FROM RECALL TO REASONING: UNDERSTANDING THE ROLE OF ASSOCIATIVE MEMORY IN HYBRID ARCHITECTURES

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Paper under double-blind review

ABSTRACT

The demand for efficient inference has driven the development of subquadratic architectures as alternatives to the Transformer, though their capacity for complex, algorithmic reasoning remains a critical open question. To investigate the effect of architectural choice on downstream reasoning performance, we conduct a controlled study of reasoning scaling laws, training from scratch multiple hybrid-attention architectures of the same size (150M parameters) across three model classes (Mamba, Gated Linear Attention, Gated Delta Net) on a unified mathematical reasoning curriculum. Furthermore, we apply parallel test-time scaling methods via majority voting, and uncover a clear trend showing an improvement in reasoning performance as we increase the amount the amount of Attention layers in the architecture. To explain this trend, we analyze the models' responses using llm-as-a-judge and categorize its errors into 8 distinct types inspired by taxonomies in math education, identifying associative recall as the primary error mode in attention-free architectures. As we move toward fully linear models without any attention layers, our findings establish a connection between the choice of architectural update rule and systematic failures on reasoning primitives such as state-tracking and associative memory. We present a principled empirical study that informs the design and evaluation of next-generation hybrid reasoning models.

1 INTRODUCTION

The frontier of artificial intelligence is increasingly defined by the capacity for complex, multi-step reasoning. While scaling Transformers has yielded remarkable results, state-of-the-art performance in domains like mathematics and science now hinges on scaling test-time compute: generating and evaluating extensive chains of thought to find a correct solution (Wei et al., 2022; DeepSeek-AI et al., 2025; Snell et al., 2024). This "slow thinking" paradigm, however, collides with the Transformer architecture's $O(N^2)$ complexity, creating a significant efficiency bottleneck (Feng et al., 2025) and motivating the development of subquadratic sequence models.

Architectures based on State Space Models (SSMs) like Mamba (Gu & Dao, 2024; Dao & Gu, 2024) and linear-recurrent variants like Gated Delta Net (Yang et al., 2025b) have emerged as leading alternatives, offering near-linear time complexity. Their strong performance on language modeling benchmarks has fueled optimism that they can serve as drop-in replacements for Transformers. However, a growing body of evidence reveals a persistent "skill gap" (Bick et al., 2025c) on tasks requiring robust in-context recall (Arora et al., 2023). This has led the community to a pragmatic solution: hybrid architectures that interleave subquadratic layers with attention. This approach, however, raises a crucial question: what capability is attention providing that recurrent mechanisms lack? Our own preliminary work offers a direct clue: we find a striking dose-response relationship where systematically increasing the proportion of attention layers improves reasoning performance (Chaudhry et al.). This finding strongly suggests that attention provides a fundamental capability that is the very subject of our investigation.

To formalize this tension, we adopt a unifying perspective (Wang et al., 2025c; Sun et al., 2024) which frames sequence models as implementations of a dynamic *associative memory* (Hopfield,

1982). From this viewpoint, the attention mechanism is a powerful, non-parametric memory that stores explicit key-value pairs, while subquadratic models are recurrent systems that compress context into a finite parametric state (a "fast weight" memory) (Schlag et al., 2021). This distinction allows us to introduce our central research question: *Does the memory compression in subquadratic architectures create a fundamental bottleneck for mathematical reasoning?*

To address this question, we conduct a systematic, controlled study of architectural scaling laws for reasoning, focusing on hybrid models and the specific role of attention within them. Whereas prior work (Wang et al., 2025a) performed an empirical study of scaling laws in language modeling, our investigation takes a complementary direction by examining the mechanisms through which attention supports reasoning in hybrid architectures. Instead of relying on distilled models, which can inherit biases from a Transformer teacher (Paliotta et al., 2025; Bick et al., 2025b; Wang et al., 2025b), we train a suite of 150M parameter models—including a Transformer baseline, Gated Linear Attention, Mamba, and a Gated DeltaNet—on a common curriculum of mathematical text and reasoning traces. We then subject these models to rigorous evaluations through *Parallel Test-Time Scaling*, measuring performance gains from majority voting over multiple sampled solutions.

Our empirical investigation uncovers a "reasoning gap": the subquadratic models underperform the Transformer model and more critically, less effectively leverage increased test-time compute. To understand the source of this reasoning gap, we employ LLM-as-a-Judge method to classify errors into core mathematical concepts that isolate core reasoning primitives like state tracking and procedural abstraction. While previous studies (Poli et al., 2024) have employed synthetic tasks to probe architectural design, we extend this mechanistic lens to hybrid attention architectures, a rapidly growing class of models. We demonstrate that the reasoning bottleneck is a manifestation of a fundamental architectural trade-off between computational efficiency and memory fidelity. This work provides a principled framework for evaluating sequence models and underscores that robust reasoning in efficient architectures requires explicit solutions to the associative memory gap.

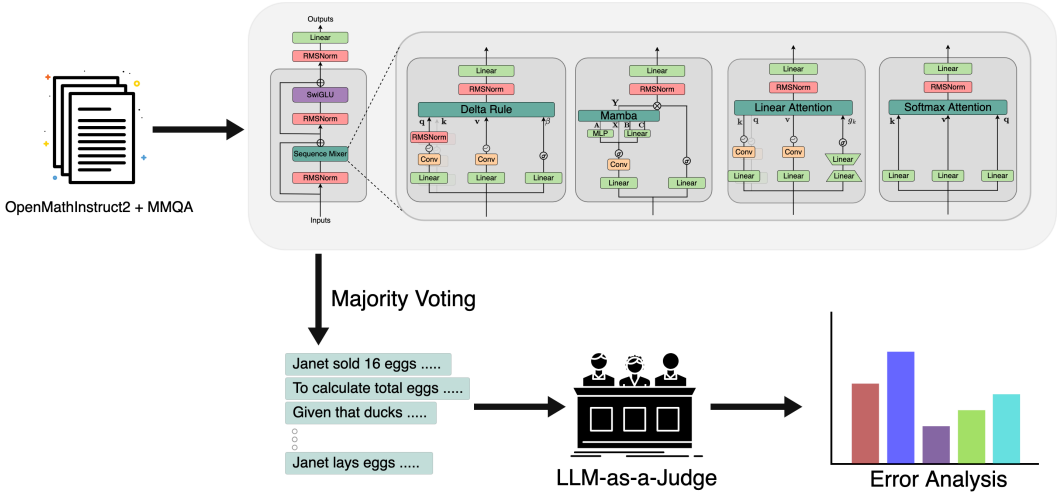


Figure 1: Overview of our experimental pipeline. Models trained on the OpenMathInstruct2 and MMQA datasets incorporate different architectural components (Gated DeltaNet, Mamba, Gated Linear Attention, and Softmax Attention). At test time, predictions are aggregated using majority voting and evaluated via an LLM-as-a-Judge framework, which classifies outputs into distinct error categories to enable fine-grained reasoning analysis.

2 Related Work

Our research is situated at the intersection of three active areas: the design of subquadratic architectures, the analysis of their reasoning and memory capabilities, and the scaling of test-time compute.

2.1 Explosion of Subquadratic Architectures

The quadratic complexity of the attention mechanism (Vaswani et al., 2017) has long been a target for optimization. Early efforts focused on linearizing attention, often by recasting it as a recurrent computation (Katharopoulos et al., 2020). This lineage has culminated in a new generation of powerful sequence models based on structured state space models (SSMs). **Mamba** introduced a selective SSM that made parameters data-dependent, dramatically improving performance on discrete data like language (Gu & Dao, 2024). Concurrently, other innovations have focused on improving the recurrent update rule itself, leading to architectures like **Gated Linear Attention (GLA)** (Yang et al., 2024a) and **Gated Delta Network (GDN)**, which incorporate a delta rule for more precise memory modifications (Yang et al., 2024b; 2025b). While these models excel in efficiency, their performance relative to Transformers is a subject of intense study.

2.2 Scaling Compute for Reasoning

The dominant paradigm for improving reasoning in LLMs is to scale compute at inference time. This can be done *sequentially*, by prompting a model to generate longer, more structured chains of thought, often guided by reinforcement learning (DeepSeek-AI et al., 2025) or budgeted thinking (Muennighoff et al., 2025). Alternatively, it can be done in *parallel*, by generating multiple solutions and using a selection mechanism like majority voting (self-consistency) (Wang et al., 2022). Some methods even propose reasoning in a latent space without generating tokens (Hao et al., 2024; Geiping et al., 2025) or apply reinforcement learning at test-time using self-generated rewards (Zuo et al., 2025b). Researchers have recently begun exploring these techniques for subquadratic models, hoping to leverage their higher throughput to outperform Transformers under a fixed time budget (Paliotta et al., 2025; Wang et al., 2025b). These works often rely on *distilling* knowledge from a powerful Transformer teacher (Bick et al., 2025b;a) or linearizing a pre-trained Transformer (Zhang et al., 2025a), which can obscure the inherent capabilities of the subquadratic architecture itself. In contrast, our work provides a systematic comparison of reasoning with these different model architectures by training models from scratch with the same dataset and systematically measuring their response to increased test-time compute.

2.3 Associative Recall Gap

Despite their efficiency, a consistent performance gap has been observed between Transformers and subquadratic models on tasks that demand robust in-context learning. Arora et al. (2023) first systematically documented this "recall gap" using an associative recall task. This gap has been framed as a trade-off between a model's state size and its recall ability (Arora et al., 2025), and has been mechanistically explained by the effectiveness of a "Gather-and-Aggregate" mechanism that is more robustly implemented by attention heads (Bick et al., 2025c). This has led to a view of Transformers as powerful associative memory systems (Zhong et al., 2025; Krotov & Hopfield, 2016; Chaudhry et al., 2024), a perspective we take inspirations from. Recent work by Okpekpe & Orvieto (2025a) adds a crucial dimension to this debate, highlighting the role of optimization stability. They demonstrate that subquadratic models like Mamba are far more sensitive to learning rate selection than Transformers on associative recall tasks, suggesting that some of the observed performance gap may be attributable to suboptimal training in addition to architectural limitations. They also uncover distinct scaling behaviors, finding that recurrent models benefit primarily from increased width (hidden state size), whereas Transformers require sufficient depth (at least two layers) to form the "induction head" circuits necessary for robust recall (Olsson et al., 2022).

2.4 Hybrid Models

The primary response to this performance gap has been the development of *hybrid models* that seek to balance efficiency and capability by interleaving attention and subquadratic layers (Lieber et al., 2024; Glorioso et al., 2024) or integrating attention in parallel heads (Dong et al., 2024). These hybrids implicitly concede that attention provides a critical function. Studies have systematically explored these hybrids and confirmed that performance on recall-intensive tasks scales directly with the proportion of attention layers (Wang et al., 2025a; Chaudhry et al.). Other lines of work aim to improve the memory of recurrent models directly through mechanisms like **test-time training** (Behrouz et al., 2024; 2025a; Oswald et al., 2025; Zhang et al., 2025b) or by framing memory

updates within a unified optimization framework (Behrouz et al., 2025b). Our work builds on these insights by explicitly studying the role of attention when reasoning with these hybrid architectures.

3 Experimental Setup

3.1 Architecture

We pretrain models of approximately 150M parameters, using the open-source OLMo codebase (OLMo et al., 2024). All model consists of 12 layers with 12 heads and a width of 768. The MLP dimension is 8x the model dimension. We use SwiGLU (Shazeer, 2020) and the Llama2 tokenizer (Touvron et al., 2023) with a vocab size of 32,000. We apply RoPE positional encoding (Su et al., 2023) to all self-attention layers and apply no positional encoding to Mamba, GLA and GDN layers. Previous works (Yang et al., 2025a) have shown that positional encodings aren't required since these architectures already represent positional information in its sequential processing. For all linear RNN layers, we apply a short convolution of size 4. For GLA and GDN, we do not apply any output gating and use a d_state of 16 for Mamba models. Appendix A.3 shows various ablation studies regarding specific architectural components of these architectures. For Mamba, we use the implementation from mamba-ssm package (Gu & Dao, 2023). For GDN and GLA, we use the implementations from flash-linear-attention library (Yang & Zhang, 2024; Yang et al., 2023; 2025b).

The pretrained models follow a striped design (Lieber et al., 2024; Glorioso et al., 2024; Ren et al., 2024), where a number of full-attention layers are interleaved in between SSM layers. In the Mamba/GDN/GLA 50 variant, attention is applied in layers 1, 3, 5, 7, 9, and 11. The 75 variant uses layers 3, 7, and 11, while the 83 variant restricts attention to layers 5 and 11. Finally, the 100 variant contains no full-attention layers.

3.2 Datasets

For pretraining, we use a mixture of OpenMathInstruct-2 (Toshniwal et al., 2024) and MetaMathQA (Yu et al., 2024). We train our models for 4 epochs on this mixture, totaling 37.1B tokens. OpenMathInstruct2 (Toshniwal et al., 2024) consists of 14M problem-solution pairs from the GSM8K and MATH Datasets and generated by Llama-3.1-405B-Instruct (Dubey et al., 2024). MetaMathQA is bootstrapped from GSM8K and MATH500 training datasets and consists of diverse reasoning traces. We do not apply any chat template to the datasets.

3.3 Hyperparameters

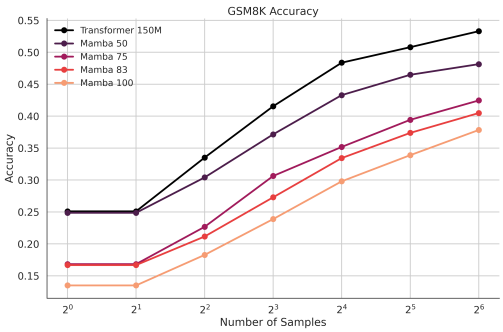
We use the AdamW optimizer (Kingma & Ba, 2017; Loshchilov & Hutter, 2019) for all of our models with a learning rate of 1e-3 and a weight decay of 0.1. We use a cosine decay scheduler to 10% of the peak learning rate and a linear warmup of 5000 steps.

3.4 Test Time Scaling

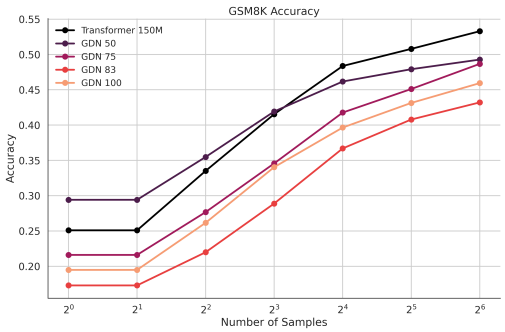
A model's capacity for reasoning can be effectively measured by its ability to improve performance when allocated more computational resources at inference time. We evaluate this on the widely-used **GSM8K** (Cobbe et al., 2021) and **MATH500** (Hendrycks et al., 2021) benchmarks, using a parallel scaling paradigm. We measure the performance gains from exploring a wide solution space using majority voting. We generate N independent solutions via sampling and report accuracy as a function of N, for $N \in \{1, \ldots, 64\}$. This tests the model's ability to converge on a correct answer through diverse reasoning paths. Specific implementation details are provided in Appendix B.4.

4 Failure Modes in Mathematical Reasoning

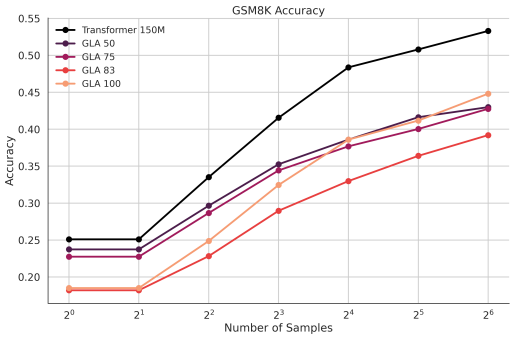
First, we train a suite of architecturally diverse yet parametrically equivalent models from scratch to create a level playing field. Second, we rigorously benchmark their reasoning capabilities by measuring their response to increased test-time compute on standard mathematical tasks. Finally, we conduct a mechanistic analysis using novel diagnostic tools to connect observed performance gaps to underlying architectural properties.



(a) Mamba



(b) GDN



(c) GLA

Figure 2: Parallel Test-Time Scaling via Majority Voting applied across architectures on GSM8K dataset. The Transformer consistently shows the best performance. As the percentage of attention layers increases, the models' performance consistently increases. However, the Transformer tends to benefit more than the other models. For GDN50, its performance starts higher than Transformer but is overtaken.

4.1 A Taxonomy of Mathematical Reasoning Errors

While prior work has compellingly linked performance gaps to behavioral failures in associative recall (Arora et al., 2023) and optimization instability (Okpekpe & Orvieto, 2025b), the relevance of these associative recall failures to real-life problems such as multi-step reasoning remains unknown. Our analysis is designed to bridge this gap by systematically analyzing the chains of thought of the different models to unpack the most common types of errors observed, and see which errors are mitigated by test-time scaling and by increasing the percentage of attention.

We classify reasoning failures into 8 distinct categories, organized by the cognitive function that fails. Errors range from foundational failures in processing the prompt's context to abstract failures in strategic planning and execution. This taxonomy is heavily inspired by similar error categorizations found in mathematics education (Radatz, 1979; Newman, 1977). In particular, working memory has been shown to be a dominant feature in mathematical reasoning which further motivates a detailed investigation into different kinds of memory-related mistakes (Ashcraft & Kirk, 2001; Raghubar et al., 2010; Geary, 2010). We combine this with the knowledge that linear attention style models struggle on tasks that require in-context associative memory, and thus develop three categories of errors. Recent work has shown that evaluations methods from childhood math education transfer well to understanding mathematical reasoning deficits in language modeling (Mishra et al., 2024).

In-Context Associative Memory Failures. These errors represent a failure to build and use a correct internal model of the problem from the prompt text. They correspond to failures on the most basic static and dynamic recall tasks.

1. **Key-Value Binding Error:** A failure at the initial "reading" phase. The model incorrectly extracts a value from the text, hallucinates a value or entity not present, or swaps values between two distinct entities. This results in a flawed internal set of facts before reasoning begins.
2. **State Tracking Error:** A failure of dynamic memory update. The model correctly calculates an intermediate value for a changing quantity but then fails to use this new value in a subsequent step, incorrectly reverting to a stale (old) value that is no longer valid.
3. **Context Synthesis Error:** A failure during a calculation step to retrieve the correct set of values from its internal model of the problem. The model's internal facts are correct, but it incorrectly gathers them, often retrieving an irrelevant distractor number instead of the required value.

Parametric Memory & Procedural Failures. These errors occur when the model fails to retrieve and apply general mathematical knowledge stored in its weights, even if the problem context is understood correctly.

4. **Procedural Retrieval Error:** The model incorrectly recalls or applies a specific, step-by-step algorithm or formula. The error is in the "how-to" knowledge for a standard mathematical process, like using the formula for area instead of perimeter, or incorrectly reversing a percentage.
5. **Conceptual Knowledge Gap:** The failure stems from a misunderstanding of an abstract mathematical definition, property, or theorem. It is not just a wrong formula, but a lack of understanding of the principles governing the problem (e.g., what is a "remainder" in a real-world context).

Logical & Execution Failures. These errors occur at the highest levels of reasoning, involving abstract planning and final execution, assuming both the internal model of the prompt and the procedural knowledge are sound.

6. **Flawed Logical Synthesis:** The model's high-level strategic plan is fundamentally invalid or nonsensical from the start. It connects facts and procedures in a sequence that does not logically address the problem's context or constraints, often by inventing an unstated goal.
7. **Calculation Error:** A simple arithmetic mistake made during the execution of an otherwise correct and logical plan. The strategy, procedure, and all variables are correct, but a basic computation (e.g., addition or multiplication) is wrong.
8. **Goal Interpretation Error:** The model executes a valid and logical sequence of steps for a subproblem but fails to answer the specific, final question asked. This often involves reporting an intermediate result as the final answer or solving for a different quantity altogether.

By evaluating our model responses against these kinds of errors, we can build a fine-grained understanding of the performance gaps observed on complex benchmarks. Full prompt templates and examples of all eight errors are provided in Appendix C.

4.2 LLM AS A JUDGE

LLM-as-a-Judge is an evaluation paradigm in which a capable LLM grades or compares system outputs under an explicit rubric, yielding scalable evaluation with strong judge–human agreement on open-ended tasks (Zheng et al., 2023; Liu et al., 2023; Gu et al., 2024). Foundational studies (MT-Bench/Chatbot Arena; G-Eval) document both effectiveness and failure modes (e.g., position/verbosity/self-preference biases) and propose mitigations such as order randomization, rubricized criteria, and multi-judge aggregation (Zheng et al., 2023; Liu et al., 2023; Shi et al., 2024; Tan et al., 2024). Judge outputs can also be repurposed as training signals (RLAIF) to supervise other models (Bai et al., 2022). For mathematical reasoning traces, step-level labels support best-of-N selection and process supervision (PRMs), which has been shown to outperform outcome-only signals on math (e.g., PRM800K; Let's Verify Step by Step) (Lightman et al., 2023). process-oriented variant.

In our setup, test-time scaling generates 64 candidate solutions; we take the first 8 (due to compute restrictions) and have an LLM judge (Gemini 2.5 Flash-Lite) produce structured error analyses that classify the single most applicable reasoning error; the full prompt appears in Appendix C (Google DeepMind, 2025). Although a response may contain multiple overlapping mistakes, the judge is instructed to choose the primary error category.

5 RESULTS

Our experiments reveal a consistent and significant gap in reasoning capabilities between the Transformer and subquadratic architectures. We first present the primary finding on standard benchmarks, demonstrating that subquadratic models fail to effectively leverage test-time compute. We then use our mechanistic analysis tools to diagnose the root cause of this gap, tracing it back to a fundamental deficiency in associative memory, a finding that both builds upon and provides a deeper explanation for the recall gap identified in prior work (Arora et al., 2023).

5.1 THE REASONING BOTTLENECK: DIMINISHING RETURNS FROM TEST-TIME COMPUTE

We begin by evaluating our 150M parameter models on the GSM8K benchmark using parallel scaling. As shown in Figure 2, a clear performance hierarchy emerges. The Transformer not only starts with a higher baseline accuracy (pass@1) but also benefits substantially from majority voting, with its performance continuing to climb steeply as the number of samples increases. In contrast, all three subquadratic architectures exhibit a much shallower scaling curve. They show modest initial gains but quickly plateau at a performance ceiling significantly below that of the Transformer. This demonstrates that simply allocating more computational "breadth" at test-time is insufficient to close the reasoning gap. We find similar results on the MATH dataset in Figure 4 of Appendix A.

To confirm this finding is not specific to one scaling method or dataset, we conducted further evaluations. When evaluated on the more challenging MATH500 benchmark, the performance delta between the Transformer and subquadratic models becomes less pronounced and even reverses for GDN hybrid models. We hypothesize that this behavior may stem from the distinctive delta-style update rule employed by Gated DeltaNet, which could confer advantages in certain reasoning regimes. Understanding this effect requires further study and represents an important direction for future work. These additional results, together with ablations on hybrid architectures that show a generally increasing performance trend as more attention layers are added, are presented in Appendix A.

5.2 DIAGNOSING THE REASONING GAP: ASSOCIATIVE MEMORY AS THE CAUSE

Having established the trend of higher Attention percentage generally increasing reasoning performance, we investigate its cause. We hypothesize that the poor performance on mathematical reasoning stems from an underlying failure of in-context associative memory. We test this directly by classifying the reasoning traces of the models into the previously mentioned categories using LLM-as-a-judge. Unlike behavioral tasks such as Multi-Query Associative Recall, which test the holistic skill of recall, our probes are meant to unpack associative recall skills specifically in mathematical reasoning. We show the results for GSM8K below and MATH in Appendix C.

	Transformer	Mamba50	Mamba75	Mamba83	Mamba100	GLA50	GLA75	GLA83	GLA100	GDN50	GDN75	GDN83	GDN100
In-Context Associative Memory Failures													
Key-Value Binding Error	**512.50**	**481.38**	**499.62**	**498.38**	**679.25**	**494.62**	**511.38**	**557.25**	**539.25**	**420.12**	**411.88**	**456.62**	**468.00**
State Tracking Error	22.75	24.00	27.75	25.25	12.25	22.62	26.00	25.25	25.12	26.00	29.00	28.00	32.12
Context Synthesis Error	93.75	111.88	112.25	123.00	68.25	109.88	103.00	108.25	116.00	113.50	124.88	120.38	113.00
Parametric Memory & Procedural Failures													
Procedural Retrieval Error	30.75	37.25	34.62	35.25	24.75	37.50	38.00	38.62	33.75	35.25	37.88	37.38	35.75
Conceptual Knowledge Gap	4.75	5.25	5.38	7.62	3.12	5.12	7.00	5.38	7.50	9.75	6.00	6.38	7.75
Logical & Execution Failures													
Flawed Logical Synthesis	254.38	271.62	227.88	237.38	324.50	273.62	285.62	277.38	206.75	266.50	190.62	198.75	185.38
Calculation Error	13.75	11.25	14.62	11.50	8.38	12.12	12.25	8.25	12.62	19.25	14.12	18.25	14.12
Goal Interpretation Error	57.75	63.75	64.50	62.00	37.38	71.62	68.25	65.75	77.88	68.12	75.75	69.38	72.75

Table 1: Error Category Decomposition for Model Responses on GSM8K dataset. We take an average across 8 generations per problem in order to account for variance in the LLM's classification. The most common error across each model is the Key-Value Binding Error (bolded).

Note that Key-Value Binding Errors constitute the vast majority of errors chosen by the model, with the second highest category being Flawed Logical Synthesis. This is for GSM8K, which are grade school problems where many of the tasks are word problems where values are associated to variables and the student is meant to do operations on them. For MATH, which constitutes substantially harder problems that require more mathematical maturity, the predominant error mode is Flawed Logical Synthesis with conceptual knowledge gaps and procedural retrieval errors increasing. In Figure 9, we find that the total errors and KV errors tend to decrease with Attention percentage in a similar fashion, with an increase in Attention generally decreasing the percentage of KV errors relative to

total errors. Note that 100% Attention corresponds to the 150M Transformer model, and we see a slight jump for KV Binding Error percentage.

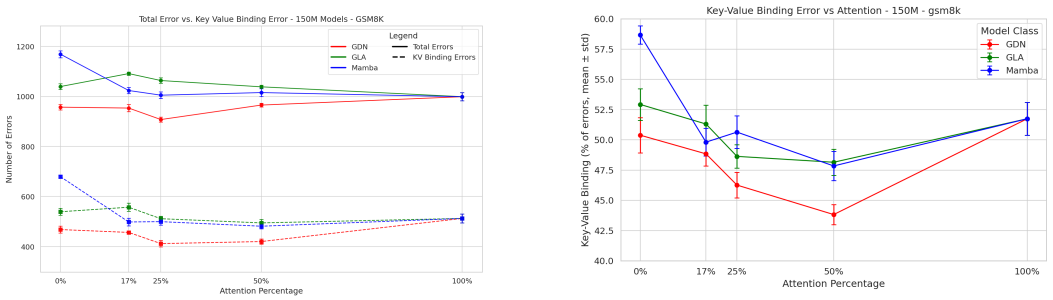


Figure 3: Key-Value Binding Errors Are Reduced by Attention. On the left, we show the total number of errors and key-value binding errors decreasing as a function of Attention for all model classes. On the right, we show that the percentage of total errors corresponding to KV binding tends to decrease with attention percentage, suggesting that the main mechanism by which Attention improves performance is by decreasing KV binding errors.

6 Discussion

Our findings establish a clear link between architectural priors, associative memory deficiency, and the capacity for algorithmic reasoning. The Transformer's non-parametric, token-to-token attention mechanism provides a high-fidelity, content-addressable memory that is resilient to distraction and scales effectively with test-time compute. In contrast, the compressed, fixed-size recurrent state of subquadratic models acts as a significant bottleneck, leading to rapid memory degradation and a hard ceiling on reasoning performance.

Our work provides a strong architectural explanation for the reasoning gap, complementing other recent findings that highlight optimization challenges in subquadratic models (Okpekpe & Orvieto, 2025b). Furthermore, we try to account for these difficulties by doing comprehensive hyperparameters sweeps and ablations as shown in Appendix A.

This study prompts a critical re-evaluation of what "efficiency" means for reasoning. In Appendix A.2, we show that even when scaling with FLOPs the hybrid models only slightly shift. This means that while subquadratic models are more efficient in FLOPs-per-token, the Transformer makes better use of each computational step, achieving higher accuracy for a given test-time compute budget. This has implications for emerging paradigms like *latent reasoning*, or "Chain-of-Continuous-Thought" (Hao et al., 2024; Geiping et al., 2025), where the success of internal reasoning steps will still hinge on the architecture's ability to maintain a high-fidelity internal state. We anticipate that pure subquadratic architectures would benefit less from these methods than pure Transformers would, although hybrid architectures could strike an interesting balance.

Our work focuses on the capabilities of the *internal*, architectural memory of sequence models. An alternative and complementary approach is to augment models with an explicit, *external* memory store. Recent work such as CAMELOT (He et al., 2024) has shown remarkable success with this paradigm, by coupling a frozen LLM with a training-free, consolidated associative memory module to enable the the model to handle arbitrarily long contexts by reading from and writing to this external store. We anticipate that the ideal reasoning architecture would elegantly unify a model with powerful internal associative memory and a persistent, external knowledge store.

Our findings might seem at odds with recent successes of "Hybrid reasoning models" like M1 (Wang et al., 2025b) and Qwen3-Next (Team, 2025), which achieve performance comparable to state-of-the-art Transformers. However, these models are not pure subquadratic architectures; they are **hybrids** that strategically interleave a significant number of attention layers within a recurrent backbone. For instance, the M1-3B model incorporates 6 attention layers among its 28 total layers. In this light, their strong performance can be understood as leveraging the attention layers to periodically compensating for the limitations of the recurrent state. Qwen3-Next on the other hand

utilized Gated Attention rather than Attention and a number of different architectural innovations, making it hard to identify the cause of their performance gain. Our paper is the first principled study into exploring these architectures' capabilities in reasoning, and highlights that associative recall is a primary cause for their performance gap.

While our experiments focused on mathematical and algorithmic tasks, the requirement for high-fidelity associative memory is a general one. Any domain that requires grounding in long, detailed contexts is likely to be affected by this architectural bottleneck, including long-form question answering, large codebase analysis, and medical or legal document processing.

7 Conclusion

In this work, we investigated the performance gap between Transformer and hybrid architectures on mathematical reasoning tasks. Through a controlled study of models trained from scratch, we demonstrated that while all architectures benefit from increased test-time compute, hybrid models exhibit sharply diminishing returns, hitting a performance ceiling that Transformers easily surpass. We performed error analysis of the reasoning traces using an LLM-as-a-judge rubric targeting primitives underlying mathematical reasoning and found that attention mainly improves performance by increasing memory fidelity and thus decreasing KV binding errors. We conclude that associative memory is not merely one skill among many but a foundational capability upon which robust, scalable reasoning is built. The path toward models that are both efficient and capable reasoners must therefore prioritize the development of architectural priors that explicitly support and preserve memory fidelity.

Our study opens several directions for further research. First, all experiments were conducted at the $\sim$150M parameter scale to enable controlled, from-scratch training; it remains an open question whether the reasoning gap we observe between Transformers and hybrid models will persist or diminish at larger model sizes. Second, while we carefully tuned optimization settings, it is possible that some of the poor performance of subquadratic models arises from sensitivity to training dynamics rather than fundamental architectural limitations. Finally, our results highlight the relative strength of Gated DeltaNet, whose delta-style update rule enables more effective use of finite recurrent memory. We believe this mechanism deserves greater focus, both as a promising architectural direction in its own right and as inspiration for designing more precise and efficient memory update rules.

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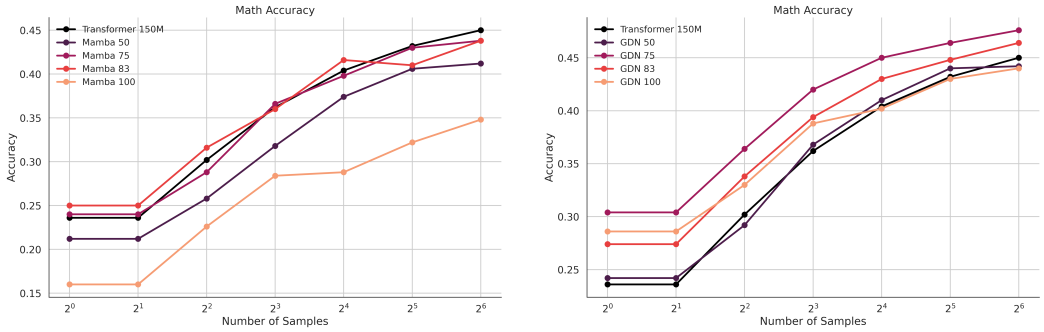
A Additional Results

This section provides supplementary results that expand upon the core findings presented in the main body of the paper, including performance on more challenging benchmarks, alternative scaling methods, and architectural ablations.

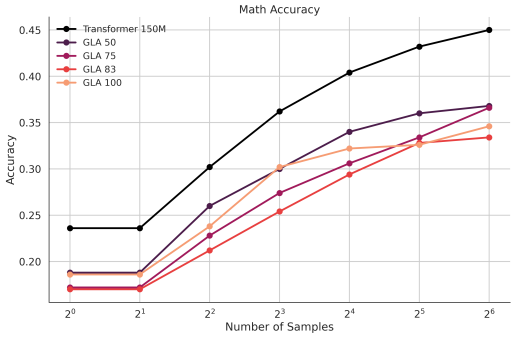
A.1 Performance on the MATH Benchmark

To validate that our findings generalize to more complex mathematical reasoning, we evaluated all 150M parameter models on the MATH benchmark (Hendrycks et al., 2021). As shown in Figure 4, the performance gap between the Transformer and subquadratic architectures is less pronounced on this more challenging dataset. The Transformer demonstrates a significant advantage in pass@128 performance for all Mamba and GLA models, but performs slightly worse than GDN hybrid models.



(a) Mamba

(b) GDN



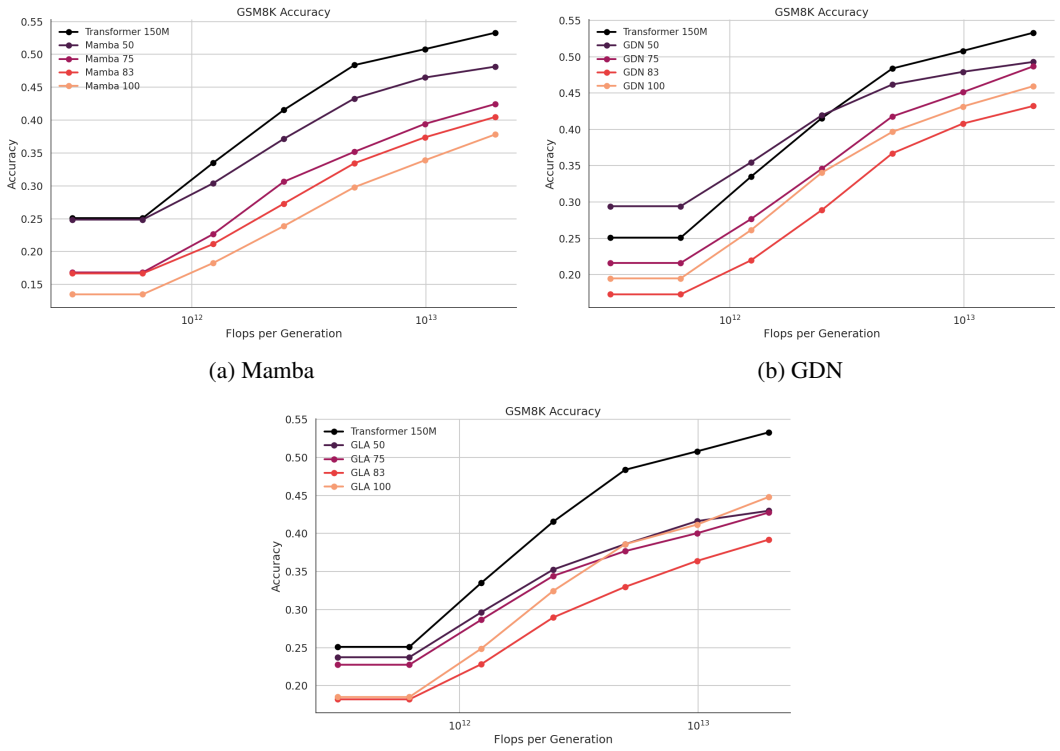
(c) GLA

Figure 4: Test-Time Scaling across architectures on MATH dataset.

A.2 Performance on GSM8K with respect to inference FLOPs

A.3 Ablation Study: Effect of architectural components

Mamba based models have shown to be hard to optimize with the optimal learning rate playing a critical role in the performance (Okpekpe & Orvieto, 2025b). We conduct ablation studies on various architectural primitives like ShortConv, dt_rank, and attention placement in hybrid models.



(a) Mamba (b) GDN

(c) GLA

Figure 5: Parallel Test-Time Scaling via Majority Voting with respect to FLOPs applied across architectures on GSM8K dataset. We get similar performance relative to the original. FLOPs are estimated using conversion values from (Wang et al., 2025a).

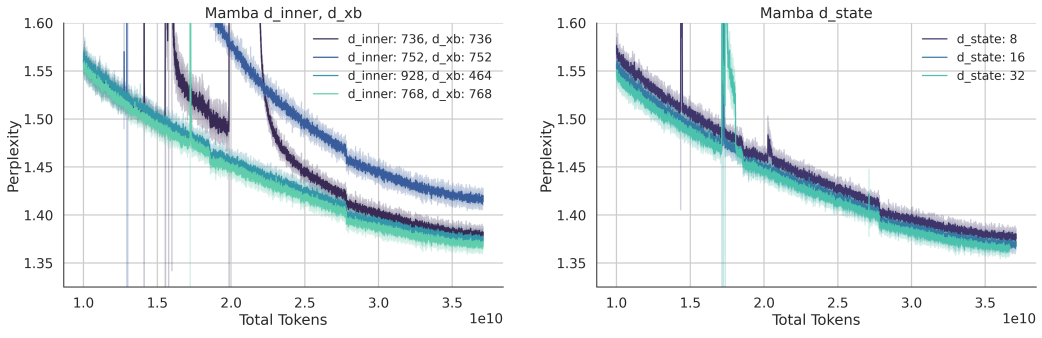


Figure 6: For mamba models, (**Right**) we sweep over some reasonable choices of d_{inner} and d_{xb}. Note that these choices are independent of the model dimension (768). (**Left**) The SSM state dimension. Mamba models are very susceptible to loss spikes.

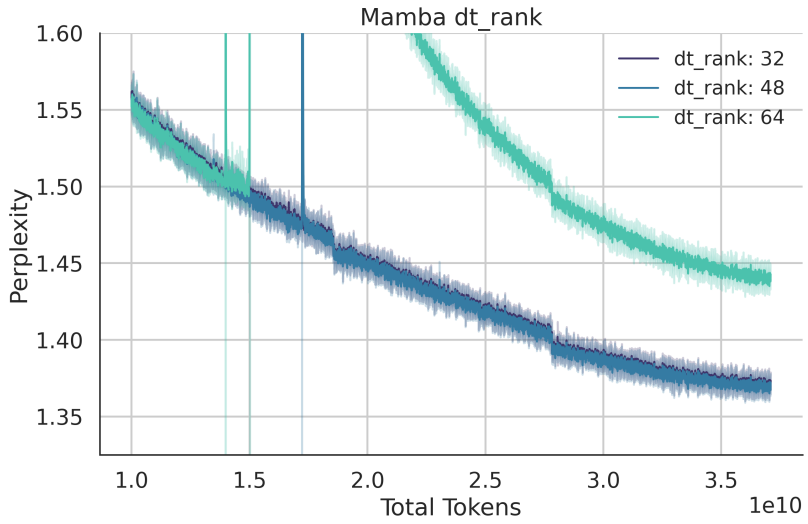


Figure 7: To understand why spikes occur in Mamba models, we tried changing the dt_rank parameter. Intuitively this controls the information written to the hidden state. We find conservative values of dt_rank reduce the instabilities. Related, Zuo et al. (2025a) reported that clipping dt to positive values also help with instabilities.

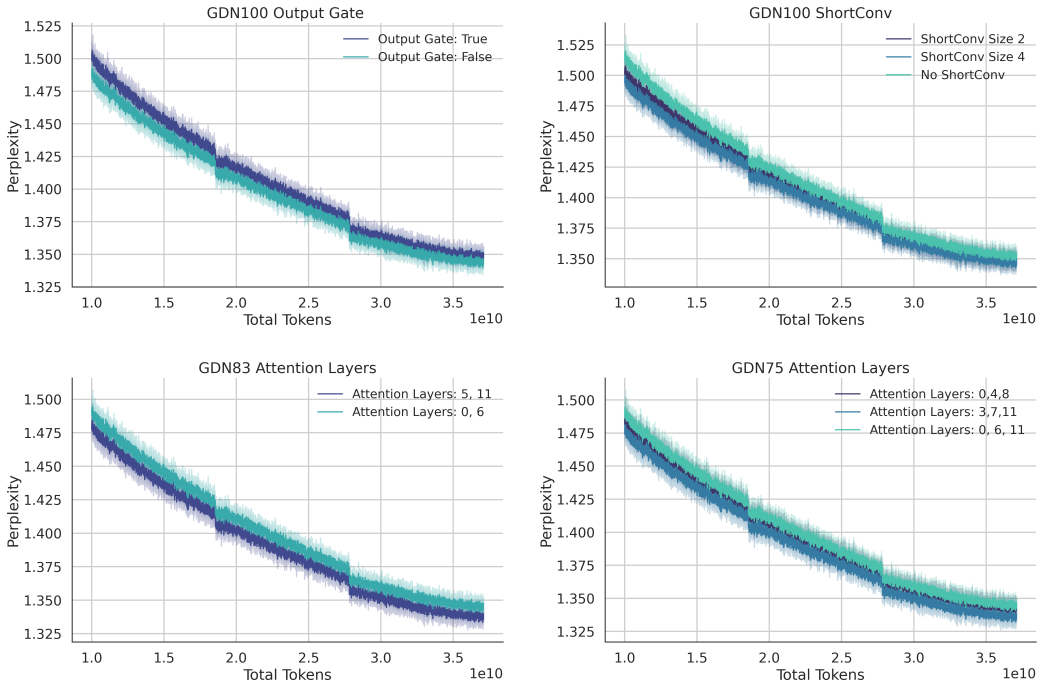


Figure 8: (**Top Left**) Sweep over the Output Gate. (**Top Right**) ShortConv is an important piece in the architectural backbone of Gated DeltaNet. While the size of the convolution does not matter, its existence is important to retain sufficient reasoning capabilities. (**Bottom Left**) For Gated DeltaNet 83, we try various permutations of inserting attention Layers. Attention in the last layer (11) greatly improves performance, also seen in (**Bottom Right**) for GDN73.

Table 2: Default hyperparameters for 150M models

Parameters	Values
Model dimension	768
Number of Layers	12
Number of Heads	12
Number of Key-Value Heads	12
MLP Ratio	8
Max Sequence length	2048
Embedding Size	32000
Activation Type	SwiGLU
Positional Embedding	RoPE
RoPE Theta	10000
Optimizer Type	AdamW
Adam Betas	(0.9, 0.95)
Learning Rate	0.001
Weight Decay	0.1
Tokenizer	Llama-2-7b

B Additional Experimental Details

This section provides a comprehensive overview of the models, datasets, and training configurations used in our study, ensuring full reproducibility.

B.1 Model Architectures

All four models were implemented in the OLMo codebase (OLMo et al., 2024) using PyTorch and open source triton implementation from Flash-Linear-Attention (Yang & Zhang, 2024).The models were scaled to approximately 150M parameters. The key architectural hyperparameters for each type of layer are detailed in Tables 2

Hyperparameter	Mamba
d_state	16
d_conv	4
expand	1
d_xb	768
d_inner	768
dt_rank	32
Total Parameters	12.1M

Table 3: Mamba configuration

Hyperparameter	GDN
expand_v	1.0
expand_k	1.0
use_gate	false
use_short_conv	true
conv_size	4
conv_bias	false
use_qk_norm	true
Total Parameters	11.5M

Table 4: GDN configuration

Hyperparameter	GLA
expand_k	1.0
expand_v	1.0
use_short_conv	true
conv_size	4
conv_bias	false
use_output_gate	false
gate_fn	swish
feature_map	null
gate_logit_normalizer	16
gate_low_rank_dim	16
clamp_min	null
fuse_norm	true
Total Parameters	11.5M

Table 5: GLA configuration

B.2 Training Datasets and Curriculum

Our training curriculum was designed to build broad mathematical competency specializing in reasoning.

Dataset Composition. The initial supervised fine-tuning dataset was a blend of the following open-source resources:

- **OMI2:** OpenMathInstruct (Toshniwal et al., 2024) consists of 14M question-answer pairs. The dataset was constructed by prompting Llama3.1-405B to 1) Generate solutions for GSM8K and MATH500, and 2) Create new question-answer pairs similar to the original datasets.
- **MMQA:** MetaMathQA (Yu et al., 2024) is generated using a novel bootstrapping method where the question answer diversity is prioritized. The diversity is particularly important in reasoning directions,

The final mixture consisted of a 1:1 ratio of OpenMathInstruct and MetaMathQA for a total of 9.3B tokens. We train our models on 4 epochs of this dataset, resulting in total 37.1B tokens.

B.3 Training Hyperparameters

Both training phases used the AdamW optimizer (Kingma & Ba, 2017; Loshchilov & Hutter, 2019). Key hyperparameters were kept consistent across all architectures and are detailed in Table 6.

Table 6: Training Hyperparameters.

Hyperparameter	Value
Optimizer	
Name	AdamW
Learning Rate	0.001
Weight Decay	0.1
Epsilon (eps)	1e-8
Decay Norm and Bias	true
Decay Embeddings	true
Scheduler	
Name	cosine_with_warmup
Warmup Steps (t_warmup)	5000
Final LR Ratio (α_f)	0.1
Warmup Min LR	0
Tokenizer	
Identifier	meta-llama/Llama-2-7b-hf

B.4 Test-Time Scaling Implementation

Parallel Scaling (Majority Voting). For all majority voting experiments, we used nucleus sampling with a temperature of 0.8 and a top-p value of 0.9 for a generation length of 1024. The final numerical answer was extracted from each of the N generated outputs using the Math_Verify library from HuggingFace.

C Error Analysis

C.1 LLM-as-a-Judge

Prompt

You are an expert AI diagnostician specializing in mathematical reasoning errors. Your task is to analyze a single generated answer and classify its primary error if it is incorrect.

— Context —

Question: {question}

Correct Answer's Final Solution: {correct_answer}

— Task —

Analyze the following generated answer:

Generated Answer: "{generated_answer}"

— Instructions —

1. First, determine if the final boxed answer in the "Generated Answer" is correct.

2. If the answer is **CORRECT**, classify the error as **"No Error"**.

3. If the answer is **INCORRECT**, identify the single, most critical reasoning error and classify it into **exactly one** of the following 8 categories based on the detailed definitions below.

— Error Taxonomy —

Group A: In-Context Associative Memory (ICAM) Failures (Errors in creating and using an internal model of the problem from the prompt text)

1. Key-Value Binding Error: A failure at the initial "reading" phase. The model incorrectly extracts a value from the text, hallucinates a value or entity not present, or swaps values between two distinct entities. This results in a flawed internal set of facts before reasoning begins.

2. State Tracking Error: A failure of dynamic memory update. The model correctly calculates an intermediate value for a changing quantity but then fails to use this new value in a subsequent step, incorrectly reverting to a stale (old) value.

3. Context Synthesis Error: A failure during a calculation step to retrieve the correct set of values from its internal model of the problem. The model's internal facts are correct, but it incorrectly gathers them, often retrieving an irrelevant distractor number instead of the required value for a specific operation.

Group B: Parametric Memory & Procedural Failures (Errors in recalling general knowledge from weights)

4. Procedural Retrieval Error: The model retrieves the wrong algorithm or a systematically flawed ("buggy") version of the correct one. The error is in the "how-to" knowledge for a standard mathematical process, like using the formula for area instead of perimeter, or incorrectly reversing a percentage.

5. Conceptual Knowledge Gap: The failure stems from a misunderstanding of an abstract mathematical definition, property, or theorem. It is not just a wrong formula, but a deeper lack of understanding of the principles governing the problem (e.g., what a "remainder" implies in a real-world context, or the definition of a prime number).

Group C: Logical & Execution Failures (Errors in high-level planning and final execution)

6. Flawed Logical Synthesis: The model's high-level strategic plan is fundamentally invalid or nonsensical from the start. It connects facts and procedures in a sequence that does not logically address the problem's context or constraints, often by inventing an unstated goal.

7. Calculation Error: A simple arithmetic mistake made during the execution of an otherwise correct and logical plan. The strategy, procedure, and all variables are correct, but a basic computation (e.g., addition, multiplication) is wrong.

8. Goal Interpretation Error: The model executes a valid and logical sequence of steps for a sub-problem but fails to answer the specific, final question asked. This often involves stopping short and reporting an intermediate result as the final answer or solving for a different quantity altogether.

Return ONLY a single, valid JSON object with this exact structure:

{

"is_correct": true/false,

"error_category": "Category Name" or "No Error",

"error_group": "A", "B", "C", "None", or "Error",

"explanation": "A brief, one-sentence explanation of the error or correctness.",

"erroneous_step_quote": "The exact quote from the generated answer where the primary mistake occurs." or null

}

C.2 Illustrative Error Types

Below are concise examples for each of the eight refined error categories. Each includes the problem context, the generated answer, and an explanation highlighting the error.

Key-Value Binding Error

Context: Janet's ducks lay 16 eggs per day. She eats 3 for breakfast and uses 4 for baking. She sells the remainder at $2 each. How much does she make?
Generated Answer: "Janet's ducks lay 61 eggs per day. She uses 3 + 4 = 7 eggs. This leaves 54 eggs to sell. 54 * $2 = $108. The answer is 108."
Explanation: The model misread the initial fact: 16 eggs became 61. All subsequent steps are logically valid, but because the first entity-value binding was wrong, the entire solution is invalid.

State Tracking Error

Context: Janet's ducks lay 16 eggs per day. She eats 3 for breakfast and uses 4 for baking. She sells the remainder at $2 each.
Generated Answer: "Janet has 16 eggs. She eats 3, leaving 13. Then she sells these 13 for $2 each. 13 * 2 = 26. The answer is 26."
Explanation: The model performed the first update ($16 \rightarrow 13$) but forgot the second update (-4 for baking). It failed to track the evolving state of the eggs.

Context Synthesis Error

Context: Janet's ducks lay 16 eggs per day. She eats 3 for breakfast and uses 4 for baking. She sells the remainder at $2 each.
Generated Answer: "Janet has 9 eggs left. She sells them at the same price as the number of eggs she eats for breakfast: $3. 9 * 3 = $27. The answer is 27."
Explanation: The model retrieved the wrong value (3, from breakfast eggs) instead of the actual price ($2). The facts exist in memory, but the wrong one was synthesized into the calculation.

Procedural Retrieval Error

Context: A rectangle has length 9 and width 2. Find the perimeter.
Generated Answer: "The perimeter is length times width. 9 * 2 = 18. The answer is 18."
Explanation: The model recalled the procedure for area instead of perimeter (2L + 2W). This is a retrieval failure of the correct formula.

Conceptual Knowledge Gap

Context: 16 people need cars that hold 3 people each. How many cars are needed?
Generated Answer: "16 ÷ 3 = 5 remainder 1. So they need 5 cars. The answer is 5."
Explanation: The procedure (division) is fine, but the model lacks conceptual understanding: the remainder implies an extra car is necessary. The correct answer is 6.

Flawed Logical Synthesis

Context: Janet's ducks lay 16 eggs per day. She eats 3 for breakfast and uses 4 for baking. She sells the remainder at $2 each.
Generated Answer: "Janet gives 4 muffins to friends. If she sold them for $2 each, she would make 8. The answer is 8."
Explanation: The model fabricated an "opportunity cost" problem not stated in the question. The plan itself is logically consistent but irrelevant to the actual goal.

Calculation Error

Context: Janet's ducks lay 16 eggs per day. She eats 3 for breakfast and uses 4 for baking. She sells the remainder at $2 each.
Generated Answer: "Janet uses 3 + 4 = 8 eggs. She has 16 - 8 = 8 left. She makes 8 * $2 = $16. The answer is 16."
Explanation: Everything is conceptually correct, but a simple addition mistake (3+4=8 instead of 7) corrupted the result.

Goal Interpretation Error

Context: Janet's ducks lay 16 eggs per day. She eats 3 for breakfast and uses 4 for baking. She sells the remainder at $2 each.
Generated Answer: "Janet uses 3 + 4 = 7 eggs. Remaining = 16 - 7 = 9. The answer is 9."
Explanation: The plan is valid but incomplete: the model stopped at the number of eggs left, instead of completing the final step (selling them for money).

C.3 Additional Error Analysis

	Transformer	Mamba50	Mamba75	Mamba83	Mamba100	GLA50	GLA75	GLA83	GLA100	GDN50	GDN75	GDN83	GDN100
In-Context Associative Memory Failures													
Key-Value Binding Error	51.62	52.12	46.00	57.62	75.50	59.62	60.88	63.50	71.00	41.88	44.75	47.88	52.38
State Tracking Error	2.00	1.62	1.75	1.88	0.38	1.12	1.38	1.38	2.25	1.75	1.62	1.75	1.62
Context Synthesis Error	10.25	8.25	12.00	14.12	10.25	8.88	9.88	12.62	11.75	11.50	10.00	12.25	10.12
Parametric Memory & Procedural Failures													
Procedural Retrieval Error	57.38	62.12	59.62	59.12	63.12	66.00	70.88	65.88	64.62	67.12	59.50	61.62	56.00
Conceptual Knowledge Gap	48.00	47.75	47.00	46.62	40.50	49.50	49.62	48.62	48.12	43.88	47.62	52.88	49.75
Logical & Execution Failures													
Flawed Logical Synthesis	**103.50**	**100.12**	**92.88**	**92.62**	**112.88**	**112.38**	**113.12**	**114.88**	**96.12**	**105.62**	**80.12**	**85.62**	**85.00**
Calculation Error	35.38	36.50	37.25	35.00	25.38	29.00	27.12	27.25	31.25	31.88	40.50	41.25	40.62
Goal Interpretation Error	32.75	36.12	33.88	28.62	34.00	39.50	38.50	34.12	37.50	30.75	32.25	32.38	33.12

Table 7: Error Category Decomposition for Model Responses on the MATH dataset. Means are averaged across 8 generations per problem. The largest value per column is in **bold**.

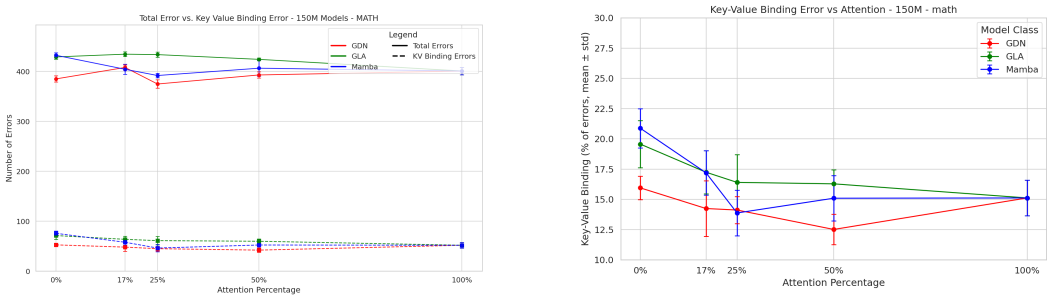


Figure 9: Key-Value Binding Errors on MATH. Increasing Attention reduces the amount to which KV Binding Errors contribute to the total amount of errors, indicating the other errors are more resilient to Attention.

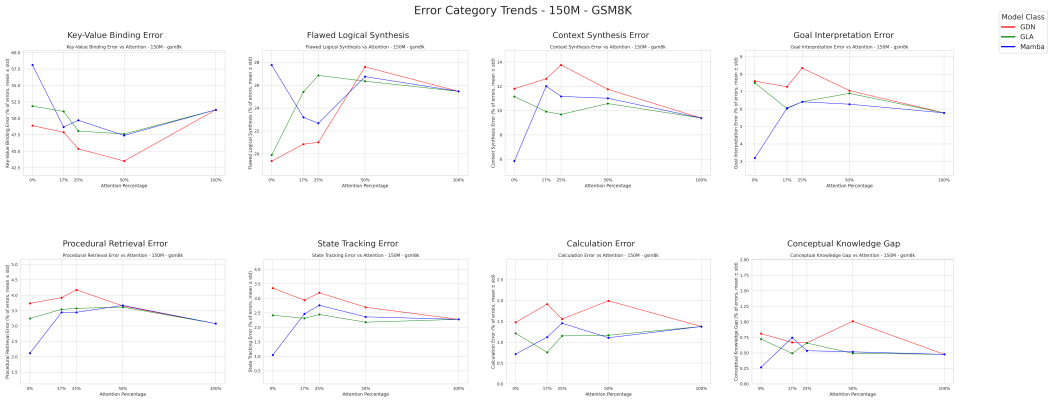


Figure 10: Composite Results on GSM8K showing different error categories as a function of Attention Ratio.

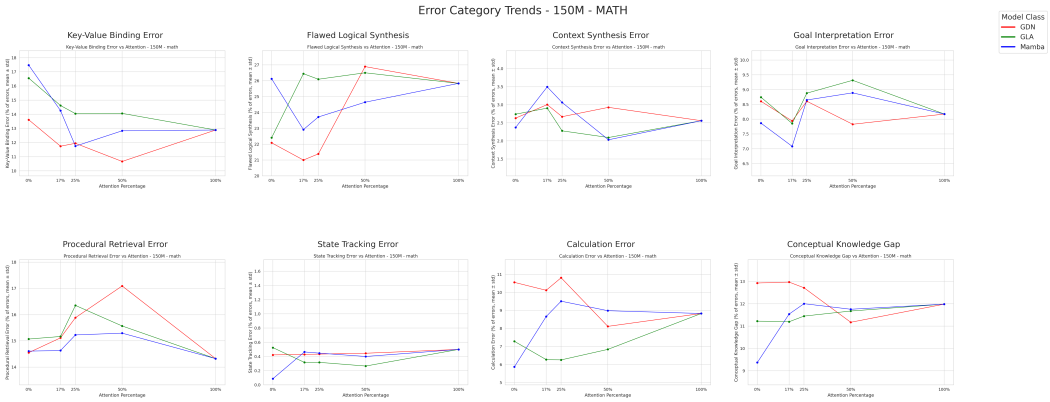


Figure 11: Composite Results on MATH showing different error categories as a function of Attention Ratio.