NLIR: Natural Language Intermediate Representation for Mechanized Theorem Proving

Laetitia Teodorescu
Adaptive ML

Guillaume Baudart
Université Paris Cité
CNRS, Inria, IRIF

Emilio Jesús Gallego Arias
Université Paris Cité
CNRS, Inria, IRIF

Marc Lelarge
DI ENS, PSL University
Inria

Abstract

Formal theorem proving is challenging for humans as well as for machines. Thanks to recent advances in LLM capabilities, we believe natural language can serve as a universal interface for reasoning about formal proofs. In this paper, 1) we introduce *Pétanque*, a new lightweight environment to interact with the Coq theorem prover; 2) we present two interactive proof protocols leveraging natural language as an intermediate representation for designing proof steps; 3) we implement beam search over these interaction protocols, using natural language to rerank proof candidates; and 4) we use Pétanque to benchmark our search algorithms. Using our method with GPT-4o we can successfully synthesize proofs for 58% of the first 100/260 lemmas from the newly published Busy Beaver proofs.

1 Introduction

The general knowledge and reasoning abilities of frontier large language models (LLMs) makes them practical as a backbone for building agents able to interact with interactive theorem provers (ITP). These agents should iteratively build proofs with help from proof engine feedback. While previous work (e.g. Yang et al. [2023]) used a costly data collection procedure to finetune modestly sized language models, we believe that reasoning in natural language before outputting tactics will lead to better and more interpretable results. Recently, Thakur et al. [2024] showed promising preliminary results by using GPT-4 as an agent proposing tactics inside a backtracking search and using rich feedback from the proof environment.

In this work, we develop infrastructure to allow communication between a GPT-4o-based agent and the Coq proof environment [The Coq Development Team, 2024]. Our key idea is to rely on natural language as much as possible when generating proofs. Using natural language leverages the strength of LLMs, and allows us to use chain-of-thought [Wei et al., 2022] by asking for an informal mathematical proof before generating the formal proof, making it more intuitive and comprehensible compared to purely automatic formal techniques. Additionally, partial proofs expressed in natural language are easier for humans to understand, adapt, or reuse, allowing for greater flexibility and collaboration between machine-generated suggestions and human mathematicians.

We present the following contributions: 1) *Pétanque*: A new fast and lightweight environment to interact with the Coq theorem prover. 2) Two interactive proof protocols both leveraging natural language reasoning: tactic-by-tactic proof construction, and hierarchical proof templating. 3) We couple both protocols with standard search algorithms leveraging feedback from the ITP and using natural language to rerank proof candidates. 4) We evaluate this agent on a new dataset of textbook

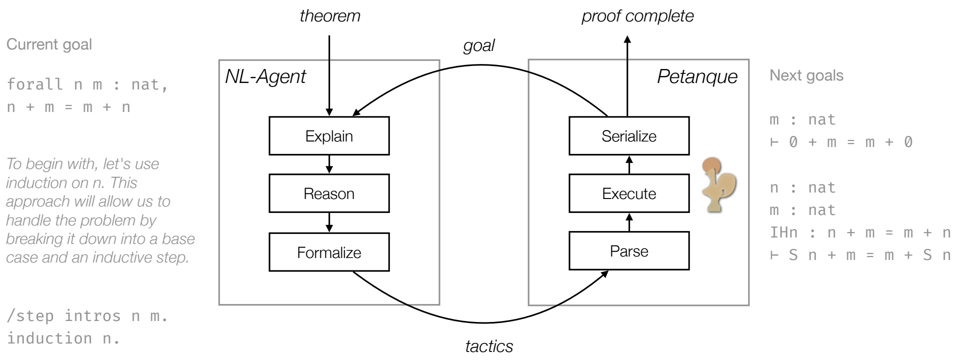


Figure 1: Tactic-by-tactic proof construction.

exercises and intermediate theorems from the recent Busy Beaver proof formalized in Coq of $\mathrm{BB}(4) = 107$, [ccz181078, 2024]. NLIR is open source (`https://github.com/llm4coq/nlir`).

2 Pétanque: a lightweight interactive environment for Coq

A common difficulty when interacting with interactive proof assistants in the context of machine learning is inadequate tooling (see for example [Reichel et al.]). Following existing work [Gallego Arias et al., 2016, Gallego Arias, 2019, Yang and Deng, 2019, Sanchez-Stern et al., 2020], we have built a new environment for machine to machine interaction for the Coq proof assistant, particularly tailored for interactive, high-throughput, low-latency learning applications. Pétanque is based on Flèche [Gallego Arias, 2024], a new document manager for Coq. We extend Flèche by enabling Pétanque to access the Coq proof engine directly without requiring edits in the associated document. This makes our environment fast and lightweight. A Python interface provides easy access to the API. See Appendix B for more information on Flèche and Pétanque.

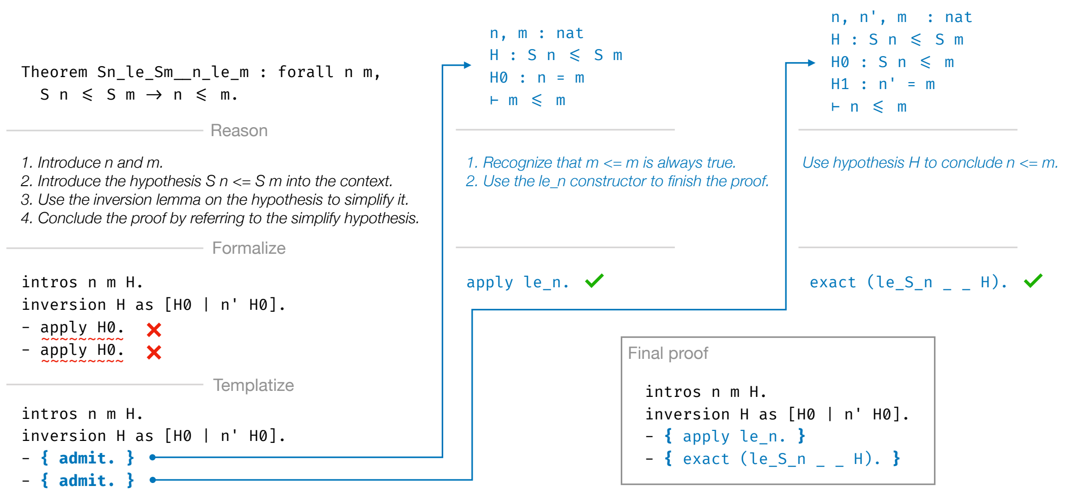
3 Proof interaction protocols

In this section, we present two approaches leveraging LLMs' ability to reason in natural language in order to find a formal proof with the help of a proof assistant. *Tactic-by-tactic proof construction* mimics the typical behavior of a standard Coq user: given the current goals, the agent generates one or several tactics that updates the goals and repeats this process until the proof is complete. By contrast, *hierarchical proof templating* tries to generate full proofs directly. Failed tactics are then replaced with *holes* to obtain a proof *template*. The agent repeats the process of filling each hole until the proof is complete. Our approach's originality is that although both protocols' inputs (goals) and outputs (tactics) are Coq code, the agent internally uses natural language as an intermediate representation to analyze the input and guide the code generation.

3.1 Tactic-by-tactic proof construction

An overview of the tactic-by-tactic proof construction agent is presented in Figure 1. Given a Coq theorem, the agent first uses natural language to describe the goal and explain how to continue the proof (chain-of-thought). The last step synthesizes the corresponding Coq tactics. For instance, in Figure 1, the goal is to prove that addition over natural numbers is commutative. The agent decides to try a proof by induction and correctly synthesizes a sequence of two tactics: `intros n m.` introduces two variables `n` and `m` of type `nat` (natural numbers), and `induction n.` starts an induction over `n`.

The tactics are sent to the Pétanque environment, which parses and executes each tactic to update the current goal. A textual representation of the new goal is then fed back to the agent to make further progress in the proof. If the execution returns an error, the current goal does not change, but we augment the prompt with the failed tactics and ask the LLM to try something else for the next attempt. For instance, in Figure 1, both tactics succeed and generate two new subgoals: the base case (for `n=0`, prove `m + 0 = 0 + m`) and the induction case (given the induction hypothesis



```
Theorem Sn_le_Sm__n_le_m : forall n m,
  S n ⩽ S m → n ⩽ m.
```

1. Introduce n and m.
2. Introduce the hypothesis S n <= S m into the context.
3. Use the inversion lemma on the hypothesis to simplify it.
4. Conclude the proof by referring to the simplify hypothesis.

```
intros n m H.
inversion H as [H0 | n' H0].
- apply H0.  ✗
- apply H0.  ✗
```

```
intros n m H.
inversion H as [H0 | n' H0].
- { admit. }
- { admit. }
```

```
n, m : nat
H : S n ⩽ S m
H0 : n = m
⊢ m ⩽ m
```

1. Recognize that m <= m is always true.
2. Use the le_n constructor to finish the proof.

```
apply le_n.  ✓
```

```
n, n', m  : nat
H : S n ⩽ S m
H0 : S n ⩽ m
H1 : n' = m
⊢ n ⩽ m
```

Use hypothesis H to conclude n <= m.

```
exact (le_S_n _ _ H).  ✓
```

```
intros n m H.
inversion H as [H0 | n' H0].
- { apply le_n. }
- { exact (le_S_n _ _ H). }
```

Figure 2: Hierarchical proof templating.

`IHn: n + m = m + n`, prove `(n + 1) + m = m + (n + 1)`). The textual representation of a goal uses the the symbol ⊢ to separate hypotheses from the conclusion, and `S n` denotes n + 1.

Model Interface. In early experiments, we observed that conversation-style reasoning often diverges: after a few rounds, the output makes very little sense, and the agent never recovers. Following [Yang et al., 2024] – and similarly to [Thakur et al., 2024] – we use a synthetic interface to summarize at each goal the global objective (initial theorem), the current goal (in the middle of a proof), and failed attempts to solve the same goal.

3.2 Hierarchical proof templating

An example execution of the hierarchical proof templating agent is presented in Figure 2. The agent pipeline is similar to the tactic-by-tactic method, but instead of focusing only on the next step, the agent generates a complete proof in natural language, before translating the proof in Coq syntax. For instance, in Figure 2, the agent uses the **`inversion`** tactics on the hypothesis `H` which generate two subgoals with a simpler hypothesis `H0`, and then tries to solve each subgoals using this `H0` hypothesis.

The Pétanque environment then repairs the proof, replacing failed tactics by *holes* which admit and close the current subgoal, removing subsequent tactics until the focus moves to the next subgoal. Pétanque then checks that the resulting *template* is correct, i.e., assuming a valid proof for each holes, the proof is complete. A textual representation of each holes is then fed back to the agent which repeat the process to fill the holes one by one. For instance, in Figure 2, **`apply`** `H0` fails on both subgoals. The agent then repeats the process for each holes, using focused fine-grain reasoning to prove the corresponding subgoal. The proof is complete when there are no more holes.

4 Proof search

We combine our interactive protocol with the classic beam search algorithm. Inspired by [Yao et al., 2023], we use the LLM to rank and sort the proposals at each step of the search. A simplified version of the code is presented on the right. At each step, `agent.generate` generates `n_actions` possible steps (tactics or proofs). Each step is then validated with `petanque.step` and the state of all the resulting candidates is stored. Then `agent.sort` calls the LLM to discuss, compare and finally rank and sort the candidates for the next step.

```python
def beam_search(n_steps, n_actions, beam_size):
  s = petanque.start(thm)
  beam = [s] # Initial state
  for step in range(n_steps):
    candidates = []
    for s in beam:
      # Try multiple actions for each state
      for a in agent.generate(s, n_actions):
        sa = petanque.step(s, a)
        if petanque.proof_finished(sa):
          return sa.proof  # Proof found!
        else:
          candidates = candidates + [sa]
    # Rank and sort candidates
    beam = agent.sort(candidates)[:beam_size]
  return None # No proof found
```

For the tactic-by-tactic agent, the state contains the current goal obtained by running all the previous steps from the initial goal, i.e., the theorem statement. At each step, `agent.generate` generates multiple possible tactics for the current goal. For each tactic proposed by the LLM, `petanque.step` executes the tactic to compute the updated state. If the tactic is invalid, we log the failure and the state is not modified.

For the template agent, the state contains a template, i.e. a proof with holes and a queue containing pointers to these holes and the associated goals. At each step, `agent.generate` generates multiple possible proofs for the first hole in the queue. For each proposed proof, `petanque.step` builds the corresponding sub-template. The updated state is obtained by replacing the current hole by the sub-template and adding the sub-templates holes to the end of the queue.

As a baseline, our naive search corresponds to a beam search with `n_action=1` and `beam_size=1` (in which case, the sorting step is useless).

5 Evaluation

Logical Foundations exercises: We extracted the exercises of *Logical Foundations* [Pierce et al., 2024], the first volume of the *Software Foundation* textbooks series that is widely used to introduce Coq. We extracted 179 exercices. Given the popularity of this textbook the risk of data leak is high. We filtered out 66 "easy" exercises that are solved with one shot prompting. This dataset thus comprises 113 exercises.

BB(4) lemmas: To avoid data leak issues, we extracted the 260 lemmas from the recent proof of BB$(4) = 107$ [ccz181078, 2024]. The repository was created in April 2024, long after the knowledge cutoff date of the current version of GPT-4o (October 2023). To provide the necessary context for the proof, for each lemma we augment the prompt with all the preceding definitions and lemmas.

Evaluation. The results are presented in the following table. We use Coq 8.19.2 and GPT-4o version 2024-05-13 for all the experiments.

	Logical Foundations				BB(4)	
	tactics		template		template	
	naive	beam	naive	beam	naive	beam
% success	30.1	46.0	(13.3) 25.6	(23.9) 38.9	(21.0) 38.0	(38.0) 58.0

For both agents, we set `n_actions=4` and `beam_size=3`, with `n_steps=30` for the tactics agent and `n_steps=10` for the template agent. While the tactics agent outperforms the template agent on the Logical Foundation benchmark, we observe that the template agent is significantly cheaper and faster than the tactics agent. By design the tactics agent requires much more interactions with the LLM to reach a full proof step by step.

To limit the costs of our experiments, we only run the template agent on the first 100 Lemmas of the BB(4) benchmark. For the template agent, the gray numbers indicate the proportion of proofs that are correct at the first try (no holes). See Appendix A and Tables 1 and 2 for more details.

6 Related work and conclusion

LLMs and theorem provers Automatic theorem-proving is a longstanding challenge in computer science [Newell et al., 1957]. Recent work has used neural models based on autoregressive language model that generate a proof tactic by tactic. Most works use finetuned LLMs [Polu and Sutskever, 2020, Han et al., 2021, Wu et al., 2022, Yang et al., 2023, First et al., 2023], trained on (goal, tactic) pairs obtained from intermediate steps of existing proofs. On the other hand, Lample et al. [2022] use online training, progressively collecting more data. Closest to our work, Thakur et al. [2024] build a tactic-by-tactic LLM agent based on GPT-4 and also use an interface to summarize past interactions. They, however, do not use proof repair or beam search. Also close to our work, Wang et al. [2024] use proof repair over hierarchical proofs in Isabelle, coupled with best-first search. Contrary to us, they use fine-tuned models and no chain-of-thought. Finally, Lin et al. [2024] propose a framework for training language models to produce informal thoughts prior to each step of a proof, thereby boosting the model's theorem-proving capabilities.

Reasoning in LLMs This work is also related to recent investigations on the reasoning abilities of LLMs [Plaat et al., 2024]. Chain-of-Thought (CoT) prompting [Wei et al., 2022] was shown to improve LLM's answers; subsequent work found that these reasoning abilities could be elicited zero-shot [Kojima et al., 2022]. Further work interleaved CoT with decision-making [Yao et al., 2022], added search and complex control flow to reasoning [Chen et al., 2022, Yao et al., 2023, Besta et al., 2024], incorporated refinement and feedback [Madaan et al., 2024, Shinn et al., 2024], and learned to generate novel reasoning traces that proved beneficial for further training [Zelikman et al., 2022, 2024]. Like our work, many of these methods – especially the ones using search and refinement – make use of LLM-based scoring or ranking functions [Zheng et al., 2023].

Conclusion In this work, we have presented a new agent for building proofs leveraging chain of thought as an intermediate representation, and generating proofs by outputting step-by-step tactics or hierarchical proof templates. We couple this with beam search and natural language reranking and obtain good performance on a new evaluation set built with the help of our novel proof environment, *Pétanque*. Future work could investigate how one could use reinforcement learning to obtain better reasoning and performance with smaller models [OpenAI, 2024].

Acknowledgements We thank Cyril Cohen and Pierre Boutillier for many insightful discussions. We also thank Alex Sanchez-Stern for his feedback on early versions of Pétanque. This work is supported by the Inria Défi LLM4Code and the project ReaLiSe, Émergence Ville de Paris 2021-2025.

References

Umut A. Acar, Guy E. Blelloch, Matthias Blume, Robert Harper, and Kanat Tangwongsan. A library for self-adjusting computation. In Nick Benton and Xavier Leroy, editors, *ML Workshop*, 2005.

Maciej Besta, Nils Blach, Ales Kubicek, Robert Gerstenberger, Michal Podstawski, Lukas Gianinazzi, Joanna Gajda, Tomasz Lehmann, Hubert Niewiadomski, Piotr Nyczyk, et al. Graph of thoughts: Solving elaborate problems with large language models. In *AAAI*, 2024.

ccz181078. `https://github.com/ccz181078/Coq-BB5/tree/main`, 2024.

Wenhu Chen, Xueguang Ma, Xinyi Wang, and William W Cohen. Program of thoughts prompting: Disentangling computation from reasoning for numerical reasoning tasks. *arXiv preprint arXiv:2211.12588*, 2022.

Emily First, Markus N. Rabe, Talia Ringer, and Yuriy Brun. Baldur: Whole-proof generation and repair with large language models. *CoRR*, abs/2303.04910, 2023.

Emilio Jesús Gallego Arias, Benoît Pin, and Pierre Jouvelot. jscoq: Towards hybrid theorem proving interfaces. In Serge Autexier and Pedro Quaresma, editors, *UITP*, 2016.

Emilio Jesús Gallego Arias. SerAPI: Machine-friendly, data-centric serialization for Coq. preprint, 01 2019. URL `https://github.com/ejgallego/coq-serapi/`.

Emilio Jesús Gallego Arias. Flèche: Incremental validation for hybrid formal documents. under revision, 2024.

Jesse Michael Han, Jason Rute, Yuhuai Wu, Edward W Ayers, and Stanislas Polu. Proof artifact co-training for theorem proving with language models. *arXiv preprint arXiv:2102.06203*, 2021.

Takeshi Kojima, Shixiang Shane Gu, Machel Reid, Yutaka Matsuo, and Yusuke Iwasawa. Large language models are zero-shot reasoners. *NeurIPS*, 2022.

Guillaume Lample, Timothee Lacroix, Marie-Anne Lachaux, Aurelien Rodriguez, Amaury Hayat, Thibaut Lavril, Gabriel Ebner, and Xavier Martinet. Hypertree proof search for neural theorem proving. *NeurIPS*, 2022.

Haohan Lin, Zhiqing Sun, Yiming Yang, and Sean Welleck. Lean-star: Learning to interleave thinking and proving. *arXiv preprint arXiv:2407.10040*, 2024.

Aman Madaan, Niket Tandon, Prakhar Gupta, Skyler Hallinan, Luyu Gao, Sarah Wiegreffe, Uri Alon, Nouha Dziri, Shrimai Prabhumoye, Yiming Yang, et al. Self-refine: Iterative refinement with self-feedback. *NeurIPS*, 2024.

Allen Newell, John Clifford Shaw, and Herbert A Simon. Empirical explorations of the logic theory machine: a case study in heuristic. In *Western joint computer conference: Techniques for reliability*, pages 218–230, 1957.

OpenAI. Learning to Reason with LLMs. `https://openai.com/o1/`, 2024.

Benjamin C. Pierce, Arthur Azevedo de Amorim, Chris Casinghino, Marco Gaboardi, Michael Greenberg, Cătălin Hriţcu, Vilhelm Sjöberg, and Brent Yorgey. *Logical Foundations*. Software Foundations. 2024. Version 6.7, `http://softwarefoundations.cis.upenn.edu`.

Aske Plaat, Annie Wong, Suzan Verberne, Joost Broekens, Niki van Stein, and Thomas Back. Reasoning with large language models, a survey. *arXiv preprint arXiv:2407.11511*, 2024.

Stanislas Polu and Ilya Sutskever. Generative language modeling for automated theorem proving. *CoRR*, abs/2009.03393, 2020.

Tom Reichel, R. Wesley Henderson, Andrew Touchet, Andrew Gardner, and Talia Ringer. Proof repair infrastructure for supervised models: Building a large proof repair dataset. In Adam Naumowicz and René Thiemann, editors, *LIPIcs*.

Alex Sanchez-Stern, Yousef Alhessi, Lawrence K. Saul, and Sorin Lerner. Generating correctness proofs with neural networks. In *MAPL@PLDI*, 2020.

Noah Shinn, Federico Cassano, Ashwin Gopinath, Karthik Narasimhan, and Shunyu Yao. Reflexion: Language agents with verbal reinforcement learning. *NeurIPS*, 2024.

Amitayush Thakur, George D. Tsoukalas, Yeming Wen, Jimmy Xin, and Swarat Chaudhuri. An in-context learning agent for formal theorem-proving. In *COLM*, 2024.

The Coq Development Team. The Coq reference manual – release 8.19.0. `https://coq.inria.fr/doc/V8.19.0/refman`, 2024.

Haiming Wang, Huajian Xin, Zhengying Liu, Wenda Li, Yinya Huang, Jianqiao Lu, Zhicheng Yang, Jing Tang, Jian Yin, Zhenguo Li, and Xiaodan Liang. Proving theorems recursively. *CoRR*, abs/2405.14414, 2024.

Jason Wei, Xuezhi Wang, Dale Schuurmans, Maarten Bosma, Brian Ichter, Fei Xia, Ed H. Chi, Quoc V. Le, and Denny Zhou. Chain-of-thought prompting elicits reasoning in large language models. In *NeurIPS*, 2022.

Yuhuai Wu, Albert Qiaochu Jiang, Wenda Li, Markus N. Rabe, Charles Staats, Mateja Jamnik, and Christian Szegedy. Autoformalization with large language models. In *NeurIPS*, 2022.

John Yang, Carlos E. Jimenez, Alexander Wettig, Kilian Lieret, Shunyu Yao, Karthik Narasimhan, and Ofir Press. Swe-agent: Agent-computer interfaces enable automated software engineering. *CoRR*, abs/2405.15793, 2024.

Kaiyu Yang and Jia Deng. Learning to prove theorems via interacting with proof assistants. In *ICML*, 2019.

Kaiyu Yang, Aidan M. Swope, Alex Gu, Rahul Chalamala, Peiyang Song, Shixing Yu, Saad Godil, Ryan J. Prenger, and Animashree Anandkumar. Leandojo: Theorem proving with retrieval-augmented language models. In *NeurIPS*, 2023.

Shunyu Yao, Jeffrey Zhao, Dian Yu, Nan Du, Izhak Shafran, Karthik Narasimhan, and Yuan Cao. React: Synergizing reasoning and acting in language models. *arXiv preprint arXiv:2210.03629*, 2022.

Shunyu Yao, Dian Yu, Jeffrey Zhao, Izhak Shafran, Tom Griffiths, Yuan Cao, and Karthik Narasimhan. Tree of thoughts: Deliberate problem solving with large language models. In *NeurIPS*, 2023.

Eric Zelikman, Yuhuai Wu, Jesse Mu, and Noah Goodman. Star: Bootstrapping reasoning with reasoning. *NeurIPS*, 2022.

Eric Zelikman, Georges Harik, Yijia Shao, Varuna Jayasiri, Nick Haber, and Noah D Goodman. Quiet-star: Language models can teach themselves to think before speaking. *arXiv preprint arXiv:2403.09629*, 2024.

Lianmin Zheng, Wei-Lin Chiang, Ying Sheng, Siyuan Zhuang, Zhanghao Wu, Yonghao Zhuang, Zi Lin, Zhuohan Li, Dacheng Li, Eric P. Xing, Hao Zhang, Joseph E. Gonzalez, and Ion Stoica. Judging LLM-as-a-Judge with MT-Bench and Chatbot Arena. In *NeurIPS*, 2023.

A Detailed results

A.1 Logical Foundations

For the template agent, the gray numbers indicate the proportion of proofs that are correct at the first try (no holes). We also report the average length of the generated proofs (number of tactics) and the size of the smallest and the biggest proofs. Details are presented in Table 1.

	tactics		template		
	naive	beam	naive	beam	total
# success	34	52	(15) 29	(27) 44	113
% success	30.1	46.0	25.6	38.9	100.0
average proof length	10.6	9.4	16.0	12.4	
(min, max) proof length	(4, 31)	(4, 53)	(4, 55)	(4, 58)	

A.2 BB(4)

For each methods, we also report the original proof sizes (mean, min, and max) on the set of lemmas that were successfully proved. Details are presented in Table 2.

	template		
	naive	beam	total
# success	(21) 38	(38) 58	100
% success	38.0	58.0	100.0
average proof length	13.7	15.4	
original average proof length	7.4	7.9	
(min, max) proof length	(3, 38)	(3, 54)	
original (min, max) proof length	(2, 34)	(2, 34)	

B From Flèche to Pétanque

In this section we will describe Pétanque, a new environment for lightweight interaction with formal proof documents. Pétanque targets machine-learning applications such as reinforcement learning and other agent-based use cases, providing *zero-overhead*, *purely functional*[1] access to Coq's proof engine, along with some utilities to implement custom proof search routines.

Flèche Pétanque is built on top of Flèche [Gallego Arias, 2024], a new document manager for Coq. Flèche is both a formal document interpreter and a build system for Coq proof documents.

A schematic view of Flèche's behavior when the document is edited is presented in Figure 3. Flèche maintains an enriched representation of Coq proof documents, including the relevant Coq states associated to the interactive proofs and their dependencies. When an edit occurs, Flèche only invalidate the parts of the document that depend on that change, following standard incremental computing practices [Acar et al., 2005].

At any point, users can query Flèche for data about the document — for example information about the current proof obligations at a given point of the document — and Flèche will compute the requested information on-demand, as fast as possible.

coq-lsp Flèche's edit / query interface accommodates seamlessly the Language Server Protocol (LSP) protocol, the standard way to provide programming language support in modern Integrated Development Enviroments (IDEs). The LSP server coq-lsp[2] built on top of Flèche thus provides continuous real-time checking for Coq documents inside popular editors such as Emacs or VSCode.

[1]computations are treated as stateless functions, i.e., for equal inputs, we obtain equal outputs

[2]`https://github.com/ejgallego/coq-lsp`

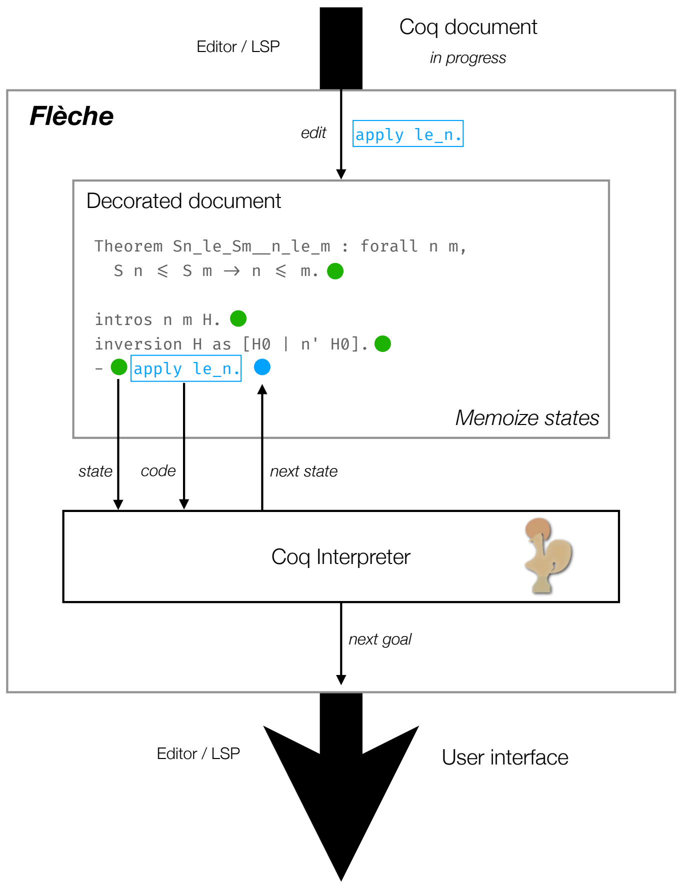


Figure 3: Flèche: a document manager for Coq. Flèche maintains a *decorated document* where each atom (definitions and proof steps) are associated with the Coq state (green dots). When an edit happens in the editor, Flèche retrieves the corresponding state, execute the code with the Coq interpreter, stores the new state (blue dot) in the decorated document, and returns the next goal that can be visualized in the editor. Communication with the editor relies on the LSP protocol.

```
class Pytanque:
  def start(self, file: str, thm: str) -> State
  def run_tac(self, state: State, tac: str) -> State
  def goals(self, state: State) -> List[Goal]
```

Figure 4: A simplified view of the pytanque API

Pétanque Unfortunately the edit/query document model turns out to be too expensive for high-throughput, proof-search applications: while Flèche invalidation on edits is very efficient, the associated overhead starts to become a problem when the edit frequency is higher than a few times per second. Moreover, using IDE protocols such as LSP means that agents need to exchange message with the server multiple times per step, which again creates non-trivial overhead. To overcome the previous problems, Pétanque provides one-shot direct access to Coq's proof state and tactic engine.

A simplified view of the Pétanque API is presented in Figure 4. Using this API, agents can perform *speculative proof checking* without altering the original document.

The `start` methods initialize a proof session where the initial Coq proof state correspond to the theorem statement `thm` in the document `file`. Then, given a state, the `run_tac` method executes a tactic `tac` and return the resulting state if successful. The `goals` method can be used to retrieve a human readable version of the proof goals (e.g., as in Figures 1 and 2).

Table 1: Detailed results for the Logical Foundations benchmark.

	tactics		template	
	naive	beam	naive	beam
Basics:andb_true_elim2	12	9	10	10
Basics:lower_letter_lowers	x	7	x	8
Basics:grade_lowered_once	11	6	10	6
Lists:eqblist_refl	x	x	x	x
Lists:count_member_nonzero	x	x	x	x
Lists:remove_does_not_increase_count	x	x	x	x
Lists:involution_injective	x	8	7	7
Lists:option_elim_hd	x	x	x	x
Lists:eqb_id_refl	x	6	14	14
Lists:update_eq	19	6	x	9
Lists:update_neq	13	6	11	7
Induction:add_comm	x	10	x	x
Induction:even_S	12	x	x	x
Induction:add_shuffle3	11	10	x	x
Induction:mul_comm	x	16	x	x
Induction:plus_leb_compat_l	x	x	x	x
Induction:mult_plus_distr_r	x	x	x	x
Induction:mult_assoc	x	11	x	x
Induction:add_shuffle3'	9	10	x	x
Induction:bin_to_nat_pres_incr	x	x	x	x
Induction:nat_bin_nat	x	x	x	x
Induction:bin_nat_bin	x	x	x	x
Imp:optimize_0plus_b_sound	x	x	x	x
Imp:pup_to_2_ceval	x	x	x	x
Imp:loop_never_stops	x	x	x	x
Imp:no_whiles_eqv	x	x	x	x
Imp:execute_app	x	x	x	x
Imp:s_compile_correct	x	x	x	x
Imp:break_ignore	4	4	4	4
Imp:while_continue	5	4	4	6
Imp:while_stops_on_break	x	4	x	x
Imp:seq_continue	x	x	x	x
Imp:seq_stops_on_break	4	5	x	x
Imp:while_break_true	4	4	4	4
Imp:ceval_deterministic	x	x	30	9
IndProp:ev_double	9	7	11	11
IndProp:ev5_nonsense	7	7	x	x
IndProp:ev'_ev	x	x	x	x
IndProp:ev_plus_plus	x	x	x	x
IndProp:total_relation_is_total	x	x	x	x
IndProp:empty_relation_is_empty	5	5	7	8
IndProp:O_le_n	4	4	10	10
IndProp:Sn_le_Sm__n_le_m	16	5	9	9
IndProp:lt_ge_cases	x	x	x	x
IndProp:le_plus_l	x	6	11	11
IndProp:plus_le	x	x	x	x
IndProp:add_le_cases	x	x	x	x
IndProp:plus_le_compat_r	x	14	x	x
IndProp:le_plus_trans	x	15	x	x
IndProp:n_lt_m__n_le_m	x	6	7	9
IndProp:plus_lt	x	x	x	x
IndProp:leb_complete	x	x	x	23
IndProp:leb_correct	x	x	x	x
IndProp:leb_true_trans	12	11	x	11
IndProp:R_equiv_fR	x	x	x	x
IndProp:subseq_refl	x	x	x	x
IndProp:subseq_app	8	4	4	4
IndProp:subseq_trans	4	4	x	6
IndProp:reflect_iff	11	12	20	18
IndProp:eqbP_practice	x	x	x	x
IndProp:merge_filter	19	4	29	4
IndProp:pal_app_rev	x	x	x	x
IndProp:pal_rev	4	4	4	4
IndProp:palindrome_converse	x	x	x	x
IndProp:pigeonhole_principle	x	x	x	x
IndProp:regex_match_correct	x	x	x	x
Poly:rev_involutive	14	9	12	12
Poly:map_rev	x	x	x	x
Poly:uncurry_curry	x	x	x	x
Poly:curry_uncurry	x	x	x	x
ImpCEvalFun:ceval__ceval_step	x	x	x	x
Logic:leb_plus_exists	x	x	x	x
Logic:In_map_iff	31	28	x	46
Logic:In_app_iff	x	x	55	x
Logic:All_In	x	x	x	x
Logic:combine_odd_even_intro	x	x	x	x
Logic:combine_odd_even_elim_odd	x	x	x	x
Logic:combine_odd_even_elim_even	x	x	x	x
Logic:eqb_neq	x	15	x	x
Logic:eqb_list_true_iff	x	x	x	x
Logic:forallb_true_iff	x	x	x	x
Logic:tr_rev_correct	x	x	x	x
Logic:excluded_middle_irrefutable	x	16	x	16
Rel:total_relation_not_partial_function	x	x	x	x
Rel:lt_trans'	18	6	x	4
Rel:lt_trans''	18	9	x	12
Rel:le_S_n	7	5	x	9
Rel:le_not_symmetric	x	7	x	7
Rel:le_antisymmetric	7	9	x	x
Rel:le_step	x	x	x	x
Rel:rtc_rsc_coincide	x	x	x	30
IndPrinciples:booltree_ind_type_correct	x	x	x	x
IndPrinciples:Toy_correct	x	x	x	x
IndPrinciples:reflect_involution	x	x	x	x
Maps:t_update_neq	x	12	10	14
Maps:t_update_permute	x	x	x	x
Tactics:rev_exercise1	9	7	17	15
Tactics:eqb_true	x	x	x	x
Tactics:plus_n_n_injective	x	x	34	x
Tactics:combine_split	x	x	21	20
Tactics:bool_fn_applied_thrice	21	16	35	x
Tactics:eqb_sym	x	x	x	17
Tactics:eqb_trans	10	x	x	x
Tactics:split_combine	x	x	x	x
Tactics:existsb_existsb'	x	x	x	x
ProofObjects:ev_8	7	7	7	7
ProofObjects:pe_implies_pi	x	12	x	11
AltAuto:ev100	x	53	58	55
AltAuto:andb3_exchange	x	4	x	4
AltAuto:andb_true_elim2	4	6	10	10
AltAuto:andb3_exchange'	x	12	x	23
AltAuto:nor_comm'	12	10	x	10
AltAuto:nor_not'	x	11	x	10

Table 2: Detailed results for the $\mathrm{BB}(4)$ benchmark.

	orig.	naive	beam
ffx_eq_x_inj	10	7	7
enc_v1_eq	6	x	x
enc_pair_inj	12	x	x
enc_list_inj	16	x	x
andb_shortcut_spec	3	7	9
orb_shortcut_spec	3	9	7
set_ins_spec	33	x	x
empty_set_WF	10	19	16
pop_back_len	8	x	20
pop_back__nth_error	15	x	54
list_eq__nth_error	34	37	44
pop_back'__push_back	6	x	x
St_enc_inj	2	5	4
St_eqb_spec	3	3	4
Sigma_eqb_spec	3	x	x
Sigma_enc_inj	2	x	x
listSigma_inj	12	38	23
map_inj	9	29	29
listT_enc_inj	7	6	6
Dir_eqb_spec	3	11	3
St_list_spec	4	x	12
Sigma_list_spec	4	13	8
Dir_list_spec	4	13	13
forallb_St_spec	9	x	14
forallb_Sigma_spec	9	18	17
forallb_Dir_spec	9	x	13
Steps_trans	9	x	x
Steps_unique	11	x	19
Steps_NonHalt	22	x	x
HaltsAt_unique	16	x	x
NonHalt_iff	27	x	x
LE_step	10	x	14
LE_Steps	10	13	12
LE_NonHalts	8	x	x
HaltTimeUpperBound_LE_NonHalt	7	x	x
LE_HaltsAtES_1	11	x	x
LE_HaltsAtES_2	14	x	x
HaltTimeUpperBound_LE_Halt	15	x	x
St_swap_swap	12	x	x
Trans_swap_swap	7	8	8
option_Trans_swap_swap	7	10	10
TM_swap_swap	8	x	15
ExecState_swap_swap	7	6	6
step_swap	18	x	48
step_halt_swap	10	x	39
Steps_swap	27	x	x
LE_swap_0	7	x	23
LE_swap	9	x	x
InitES_swap	8	x	15
HaltsAt_swap_0	15	x	17
HaltsAt_swap	9	31	30
HaltTimeUpperBound_LE_swap	10	x	x
HaltTimeUpperBound_LE_swap_InitES	5	x	x
Trans_rev_rev	7	6	8
option_Trans_rev_rev	8	11	10
TM_rev_rev	7	8	11
Tape_rev_rev	7	12	9
ExecState_rev_rev	7	6	6
fext_inv	3	5	5
step_rev	44	x	x
step_halt_rev	11	x	x
Steps_rev	27	x	x
LE_rev_0	7	19	19
LE_rev	9	x	x
InitES_rev	3	8	6
HaltsAt_rev_0	15	20	18
HaltsAt_rev	9	x	x
HaltTimeUpperBound_LE_rev	10	x	x
HaltTimeUpperBound_LE_rev_InitES	5	x	x
Trans_swap_id	10	x	x
isUnusedState_spec	58	x	x
step_UnusedState	11	13	17
Steps_UnusedState	15	x	x
HaltTimeUpperBound_LE_HaltsAtES_UnusedState	68	x	x
TM0_LE	7	x	x
UnusedState_TM0	10	12	21
UnusedState_dec	4	x	12
HaltTimeUpperBound_LE_HaltAtES_MergeUnusedState	31	x	x
St_to_nat_inj	4	5	5
St_suc_le	4	x	3
St_suc_eq	5	x	14
St_suc_neq	3	17	8
HaltTimeUpperBound_LE_HaltAtES_UnusedState_ptr	21	x	x
HaltsAtES_Trans	9	x	25
UnusedState_upd	68	x	x
UnusedState_ptr_upd	97	x	x
isHaltTrans_0	3	21	18
CountHaltTrans_upd	7	x	x
CountHaltTrans_0_NonHalt	21	x	x
Trans_list_spec	6	x	8
St_leb_spec	13	x	10
TM_simplify_spec	6	9	7
TM_upd'_spec	5	9	8
nat_eqb_spec	3	11	11
TNF_Node_expand_spec	64	x	x
TNF_Node_NonHalt	6	x	9
HaltDecider_cons_spec	7	16	39
SearchQueue_upd_spec	74	x	x
SearchQueue_upd_bfs_spec	30	x	x
SearchQueue_reset_spec	13	29	26