time Fourier transform, frames.**

I. Introduction

This article draws inspiration from the paper *'How large are the spectral gaps?'* by Iosevich and Pedersen [12], as well as the recent work by Papageorgiou and van Velthoven [19], who gave quantitative estimates for relative denseness for exponential frames and coherent frames on groups of polynomial growth, respectively. The main objective of this paper is to obtain a similar result for the wavelet transform using a specific family of orthogonal functions as wavelets. In particular, we establish an upper bound on the radius of any ball in wavelet phase space that does not intersect the support of a sampling measure. Our proof follows a different approach than [19] since the $(ax+b)$-group underlying the wavelet transform exhibits exponential growth. In addition, we discuss sampling measures for the short-time Fourier transform (STFT) and provide a bound which, in principle, could readily be obtained using the ideas of [19]. Given that the proof strategy closely mirrors that of the wavelet transform and that our elementary approach provides explicit constants, we chose to include the statement in this article.

Let (X,ν) be a metric measure space and $\mathcal{H}$ be a closed subspace of $L^2(X,\nu)$. A measure μ on X is called a *sampling measure* if there are constants $A,B>0$ s.t.

$$A\|F\|_{L^2_\nu}^2 \leq \int_X |F(z)|^2 d\mu(z) \leq B\|F\|_{L^2_\nu}^2, \quad F\in\mathcal{H}.$$

As we already pointed out, we focus on the cases when $\mathcal{H}$ is the range of the wavelet transform or of the STFT. A measure μ is called (γ,R)*-dense* if $\mu(B_R(z))\geq\gamma>0$ for every $z\in X$, and *relatively dense* if there exist $R>0$ and $\gamma>0$ s.t. μ is (γ,R)-dense.

There are various qualitative and quantitative results relating sampling measures and relative denseness. For one, it is well-known that the support of a sampling measure for the wavelet transform as well as the STFT is necessarily relatively dense, see, e.g., [7], [14], [16], [18]. If a sampling measure for the STFT is discrete, i.e., $\mu=\sum_{\lambda\in\Lambda}\delta_\lambda$ with $\Lambda\subset\mathbb{R}^{2d}$ discrete, then the density theorem for Gabor frames provides a more refined analysis assuring that the lower Beurling density of Λ is greater or equal than one for every window function [11]. For the wavelet transform, however, no such general result is known and all existing density statements either require a particular structure of the sampling points and / or choice of the wavelet function [5], [20], [22], [23]. See [15] for a discussion of the difficulties that occur in this setting.

Previous quantitative results on relative denseness primarily focused on estimates of the sampling constants A and B in terms of R and γ. This approach was studied, e.g., in [4], [6], [10], [13] for $\mu=\chi_\Omega\nu$ (here χ_Ω denotes the characteristic function of a measurable set Ω), or in [24] for a discrete measure μ leading to explicit frame bounds. Here, we study the converse problem of providing an upper bound on R in terms of A and B given that μ is not (γ,R)-dense.

We prove our main estimates in Theorem 3 for a family of orthogonal wavelets ψ_n^α, $n\in\mathbb{N}_0$, $\alpha>0$, defined in Fourier domain in terms of the generalized Laguerre polynomials (see equation (4)). The ranges of the respective wavelet transforms play important roles in various fields as they may be identified with Bergman spaces of analytic functions on the upper half-plane and the unit disk ($n=0$), and with hyperbolic Landau level spaces in quantum mechanics, see [1], [2], [6] for detailed discussions of these connections. It seems that the statement of Theorem 3 was previously unknown, even in the literature on Bergman spaces. The only result, that we are aware of, that exhibits a conceptual similarity, is [21, Lemma 4.1].

Our proofs follow the general strategy developed in [12] and rely on explicit formulas of the wavelet transform and the STFT. This approach allowed us to derive elementary bounds on the quotient of, e.g., two wavelet transforms

$$z\mapsto W_{\psi_n^\alpha}\psi_0^\alpha(z)/W_{\psi_n^\alpha}\psi_0^\alpha(w^{-1}\cdot z),\quad z,w\in\mathbb{C}^+,\ z\neq w,$$

on certain regions in phase space which is then partitioned accordingly.

II. Sampling Measures for the Wavelet Transform

A. Basic Wavelet Theory

We use the standard notation $\mathcal{H}^2(\mathbb{C}^+)$ for the *Hardy space* of analytic functions in $\mathbb{C}^+$ equipped with the norm

$$\|f\|_{\mathcal{H}^2}^2 := \sup_{0<s<\infty}\int_{-\infty}^{\infty}|f(x+is)|^2\,dx<\infty.$$

Let $z = x + is \in \mathbb{C}^+$. The *time-scale shift* $\pi(z)$ of a function $\psi \in H^2(\mathbb{C}^+)$ is defined as

$$\pi(z)\psi(t) := T_x D_s \psi(t) = s^{-\frac{1}{2}}\psi(s^{-1}(t-x)).$$

One may identify $\mathbb{C}^+$ with the $(ax+b)$-group via the multiplication

$$z \cdot w = x + sx' + iss', \quad z = x + is,\ w = x' + is' \in \mathbb{C}^+. \quad (1)$$

Then the neutral element is i, and the inverse element of z is $z^{-1} = -x/s + i/s$. Throughout this paper "$\cdot$" will exclusively be used to denote the group multiplication (1). The *wavelet transform* of $f \in \mathcal{H}^2(\mathbb{C}^+)$ with respect to a wavelet ψ is defined as

$$W_\psi f(z) = \langle f, \pi(z)\psi\rangle.$$

A wavelet ψ is called *admissible* if

$$0 < \int_{\mathbb{R}^+} \left|\widehat{\psi}(\xi)\right|^2 \frac{d\xi}{\xi} =: C_\psi < \infty, \quad (2)$$

where $\widehat{\psi}$ denotes the Fourier transform of ψ. For an admissible wavelet ψ, the wavelet transform $W_\psi : \mathcal{H}^2(\mathbb{C}^+) \to L^2(\mathbb{C}^+, s^{-2}dz)$ is a constant multiple of an isometry, i.e.,

$$\int_{\mathbb{C}^+} |W_\psi f(z)|^2 s^{-2} dz = C_\psi \|f\|^2_{\mathcal{H}^2}, \quad (3)$$

where dz denotes the Lebesgue measure on $\mathbb{C}^+$. A family of vectors $\{\pi(\lambda)\psi\}_{\lambda\in\Lambda} \subset \mathcal{H}^2(\mathbb{C}^+)$ is called a *wavelet frame* if there exist constants $A, B > 0$ s.t. for every $f \in \mathcal{H}^2(\mathbb{C}^+)$

$$A\|f\|^2_{\mathcal{H}^2} \le \sum_{\lambda\in\Lambda} |W_\psi f(\lambda)|^2 \le B\|f\|^2_{\mathcal{H}^2}.$$

A measure μ is called a *sampling measure* for the wavelet transform W_ψ if for every $f \in \mathcal{H}^2(\mathbb{C}^+)$

$$A\|f\|^2_{\mathcal{H}^2} \le \int_{\mathbb{C}^+} |W_\psi f(z)|^2 d\mu(z) \le B\|f\|^2_{\mathcal{H}^2}.$$

In this terminology, a wavelet frame $\{\pi(\lambda)\psi\}_{\lambda\in\Lambda}$ corresponds to the sampling measure $\mu = \sum_{\lambda\in\Lambda}\delta_\lambda$, where δ_λ denotes the Kronecker delta.

B. Pseudohyperbolic Metric and Möbius Transform

The *pseudohyperbolic metric* on $\mathbb{C}^+$ is given by

$$\rho_{\mathbb{C}^+}(z, w) := \left|\frac{z-w}{z-\overline{w}}\right|, \qquad z, w \in \mathbb{C}^+,$$

and the *pseudohyperbolic disk* of radius $R > 0$ centered at $z \in \mathbb{C}^+$ is denoted by $\mathcal{D}_R(z) := \{\omega \in \mathbb{C}^+ :\ \rho_{\mathbb{C}^+}(z, w) < R\}$. Note that $\rho_{\mathbb{C}^+}$ only takes values in the half open interval $[0, 1)$ and that $\rho_{\mathbb{C}^+}(z, w) = \rho_{\mathbb{C}^+}(z^{-1} \cdot w, i)$.

Let $\mathbb{D}_R(z) \subset \mathbb{C}$ denote the Euclidean disk of radius $R > 0$ centered at $z \in \mathbb{C}$. We write for short $\mathbb{D}_R := \mathbb{D}_R(0)$ and $\mathbb{D} := \mathbb{D}_1$. The Möbius transform $T : \mathbb{D} \to \mathbb{C}^+$,

$$T(u) := i\frac{1+u}{1-u}, \quad u \in \mathbb{D},$$

is bijective and maps the pseudohyperbolic distance in $\mathbb{C}^+$ to the pseudohyperbolic distance in $\mathbb{D}$, i.e., for $u, v \in \mathbb{D}$

$$\rho_{\mathbb{C}^+}(T(u), T(v)) = \left|\frac{v-u}{1-\overline{u}v}\right| =: \rho_{\mathbb{D}}(u, v).$$

C. Quantitative Bounds for Gaps in $\mathbb{C}^+ \backslash \mathrm{supp}(\mu)$

We will establish our main result for wavelets chosen from the orthonormal basis $\{\psi_n^\alpha\}_{n\in\mathbb{N}_0} \subset \mathcal{H}^2(\mathbb{C}^+)$, $\alpha > 0$, which is defined in the Fourier domain via

$$\widehat{\psi_n^\alpha}(t) := \sqrt{\frac{2^{\alpha+2}\pi n!}{\Gamma(n+\alpha+1)}}\ t^{\frac{\alpha}{2}} e^{-t} L_n^\alpha(2t), \quad t > 0, \quad (4)$$

where L_n^α denotes the generalized Laguerre polynomial of degree n, see, e.g., [17, Chapter 18]

$$L_n^\alpha(t) = \sum_{k=0}^n \frac{(-1)^k}{k!}\binom{n+\alpha}{n-k} t^k, \quad t > 0.$$

We need an explicit formula for the inner products of time-scale shifted versions of ψ_0^α and ψ_n^α which can be found, e.g., in [6, Proposition 1].

***Proposition** 1: Let $\alpha > 0$, and $z = x + is$ as well as $w = x' + is'$ be in $\mathbb{C}^+$. For every $n \in \mathbb{N}_0$, one has*

$$\langle \pi(w)\psi_n^\alpha, \pi(z)\psi_0^\alpha\rangle = c_n^\alpha \left(\frac{z-w}{z-\overline{w}}\right)^n \left(\frac{2\sqrt{ss'}}{i(\overline{w}-z)}\right)^{\alpha+1},$$

where $c_n^\alpha = \sqrt{\Gamma(n+\alpha+1)/\Gamma(\alpha+1)n!}$.

Upon applying the Möbius transform, a straightforward computation shows that for $u, v \in \mathbb{D}$

$$\begin{aligned}&\left|\langle \pi(T(v))\psi_0^\alpha, \pi(T(u))\psi_n^\alpha\rangle\right| \\ &\qquad = c_n^\alpha\, \rho_{\mathbb{D}}(u, v)^n \left(\frac{(1-|u|^2)(1-|v|^2)}{|1-\overline{u}v|^2}\right)^{\frac{\alpha+1}{2}}.\end{aligned}$$

In particular, $\langle \pi(T(v))\psi_0^\alpha, \pi(T(u))\psi_n^\alpha\rangle$ is nonzero whenever $u \ne v$ which implies that the following auxiliary function is well-defined for $u \ne v$

$$\begin{aligned}H_n^\alpha(u, v) &:= \left|\frac{\langle \pi(T(0))\psi_0^\alpha, \pi(T(u))\psi_n^\alpha\rangle}{\langle \pi(T(v))\psi_0^\alpha, \pi(T(u))\psi_n^\alpha\rangle}\right|^2 \\ &= \left(\frac{\rho_{\mathbb{D}}(0, u)}{\rho_{\mathbb{D}}(u, v)}\right)^{2n}\left(\frac{|1-\overline{u}v|^2}{1-|v|^2}\right)^{\alpha+1} \\ &= \frac{|u|^{2n}|1-\overline{u}v|^{2(n+\alpha+1)}}{|u-v|^{2n}(1-|v|^2)^{\alpha+1}}.\end{aligned}$$

Note that this expression is rotationally invariant, i.e., $H_n^\alpha(ue^{i\varphi}, ve^{i\varphi}) = H_n^\alpha(u, v)$ for any $\varphi \in (0, 2\pi]$.

***Lemma** 2: Let $v = |v|$ and $u = |u|e^{-i\varphi}$ with $0 < |v| < R \le |u| < 1$. If $|\varphi| \le (1 - |v|R)$, then*

$$H_n^\alpha(u, v) \le 2^{n+\alpha+1}\frac{(1-|v|R)^{2n+\alpha+1}}{(1-|v|/R)^{2n}}.$$

Proof: First, we note that $|1 - \beta e^{i\varphi}|^2 = 1 + \beta^2 - 2\beta\cos\varphi$. Therefore,

$$\begin{aligned}H_n^\alpha(u, v) &= \frac{|u|^{2n}\left|1-|uv|e^{i\varphi}\right|^{2(n+\alpha+1)}}{\left||u|e^{-i\varphi}-|v|\right|^{2n}(1-|v|^2)^{\alpha+1}} \\ &\le \frac{\left|1-|uv|e^{i\varphi}\right|^{2(n+\alpha+1)}}{(1-|v|/|u|)^{2n}(1-|v|R)^{\alpha+1}}\end{aligned}$$

$$\leq \frac{(1+|uv|^2-2|uv|\cos\varphi)^{n+\alpha+1}}{(1-|v|/R)^{2n}\,(1-|v|R)^{\alpha+1}}. \tag{5}$$

Observe that $2(1-\cos\varphi) \leq \varphi^2$ for every $\varphi \in \mathbb{R}$. Using the assumption on φ we hence deduce that

$$\begin{aligned} 1+|uv|^2-2|uv|\cos\varphi &= (1-|uv|)^2 + 2|uv|(1-\cos\varphi) \\ &\leq (1-|v|R)^2 + \varphi^2 \\ &\leq 2(1-|v|R)^2. \end{aligned}$$

Plugging this estimate into (5) shows

$$\begin{aligned} H_n^\alpha(u,v) &\leq \frac{2^{n+\alpha+1}(1-|v|R)^{2(n+\alpha+1)}}{(1-|v|/R)^{2n}\,(1-|v|R)^{\alpha+1}} \\ &= \frac{2^{n+\alpha+1}(1-|v|R)^{2n+\alpha+1}}{(1-|v|/R)^{2n}}, \end{aligned}$$

which concludes the proof. □

With this auxiliary result in place, we are now ready to prove our main result.

Theorem 3: *Let $n \in \mathbb{N}_0$, $\alpha > 0$, and μ be a sampling measure for the wavelet transform with wavelet ψ_n^α and sampling constants $A, B > 0$.*

If there exists $z \in \mathbb{C}^+$ s.t. $\mu(\mathcal{D}_R(z)) = 0$, then

$$R \leq 1 - \left(\frac{C_{n,\alpha}}{\pi}\,\frac{A}{B}\right)^{1/\alpha}, \tag{6}$$

where

$$C_{n,\alpha} = \begin{cases} 4^{-(\alpha+1)}, & n = 0, \\ 6^{-(2n+\alpha+1)}, & n \in \mathbb{N}. \end{cases}$$

Proof: We partition $\mathbb{D}\backslash\{0\}$ into K circular sectors

$$S_k = \Big\{re^{i\varphi} \in \mathbb{D}\backslash\{0\} :\ \frac{\pi(2k-1)}{K} \leq \varphi < \frac{\pi(2k+1)}{K}\Big\},$$

$k = 0, ..., K-1$, and pick K points $v_k = re^{2\pi ik/K} \in S_k$ with $r < R$. The appropriate choice of K and r in terms of R will be determined later. Moreover, we set $P_k = T(S_k)$ and point out that by a straightforward computation $w \in (z\cdot P_k)\backslash\mathcal{D}_R(z)$ if and only if $T^{-1}(z^{-1}\cdot w) \in S_k\backslash\mathbb{D}_R$. Therefore, if μ is a sampling measure s.t. $\mu(\mathcal{D}_R(z)) = 0$, then

$$\begin{aligned} A &= A\|\pi(z)\psi_0^\alpha\|_{\mathcal{H}^2}^2 \\ &\leq \int_{\mathbb{C}^+} |\langle\pi(z)\psi_0^\alpha, \pi(w)\psi_n^\alpha\rangle|^2 d\mu(w) \\ &= \int_{\mathbb{C}^+} |\langle\psi_0^\alpha, \pi(z^{-1}\cdot w)\psi_n^\alpha\rangle|^2 d\mu(w) \\ &= \sum_{k=1}^K \int_{z\cdot P_k} \left|\frac{\langle\psi_0^\alpha, \pi(z^{-1}\cdot w)\psi_n^\alpha\rangle}{\langle\pi(T(v_k))\psi_0^\alpha, \pi(z^{-1}\cdot w)\psi_n^\alpha\rangle}\right|^2 \times \\ &\quad \times \left|\langle\pi(T(v_k))\psi_0^\alpha, \pi(z^{-1}\cdot w)\psi_n^\alpha\rangle\right|^2 d\mu(w) \\ &= \sum_{k=1}^K \int_{z\cdot P_k} H_n^\alpha\big(T^{-1}(z^{-1}\cdot w), v_k\big)\times \\ &\quad \times \left|\langle\pi(z\cdot T(v_k))\psi_0^\alpha, \pi(w)\psi_n^\alpha\rangle\right|^2 d\mu(w) \\ &\leq \sum_{k=1}^K \sup_{u\in S_k\backslash\mathbb{D}_R} H_n^\alpha(u, v_k)\times \\ &\quad \times \int_{\mathbb{C}^+} \left|\langle\pi(z\cdot T(v_k))\psi_0^\alpha, \pi(w)\psi_n^\alpha\rangle\right|^2 d\mu(w) \\ &\leq KB \sup_{u\in S_0\backslash\mathbb{D}_R} H_n^\alpha(u, v_0), \end{aligned}$$

where we used that $\left\|\pi(z)\psi_0^\alpha\right\|_{\mathcal{H}^2} = 1$ and the rotational invariance of H_n^α in the final step.

Let us choose $r = R^\kappa$ for $\kappa > 1$, and $K = \lceil \pi/(1-rR)\rceil$. Any $u = |u|e^{i\varphi} \in S_0\backslash\mathbb{D}_R$ satisfies $|u| \geq R$ as well as $|\varphi| \leq \pi/K \leq (1-rR)$. We may thus apply Lemma 2 to show that

$$\begin{aligned} A &\leq \left\lceil \frac{\pi}{1-R^{1+\kappa}} \right\rceil 2^{n+\alpha+1}\, B\, \frac{(1-R^{1+\kappa})^{2n+\alpha+1}}{(1-R^{\kappa-1})^{2n}} \\ &\leq \pi\, 2^{n+\alpha+2}\, B\, \frac{(1-R^{1+\kappa})^{2n+\alpha}}{(1-R^{\kappa-1})^{2n}}. \end{aligned}$$

This inequality simplifies to $A \leq \pi\, 2^{\alpha+2}B(1-R^{1+\kappa})^\alpha$ if $n = 0$. Since the right hand side is continuous for $\kappa \geq -1$, we conclude that the inequality holds in the limit $\kappa \to 1$. Using $1 - R^2 \leq 2(1-R)$ and solving for R thus shows $R \leq 1 - \big(A/(\pi 4^{\alpha+1}B)\big)^{1/\alpha}$.

For $n \in \mathbb{N}$, we choose $\kappa = 2$ and note that $1 - R^3 \leq 3(1-R)$which allows us to conclude that $R \leq 1 - \big(A/(4\pi 6^{2n+\alpha}B)\big)^{1/\alpha}$. □

Remark 4: *(1). One could improve the constants in Theorem 3 by optimizing the choice of r in our proof. However, due to page limitations, we chose not to pursue this idea.*

(2). If $\mu = \sum_{\lambda\in\Lambda}\delta_\lambda$ for some discrete set $\Lambda \subset \mathbb{C}^+$, then μ is a sampling measure if and only if $\{\pi(\lambda)\psi_n^\alpha\}_{\lambda\in\Lambda}$ is a wavelet frame. Theorem 3 thus provides a bound on the maximal radius of a pseudohyperbolic disk that does intersect Λ.

(3). Our approach is independent of the Hilbert space structure and a slight adaptation would therefore also provide bounds on the gaps of L^p-sampling measures where the L^p_μ-norm of $W_{\psi_n^\alpha} f$ is compared to the L^p-co-orbit space norm of f. The resulting bound on R is then dependent on p. We refer to [8] for an introduction to co-orbit theory.

III. Bergman Spaces

Let $\mathcal{A}_\alpha(\mathbb{C}^+)$ be the space of holomorphic functions on the upper half plane that satisfy

$$\|F\|_{\mathcal{A}_\alpha}^2 := \int_{\mathbb{C}^+} |F(z)|^2 s^\alpha dz < \infty.$$

A measure μ is called a *sampling measure* for $\mathcal{A}_\alpha(\mathbb{C}^+)$ if

$$A\|F\|_{\mathcal{A}_\alpha}^2 \leq \int_{\mathbb{C}^+} |F(z)|^2 s^\alpha d\mu(z) \leq B\|F\|_{\mathcal{A}_\alpha}^2$$

The *Bergman transform* $\mathrm{B}_\alpha : \mathcal{H}^2(\mathbb{C}^+) \to \mathcal{A}_\alpha(\mathbb{C}^+)$

$$\mathrm{B}_\alpha f(z) = s^{-\frac{\alpha}{2}-1} W_{\psi_0^{\alpha+1}} f(z)$$

is a constant multiple of an isometric isomorphism. This property immediately yields the following corollary of Theorem 3.

Corollary 5: *Let μ be a sampling measure for $\mathcal{A}_\alpha(\mathbb{C}^+)$ with constants $A, B > 0$. If $\mu(\mathcal{D}_R(z)) = 0$ for some $z \in \mathbb{C}^+$, then*

$$R \leq 1 - \left(\frac{1}{4^{\alpha+1}\pi} \; \frac{A}{B} \right)^{1/\alpha}, \tag{7}$$

To the best of our knowledge, this result is not known even for the special case that $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a discrete measure , i.e., if Λ is a *set of stable sampling* for $\mathcal{A}_\alpha(\mathbb{C}^+)$.

IV. Sampling Measures for the short-time Fourier transform

The *short-time Fourier transform (STFT)* of a function $f \in L^2(\mathbb{R})$ using a window $g \in L^2(\mathbb{R})$ is given by

$$V_g f(z) = \langle f, \pi(z) g \rangle, \quad z \in \mathbb{C},$$

where $\pi(z)g(t) = e^{2\pi i \xi t} g(t-x)$, $z = x + i\xi$. The STFT has, among other useful features, the covariance property

$$V_g(\pi(w)f)(z) = e^{-2\pi i x'(\xi-\xi')} V_g f(z-w), \tag{8}$$

$w = x' + i\xi'$, and satisfies *Moyal's formula*

$$\int_{\mathbb{C}} |V_g f(z)|^2 dz = \|f\|_2^2 \|g\|_2^2, \quad f, g \in L^2(\mathbb{R}).$$

For a thorough introduction to time-frequency analysis we refer to [9]. A measure μ is called a *sampling measure* for the STFT if there exist $A, B > 0$ s.t. for any $f \in L^2(\mathbb{R})$

$$A\|f\|_2^2 \leq \int_{\mathbb{C}} |V_g f(z)|^2 d\mu(z) \leq B\|f\|_2^2.$$

If $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$ (for $\Lambda \subset \mathbb{C}$ discrete) is a sampling measure, then $\{\pi(\lambda)g\}_{\lambda\in\Lambda}$ forms a *Gabor frame*, i.e., for $f \in L^2(\mathbb{R})$

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2.$$

If $\{\pi(\lambda)g\}_{\lambda\in\Lambda}$ is a Gabor frame and if $|V_g g(z)|^2 \lesssim (1+|z|)^{-(2+\sigma)}$, for some $\sigma > 0$, then Theorem 1.3 in Papageorgiou and van Velthoven's paper [19] established quantitative bounds for the maximal radius of balls that do not intersect Λ. Their proof can be readily adapted for windows with exponential or Gaussian decay, as well as for general sampling measures which would improve the upper bound on the radius in [19, Theorem 1.3] from a polynomial dependence to a logarithmic one (compare to Theorem 7 below). We nevertheless chose to include our proof of Theorem 7 based on the elementary ideas developed in Section II as it directly provides explicit constants.

We assume that the window function is chosen from the family of Hermite functions $h_n(t) := c_n e^{-\pi t^2} \frac{d}{dt} e^{-2\pi t^2}$, where c_n is defined s.t. $\|h_n\|_2 = 1$. It is well-known that $|V_{h_n} h_0(z)| = C_n |z|^n e^{-\pi|z|^2/2}$, for some $C_n \in \mathbb{R}^+$, see, e.g., [4]. Let us define the auxiliary function

$$H_n(z,w) = \left| \frac{\langle \pi(0)h_0, \pi(z)h_n\rangle}{\langle \pi(w)h_0, \pi(z)h_n\rangle} \right|^2$$

$$= \frac{|z|^{2n}}{|z-w|^{2n}} e^{-\pi(|z|^2 - |z-w|^2)}, \tag{9}$$

where we used (8). Note that, just like in the wavelet case, H_n is rotationally invariant, i.e., $H_n(ze^{i\varphi}, we^{i\varphi}) = H_n(z,w)$, $\varphi \in (0, 2\pi]$.

Lemma 6: *Let $n \in \mathbb{N}_0$, $w = R/2$, and $z = re^{i\varphi}$ with $r \geq R$ and $|\varphi| \leq \pi/5$. Then*

$$H_n(z,w) \leq 4^n e^{-\pi R^2/2}.$$

Proof: First, we note that

$$\begin{aligned} |z-w|^2 &= R^2/4 + r^2 - Rr\cos\varphi \\ &\geq R^2/4 + r^2 - Rr = (r - R/2)^2 \geq r^2/4. \end{aligned}$$

Second, if $|\varphi| \leq \pi/5$, then $\cos\varphi \geq 3/4$. Consequently,

$$\begin{aligned} |z|^2 - |z-w|^2 &= r^2 - R^2/4 - r^2 + Rr\cos\varphi \\ &\geq -R^2/4 + 3rR/4 \geq R^2/2. \end{aligned}$$

Plugging the previous two estimates into (9) yields

$$H_n(z,w) \leq \frac{r^{2n}}{r^{2n}/4^n} e^{-\pi R^2/2} = 4^n e^{-\pi R^2/2},$$

which was to be shown. □

Theorem 7: *Let $n \in \mathbb{N}_0$, and μ be a sampling measure for the STFT with window h_n and sampling bounds $A, B > 0$. If $\mu(\mathbb{D}_R(z)) = 0$ for some $z \in \mathbb{C}$, then*

$$R^2 \leq \frac{2}{\pi} \log\left(4^n 5 \; \frac{B}{A}\right). \tag{10}$$

Proof: We proceed similarly to the proof of Theorem 3. First, we divide $\mathbb{C}\backslash\{0\}$ into 5 segments

$$S_k = \left\{ re^{i\varphi} : \; r > 0, \; \pi(2k-1)/5 \leq \varphi \leq \pi(2k+1)/5 \right\},$$

and choose the points $u_k = R/2\, e^{2\pi i k/5}$, $k = 0, 1, .., 5$. If $\mu(\mathbb{D}_R(z)) = 0$, then an application of (8), the rotational invariance of H_n and Lemma 6 shows

$$\begin{aligned} A &= A\|\pi(z)h_0\|_2^2 \\ &\leq \int_{\mathbb{C}} \left|\langle \pi(z)h_0, \pi(w)h_n\rangle\right|^2 d\mu(w) \\ &= \int_{\mathbb{C}} \left|\langle h_0, \pi(w-z)h_n\rangle\right|^2 d\mu(w) \\ &= \sum_{k=1}^{5} \int_{z+S_k} H_n(w-z, u_k) \left|\langle \pi(u_k)h_0, \pi(w-z)h_n\rangle\right|^2 d\mu(w) \\ &\leq \sum_{k=1}^{5} \sup_{\gamma\in S_k\backslash\mathbb{D}_R} H_n(\gamma, u_k) \int_{\mathbb{C}} \left|\langle \pi(z+u_k)h_0, \pi(w)h_n\rangle\right|^2 d\mu(w) \\ &\leq 5B \sup_{\gamma\in S_0\backslash\mathbb{D}_R} H_n(\gamma, u_0) \leq 4^n 5 B e^{-\pi R^2/2}. \end{aligned}$$

Solving for R then completes the proof. □

Remark 8: *Like in Section III, the bounds of Theorem 7 can be directly translated to bounds for sampling measures on the Bargmann-Fock space of entire functions ($n = 0$), and to (true) polyanalytic Bargmann-Fock spaces ($n \geq 1$), see [3].*

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