# Frames and phase retrieval for vector bundles


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***Abstract*—A frame for a Hilbert space $H$, like an ortho-normal basis, gives a continuous, linear, and stable reconstruction formula for any vector $x \in H$. However, the redundancy of frames allows for more adaptability to different applications. For example, in order to do phase retrieval to recover a vector from only the magnitudes of a collection of linear measurements (such as in coherent diffraction imaging), one must use a frame because a basis cannot recover any loss of information. Frames are also useful when working with a coordinate system for a vector bundle which moves continuously over a manifold. Although topological restrictions often prevent the existence of a continuously moving basis for a vector bundle, every vector bundle over a paracompact manifold has a moving redundant frame. We consider a combination of these two situations where one must recover a section of a vector bundle (up to an equivalence relation) from only the magnitudes of a collection of linear measurements on each fiber. Furthermore, we consider how to approximate a section from only finitely many samples.**



***Index Terms*—Frame theory, moving Parseval frames, sampling, perturbations.**


## I. Introduction

Let $(f_i)_{i\in I}$ be a family of vectors in a Hilbert space $H$. We say that $(f_i)_{i\in I}$ is a *frame* if there exist constants $0 < A \leq B < \infty$ such that

$$A\|v\|^2 \leq \sum_{i\in I} |\langle v, f_i\rangle|^2 \leq B\|v\|^2 \text{ for all } v \in H.$$

We call $A$ and $B$ the lower and upper *frame bounds*, respectively. If $A = B = 1$, then $(f_i)_{i\in I}$ is called a **Parseval frame**. A frame is called **equi-norm** if all the vectors have the same norm. Parseval frames can be characterized as orthogonal projections of ortho-normal bases for higher-dimensional spaces [28]. That is, projecting an ortho-normal basis onto a subspace gives a Parseval frame for that subspace, and every Parseval frame can be constructed in such a way. The famous Parseval's identity for ortho-normal bases then holds as well for Parseval frames. That is, if $(f_i)_{i\in I}$ is a collection of vectors in a Hilbert space $H$ then $(f_i)_{i\in I}$ is a Parseval frame of $H$ if and only if

$$v = \sum_{i\in I} \langle v, f_i\rangle f_i \text{ for all } v \in H. \tag{1}$$

Instead of working with a single fixed Hilbert space, we consider a vector bundle over a manifold. Intuitively, we think of a manifold $M$ as a topological space which is locally homeomorphic to Euclidean space. Given manifolds $E$ and $M$, a rank $n$ vector bundle $\pi : E \to M$ is a continuous map such that for each $x \in M$, the fiber $\pi^{-1}(x)$ takes on the structure of an $n$-dimensional vector space. As we will be working with frames, we add the additional assumption that each fiber is a Hilbert space and the inner product $\langle\cdot,\cdot\rangle$ varies continuously over the manifold. Further, we fix a metric $d$ which generates the topology on $M$. A section of a vector bundle $\pi : E \to M$ is a continuous map $f : M \to E$ such that for all $x \in M$, $f(x)$ is a vector in the fiber $\pi^{-1}(x)$. We now consider how the idea of a Parseval frame for a single fixed Hilbert space can be extended to a Parseval frame which moves continuously over a vector bundle, which has previously been studied in [7], [22], [23], [29], [30].

**Definition 1.** *Let $\pi : E \to M$ be a rank $n$-vector bundle over a smooth manifold $M$ with a given inner product $\langle\cdot,\cdot\rangle$. Let $k \geq n$, and $f_i : M \to E$ be a section of $\pi$ for all $1 \leq i \leq k$. We say that $(f_i)_{i=1}^k$ is a **moving Parseval frame** for $\pi$ if for all $x \in M$ we have that $(f_i(x))_{i=1}^k$ is a Parseval frame for the fiber $\pi^{-1}(x)$. That is, for all $x \in M$, the reconstruction formula gives,*

$$v = \sum_{i=1}^k \langle v, f_i(x)\rangle f_i(x) \textit{ for all } v \in \pi^{-1}(x). \tag{2}$$

The following theorem extends the characterization of Parseval frames as orthogonal projections of ortho-normal bases to the continuous setting of vector bundles.


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**Theorem 1.** *[23] Let $\pi_1 : E_1 \to M$ be a rank $n$-vector bundle over a paracompact manifold $M$ with a moving Parseval frame $(f_i)_{i=1}^k$. Then, there exists a rank $k-n$ vector bundle $\pi_2 : E_2 \to M$ with a moving Parseval frame $(g_i)_{i=1}^k$ so that $(f_i \oplus g_i)_{i=1}^k$ is a moving ortho-normal basis for the vector bundle $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to M$.*

Hence, a moving Parseval frame of $k$-sections may be lifted to a moving ortho-normal basis of a rank $k$ vector bundle. By identifying the moving ortho-normal basis with a fixed ortho-normal basis for a $k$-dimensional Hilbert space $H^k$, we may treat each fiber $\pi^{-1}(x)$ as a subspace of $H^k$. This allows for Parseval frames to measure the distance between vectors in different fibers. Let $(f_i)_{i=1}^k$ be a moving Parseval frame for a vector bundle $\pi : E \to M$. For each $x, y \in M$, $v_x \in \pi^{-1}(x)$, and $u_y \in \pi^{-1}(y)$ we consider the distance between $v_x$ and $u_y$ to be

$$\|v_x - u_y\| = \Big( \sum_{i=1}^k |\langle v_x, f_i(x)\rangle - \langle u_y, f_i(y)\rangle|^2 \Big)^{1/2}. \quad (3)$$

Note that as $(f_i)_{i=1}^k$ is a moving Parseval frame, we have that (3) corresponds to the distance induced by the norm on the fiber $\pi^{-1}(x)$ when $x = y$.

## II. Sampling and sections of vector bundles

In applications, one must often approximate a function using only finitely many samples from the domain, such as in quadrature rules for numerical integration, and there has been many recent advances in this direction in approximation theory and frame theory [9], [10], [17], [21], [24], [25], [35], [32]. We consider how a moving Parseval frame for a vector bundle $\pi : E \to M$ can be used to approximate a section from finitely many samples in the manifold $M$. That is, given finitely many sampling points $(x_j)_{1\le j\le N} \subseteq M$ , how can we approximate a section $f$ from the frame coefficients $(\langle f(x_j), f_i(x_j)\rangle)_{1\le i\le k; 1\le j\le N}$?

Note that we cannot approximate every section from finitely many sampling points as given any finite set $(x_j)_{j=1}^N \subseteq M$ we can choose a section $f : M \to E$ which gets arbitrarily large but that $f(x_j) = 0$ for all $1 \le j \le N$. Because of this, we will restrict ourselves to only considering sections with a given bound on their modulus of continuity. This restricts how much a section can change between different sampling points.

**Definition 2.** *Let $M$ be a compact manifold endowed with a metric $\rho$, and let $\pi : E \to M$ be a vector bundle with a moving Parseval frame. The* ***modulus of uniform continuity*** *of a section $f : M \to E$, is the function $\omega_f : (0,\infty) \to [0,\infty]$ given by*

$$\omega_f(\delta) = \sup\{\|f(y) - f(x)\| : x, y \in M, \rho(x, y) \le \delta\}. \quad (4)$$

The remark after Theorem 1 ensures that (4) is well-defined as we can consider $f(x)$ and $f(y)$ to be elements of the same Hilbert space.

By enforcing a bound on the modulus of uniform continuity of $f$, we can guarantee that if $x$ is sufficiently close to a sampling point $x_j$ then the frame coefficients for $f(x)$ will be closely approximated by the frame coefficients for $f(x_j)$. That is, we can approximate $f(x)$ with $\sum_{i=1}^k \langle f(x_j), f_i(x_j)\rangle f_i(x) \in \pi^{-1}(x)$. Obtaining estimates for this depends on the idea of perturbations of frames as developed in [14]. Given $\varepsilon > 0$, we say that a family of vectors $(g_i)_{i=1}^k$ is an $\varepsilon$-perturbation of a frame $(f_i)_{i=1}^k$ for a Hilbert space $H$ if $\sum_{i=1}^k \|f_i - g_i\|^2 < \varepsilon$. We will need the following perturbation theorem.

**Lemma 1.** *Let $\varepsilon > 0$ and let $(f_i)_{i=1}^k$ and $(g_i)_{i=1}^k$ be frames of a Hilbert space $H$ which satisfy $\sum_{i=1}^k \|f_i - g_i\|^2 < \varepsilon$. Let $B$ be an upper frame bound for $(f_i)_{i=1}^k$. Then for all $v, u \in H$ we have that*

$$\Big\| \sum_{i=1}^k \langle v, f_i\rangle f_i - \sum_{i=1}^k \langle u, g_i\rangle f_i \Big\| \le B^{1/2}\|v - u\| + B\|u\|\varepsilon^{1/2}.$$

*Proof.*

$$\begin{aligned}
&\Big\| \sum \langle v, f_i\rangle f_i - \sum \langle u, g_i\rangle f_i \Big\| \\
&\le \Big\| \sum \langle v - u, f_i\rangle f_i \Big\| + \Big\| \sum \langle u, f_i - g_i\rangle f_i \Big\| \\
&\le B\|v - u\| + B\Big( \sum |\langle u, g_i - f_i\rangle|^2 \Big)^{1/2} \\
&\le B\|v - u\| + B\|u\|\Big( \sum \|g_i - f_i\|^2 \Big)^{1/2} \\
&\le B\|v - u\| + B\|u\|\epsilon^{1/2}.
\end{aligned}$$

□

Lemma 1 allows us to use the values $(\langle f(x_j), f_i(x_j)\rangle)_{1\le i\le k}$ to approximate $f(x)$ when $x$ is in a small neighborhood of $x_j$. A partition of unity can be used to combine these local approximations into a global approximation over the manifold.

**Definition 3.** *Given an open cover $(U_j)_{j=1}^N$ of a compact manifold $M$, we say that a sequence of continuous functions $(\Psi_j)_{j=1}^N$ from $M$ to $[0,1]$ is a* ***partition of unity*** *subordinate to $(U_j)_{j=1}^N$ if supp$(\Psi_j) \subseteq U_j$ for all $1 \le j \le N$ and $\sum_{j=1}^N \Psi_j(x) = 1$ for all $x \in M$.*

We now use Lemma 1 with a partition of unity argument to obtain the following theorem, which gives a linear approximation formula for a section $f$ from the sampled frame coefficients $(\langle f(x_j), f_i(x_j)\rangle)_{1\le i\le k; 1\le j\le N}$. If $x \in M$ and $\delta > 0$, then we denote $U_\delta(x) \subseteq M$ to be the open ball of radius $\delta$ centered at $x$.

**Theorem 2.** *Let $(f_i)_{i=1}^k$ be a moving Parseval frame for a vector bundle $\pi : E \to M$ over a compact manifold $M$ and let $\varepsilon, \delta' > 0$. Choose $\delta > 0$ such that $\delta' \ge \delta$ and $\max_{1\le i\le k} \omega_{f_i}(\delta') < k^{-1/2}\varepsilon/2$. Choose any $\delta$-dense subset $(x_j)_{j=1}^N \subseteq M$ and a partition of unity $(\Psi_j)_{j=1}^N$ which is subordinate to $(U_\delta(x_j))_{j=1}^N$. Then for any section $f$ of $\pi$ satisfying $\omega_f(\delta) < \varepsilon/2$ and $\sup_{y\in M} \|f(y)\| \le 1$, we have for all $x \in M$ that*

$$\Big\| f(x) - \sum_{j=1}^N \Psi_j(x) \Big( \sum_{i=1}^k \langle f(x_j), f_i(x_j)\rangle f_i(x) \Big) \Big\| < \varepsilon.$$

*Proof.* Consider $x \in M$ and $1 \le j \le N$ such that $d(x, x_j) < \delta$. As $\max_{1\le i\le k} \omega_{f_i}(\delta) < k^{-1/2}\varepsilon/2$, we have $\sum_{i=1}^k \|f_i(x) - f_i(x_j)\|^2 < \varepsilon^2/4$. Hence, $(f_i(x_j))_{i=1}^k$ is an $(\varepsilon^2/4)$-perturbation of $(f_i(x))_{i=1}^k$. Further, $\omega_f(\delta) < \varepsilon/2$ and hence $\|f(x) - f(x_j)\| < \varepsilon/2$. As $(f_i(x))_{i=1}^k$ is a Parseval frame, $f(x) = \sum_{i=1}^k \langle f(x), f_i(x)\rangle f_i(x)\rangle$. Lastly, we assumed $\|f(x_j)\| \le 1$. Lemma 1 now gives that

$$\big\|f(x) - \sum_{i=1}^k \langle f(x_j), f_i(x_j)\rangle f_i(x)\big\| < \varepsilon/2 + \varepsilon/2. \quad (5)$$

As $M$ is a compact manifold, we may choose $(\Psi_j)_{j=1}^N$ to be a partition of unity subordinate to $(U_\delta(x_j))_{j=1}^N$. This gives, for $x \in M$ and $1 \le j \le N$ that if $\|x - x_j\| < \delta$ then (5) is satisfied, and if $\|x - x_j\| \ge \delta$ then $\Psi(x) = 0$. As $\sum_{j=1}^N \Psi_j(x) = 1$, we have

$$\begin{aligned}&\big\|f(x) - \sum_{j=1}^N \Psi_j(x) \sum_{i=1}^k \langle f(x_j), f_i(x_j)\rangle f_i(x)\big\| \\ &\le \sum_{j=1}^N \Psi_j \|f(x) - \sum_{i=1}^k \langle f(x_j), f_i(x_j)\rangle f_i(x)\| < \varepsilon\end{aligned}$$

□

## III. Phase retrieval for sections

In applications such as coherent diffraction imaging, one must use phase retrieval to recover a vector (up to a unimodular scalar) from only the magnitudes of its frame coefficients [11], [18], [19], [33]. Let $(f_i)_{i\in I}$ be a frame for a Hilbert space $H$. Note that if $v \in H$ and $|\lambda| = 1$ then $(|\langle v, f_i\rangle|)_{i\in I} = (|\langle \lambda v, f_i\rangle|)_{i\in I}$. Thus, we consider the equivalence relation $\sim$ on $H$ given by $v \sim \lambda v$ for all $|\lambda| = 1$. We say that a frame $(f_i)_{i\in I}$ *does phase retrieval* for a Hilbert space $H$ if whenever $v, u \in H$ are such that $(|\langle v, f_i\rangle|)_{i\in I} = (|\langle u, f_i\rangle|)_{i\in I}$ we have that $v \sim u$. This definition naturally extends to moving Parseval frames on vector bundles. Let $(f_i)_{i=1}^k$ be a moving Parseval frame for a vector bundle $\pi : E \to M$. We say that $(f_i)_{i=1}^k$ does *phase retrieval* for $\pi$ if for all $x \in M$ we have that $(f_i(x))_{i=1}^k$ does phase retrieval for the fiber $\pi^{-1}(x)$. Given sections $f$ and $g$ of $\pi : E \to M$ we let $f \sim_M g$ if for all $x \in M$, $f(x) \sim g(x)$. Implementing phase retrieval for a section $f$ is then recovering the equivalence class $[f]_{\sim_M}$ from the phaseless measurements $(|\langle f(x), f_i(x)\rangle|)_{x\in M; 1\le i\le k}$.

A frame theoretic approach to phase retrieval was originally developed in [5], and has grown into rich area of study [1], [2], [8], [12], [16], [27], [33], [36]. In this section we will first show how some of the fundamental results on phase retrieval for Hilbert spaces can be naturally extended to the vector bundle setting. We will then show how other results on phase retrieval do not immediately generalize and instead become interesting topological questions.

As any application involves error, it is essential that phase retrieval not only be possible but that it be stable as well. Given a Hilbert space $H$, we consider the quotient metric $d_\sim$ on $H/\sim$ by $d_\sim([v]_\sim, [u]_\sim) = \min_{|\lambda|=1} \|v - \lambda u\|$. A frame $(f_i)_{i\in I}$ for a Hilbert space $H$ is said to do $C$-stable phase retrieval if the map $(|\langle v, f_i\rangle|)_{i\in I} \mapsto [x]_\sim$ is $C$-Lipschitz. That is, for all $v, u \in H$ we have that $\min_{|\lambda|=1} \|v - \lambda u\| \le C(\sum_{i\in I} ||\langle v, f_i\rangle| - |\langle u, f_i\rangle||^2)^{1/2}$. A frame for a finite dimensional Hilbert space does phase retrieval if and only if it does $C$-stable phase retrieval for some $C > 0$ [6], [8]. Furthermore, if $(f_i)_{i\in I}$ is a frame for a finite dimensional Hilbert space which does phase retrieval then there exists $\varepsilon > 0$ and $C > 0$ so that every $\varepsilon$-perturbation of $(f_i)_{i\in I}$ does $C$-stable phase retrieval [3], [4]. The following extension to the vector bundle setting follows by compactness.

**Proposition 1.** *Let $(f_i)_{i=1}^k$ be a moving Parseval frame for a vector bundle $\pi : E \to M$ over a compact manifold $M$. Then $(f_i)_{i=1}^k$ does phase retrieval for the vector bundle if and only if there exists $C > 0$ so that for all $x \in M$, $(f_i(x))_{i=1}^k$ does $C$-stable phase retrieval for the fiber $\pi^{-1}(x)$.*

We now consider the problem of recovering a section of a vector bundle from finitely many samples. Let $\pi : E \to M$ be a vector bundle with a moving Parseval frame $(f_i)_{i=1}^k$. Previously, we considered how to recover a section $f : M \to E$ from the sampled values $(\langle f(x_j), f_i(x_j)\rangle)_{1\le i\le k; 1\le j\le N}$. Our goal now is to recover $[f]_{\sim_M}$ from $(|\langle f(x_j), f_i(x_j)\rangle|)_{1\le i\le k; 1\le j\le N}$.

The first obstacle is that for each $x \in M$, the equivalence class $[f(x)]_\sim$ is not a vector and hence we cannot use a partition of unity to add local reconstructions together. This is overcome by identifying an equivalence class $[v]_\sim$ with the rank 1 operator $vv^*$. Recall that a frame $(f_i)_{i\in I}$ for a Hilbert space $H$ does phase retrieval if whenever $|\langle v, f_i\rangle| = |\langle u, f_i\rangle|$ for all $i \in I$ we have that $v = \lambda u$ for some $|\lambda| = 1$. Note that the Hilbert-Schmidt inner product on the space of bounded operators gives that $\langle vv^*, f_i f_i^*\rangle_{HS} = |\langle v, f_i\rangle|^2$. Further, $\langle vv^*, f_i f_i^*\rangle_{HS} = \langle uu^*, f_i f_i^*\rangle_{HS}$ if and only if $\langle vv^* - uu^*, f_i f_i^*\rangle_{HS} = 0$. It follows that $(f_i)_{i\in I}$ does phase retrieval for $H$ if and only if whenever $F : H \to H$ is a self-adjoint operator with $1 \le \mathrm{rank}(F) \le 2$ we have that $\langle F, f_i f_i^*\rangle_{HS} \ne 0$ for some $i \in I$. Thus, recovering $[f]_{\sim_M}$ is equivalent to recovering $F$ where for all $x \in M$, $F(x) = f(x)f(x)^*$ is a positive rank one operator on $\pi^{-1}(x)$.

There are multiple algorithms for performing phase retrieval for a fixed Hilbert space, such as phase lift [13] or Griffin-Lim [34]. When doing phase retrieval for a vector bundle from samples, we assume that one is able to do phase retrieval for each of the sampled fibers. Our goal is to show how these samples can be combined to approximately do phase retrieval over the vector bundle. That is, given a moving Parseval frame $(f_i)_{i=1}^k$ for a vector bundle $\pi : E \to M$ and finitely many sampling points $(x_j)_{j=1}^N \subseteq M$ our goal is to approximate a continuously moving rank 1 positive operator $F$ from the values $(\langle F(x_j), f_i(x_j) f_i^*(x_j)\rangle_{HS})_{1\le i\le k; 1\le j\le N}$. Note that using a partition of unity to sum a collection of rank 1 positive operators will likely result in a positive operator which has rank greater than 1. However, if a collection of positive rank one operators are all sufficiently close together, then any convex combination can be closely approximated by a positive rank 1 operator. Given a positive finite rank operator $G$ whose largest eigenvalue is $\lambda_1$, we denote $R_1(G)$ to be a positive rank 1 operator such that $\lambda_1$ is the largest eigenvalue of $R_1(G)$ and the corresponding eigenspace for $R_1(G)$ is contained in the corresponding eigenspace for $G$. Note that $R_1(G)$ is not uniquely defined when the eigenspace corresponding to $\lambda_1$ has dimension greater than 1. However, if $G$ continuously varies over a domain $X$ where the eigenspace corresponding to the largest eigenvalue of $G(x)$ always has dimension 1, then $R_1(G)$ continuously varies over $X$ as well. Because of this, we will threshold our approximation to always guarantee that we only consider the case where the eigenspace corresponding to the largest eigenvalue of $G(x)$ always has dimension 1. Given any $\gamma > 0$, we let $\tau_\gamma : [0, \infty) \to [0, \infty)$ be an increasing smooth function such that $\tau_\gamma(t) = 0$ for all $t \in [0, \gamma]$ and $\tau_\gamma(t) = t$ for all $t \in [2\gamma, \infty)$.

Note that if $F$ is a rank one operator on a Hilbert space $H$ then $F = vv^*$ for some $v \in H$. Further, if $(f_i)_{i=1}^k$ is a Parseval frame of $H$ then $v = \sum \langle v, f_i \rangle f_i$ and hence $F = \sum_{1 \le i,i' \le k} \langle v, f_i \rangle \langle f_{i'}, v \rangle f_i f_{i'}^*$. As the map $v \mapsto vv^*$ is continuous, we may apply a similar proof to Theorem 2 to obtain the following.

**Theorem 3.** *Let $(f_i)_{i=1}^k$ be a moving Parseval frame for a vector bundle $\pi : E \to M$ over a compact manifold $M$ such that for all $x \in M$, $(f_i(x))_{i=1}^k$ does phase retrieval for the fiber $\pi^{-1}(x)$. Then for every $\varepsilon > 0$ there exists $\varepsilon' > 0$ so that for all $\delta' > 0$ there exists a collection of sampling points $(x_j)_{j=1}^N \subseteq M$, and a partition of unity $(\Psi_j)_{j=1}^N$ so that the following hold. Let $F$ be a continuously moving rank 1 positive operator satisfying $\omega_F(\delta') < \varepsilon'$ and $\sup_{y \in M} \|f(y)\|_{HS} \le 1$. For each $1 \le j \le N$, the frame $(f_i)_{i=1}^k$ does phase retrieval for $\pi^{-1}(x_j)$, and thus we may obtain a vector $v_j \in \pi^{-1}(x_j)$ such that $F(x_j) = v_j v_j^*$. Let $F_j$ be the continuously moving rank 1 positive operator given by $F_j(x) = \sum_{1 \le i,i' \le k} \langle v_j, f_i(x_j) \rangle \langle f_{i'}(x_j), v_j \rangle f_i(x) f_{i'}^*(x)$. Then for all $x \in M$ we have that if $\| \sum_{j=1}^N \Psi_j(x) F_j(x) \| \ge \varepsilon/2$ then $R_1 \sum_{j=1}^N \Psi_j(x) F_j(x)$ is uniquely defined and*

$$\Big\| F(x) - \tau_{\varepsilon/2}\Big( \Big\| \sum_{j=1}^N \Psi_j(x) F_j(x) \Big\| \Big) R_1 \sum_{j=1}^N \Psi_j(x) F_j(x) \Big\| < \varepsilon.$$

Our previous results have considered cases when the existing theory of phase retrieval for a single Hilbert space can be naturally extended to do phase retrieval for a section of a vector bundle. We now introduce some topics where the existing theory runs into some interesting topological obstructions when one generalizes to the vector bundle setting.

The foundational paper [5] characterizes the frames which do phase retrieval for $\mathbb{R}^n$ as those satisfying the complement property. That is, a frame $(f_i)_{i=1}^m$ does phase retrieval for $\mathbb{R}^n$ if and only if for every $I \subseteq \{1, ..., m\}$ either $(f_i)_{i \in I}$ or $(f_i)_{i \in I^c}$ is a frame for $\mathbb{R}^n$. It follows immediately that there exists a frame of $m$ vectors for $\mathbb{R}^n$ which does phase retrieval if and only $m \ge 2n - 1$.

Note that if $(f_i)_{i=1}^{2n-1}$ is a moving Parseval frame for a rank $n$ vector bundle $\pi : E \to M$ then $(f_i)_{i=1}^{2n-1}$ does phase retrieval if and only if every $n$-element subset $(f_{j_i})_{i=1}^n$ is a moving basis of $\pi$. This gives the following proposition.

**Proposition 2.** *A real rank $n$ vector bundle has a moving Parseval frame of $2n - 1$ vectors which does phase retrieval if and only if the vector bundle has a moving basis.*

This naturally leads to the following questions. Given a rank $n$ vector bundle $\pi : E \to M$, for what $m \ge 2n - 1$ does there exist a moving Parseval frame of $m$ sections which does phase retrieval? For what $m \ge 2n - 1$ does there exist a moving equi-norm Parseval frame of $m$ sections which does phase retrieval? To motivate these as interesting topological problems, we present solutions for the case of the tangent space for the Möbius strip or Klein bottle.

We consider the Möbius strip $M$ to be the square $[0, 1] \times [0, 1]$ where each bottom point $(x, 0)$ is identified with the top point $(1 - x, 1)$. As the manifold is non-orientable, the tangent space $TM$ does not have a moving basis or a moving equi-norm Parseval frame of 3 vectors. However, $TM$ has a moving Parseval frame of 3 vectors as one can immerse $M$ in $\mathbb{R}^3$ and then project an ortho-normal for $\mathbb{R}^3$ onto $TM$. Equi-norm Parseval frames are particularly desirable in applications as they minimize mean squared error due to noise [26], and it is shown in [23] that $TM$ has a moving equi-norm Parseval frame of $m$ vectors for all $m \ge 4$.

As $M$ is non-orientable, $TM$ does not have a moving Parseval frame of 3 vectors which does phase retrieval by Proposition 2. However, we can build a Parseval frame $(f_i)_{i=1}^4$ which does phase retrieval in the following way. We first define the frame at the bottom of the square. Let $f_1((x, 0)) = (0, \sqrt{\frac{2}{3}})$, $f_2((x, 0)) = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{6}})$, $f_3((x, 0)) = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{6}})$, $f_4((x, 0)) = (0, 0)$. In order for $(f_i)_{i=1}^4$ to be continuous, this choice requires that $f_1((x, 1) = f_1((x, 0))$, $f_2((x, 1)) = f_3((x, 0))$, $f_3((x, 1)) = f_2((x, 0))$, $f_4((x, 0)) = (0, 0)$. To do this, we can divide $[0, 1] \times [0, 1]$ into 3 strips $[0, 1] \times [0, 1/3)$, $[0, 1] \times [1/3, 2/3)$, and $[0, 1] \times [2/3, 1]$. Over the first strip, we switch the places of $f_2$ and $f_4$ while keeping the other vectors fixed. In the second strip we switch $f_2$ and $f_3$, and in the third we switch $f_3$ and $f_4$. This results in $(f_i)_{i=1}^4$ being continuously defined over the entire manifold.

Note that the above example is not an equi-norm frame. We now claim that if $(f_i)_{i=1}^4$ is a moving equi-norm Parseval frame of four vectors for $TM$, then $(f_i)_{i=1}^4$ cannot do phase retrieval. Indeed, as $M$ is non-orientable we may assume without loss of generality that there exists $t \in M$ so that $\mathrm{span}(f_1(t)) = \mathrm{span}(f_2(t))$. Every equi-norm Parseval frame of four vectors for $\mathbb{R}^2$ is a union of 2 scaled ortho-normal bases. Thus, we must also have that $\mathrm{span}(f_3(t)) = \mathrm{span}(f_4(t))$. We have that $(f_i(t))_{i=1}^4$ fails the complement property and hence does not do phase retrieval.

On the other hand, $TM$ has a moving equi-norm Parseval frame of five vectors and every equi-norm Parseval frame of five vectors for $\mathbb{R}^2$ necessarily does phase retrieval.

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