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HOMOLOGICAL DIMENSIONS AND SEMIDUALIZING COMPLEXES

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ABSTRACT. For finite modules over a local ring and complexes with finitely generated homology, we consider several homological invariants sharing some basic properties with projective dimension.

In the second section, we introduce the notion of a *semidualizing* complex, which is a generalization of both a dualizing complex and a *suitable* module. Our goal is to establish some common properties of such complexes and the homological dimension with respect to them. Basic properties are investigated in Sec. 2.1. In Sec. 2.2, we study the structure of the set of semidualizing complexes over a local ring, which is closely related to the conjecture of Avramov–Foxby on the transitivity of the G-dimension. In particular, we prove that, for a pair of semidualizing complexes X_1 and X_2 such that $G_{X_2} \dim X_1 < \infty$, we have $X_2 \simeq X_1 \otimes^L_R \mathbf{R}\operatorname{Hom}_R(X_1, X_2)$. Specializing to the case of semidualizing modules over Artinian rings, we obtain a number of quantitative results for the rings possessing a configuration of semidualizing modules of special form. For the rings with $\mathfrak{m}^3 = 0$, this condition reduces to the existence of a nontrivial semidualizing module, and we prove a number of structural results in this case.

In the third section, we consider the class of modules that contains the modules of finite CI-dimension and enjoys some nice additional properties, in particular, good behavior in short exact sequences.

In the fourth section, we introduce a new homological invariant, CM-dimension, which provides a characterization for Cohen–Macaulay rings in precisely the same way as projective dimension does for regular rings, CI-dimension for locally complete intersections, and G-dimension for Gorenstein rings.

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Introduction

Homological methods in commutative algebra are one of the most powerful tools for a researcher. An important theorem on the localizability of the property of regularity of a local ring is an example of an assertion that can be rather easily proved by means of homological methods, but a proof using only classical methods is unknown.

The main idea in the proof of this theorem is that projective dimension characterizes regular rings in the following sense: every module over a regular ring has a finite projective dimension and, conversely, finiteness of the projective dimension of the quotient field implies that the ring is regular.

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The basic idea of M. Auslander, presented in the beginning of the 1960s by him, is that modules of finite projective dimension over a ring (not necessarily regular) have almost the same behavior as modules over regular rings.

As an example of a property that has been generalized from the modules over regular rings to the modules of finite projective dimension in the context of this approach (see [34]), one can give the following statement: if a sequence of elements of a ring R is regular w.r.t. a module M, then the sequence is regular w.r.t. the ring R.

The main motivation for studying homological dimensions characterizing other types of rings, which are important in algebraic geometry, i.e., locally complete intersections, Gorenstein rings, and Cohen–Macaulay rings, is the search for a reasonable description of modules having properties similar to those of modules over rings of the corresponding types.

Such classes of modules appeared several times in problems of commutative algebra.

For Gorenstein rings, the corresponding class was considered by M. Auslander and M. Bridger [3] and was defined as follows. Let $\mathrm{G\text{-}dim}\, M = 0$ if the natural map $M \to \mathrm{Hom}(\mathrm{Hom}(M, R), R)$ is an isomorphism and $\mathrm{Ext}^i_R(M, R) = 0 = \mathrm{Ext}^i_R(M^*, R)$ when $i > 0$. The infimum of the lengths of resolvents of the module M composed of modules P with $\mathrm{G\text{-}dim}\, P = 0$ will be denoted by $\mathrm{G\text{-}dim}\, M$.

For complete intersections, the corresponding class of modules, called modules of finite virtual projective dimension (vpd), appeared in the paper of L. L. Avramov [6] studying the properties of the Betti numbers of modules of infinite projective dimension. The dimension $\mathrm{vpd}_R\, M$ is assumed to be finite if there exists a surjective ring homomorphism $S \to \hat{R}$, where $\hat{R}$ is the completion of R in the $\mathfrak{m}$-adic topology such that the kernel of the homomorphism is generated by a regular sequence and $\mathrm{pd}_S(M \otimes_R \hat{R}) < \infty$. For a (possibly) more general class of modules of finite CI-dimension (see Definition 3.2), considered in [9] and characterizing complete intersections, several facts were proved, whose validity for virtual projective dimension is unknown, in particular the good behavior under localization. In addition, in some problems, modules of finite CI-dimension really demonstrate a behavior similar to that of modules over complete intersections. The most important example is provided by the asymptotic properties of free resolvents, which is the main point in [9]. Let us also mention [2, 12, 30], where the formula of depth has been carried over from the modules over complete intersections to the modules of finite CI-dimension, and [2], where the Auslander criterion of freedom has been generalized.

The mentioned dimensions are related to the inequalities

$$\mathrm{pd}_R\, M \le \mathrm{CI\text{-}dim}_R\, M \le \mathrm{G\text{-}dim}_R\, M.$$

A particular case of this (for $M = k$) is the following statement about the ring R:

$$R \text{ is regular} \implies R \text{ is a complete intersection} \implies R \text{ is a Gorenstein ring.}$$

All necessary prerequisites from commutative and homological algebra can be found in Sec. 1.

We shall say that a *generalized homological dimension* characterizing a class of rings Ω is given if for any ring R a class of modules H_R and a map $\mathrm{H\text{-}dim}_R$ from H_R to $\mathbb{Z}$ are given. Let us give some restrictions that are reasonable to apply in order to obtain a meaningful notion.

I. $k = R/\mathfrak{m} \in \mathrm{H}_R \iff$ any R-module$M \in \mathrm{H}_R \iff R \in \Omega$.
II. $M \in \mathrm{H}_R \implies \mathrm{H\text{-}dim}_R\, M + \mathrm{depth}\, M = \mathrm{depth}\, R$.
III. Let x be an R- and M-regular element. Then $M \in \mathrm{H}_R$ if and only if $M/xM \in \mathrm{H}_{R/xR}$, and under these two conditions $\mathrm{H\text{-}dim}_R\, M = \mathrm{H\text{-}dim}_{R/xR}\, M/xM$.
IV. $M \in \mathrm{H}_R \implies M_\mathfrak{p} \in \mathrm{H}_{R\mathfrak{p}}$ and $\mathrm{H\text{-}dim}_R\, M \ge \mathrm{H\text{-}dim}_{R\mathfrak{p}}\, M_\mathfrak{p}$.
V. If in a short exact sequence $0 \to M \to N \to K \to 0$ two of the three modules belong to H_R, then the third one belongs to H_R as well; if this exact sequence splits, then $N \in \mathrm{H}_R$ implies that $M \in \mathrm{H}_R$ and $K \in \mathrm{H}_R$.

Note that projective dimension and G-dimension satisfy all these conditions. For virtual projective dimension properties I and II have been proved, and for CI-dimension properties I–IV have been demonstrated.

The main object of investigation for us is G-dimension. Although this notion was introduced almost 40 years ago, a renewed interest in it has arisen, especially in the last decade, when many papers were devoted to this subject (see, e.g., [13]). An important set of unsolved problems related to G-dimension consists of the questions of its behavior under a change of the ring, in particular the so-called conjecture of transitivity of G-dimension.

Conjecture 0.1 ([8])**.** If $\phi\colon S \to R$ is a finite local homomorphism of finite G-dimension (i.e., $\text{G-dim}_S R < \infty$), then for any R-module M such that $\text{G-dim}_R M < \infty$, we have $\text{G-dim}_S M < \infty$.

A close problem was studied in [24]. It turns out that for homomorphisms ϕ of finite G-dimension of a special form, more precisely, surjective ones satisfying the condition $\operatorname{grade}(S/\operatorname{Ker}\phi) = \text{G-dim}_S(S/\operatorname{Ker}\phi)$, over the ring R there is a natural way to define a "suitable" module (which was also studied by H.-B. Foxby [15]) independently, and for the introduced analogue of G-dimension w.r.t. such modules a result like a ring change is true (see Theorem 1.54).

Trivial examples of "suitable" modules are the free module of rank 1 and the dualizing module when it exists.

For G_K-dimension w.r.t. a "suitable" module K, the analogues of Properties I–V hold true, where the analogue of Property I is the assertion about the equivalence between the finiteness of the G_K-dimension of the quotient field and the condition that the module K is dualizing.

In Sec. 2, we consider complexes that are a generalization of both dualizing complexes and "suitable" (see [24]) modules. These complexes were independently introduced by the author in [19] under the name of "suitable" and by L. V. Christensen in [14] under the name of "semidualizing." Since the latter name fits better and is more popular in the literature, we are going to use it.

A complex is called semidualizing (see Definition 2.1) if its homology is finitely generated and the natural morphism $R \to \mathbf{R}\operatorname{Hom}(X, X)$ is an isomorphism in the derived category. Such complexes naturally appear in the study of local homomorphisms of rings $\phi\colon S \to R$ of finite G-dimension. In such a case, the R-complex $\mathbf{R}\operatorname{Hom}_S(R, S)$ is semidualizing. Trivial examples of semidualizing complexes are the free module of rank 1 and the dualizing complex when it exists. If a semidualizing complex has just one nonzero homology, then, up to a translation, this complex is isomorphic to a suitable module as an object of the corresponding derived category.

In Sec. 2.1, the following results appear. We construct a theory of G-dimension w.r.t. a semidualizing complex in which the analogues of Properties I–V are true. For the constructed dimension, there is a result about a ring change that is similar to [24, Proposition 5] (Theorem 2.10). We reformulate the Avramov–Foxby conjecture in terms of semidualizing complexes.

It is interesting whether it is possible to obtain any semidualizing complex by means of the induction described above. For dualizing complexes, such a question forms the Sharp conjecture.

Conjecture 0.2 ([36])**.** If R is a local ring and X is a dualizing complex, then there exist a ring S and a surjective homomorphism $\phi\colon S \to R$ such that $\mathbf{R}\operatorname{Hom}_S(R, S) \simeq X$.

This conjecture was proved by T. Kawasaki [31]. Here we show that a similar statement is also true for suitable modules (Corollary 4.8), i.e., for semidualizing complexes with a unique nonzero homology.

In Sec. 2.2, we study the problem of existence of nontrivial semidualizing complexes (different from the free module of rank 1 and the dualizing complex). For modules, the corresponding problem was proposed by E. S. Golod [25]. The first nontrivial example of a suitable module was constructed by Foxby [16]. It is easy to show that the Cohen–Macaulay type (see Definition 1.31) of a suitable module must be a divisor of the Cohen–Macaulay type of the ring. In the present paper, for a given type m we construct an example of a ring over which, for any divisor m, there exists a corresponding semidualizing module (Example 2.21). For this example, the structure of the set of constructed semidualizing modules coincides with the structure of the set of all subsets of a finite set of the power equal to the number of prime divisors of m.

Hypothetically, also in the general case there should be a similar structure. Let K_I be a suitable module corresponding to some subset $I \subset \{1, \dots, n\}$. For Example 2.21, the following statement is true: $I \subset J$

if and only if G_{K_J}-$\dim K_I < \infty$. For the latter property, there is an analogue in the general case that allows us to introduce a natural binary relation on the set of semidualizing complexes. The relation is symmetric and reflexive; its transitivity is a conjecture and is closely related to the problem of transitivity of G-dimension formulated earlier. Nevertheless, the search for analogues of the relations satisfied in the example described allows one to obtain several important corollaries for the general situation. The most interesting is the analogue of $I \subset J \implies K_J \simeq K_I \otimes K_{J\setminus I}$, which, in the general case, gives a way to construct an isomorphism $X_2 \simeq X_1 \otimes^L_R \mathbf{R}\operatorname{Hom}_R(X_1, X_2)$ in the derived category if a pair of semidualizing complexes X_1 and X_2 satisfying G_{X_2}-$\dim X_1 < \infty$ is given.

After that, we consider the classification problem for semidualizing complexes over Cohen–Macaulay rings. For such rings, any semidualizing complex is a suitable module in the sense of the corresponding derived category (Proposition 2.32). A classification of suitable modules over complete rings can be reduced to the case of a ring of depth 0 (Proposition 2.33); in particular, for Cohen–Macaulay rings to the case of Artinian rings. For Artinian rings, we consider bases of semidualizing modules similar to the collection $K_{\{1\}}, \dots, K_{\{n\}}$ in Example 2.21. The length of such a basis is naturally bounded in terms of the minimal degree of a maximal ideal becoming 0 (Proposition 2.39), and at this moment the Betti series of the ring becomes rational. This restriction is also strict and is an equality for Example 2.21. After that, we deal with rings having this restriction as an equality (Definition 2.43). Numerical invariants (Betti and Bass series of the quotient field) of such rings appear to coincide with the corresponding numerical invariants of the ring from Example 2.21. In the case of rings with $\mathfrak{m}^3 = 0$, we also have the Koszul property (Proposition 2.57) and the equality of the length of the ring and of the nontrivial semidualizing modules (Proposition 2.52).

In Sec. 3, we consider alternative (using zero-dimensional modules) approaches to defining a dimension characterizing complete intersections. As a result, we obtain an extension of the class of modules of finite CI-dimension satisfying Property V. Further, we give a proof, simpler than in [5], of the theorem that a localization of a complete intersection is a complete intersection.

In Sec. 4, we study the dimension characterizing Cohen–Macaulay rings for which Properties I–IV and the following statement hold true: the class of modules of finite CM-dimension contains the class of modules of finite G_K-dimension for any suitable module K. A new characterization of suitable modules is used in the proof by means of G-Gorenstein connected ideals.

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1. Preliminary Facts

All rings are assumed to be commutative, Noetherian, and local, and modules are assumed to be finitely generated. The maximal ideal of a ring R is denoted by $\mathfrak{m}$, and the quotient field is denoted by $k \cong R/\mathfrak{m}$.

1.1. Basic Classes of Local Rings.

Definition 1.1. An element $x \in \mathfrak{m}$ is called *regular w.r.t. a module M* or *M-regular* if for any nonzero $a \in M$ we have $xa \neq 0$.

Definition 1.2. A sequence $(\boldsymbol{x}) = (x_1, \dots, x_n) \in \mathfrak{m}$ is called *M-regular* if for any $i \in \{1, \dots, n\}$ the element x_i is regular w.r.t. the module $M/(x_1, \dots, x_{i-1})M$.

Definition 1.3. We call a sequence $(\boldsymbol{x}) = (x_1, \dots, x_n) \in \mathfrak{m}$ *maximal M-regular* if $\mathfrak{m}R$ does not have $M/(x_1, \dots, x_n)M$-regular elements.

Definition 1.4. The length of a maximal M-regular sequence is uniquely determined and is called the *depth* of M (the notation is $\operatorname{depth} M$).

Definition 1.5. A ring R is called *regular* if its maximal ideal is generated by a regular sequence.

Definition 1.6. A ring R is referred to as a *locally complete intersection* if its completion $\hat{R}$ is isomorphic to a factor-ring of a regular ring by a regular sequence.

Definition 1.7. We call a ring R a *Cohen–Macaulay ring* if its depth $\operatorname{depth} R$ is equal to the Krull dimension $\operatorname{Krull\,dim} R$.

Definition 1.8. A ring R is called a *Gorenstein ring* if one of the following equivalent conditions is satisfied:

(1) R is Cohen–Macaulay and the factor-ring of R by a maximal regular sequence is an injective module over itself;
(2) the injective dimension of R as a module over itself is finite.

Proposition 1.9. *One has the following implications*:

A ring R is regular $\Longrightarrow$ R is a locally full intersection

$\Longrightarrow$ R is a Gorenstein ring $\Longrightarrow$ R is a Cohen–Macaulay ring.

1.2. Commutative Algebra of Complexes.

Definition 1.10. A *complex* X is a collection of modules X_i and homomorphisms $\partial_i^X \colon X_i \to X_{i-1}$ such that $\partial_i^X \partial_{i+1}^X = 0$. We say that the ith *homology* of the complex X is the module $\mathrm{H}_i(X) = \ker \partial_i^X / \operatorname{im} \partial_{i+1}^X$.

The homologies of all complexes under consideration are assumed to be finitely generated. We identify M with the complex X having $X_i = 0$ for $i \neq 0$ and $X_0 \simeq M$.

The following numbers reflect the locations of zero homologies of a complex X.

Definition 1.11. The *supremum*, *infimum*, and *amplitude* of a complex X are defined by the following equalities, respectively:

$$\sup(X) = \sup\{i \mid \mathrm{H}_i(X) \neq 0\},$$
$$\inf(X) = \inf\{i \mid \mathrm{H}_i(X) \neq 0\},$$
$$\operatorname{amp}(X) = \sup(X) - \inf(X).$$

Definition 1.12. A complex is called *bounded* (*bounded from above, bounded from below*) if $\operatorname{amp}(X) < \infty$ (respectively, $\sup(X) < \infty$, $\inf(X) > -\infty$).

Definition 1.13. A complex is called *acyclic* if $\mathrm{H}_i(X) = 0$ for any i.

Remark 1.14. For an acyclic complex, we have

$$\sup(X) = -\infty, \quad \inf(X) = \infty, \quad \operatorname{amp}(X) = -\infty.$$

Definition 1.15. A *morphism of complexes* $\alpha \colon X \to Y$ is a collection of maps $\alpha_i \colon X_i \to Y_i$ such that $\partial_i^Y \alpha_i - \alpha_{i-1}\partial_i^X = 0$. A morphism of complexes α induces a map in homologies $\mathrm{H}_i(\alpha) \colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$.

Definition 1.16. A morphism of complexes $\alpha \colon X \to Y$ is called an *isomorphism* if α_i is an isomorphism for all i. The corresponding complexes X and Y are called *isomorphic* (notation $X \cong Y$).

Definition 1.17. A morphism of complexes $\alpha \colon X \to Y$ is called a *quasi-isomorphism* if $\mathrm{H}_i(\alpha)$ is an isomorphism for all i. The corresponding complexes X and Y are called quasi-isomorphic (notation $X \simeq Y$).

Definition 1.18. The *shift* of a complex X (denoted by ΣX) is the complex with $(\Sigma X)_i = X_{i-1}$ and $\partial_{i+1}^{\Sigma X} = -\partial_i^X$.

Remark 1.19. The map Σ^X sending $f \in X_i$ into $f \in (\Sigma X)_{i+1}$ is an isomorphism of complexes.

Definition 1.20. The *cone* of a morphism of complexes $\phi\colon X \to Y$ is the complex $\operatorname{cone}(\phi)$ with

$$\operatorname{cone}(\phi)_i = Y_i \oplus (\Sigma X)_i, \quad \partial_i^{\operatorname{cone}(\phi)} = \begin{pmatrix} \partial_i^Y & (\Sigma^Y)_i^{(-1)}\Sigma(\phi)_i \\ 0 & \partial_i^{\Sigma X} \end{pmatrix}.$$

Proposition 1.21. *The following two conditions on a morphism $\phi\colon X \to Y$ are equivalent*:
(1) *ϕ is a quasi-isomorphism*;
(2) *$\operatorname{cone}(\phi)$ is acyclic.*

Definition 1.22. Given complexes X and Y, we define the complex $U = X \otimes Y$ in the following way:

$$U_n = \sum_i X_i \otimes Y_{n-i}, \quad \partial_n^U(x_i \otimes y_{n-i}) = \partial_i^X(x_i) \otimes y_{n-i} + (-1)^i x_i \otimes \partial_{n-i}^Y(y_{n-i}).$$

Definition 1.23. Given complexes X and Y, we define the complex $V = \operatorname{Hom}(X, Y)$ as follows:

$$V_n = \prod_{j-i=n} \operatorname{Hom}(X_i, Y_j), \quad \partial_n^V(f)_i = \partial_i^Y f_i - (-1)^n f_{i-1}\partial_i^X.$$

Morphisms from X into Y correspond to cycles in the complex $\operatorname{Hom}(X, Y)$ in the sense of Definition 1.15.

We deal with derived categories $\mathcal{D}_b^f(R)$ ($\mathcal{D}_+^f(R)$, $\mathcal{D}_-^f(R)$), i.e., with categories of bounded (bounded from above, bounded from below) complexes with finitely generated modules of homologies, localized w.r.t. the class of quasi-isomorphism (see [17, 28]).

By $\mathbf{R}\operatorname{Hom} R(\cdot,\cdot)$ ($\otimes_R^L$) we denote the right (left) derived functor of the functor of homomorphisms (tensor products) of complexes. According to [7, 37], the bounding conditions on arguments are not necessary.

Definition 1.24. For complexes X and Y, we let

$$\operatorname{Ext}_R^i(X, Y) = \mathrm{H}_{-i}(\mathbf{R}\operatorname{Hom}_R(X, Y))), \quad \operatorname{Tor}_i^R(X, Y) = \mathrm{H}_i(X \otimes_R^L Y).$$

Definition 1.25. For a complex $X \in \mathcal{D}_+^f(R)$, we define the Betti numbers of X as follows:

$$\beta_i^R(X) = \dim_k(\operatorname{Tor}_i^R(X, k)) = \dim_k(\operatorname{Ext}_R^i(X, k)).$$

The corresponding generating function is of the form

$$\mathrm{P}_X^R(t) = \sum_i \beta_i^R(X)t^i.$$

Remark 1.26. $\beta_i^R(X) = \operatorname{rank} P_i(X)$, where $P(X)$ is a minimal projective resolvent of X.

Definition 1.27. For a complex $X \in \mathcal{D}_-^f(R)$, we define the Bass numbers of X in the following way:

$$\mu_R^i(X) = \dim_k(\operatorname{Ext}_i^R(k, X)).$$

The corresponding generating function has the form

$$\mathrm{I}_R^X(t) = \sum_i \mu_R^i(X)t^i.$$

Remark 1.28. The number $\mu_R^i(X)$ equals the number of direct summands of the form $\mathrm{E}(k)$ (the injective envelope of the quotient field) in the ith component of the minimal injective resolvent of X.

Definition 1.29. For a complex $X \in \mathcal{D}_-^f(R)$, let us define the depth of X according to the following formula:

$$\operatorname{depth} X = \inf\{n \mid \operatorname{Ext}_R^i(k, X) \neq 0\}.$$

Remark 1.30. For modules, the definitions of depth 1.4 and 1.29 are equivalent.

Definition 1.31. For a complex $X \in \mathcal{D}^f_-(R)$, we define the *Cohen–Macaulay type* as

$$\dim_k(\operatorname{Ext}_R^{\operatorname{depth} X}(k, X)).$$

We shall need the following properties, which guarantee the preservation of the properties of being finitely generated under the action of some functors (see [7]).

Proposition 1.32. *If $X \in \mathcal{D}^f_+(R)$ and $Y \in \mathcal{D}^f_+(R)$, then $X \otimes^L_R Y \in \mathcal{D}^f_+(R)$.*

Proposition 1.33. *If $X \in \mathcal{D}^f_+(R)$ and $Y \in \mathcal{D}^f_-(R)$, then $\mathbf{R}\operatorname{Hom}_R(X, Y) \in \mathcal{D}^f_-(R)$.*

The behavior of the functors $\mathbf{R}\operatorname{Hom}_R(\cdot,\cdot)$ and $\otimes^L_R$ under localization can be described in the following way.

Proposition 1.34. *If $X \in \mathcal{D}^f(R)$ and $Y \in \mathcal{D}^f(R)$, then*

$$(X \otimes^L_R Y)_{\mathfrak{p}} \simeq (X_{\mathfrak{p}} \otimes^L_{R_{\mathfrak{p}}} Y_{\mathfrak{p}})_{\mathfrak{p}}.$$

Proposition 1.35. *If $X \in \mathcal{D}^f_+(R)$ and $Y \in \mathcal{D}^f_-(R)$, then*

$$\mathbf{R}\operatorname{Hom}_R(X, Y)_{\mathfrak{p}} \simeq \mathbf{R}\operatorname{Hom}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}).$$

Note that the conditions for X and Y are essential, since for modules that are not finitely generated, in general, $\operatorname{Hom}_R(M, N)_{\mathfrak{p}} \not\cong \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.

Definition 1.36. For $X \in \mathcal{D}^f_b(R)$, we define the projective dimension as

$$\operatorname{pd}_R X = \inf\{\sup(Z) \mid Z \simeq X,\ Z_i = 0 \text{ for } |i| \gg 0,\ Z_i \text{ is projective}\}.$$

Proposition 1.37 ([7, 2.10])**.** $\operatorname{pd}_R X = \sup(X \otimes^L_R k)$.

Proposition 1.38 (the Auslander–Buchsbaum formula)**.** *If $\operatorname{pd}_R X < \infty$, then*

$$\operatorname{pd}_R X + \operatorname{depth} X = \operatorname{depth} R.$$

Definition 1.39. For $X \in \mathcal{D}^f_b(R)$, we define the injective dimension as

$$\operatorname{id}_R X = \inf\{-\inf(Z) \mid Z \simeq X,\ Z_i = 0 \text{ for } |i| \gg 0,\ Z_i \text{ is injective}\}.$$

Proposition 1.40 ([7, 2.10])**.** $\operatorname{id}_R X = -\inf(\mathbf{R}\operatorname{Hom}(k, X))$.

Definition 1.41. A complex $X \in \mathcal{D}^f_b(R)$ is called dualizing if $\operatorname{id}_R X < \infty$ and the natural morphism of complexes

$$R \xrightarrow{\alpha_R} \mathbf{R}\operatorname{Hom}_R(X, X)$$

is a quasi-isomorphism.

Proposition 1.42. *If a complex X is dualizing, then for any $M \in \mathcal{D}^f_b(R)$ the natural morphism*

$$M \xrightarrow{\alpha_M} \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M, X), X)$$

is a quasi-isomorphism.

Proposition 1.43. *If R is a Gorenstein ring, then the free module of rank 1 is a dualizing complex.*

Proposition 1.44. *If $R \simeq S/I$, where S is a Gorenstein ring, then the R-complex $\mathbf{R}\operatorname{Hom}_S(R, S)$ is dualizing.*

Proposition 1.45 ([28, V.3.4])**.** *The following two conditions for a complex $X \in \mathcal{D}^f_-(R)$ are equivalent:*

(1) $\mathrm{I}^X_R(t) = t^n$;
(2) *the complex X is dualizing.*

1.3. Suitable Modules. Let us list, following [24], the basic facts about G_K-dimension and G_K-perfect modules that are necessary for further considerations.

Fix a module K. For any module P, we denote the module $\mathrm{Hom}_R(P,K)$ by P^*. We say that a module P is K-reflexive if the canonical homomorphism $P \to P^{**}$ is a bijection.

Definition 1.46. For K-reflexive modules P over a ring R such that for all $i>0$ we have $\mathrm{Ext}^i_R(P,K) = 0 = \mathrm{Ext}^i_R(P^*,K)$, we set $\mathrm{G}_K\text{-}\dim_R P = 0$;

$$\mathrm{G}_K\text{-}\dim_R M = \inf\{n \mid \text{there exists an exact sequence}$$
$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0, \text{ where } \mathrm{G}_K\text{-}\dim_R P_i = 0\}.$$

In the case where $\mathrm{G}_K\text{-}\dim M$ is finite, this can be expressed in the following way.

Proposition 1.47. *If* $\mathrm{G}_K\text{-}\dim M < \infty$, *then* $\mathrm{G}_K\text{-}\dim M = \sup\{n \mid \mathrm{Ext}^n_R(M,K) \neq 0\}$.

If $\mathrm{G}_K\text{-}\dim_R R = 0$, then the module K is called suitable. In other words, K is suitable if and only if $\mathrm{Hom}_R(K,K) \simeq R$ and for any $i>0$ we have $\mathrm{Ext}^i_R(K,K) = 0$. For example, a free module of rank 1 is suitable (the corresponding dimension coincides with the classical G-dimension of the module M; we denote it by $\mathrm{G}\text{-}\dim M$ for the sake of brevity), and so is the dualizing module. For G_K-dimension w.r.t. a suitable module K, the following analogue of the Auslander–Buchsbaum formula holds true.

Proposition 1.48. *If* $\mathrm{G}_K\text{-}\dim M < \infty$, *then* $\mathrm{G}_K\text{-}\dim M + \operatorname{depth} M = \operatorname{depth} R$.

Moreover, we have the following statement.

Proposition 1.49. *The following three conditions are equivalent*:

(1) *K is a dualizing module*;
(2) *for any M, we have* $\mathrm{G}_K\text{-}\dim M < \infty$;
(3) $\mathrm{G}_K\text{-}\dim k < \infty$.

Definition 1.50. $\operatorname{grade} M = \inf\{i \mid \mathrm{Ext}^i_R(M,R) \neq 0\}$.

Proposition 1.51. *If I is an ideal of R, then* $\operatorname{grade} R/I$ *equals the length of a maximal R-regular sequence in I.*

Proposition 1.52. $\operatorname{grade} M \leq \mathrm{G}_K\text{-}\dim M$.

Definition 1.53. If $\operatorname{grade} M = \mathrm{G}_K\text{-}\dim M$, then the module M is called G_K*-perfect*. An ideal I of a ring R is called G_K*-perfect* if R/I is a G_K-perfect module.

The meaning of the notion of "G_K-perfect ideal" is given by the following theorem reflecting the behavior of G-dimension under a ring change.

Theorem 1.54 ([24, Proposition 5])**.** *Let I be a G_K-perfect ideal and K be a suitable R-module. Then* $\mathrm{Ext}^{\operatorname{grade} R/I}_R(R/I,K)$ *is a suitable R/I-module and for any R/I-module M*

$$\mathrm{G}_K\text{-}\dim_R M < \infty \iff \mathrm{G}_{K'}\text{-}\dim_{R/I} M < \infty,$$

where $K' = \mathrm{Ext}^{\operatorname{grade} R/I}_R(R/I,K)$. *Under the finiteness condition, these two dimensions satisfy the following equality:*

$$\mathrm{G}_K\text{-}\dim_R M = \operatorname{grade} R/I + \mathrm{G}_{K'}\text{-}\dim_{R/I} M.$$

Later (see Remark 2.11), we shall give a proof of this theorem that differs from the one in [24].

Definition 1.55. Let, under the assumptions of Theorem 1.54, $K' \simeq R/I$. In this case, we call the ideal I a G_K-Gorenstein ideal. The simplest example is the ideal generated by a regular sequence.

Proposition 1.56. *The following two conditions for an ideal I are equivalent*:

(1) *an ideal I is* G-*perfect*;

(2) *for $k \neq \operatorname{grade} R/I$ we have $\operatorname{Ext}_R^k(R/I, R) = 0$; further, $\operatorname{Ext}_R^{\operatorname{grade} R/I}(R/I, R)$ is a suitable R/I-module.*

Lemma 1.57. *If $\mathfrak{a}$ is a* G-*Gorenstein ideal, $\mathfrak{a} \subset I$, and $\operatorname{grade} R/I = \operatorname{grade} R/\mathfrak{a}$, then*

$$\operatorname{Ext}_R^{i+\operatorname{grade} R/I}(R/I, R) \cong \operatorname{Ext}_{R/\mathfrak{a}}^i(R/I, R/\mathfrak{a})$$

for all $i > 0$. In particular,

$$\operatorname{Ext}_R^{\operatorname{grade} R/I}(R/I, R) \cong (\mathfrak{a} : I)/\mathfrak{a}.$$

Definition 1.58. Ideals I and J are called directly G-*connected* if there exists a G-Gorenstein ideal $\mathfrak{a}$ such that $I = (\mathfrak{a} : J)$ and $J = (\mathfrak{a} : I)$. In this case,

$$\operatorname{grade} R/I = \operatorname{grade} R/\mathfrak{a} = R/\operatorname{grade} J.$$

It follows from Lemma 1.57 that ideals I and J are directly G-connected through a G-Gorenstein ideal $\mathfrak{a}$ if and only if

$$\operatorname{Ext}_R^{\operatorname{grade} R/J}(R/J, R) \cong I/\mathfrak{a}, \quad \operatorname{Ext}_R^{\operatorname{grade} R/I}(R/I, R) \cong J/\mathfrak{a}.$$

Below (Theorem 4.3), we obtain one more equivalent condition characterizing G-perfect ideals.

2. Semidualizing Complexes and G_X-Dimension

2.1. Basic Notions. In this section, we define semidualizing complexes and G-dimension w.r.t. such complexes and describe their essential properties.

Definition 2.1. A complex $X \in \mathcal{D}_b^f(R)$ is called semidualizing if the natural morphism of complexes

$$R \xrightarrow{\alpha_R} \mathbf{R}\operatorname{Hom}_R(X, X)$$

is a quasi-isomorphism.

Example 2.2. The easiest examples of semidualizing complexes are the free module of rank 1 and the dualizing complex when it exists. In what follows, such semidualizing complexes are called *trivial*.

The existence of a nontrivial semidualizing complex X over a ring provides a very strong restriction on its Bass numbers.

Proposition 2.3. *If X is a semidualizing complex over a ring R, then*

$$\mathrm{P}_R^X(t)\,\mathrm{I}_X^R(t) = \mathrm{I}_R^R(t).$$

Proof. We have

$$\mathbf{R}\operatorname{Hom}_R(k, R) \simeq \mathbf{R}\operatorname{Hom}_R(k, \mathbf{R}\operatorname{Hom}(X, X)) \simeq \mathbf{R}\operatorname{Hom}_R(k \otimes_R^L X, X) \simeq \mathbf{R}\operatorname{Hom}_k(k \otimes_R^L X, \mathbf{R}\operatorname{Hom}_R(k, X)).$$

Computing the dimension of the corresponding vector spaces, we obtain the equality we are looking for. □

Given $M \in \mathcal{D}_b^f(R)$, we write $M_X^{**} = \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M, X), X)$ for the sake of brevity.

Definition 2.4. Let $M \xrightarrow{\alpha_M} M_X^{**}$ be a quasi-isomorphism. Set

$$\mathrm{G}_X\text{-}\dim M = -\inf(\operatorname{Hom}(M, X)) + \inf(X).$$

Remark 2.5. In contrast to classical homological dimensions of modules, the introduced invariant can be negative.

Let us get a connection between the given definition and Definition 1.46.

Lemma 2.6. *Let, in a short exact sequence*

$$0 \to M \to N \to K \to 0,$$

two of the three modules have finite G_X*-dimension w.r.t. a semidualizing complex* X*. The third one has the same property.*

Proof. Let I be an injective resolvent of the complex X. The statement evidently follows from the exact sequence

$$0 \to \mathrm{Hom}(K, I) \to \mathrm{Hom}(N, I) \to \mathrm{Hom}(M, I) \to 0$$

and the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & K & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha_M} & & \Big\downarrow{\scriptstyle \alpha_N} & & \Big\downarrow{\scriptstyle \alpha_K} & & \\ 0 & \longrightarrow & M_X^{**} & \longrightarrow & N_X^{**} & \longrightarrow & K_X^{**} & \longrightarrow & 0 \end{array}$$

The lemma is proved. $\square$

Lemma 2.7. *Let, in a short exact sequence*

$$0 \to M \to N \to K \to 0,$$

one have

$$\mathrm{G}_X\text{-}\dim N = 0, \quad 0 < \mathrm{G}_X\text{-}\dim K < \infty.$$

Then

$$\mathrm{G}_X\text{-}\dim M = \mathrm{G}_X\text{-}\dim K - 1.$$

The proof is similar to that of Lemma 2.6.

Theorem 2.8. *Let a semidualizing complex* X *have one nonzero homology in degree* 0*. Then* $K \simeq \mathrm{H}_0(X)$ *is a suitable module and* $\mathrm{G}_X\text{-}\dim M = \mathrm{G}_K\text{-}\dim M$*.*

Proof. If the complex X has just one nonzero homology, then the condition of $\alpha_R \colon R \to \mathbf{R}\,\mathrm{Hom}_R(X, X)$ being a quasi-isomorphism can be reduced to the following:

(1) $R \xrightarrow{\mathrm{H}_0(\alpha_R)} \mathrm{Hom}_R(\mathrm{H}_0(X), \mathrm{H}_0(X))$;
(2) $\mathrm{Ext}_R^i(\mathrm{H}_0(X), \mathrm{H}_0(X)) = 0$ for $i > 0$.

We use induction in that one of these dimensions that is finite. It is easy to see that $\mathrm{G}_K\text{-}\dim M = 0$ if and only if $\mathrm{G}_X\text{-}\dim M = 0$. Indeed, if one of the dimensions equals zero, then we have $\mathrm{Ext}_R^n(M, K) = 0$ with $n > 0$. Hence, $M \xrightarrow{\alpha_M} M_X^{**}$ is a quasi-isomorphism if and only if the canonical homomorphism $M \to M_K^{**}$ is an isomorphism, which is what we claimed. Now let one of the dimensions of the module M be finite and greater than 0. Then we cover the module M by a free module and apply the inductive assumption and Lemma 2.7 to the kernel of this covering. $\square$

Remark 2.9. It follows from Lemma 2.6 that, in particular, if $\mathrm{pd}\, M < \infty$, then $\mathrm{G}_X\text{-}\dim M < \infty$ for any semidualizing complex X.

Theorem 2.10. *Let* X *be a semidualizing complex over* S*,* R *be a finite* (*finitely generated as an* S*-module*) S*-algebra, and* $X' = \mathbf{R}\,\mathrm{Hom}_S(R, X)$*. Then*

(1) $\mathrm{G}_X\text{-}\dim_S R < \infty$ *if and only if* X' *is a semidualizing complex over* R;
(2) *under the condition of* $\mathrm{G}_X\text{-}\dim_S R < \infty$ *for an* R*-complex* M*, the finiteness of the* G_X*-dimension over the ring* S *is equivalent to the finiteness of the* $\mathrm{G}_{X'}$*-dimension over the ring* R*. Moreover,*

$$\mathrm{G}_X\text{-}\dim_S M = \mathrm{G}_{X'}\text{-}\dim_R M + \mathrm{G}_X\text{-}\dim_S R.$$

Proof. The functor

$$\mathbf{R}\operatorname{Hom}_R(\cdot, X') \simeq \mathbf{R}\operatorname{Hom}_R(\cdot, \mathbf{R}\operatorname{Hom}_S(R, X))$$

from the category $\mathcal{D}_+^f(R)$ into $\mathcal{D}_-^f(R)$ is isomorphic to the functor $\mathbf{R}\operatorname{Hom}_S(\cdot, X)$. Hence

$$\mathbf{R}\operatorname{Hom}_R(X', X') \simeq \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_S(R, X), X),$$

and we get the first statement of the theorem. The equivalence of the inequalities $\mathrm{G}_X\text{-}\dim_S M < \infty$ and $\mathrm{G}_{X'}\text{-}\dim_R M < \infty$ can be proved similarly.

Now, for $M \in \mathcal{D}_b^f(R)$, let us have $\mathrm{G}_X\text{-}\dim_S M < \infty$. Then

$$\begin{aligned}\mathrm{G}_X\text{-}\dim_S M &= -\inf(\mathbf{R}\operatorname{Hom}_S(M, X)) + \inf(X)\\ &= -\inf(\mathbf{R}\operatorname{Hom}_S(M, X)) + \inf(X') - \inf(X') + \inf(X)\\ &= -\inf(\mathbf{R}\operatorname{Hom}_R(M, X')) + \inf(X') - \inf(X') + \inf(X)\\ &= \mathrm{G}_{X'}\text{-}\dim_R M + \mathrm{G}_X\text{-}\dim_S R.\end{aligned}$$

The theorem is proved. □

Remark 2.11. Let a semidualizing complex X have just one nonzero homology that is a suitable module and let $\mathfrak{a}$ be a G_K-perfect ideal, where $K \simeq \mathrm{H}_0(X)$. In this case, the semidualizing complex $X' \simeq \mathbf{R}\operatorname{Hom}_S(R, X)$ also has one nonzero homology that is an $R/\mathfrak{a}$-suitable module. Applying Theorem 2.10, we obtain the statement of Theorem 1.54.

We describe semidualizing complexes X such that for any R-module M we have $\mathrm{G}_X\text{-}\dim M < \infty$.

Theorem 2.12. *Let X be a semidualizing complex over a ring R and $\mathrm{G}_X\text{-}\dim_R k < \infty$. Then X is a dualizing complex and for any R-complex $M \in \mathcal{D}_b^f(R)$ we have*

$$\mathrm{G}_X\text{-}\dim_R M < \infty.$$

Proof. From Theorem 2.10, we see that $\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{m}, X)$ is a semidualizing complex over the vector space k, whence there exists just one value $i = i_0$ such that $\mu^i(\mathfrak{m}, X)$ is not zero. Moreover, $\mu^{i_0}(\mathfrak{m}, X) = 1$. According to Proposition 1.45, this property of the Bass numbers characterizes dualizing complexes. The converse easily follows from Proposition 1.42. □

Further we consider the behavior of G_X-dimension under localization.

Theorem 2.13. *Let X be a semidualizing complex over a ring R and let M be an R-module. Then $X_\mathfrak{p}$ is a semidualizing complex over the ring $R_\mathfrak{p}$ and the finiteness of $\mathrm{G}_X\text{-}\dim_R M$ implies the finiteness of $\mathrm{G}_{X_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}} M_\mathfrak{p}$.*

Proof. From Proposition 1.35 we see that the $R_\mathfrak{p}$-complex $X_\mathfrak{p}$ is semidualizing and that applying localization to a quasi-isomorphism $M \to \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M, X), X)$ one can get the required quasi-isomorphism

$$M_\mathfrak{p} \to \mathbf{R}\operatorname{Hom}_{R_\mathfrak{p}}(\mathbf{R}\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, X_\mathfrak{p}), X_\mathfrak{p}).$$

The theorem has been proved. □

We prove an analogue of the Auslander–Buchsbaum formula for G_X-dimension.

Theorem 2.14. *Let X be a semidualizing complex over a ring R, and, for $M \in \mathcal{D}_+^f(R)$, let us have $\mathrm{G}_X\text{-}\dim_R M < \infty$. Then*

$$\mathrm{G}_X\text{-}\dim_R M + \operatorname{depth} M = \operatorname{depth} R.$$

Proof. We have

$$\begin{aligned}\operatorname{depth} M &= -\sup(\mathbf{R}\operatorname{Hom}_R(k, M))\\ &= -\sup(\mathbf{R}\operatorname{Hom}_R(k, \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M, X), X)))\\ &= -\sup(\mathbf{R}\operatorname{Hom}_R(k \otimes^L_R \mathbf{R}\operatorname{Hom}_R(M, X), X))\\ &= -\sup(\mathbf{R}\operatorname{Hom}_k(k \otimes^L_R \mathbf{R}\operatorname{Hom}_R(M, X), \mathbf{R}\operatorname{Hom}_R(k, X)))\\ &= -(\sup(\mathbf{R}\operatorname{Hom}_R(k, X)) - \inf(\mathbf{R}\operatorname{Hom}_R(M, X))).\end{aligned}$$

Similarly, we get

$$\operatorname{depth} R = -(\sup(\mathbf{R}\operatorname{Hom}_R(k, X)) - \inf(X)).$$

Then

$$\operatorname{depth} R - \operatorname{depth} M = \inf(X) - \inf(\mathbf{R}\operatorname{Hom}_R(M, X)) = \mathrm{G}_X\text{-}\dim M.$$

We have proved the theorem. ☐

Remark 2.15. Note that the proof of the Auslander–Buchsbaum formula for the classical G-dimension of modules that appeared in [3] has a big gap (see [32]). It seems that the first correct proof is in [32] and it is much more complicated than what we give here. A simpler way is presented in [13], but also in this case the moving from modules to complexes simplifies the proof significantly.

Consider the following situation: a ring R is an S-algebra, and M is an R-module of finite R-projective dimension. Then it is easy to see that the projective dimension of M over S is also finite and we have the equality

$$\operatorname{pd}_S M = \operatorname{pd}_R M + \operatorname{pd}_S R.$$

In [8, Remark 4.8], it was conjectured that the analogue of this statement is true for G-dimension. Using Theorem 2.10, we propose the following conjecture, which may be more general.

Conjecture 2.16. Let X be a semidualizing complex over a ring R. Then

$$\mathrm{G}_X\text{-}\dim M \leq \mathrm{G}\text{-}\dim M,$$

and if the right-hand side is finite, then the equality holds.

We show that from this conjecture for semidualizing complexes of a special form the required assertion follows.

Corollary 2.17. *Let R be a finite S-algebra,* $\mathrm{G}\text{-}\dim_S R < \infty$*, and $M \in \mathcal{D}^f_b(R)$. Then under the assumption of validity of Conjecture* 2.16 *for the ring R one has*

$$\mathrm{G}\text{-}\dim_R M < \infty \implies \mathrm{G}\text{-}\dim_S M < \infty,$$

and under the condition of finiteness the following formula holds:

$$\mathrm{G}\text{-}\dim_S M = \mathrm{G}\text{-}\dim_R M + \mathrm{G}\text{-}\dim_S R. \tag{1}$$

Proof. Consider the complex $X' = \mathbf{R}\operatorname{Hom}_S(R, S)$. According to Theorem 2.10, X is a semidualizing R-complex, whence, if Conjecture 2.16 is true, then we have $\mathrm{G}_X\text{-}\dim_R M < \infty$. Applying now Theorem 2.10, we get $\mathrm{G}\text{-}\dim_S M < \infty$. Equality (1) obviously follows from Theorem 2.14. ☐

The following problem is also interesting: Is it true that all semidualizing complexes over a ring R are just the complexes of the form $X = \mathbf{R}\operatorname{Hom}_S(R, S)$, where R is a quotient-ring of S?

For dualizing complexes, the corresponding problem was given by R. Sharp [36], and a positive answer to it appeared in [31], where it was shown that any ring over which there exists a dualizing complex is a quotient-ring of a Gorenstein ring. Below (Corollary 4.8), it is shown that the answer to this question is also positive in the case where the complex X is semidualizing with a unique nonzero homology.

2.2. The Structure of the Set of Semidualizing Complexes. We begin by presenting a construction of a ring with a big number of nontrivial suitable modules, which provides the foundation for studying the structure of the set of semidualizing complexes in the general case.

Let $S_1, \dots, S_n$ be finite local algebras over a local ring R. We denote the maximal ideal in S_i by $\mathfrak{m}_i$, and $\mathfrak{m}$ denotes the maximal ideal of the ring R. Consider the R-algebra $S = S_1 \otimes_R S_2 \otimes_R \dots \otimes_R S_n$.

For any $P \subset \{1, \dots, n\}$, set $S_P = \bigotimes_R S_i$, where i runs through the set P. We denote $\bar{P} = \{1, \dots, n\} \setminus P$.

Proposition 2.18. *Let, for any i, the algebra S_i be free as an R-module and let $S_i/\mathfrak{m}_i \cong R/\mathfrak{m}$. Then the algebra S is local, for any $P \subset \{1, \dots, n\}$ the S-module $K_P = \operatorname{Hom}_{S_{\bar{P}}}(S, S_{\bar{P}})$ is suitable, and*

$$\operatorname{type} K_P = (\operatorname{type} R)^n \prod_{i \in \bar{P}} \dim_{S_i/\mathfrak{m}_i} \operatorname{Hom}_{S_i/\mathfrak{m}S_i}(S_i/\mathfrak{m}_i, S_i/\mathfrak{m}S_i).$$

If, moreover, none of the rings $S_i/\mathfrak{m}S_i$ is a Gorenstein ring, then the modules K_P are pairwise nonisomorphic.

Proof. Consider the ideal generated by all $\mathfrak{m}_i$ in S. The condition of the proposition provides its maximality. On the other hand, any other maximal ideal must intersect each S_i giving the ideal $\mathfrak{m}_i$. The suitability of the modules K_P follows from the fact that S is a free $S_{\bar{P}}$-module and from Theorem 2.10. We verify that they are not isomorphic. Let P and Q be two different subsets of $\{1, \dots, n\}$. Without loss of generality, we assume that there exists $i \in P \setminus Q$. We prove that K_P and K_Q are not isomorphic as S_i-modules. The S_i-module K_P is isomorphic to the module $\operatorname{Hom}_R(S_i, R)^l$, and the S_i-module K_Q is isomorphic to the module S_i^l, where $l = \operatorname{rank}_{S_i} S$. Over the ring $S_i/\mathfrak{m}S_i$, a minimal system of generators of the module $K_P/\mathfrak{m}K_P$ consists of $l \cdot \dim_{S_i/\mathfrak{m}_i} \operatorname{Hom}_{S_i/\mathfrak{m}S_i}(S_i/\mathfrak{m}_i, S_i/\mathfrak{m}S_i)$ elements (which is not equal to l, since the ring $S_i/\mathfrak{m}S_i$ is not a Gorenstein ring), and a minimal system of generators of the module $K_Q/\mathfrak{m}K_Q$ has l elements. One can calculate the type of K_P from this. ☐

Remark 2.19. The module $K_P = \operatorname{Hom}_{S_{\bar{P}}}(S, S_{\bar{P}})$ can be represented in the following form:

$$\bigotimes_{i \in P} \operatorname{Hom}_R(S_i, R) \otimes \bigotimes_{i \in \bar{P}} S_i.$$

Proposition 2.20. *The Cohen–Macaulay type of a suitable S-module K is a divisor of the Cohen–Macaulay type of the ring.*

Proof. According to Proposition 2.3, we have $\mathrm{P}_S^K(t)\,\mathrm{I}_K^S(t) = \mathrm{I}_S^S(t)$. Comparing the leading coefficients, we obtain that the Cohen–Macaulay type of the ring R is the product of the Cohen–Macaulay type of the module K and the number of its generators. ☐

In [14], for any natural i, an example of a ring with type 2^{2^i} and suitable modules of all types admissible by Proposition 2.20 were constructed.

Here, for any given natural m, we construct an example of a ring S with Cohen–Macaulay type m over which, for any divisor of m, there exists a suitable module with the corresponding Cohen–Macaulay type.

Example 2.21. In Proposition 2.18, let $R = k$. Represent the natural number m as the product of primes:

$$m = \prod_{i=1}^{n} p_i.$$

Consider the ring

$$T_m = \bigotimes_{i=1}^{n} k \ltimes k^{p_i},$$

where k is a field. The type of the ring T_m equals m, and, according to what has been proved, for any divisor a of the number m there exists a suitable T_m-module K with type a.

Remark 2.22. Under the conditions of Proposition 2.18, the Betti and Bass numbers of the module K_P depend only on the collection of isomorphism classes of the rings S_i, where $i \in P$. That is why if there are isomorphic rings among the rings S_i, then the modules K_P cannot be separated by these invariants, in general.

Remark 2.23. As shown in [35, Corollary 4.9], for the ring T_m from Example 2.21 the set of suitable modules just consists of the modules constructed in Proposition 2.18.

The problem whether this construction is universal at least for the case of finite-dimensional algebras over a field is interesting. All other examples known to the author are like those.

The following consideration provides the basis for further investigations of the set of semidualizing complexes over a ring.

Proposition 2.24. *Under the conditions of Example* 2.21, $\mathrm{G}_{K_I}\text{-}\dim K_J < \infty$ *if and only if* $J \subset I$.

Proof. All of the statements follow from the isomorphism

$$\mathbf{R}\operatorname{Hom}_S\left(\bigotimes_{1\le i\le n}{}_R M_i, \bigotimes_{0\le i\le n}{}_R N_i\right) \simeq \bigotimes_{1\le i\le n}{}_R \mathbf{R}\operatorname{Hom}_{S_i}(M_i, N_i).$$

The proposition is proved. ◻

Proposition 2.25. *Under the assumptions of Example* 2.21, $J \subset I$ *if and only if*

$$K_J \otimes K_{I\setminus J} \simeq K_I.$$

The proof evidently follows from Remark 2.19.

We show that, in the general case, we have a similar decomposition.

Theorem 2.26. *If* X_1 *and* X_2 *are semidualizing complexes over* R *such that* $\mathrm{G}_{X_2}\text{-}\dim X_1 < \infty$ *and the complex* $M \in \mathcal{D}^f_+(R)$ *has finite* G*-dimension w.r.t.* X_1 *and* X_2*, then the composition morphism*

$$\varphi\colon \mathbf{R}\operatorname{Hom} R(M, X_1) \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_1, X_2) \to \mathbf{R}\operatorname{Hom} R(M, X_2)$$

is a quasi-isomorphism.

Proof. It suffices to show that the complex $\operatorname{cone}\varphi$ is acyclic. From the commutative diagram

$$\begin{array}{ccc}
\begin{array}{c}\mathbf{R}\operatorname{Hom} R(\mathbf{R}\operatorname{Hom} R(M, X_1) \\ \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_1, X_2), X_2)\end{array} & \xleftarrow{\mathbf{R}\operatorname{Hom} R(\varphi, X_2)} & \mathbf{R}\operatorname{Hom} R(\mathbf{R}\operatorname{Hom} R(M, X_2), X_2) \\
\downarrow\simeq & & \uparrow\simeq \\
\begin{array}{c}\mathbf{R}\operatorname{Hom} R(\mathbf{R}\operatorname{Hom} R(M, X_1), \\ \mathbf{R}\operatorname{Hom} R(\mathbf{R}\operatorname{Hom} R(X_1, X_2), X_2))\end{array} & & M \\
\uparrow\simeq & & \| \\
\mathbf{R}\operatorname{Hom} R(\mathbf{R}\operatorname{Hom} R(M, X_1), X_1) & \xleftarrow{\simeq} & M
\end{array}$$

it follows that $\mathbf{R}\operatorname{Hom} R(\varphi, X_2)$ is a quasi-isomorphism. Hence, the complex $\mathbf{R}\operatorname{Hom} R(\operatorname{cone}\varphi, X_2)$ is acyclic. Since the complexes $\mathbf{R}\operatorname{Hom} R(M, X_1) \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_1, X_2)$ and $\mathbf{R}\operatorname{Hom} R(M, X_2)$ are bounded from below, the complex $\operatorname{cone}\varphi$ is also bounded from below. If $\mathrm{H}(\operatorname{cone}\varphi) \neq 0$, then $\inf \operatorname{cone}\varphi$ is finite. We have

$$\begin{aligned}
-\infty &= \sup \mathbf{R}\operatorname{Hom} R(k, \mathbf{R}\operatorname{Hom} R(\operatorname{cone}\varphi, X_2)) = \sup \mathbf{R}\operatorname{Hom} R(k \otimes^L_R \operatorname{cone}\varphi, X_2) \\
&= \sup \mathbf{R}\operatorname{Hom} k(k \otimes^L_R \operatorname{cone}\varphi, \mathbf{R}\operatorname{Hom} R(k, X_2)) = \sup \mathbf{R}\operatorname{Hom} R(k, X_2) - \inf \operatorname{cone}\varphi.
\end{aligned}$$

This leads to a contradiction, since the last expression is finite. ◻

Definition 2.27. Nonisomorphic semidualizing complexes $X_0, X_1, \dots, X_n$ form a *chain of length n* if $\mathrm{G}_{X_i}\text{-}\dim X_{i-1} < \infty$ for all $i = 1, \dots, n$.

Corollary 2.28. *If semidualizing complexes $X_0, X_1, \dots, X_n$ form a chain, then there is a quasi-isomorphism*

$$X_n \simeq X_0 \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_0, X_1) \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_1, X_2) \otimes^L_R \cdots \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_{n-1}, X_n). \tag{2}$$

Proof. For each i, apply Theorem 2.26 with $M = R$ to the semidualizing complexes X_i and X_{i-1}. ☐

Remark 2.29. If the semidualizing complexes $X_0, X_1, \dots, X_n \simeq X_0$ form a chain, then, according to Corollary 2.28, one has the quasi-isomorphism

$$X_0 \simeq X_0 \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_0, X_1) \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_1, X_2) \otimes^L_R \cdots \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_{n-1}, X_n).$$

Hence, for any i we get

$$\mathbf{R}\operatorname{Hom} R(X_i, X_{i+1}) \simeq R.$$

Dualizing w.r.t. X_{i+1}, we obtain $X_i \simeq X_{i+1}$. For the case of chains of length 1 with $X_0 \simeq R$, this was shown in [14, Proposition 8.3, (iii) ⇒ (ii)], and for arbitrary chains of length 1 this was shown in [1, Theorem 5.5].

Proposition 2.30. *If the complexes X_1, $X_1 \otimes^L_R X_2$ are semidualizing over R, then*

$$\varphi\colon X_1 \to \mathbf{R}\operatorname{Hom} R(X_2, X_1 \otimes^L_R X_2)$$

is a quasi-isomorphism. If X_2 is also semidualizing, then

$$\psi\colon X_2 \to \mathbf{R}\operatorname{Hom} R(X_1, X_1 \otimes^L_R X_2)$$

is a quasi-isomorphism. In particular, $\mathrm{G}_{X_1 \otimes^L_R X_2}\text{-}\dim X_1 < \infty$.

Proof. Similarly to the proof of Theorem 2.26, we note that $\operatorname{cone}\varphi$ is bounded from above. Thus, if $\operatorname{cone}\varphi$ has nonzero homologies, then $\mathbf{R}\operatorname{Hom} R(X_1, \operatorname{cone}\varphi)$ also has nonzero homologies. Contradiction. ☐

Remark 2.31. Many problems concerning the structure of the set of semidualizing complexes are still unsolved. We point out the more interesting of them.

Transitivity: if a triple of semidualizing complexes X_1, X_2, X_3 is such that $\mathrm{G}_{X_3}\text{-}\dim X_2 < \infty$ and $\mathrm{G}_{X_2}\text{-}\dim X_1 < \infty$, then does this imply that $\mathrm{G}_{X_3}\text{-}\dim X_1 < \infty$?

Existence of "union": does there exist for a pair of semidualizing complexes X_1, X_2 a third semidualizing complex X_3 such that $\mathrm{G}_{X_3}\text{-}\dim X_2 < \infty$ and $\mathrm{G}_{X_3}\text{-}\dim X_1 < \infty$? (Note that this is trivially true when there is a dualizing complex over the ring.)

Consider the classification problem of semidualizing complexes over Cohen–Macaulay rings. Semidualizing complexes with more than one nonzero homology can be found only in rings sufficiently far from regular. More precisely, one has the following statement.

Proposition 2.32. *Let X be a semidualizing complex over R and $\operatorname{amp}(X) > 0$. Then R is not a Cohen–Macaulay ring.*

Proof. Assume, under the assumptions of the proposition, that R is a Cohen–Macaulay ring. Let I be an injective resolvent of the complex X. If x is an R-regular element, then from the exact sequence of complexes

$$0 \to \operatorname{Hom}_R(R/xR, I) \to I \xrightarrow{x} I \to 0$$

we see that $\operatorname{amp}(\operatorname{Hom}_R(R/xR, X)) \geq \operatorname{amp}(X)$. Applying induction on depth, we may assume that R is Artinian. We have

$$0 = \operatorname{amp}(\mathbf{R}\operatorname{Hom}(X, X)) > \operatorname{amp}(X),$$

since $\mathrm{H}_0(\mathbf{R}\operatorname{Hom}(X, X)) \simeq R$, and

$$\mathrm{H}_{\operatorname{amp}(x)}(\mathbf{R}\operatorname{Hom}(X, X)) \simeq \operatorname{Hom}(\mathrm{H}_{\inf X}(X), \mathrm{H}_{\sup X}(X)) \neq 0,$$

because over an Artinian ring $\operatorname{Hom}(M, N) \neq 0$. ☐

Proposition 2.33. *Let R be a complete local ring and x be an R-regular element. Then it is possible to obtain a one-to-one correspondence between the sets of isomorphism classes of suitable modules over the rings R and $R/(x)$.*

Proof. If a module K is suitable over R, then it is known [24] that the module $K' = K/(x)K$ is suitable over $R/(x)$. On the other hand, the fact that for the suitable $R/(x)$-module K' by definition we have $\operatorname{Ext}^2_{R/(x)}(K', K') = 0$ implies [4, Proposition 1.7] that there exists an R-module K such that $K' = K/(x)K$ and $\operatorname{Tor}_i^R(R/(x), K) = 0$ when $i > 0$. Its suitability can be checked directly, and uniqueness follows from the equality $\operatorname{Ext}^1_{R/(x)}(K', K') = 0$ (see [4, Proposition 2.5]). $\square$

Remark 2.34. As the two previous propositions show, the classification of semidualizing complexes over complete Cohen–Macaulay rings is reduced to the classification of suitable modules over Artinian rings.

Below, we consider the case of Artinian rings. There are reasons to think that the ring from Example 2.21 has maximally possible chains of suitable modules in some sense. We formalize this consideration later.

Definition 2.35. Modules $K_1, K_2, \dots, K_n$ are called *weakly* Tor-*independent* if

$$\operatorname{amp}\left(\bigotimes_{1\le i\le n}^{L} K_i\right) = 0.$$

Definition 2.36. Modules $K_1, K_2, \dots, K_n$ are called *strongly* Tor-*independent* if for any subset $I \subset \{1, \dots, n\}$ we have

$$\operatorname{amp}\left(\bigotimes_{i\in I}^{L} K_i\right) = 0.$$

Remark 2.37. In the case $n = 2$, both notions are equivalent to the classical Tor-independence, i.e., to the condition that $\operatorname{Tor}_i^R(K_1, K_2)$ is zero for $i > 0$.

Remark 2.38. It is unknown whether the weak Tor-independence implies the strong one for $n > 2$.

Theorem 2.39. *If modules $K_1, K_2, \dots, K_n$ are not free and strongly* Tor-*independent, then $\mathfrak{m}^n \neq 0$. If, under the same assumptions, $\mathfrak{m}^{n+1} = 0$, then the Betti series of k is of the form*

$$\frac{1}{\prod_{i=1}^{n}(1 - d_i t)}$$

for some natural d_i.

Proof. We denote $Y_i = \operatorname{Syz}_1(K_i)$. Note that if we choose for each i a module $X_i \in \{K_i, Y_i\}$, then the modules X_i are still strongly Tor-independent. We assume that $\mathfrak{m}^n = 0$. We prove by induction that

$$\mathfrak{m}^{n-j} \otimes \bigotimes_{1\le i\le j} Y_i = 0.$$

If $j = 1$, then this follows from $Y_1 \subset \mathfrak{m}R^{\beta_0^R(K_1)}$. Let the induction assumption be proved for $j = l$. Consider the exact sequence

$$0 \to Y_{l+1} \to R^{\beta_0^R(K_{l+1})} \to K_{l+1} \to 0$$

and tensor it by $\bigotimes_{1\le i\le l} Y_i$. Using the strong Tor-independence, we get

$$\bigotimes_{1\le i\le l+1} Y_i \subset \mathfrak{m}\left(\bigotimes_{1\le i\le l+1} Y_i\right)^{\beta_0^R(K_{l+1})},$$

which implies what is claimed by the induction assumption. Using the proved statement for $j = n-1$, we obtain

$$\mathfrak{m}\Bigl(\bigotimes_{1\le i\le n-1} Y_i\Bigr) = 0,$$

i.e., $\bigotimes\limits_{1\le i\le n-1} Y_i$ is a vector space over the quotient field of R. Since

$$\operatorname{Tor}_1^R\Bigl(\bigotimes_{1\le i\le n-1} Y_i, K_n\Bigr) = 0,$$

the module K_n is free. We arrive at a contradiction, whence $\mathfrak{m}^n \neq 0$.

If now $\mathfrak{m}^{n+1} = 0$, then a similar consideration shows that

$$\mathfrak{m}^2\Bigl(\bigotimes_{1\le i\le n-1} Y_i\Bigr) = 0, \quad \mathfrak{m}\Bigl(\bigotimes_{1\le i\le n} Y_i\Bigr) = 0.$$

The first equality implies the existence of an exact sequence of the form

$$0 \to k^{a_n} \to \bigotimes_{1\le i\le n-1} Y_i \to k^{b_n} \to 0.$$

Tensoring it by K_n and using the equalities

$$\operatorname{Tor}_j^R\Bigl(\bigotimes_{1\le i\le n-1} Y_i, K_n\Bigr) = 0, \quad j > 0,$$

from the long exact sequence of Tor-complexes, we obtain the isomorphisms

$$\operatorname{Tor}_i^R(K_n, k)^{a_n} \simeq \operatorname{Tor}_{i+1}^R(K_n, k)^{b_n}$$

for all $i > 0$. From this it follows that

$$\operatorname{Tor}_i^R(Y_n, k)^{a_n} \simeq \operatorname{Tor}_{i+1}^R(Y_n, k)^{b_n}, \quad i \ge 0.$$

Thus, the Betti series of Y_n has the form

$$\mathrm{P}_{Y_n}^R(t) = \frac{c_n}{1-(a_n/b_n)t}.$$

Similarly, we get the Betti series for all modules Y_i. Using the strong Tor-independence, we obtain the equality

$$\mathrm{P}^R_{\bigotimes\limits_{1\le i\le n} Y_i}(t) = \frac{\prod\limits_{i=1}^{n} c_i}{\prod\limits_{i=1}^{n}(1-(a_i/b_i)t)}.$$

It remains to note that $\bigotimes\limits_{1\le i\le n} Y_i$ is a vector space over k and $\beta_0^R(k) = 1$. $\square$

Conjecture 2.40. If suitable modules $K_0, K_1, \dots, K_n$ form a chain, then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1} = 0$, then the Betti series of k is

$$\frac{1}{\prod\limits_{i=1}^{n}(1-d_i t)}$$

for some d_i.

Remark 2.41. By Corollary 2.28, the conditions of the conjecture imply weak Tor-independence of the modules

$$K_0, \operatorname{Hom}_R(K_0, K_1), \operatorname{Hom}_R(K_1, K_2), \dots, \operatorname{Hom}_R(K_{i-1}, K_i)$$

for all $i \le n$. This condition is intermediate between the weak and strong Tor-independence of the modules

$$K_0, \operatorname{Hom}_R(K_0, K_1), \operatorname{Hom}_R(K_1, K_2), \dots, \operatorname{Hom}_R(K_{n-1}, K_n).$$

Theorem 2.42. *Conjecture* 2.40 *is true for* $n \le 3$.

Proof. First note that if the suitable modules $K_0, K_1, \dots, K_n$ form a chain, then the modules $R, K_1, \dots, K_{n-1}, D$ (where D is dualizing) also form a chain. Thus, one can assume the chain to be of the described form. For $n = 1$, the statement is trivially true. The existence of two nonisomorphic suitable modules now implies $\mathfrak{m} \neq 0$, and the statement about the Betti series of the quotient field holds for all rings with $\mathfrak{m}^2 = 0$. For $n = 2$, the modules K_1 and $\operatorname{Hom}(K_1, D)$ are Tor-independent and we are under the conditions of Theorem 2.39. According to Theorem 2.39, for $n = 3$ it suffices to show the strong Tor-independence of the modules K_1, $\operatorname{Hom}_R(K_1, K_2)$, and $\operatorname{Hom}_R(K_2, D)$. The weak Tor-independence follows from 2.28. To prove that any two of these modules are Tor-independent, we apply Theorem 2.26 to the triples $(M, X_1, X_2) = (R, K_1, K_2)$, (K_1, K_2, D), and $(\operatorname{Hom}_R(K_1, K_2), K_2, D)$. □

Definition 2.43. An Artinian ring R is called $\mathrm{SD}(n)$-complete if the following conditions hold:

(1) $\mathfrak{m}^{n+1} = 0$;
(2) over the ring there exist strongly Tor-independent nonfree suitable modules $K_1, K_2, \dots, K_n$ such that, for any subset $I \subset \{1, \dots, n\}$, the module $\bigotimes_{i \in I} K_i$ is suitable.

Remark 2.44. If a set of suitable modules satisfies the second condition of Definition 2.43, then the suitable modules $X_0 = R$, $X_k = \bigotimes_{1 \le i \le k} K_i$ form a chain by Proposition 2.30.

Example 2.45. All non-Gorenstein rings with $\mathfrak{m}^2 = 0$ are SD(1)-complete. A ring with $\mathfrak{m}^3 = 0$ is SD(2)-complete if and only if there exists a nontrivial suitable module over it. The ring $\bigotimes_k^{1 \le i \le n} k \ltimes k^{a_i}$, where $a_i > 1$, is $\mathrm{SD}(n)$-complete (see Example 2.21).

Proposition 2.46. *For an* $\mathrm{SD}(n)$*-complete ring* R*, the module* $\bigotimes K_i$ *is dualizing.*

Proof. Assume that $\bigotimes K_i$ is not dualizing. We show that the suitable modules $K_1, K_2, \dots, K_n$, $\operatorname{Hom}\left(\bigotimes_R K_i, D\right)$ are strongly Tor-independent. Setting

$$X_1 = \bigotimes_{i \in I} K_i, \quad X_2 = \bigotimes_{i \notin I} K_i,$$

we obtain the following isomorphisms:

$$\begin{aligned} &X_1 \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_1 \otimes^L_R X_2, D) \\ &\qquad \simeq \mathbf{R}\operatorname{Hom} R(X_2, X_1 \otimes^L_R X_2) \otimes^L_R \mathbf{R}\operatorname{Hom} R(X_1 \otimes^L_R X_2, D) \simeq \mathbf{R}\operatorname{Hom} R(X_2, D). \end{aligned}$$

Here, Proposition 2.30 provides the first isomorphism and Theorem 2.26 provides the second one. Thus,

$$\operatorname{amp}\left(\bigotimes_{i \in I}^L K_i \otimes^L \operatorname{Hom}\left(\bigotimes_R^L K_i, D\right)\right) = \operatorname{amp}\left(\operatorname{Hom}\left(\bigotimes_{i \notin I}^L K_i, D\right)\right) = 0,$$

as required. Now, using the first condition of the definition of an $\mathrm{SD}(n)$-complete ring, we arrive at a contradiction with Theorem 2.39. □

Remark 2.47. If the conditions of Conjecture 2.40 hold for a ring R, $n \le 3$, and $m^{n+1} = 0$, then R is $\mathrm{SD}(n)$-complete.

We study the case of $\mathrm{SD}(n)$-complete Artinian rings below. First, we prove several auxiliary assertions about the modules M being annihilated by the square of the maximal ideal with finite G_K-dimension w.r.t. some suitable nondualizing module.

Proposition 2.48. *If R is Artinian, $\mathfrak{m}^2 M = 0$, and $\mathrm{G}_K\text{-}\dim M = 0$, then there exists a natural c such that for the Bass numbers $\mu^i(K)$ we have $\mu^{i+1}(K) = c\mu^i(K)$ for any $i > 0$.*

Proof. From the short exact sequence

$$0 \to k^a \to M \to k^b \to 0,$$

writing down the long exact sequence for the functor $\mathrm{Ext}^i_R(-, K)$ and using $\mathrm{Ext}^i_R(M, K) = 0$ for all $i > 0$, we obtain the isomorphisms $\mathrm{Ext}^i_R(k, K)^a \simeq \mathrm{Ext}^{i+1}_R(k, K)^b$ for all $i > 0$. Since K is not dualizing, we have $\mathrm{Ext}^1_R(k, K) \neq 0$. Hence, $(a/b)^n \dim_k(\mathrm{Ext}^1_R(k, K)) = \dim_k(\mathrm{Ext}^{n+1}_R(k, K))$ is a natural number for any $n \geq 0$. Thus, b divides a. □

Proposition 2.49. *If a ring R is Artinian, $\mathfrak{m}^2 M = 0$, and $\mathrm{G}_K\text{-}\dim M = 0$, then $\mathrm{l}(M^*_K) = \mathrm{l}(M)$ and $\mu^1(K) = \mu^0(K)^2 - 1$.*

Proof. From the short exact sequence

$$0 \to k^a \to M \to k^b \to 0,$$

applying the functor $\mathrm{Hom}(-, K)$ and using $\mathrm{Ext}^i_R(M, K) = 0$ for all $i > 0$, we get the short exact sequence

$$0 \to k^{b\mu^0(K)} \to M^*_K \to k^{a\mu^0(K) - b\mu^1(K)} \to 0.$$

The computation of the lengths gives

$$\mathrm{l}(M^*_K) = (a+b)\mu^0(K) - b\mu^1(K) = \mathrm{l}(M)\mu^0(K) - b\mu^1(K). \tag{3}$$

Similarly, starting from the exact sequence

$$0 \to k^{b\mu^0(K)} \to M^*_K \to k^{a\mu^0(K) - b\mu^1(K)} \to 0,$$

we obtain the equality

$$\mathrm{l}(M^{**}_K) = \mathrm{l}(M^*_K)\mu^0(K) - (a\mu^0(K) - b\mu^1(K))\mu^1(K). \tag{4}$$

In addition, we have

$$a + b = \mathrm{l}(M) = \mathrm{l}(M^{**}_K). \tag{5}$$

Eliminating a and b from these formulas, we obtain

$$\mathrm{l}(M^*_K)\mu^1(K) = \mathrm{l}(M)(\mu^0(K)^2 - 1)$$

and

$$\mathrm{l}(M)\mu^1(K) = \mathrm{l}(M^*_K)(\mu^0(K)^2 - 1),$$

from which we obtain the required equalities. □

Remark 2.50. Let R be an $\mathrm{SD}(n)$-complete ring and $K_1, K_2, \ldots, K_n$ be the corresponding set of nontrivial suitable modules. Denoting $Y_i = \mathrm{Syz}_1(K_i)$, from the proof of Theorem 2.39 we deduce that for any $i \in \{1, \ldots, n\}$ the module $\bigotimes\limits_{j \neq i} Y_j$ is annihilated by the square of the maximal ideal and has finite $\mathrm{G}_{K_{-i}}$-dimension, where $K_{-i} = \bigotimes\limits_{j \neq i} K_j$.

The following proposition shows that in the general case the Betti numbers of suitable modules over $\mathrm{SD}(n)$-complete rings behave as in Example 2.21.

Proposition 2.51. *Let R be an* $\mathrm{SD}(n)$*-complete ring and $K_1, K_2, \ldots, K_n$ be the corresponding set of nontrivial suitable modules. For each i, the Bass series for K_{-i} is*

$$\mathrm{I}^{K_{-i}}(t) = \frac{\mu^0(K_{-i}) - t}{1 - \mu^0(K_{-i})t},$$

and the Betti series for K_i is

$$\mathrm{P}_{K_i}(t) = \frac{\beta_0(K_i) - t}{1 - \beta_0(K_i)t}.$$

Proof. Using Propositions 2.48 and 2.49, we obtain the following expression for the Bass series of K_{-i}:

$$\mathrm{I}^{K_{-i}}(t) = \frac{\mu^0(K_{-i}) - \mu^0(K_{-i})ct + \mu^0(K_{-i})^2 t - t}{1 - ct},$$

where $\mu^{j+1}(K_{-i}) = c\mu^j(K_{-i})$ for $j > 0$. It remains to show that $c = \mu^0(K_{-i})$. By Remark 2.50, there exists an R-module M that is annihilated by $\mathfrak{m}^2$ and has finite $\mathrm{G}_{K_{-i}}$-dimension. Dualizing the exact sequence

$$0 \to k^a \to M \to k^b \to 0,$$

we obtain the exact sequence

$$0 \to k^{b\mu^0(K_{-i})} \to M^* \to k^{a\mu^0(K_{-i}) - b\mu^1(K_{-i})} \to 0.$$

As in the proof of Proposition 2.48, from these two exact sequences we get the following:

$$\frac{a}{b} = c = \frac{b\mu^0(K_{-i})}{a\mu^0(K_{-i}) - b\mu^1(K_{-i})}.$$

Substituting $\mu^1(K_{-i}) = \mu^0(K_{-i})^2 - 1$ from Proposition 2.49 and transforming the expression, we obtain the equality

$$(\mu^0(K_{-i})b - a)(b - \mu^0(K_{-i})a) = 0.$$

Since a/b is a natural number, we have $a = \mu^0(K_{-i})b$. From Proposition 2.46, we see that the module $K_i \otimes K_{-i}$ is dualizing, which is why from the isomorphism $\mathbf{R}\operatorname{Hom} R(K_i, K_i \otimes K_{-i}) \simeq K_{-i}$ we obtain $\mathrm{P}_{K_i}(t) = \mathrm{I}^{K_{-i}}(t)$. $\square$

Below, we consider the case of SD(2)-complete rings, i.e., the rings with $\mathfrak{m}^3 = 0$ over which there exists a nontrivial suitable module K.

Proposition 2.52. *If R is an* $\mathrm{SD}(2)$*-complete ring and K is a nontrivial suitable module over it, then* $\mathrm{l}(K) = \mathrm{l}(R)$.

Proof. Dualizing w.r.t. K the exact sequence

$$0 \to \mathrm{Syz}_1(K) \to R^{\beta_0(K)} \to K \to 0,$$

we obtain the sequence

$$0 \to R \to K^{\beta_0(K)} \to \mathrm{Syz}_1(K)^* \to 0.$$

From Proposition 2.49 it follows that $\mathrm{l}(\mathrm{Syz}_1(K)) = \mathrm{l}(\mathrm{Syz}_1(K)^*)$. The computation of the lengths yields

$$\beta_0(K)\,\mathrm{l}(R) - \mathrm{l}(K) = \beta_0(K)\,\mathrm{l}(K) - \mathrm{l}(R),$$

from which it follows that $\mathrm{l}(K) = \mathrm{l}(R)$. $\square$

Conjecture 2.53. For any suitable module K over an Artinian ring R, we have $\mathrm{l}(K) = \mathrm{l}(R)$.

Remark 2.54. Any finite algebra with $\mathfrak{m}^3 = 0$ is naturally graduated (see [40, proof of Theorem 3.1, step 7]). On any R-module that is annihilated by $\mathfrak{m}^2$ there exists a structure of a graduated module.

Lemma 2.55. *If R is an* $\mathrm{SD}(2)$*-complete ring and K is a nontrivial suitable module over it, then* $\operatorname{socle} R = \mathfrak{m}^2$ *for all $i \geq 2$ and* $\mathfrak{m}\,\mathrm{Syz}_i(K) = \mathfrak{m}^2 R^{\beta_{i-1}(K)}$*. In particular, there is a natural graduation on the minimal free resolvent of the module* $\mathrm{Syz}_1(K)$.

Proof. Applying [29, Remark 2.4] to $M = K$ and $N = \operatorname{Hom}(K, D)$, where D is the dualizing module and K is nontrivial semidualizing, we obtain $\operatorname{socle} R = \mathfrak{m}^2$. To obtain the second equality, we argue as in [29, Remark 2.4]. The inclusion $\mathfrak{m}\operatorname{Syz}_i(K) \subset \mathfrak{m}^2 R^{\beta_{i-1}(K)}$ is obvious. To prove the converse inclusion, we assume that there exists $x \in \mathfrak{m}^2 R^{\beta_{i-1}(K)} \setminus \mathfrak{m}\operatorname{Syz}_i(K)$. As $\operatorname{Syz}_{i-1}(K)$ is annihilated by $\mathfrak{m}^2$, we have $x \in \operatorname{Syz}_i(K) \setminus \mathfrak{m}\operatorname{Syz}_i(K)$. For $i > 0$, the equality $\operatorname{Tor}_i^R(\operatorname{Syz}_i(K), \operatorname{Hom}(K, D)) = 0$ holds, whence $\operatorname{Syz}_i(K)$ does not have k as a direct summand. Thus, x is not annihilated by m, a contradiction. □

Proposition 2.56. *If R is an* SD(2)*-complete ring and K is a nontrivial suitable module over it, then the following equalities hold*:

(1) $\dim_k \mathfrak{m}^2 = \mu^0(K)\beta_0(K)$;
(2) $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \mu^0(K) + \beta_0(K)$;
(3) $\dim_k \mathfrak{m}^2 K = \mu^0(K)$.

Proof. The first equality follows from $\operatorname{Hom}(K, K) \simeq R$ (hence, $\dim_k \operatorname{socle} K \dim_k K/\mathfrak{m}K = \dim_k \operatorname{socle} R$) and Lemma 2.55. To prove the second one, consider the sequence

$$0 \to \operatorname{Syz}_2(K)/\mathfrak{m}\operatorname{Syz}_2(K) \to (R/\mathfrak{m}^2 R)^{\beta_1(K)} \to \operatorname{Syz}_1(K) \to 0,$$

which is exact by Lemma 2.55. The computation of the lengths gives

$$\begin{aligned} &\dim_k \operatorname{Syz}_2(K)/\mathfrak{m}\operatorname{Syz}_2(K) \\ &\quad = \beta_1(K)(1 + \dim_k \mathfrak{m}/\mathfrak{m}^2) - \dim_k \operatorname{Syz}_1(K)/\mathfrak{m}\operatorname{Syz}_1(K) - \dim_k \mathfrak{m}\operatorname{Syz}_1(K) \\ &\quad = \beta_1(K)(1 + \dim_k \mathfrak{m}/\mathfrak{m}^2) - \beta_1(K) - \beta_1(K)\mu^0(K) = \beta_1(K)(\dim_k \mathfrak{m}/\mathfrak{m}^2 - \mu^0(K)), \end{aligned} \tag{6}$$

where in the second equality we use the equality

$$\dim_k \mathfrak{m}\operatorname{Syz}_1(K) = \dim_k(\operatorname{Syz}_1(K)/\mathfrak{m}\operatorname{Syz}_1(K))\mu^0(K)$$

(see the proof of Proposition 2.51). On the other hand, from Lemma 2.55 we have

$$\dim_k \mathfrak{m}\operatorname{Syz}_2(K) = \beta_1(K)\dim_k \mathfrak{m}^2 = \beta_1(K)\mu^0(K)\beta_0(K). \tag{7}$$

Finally, note that the module $\operatorname{Syz}_2(K)$ also has finite G_K-dimension, from which, as in the proof of Proposition 2.51, we obtain

$$\dim_k \mathfrak{m}\operatorname{Syz}_2(K) = \mu^0(K)\dim_k \operatorname{Syz}_2(K)/\mathfrak{m}\operatorname{Syz}_2(K). \tag{8}$$

Combining (6), (7), and (8), we get

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \mu^0(K) + \beta_0(K).$$

To obtain the second equality, consider the short exact sequence

$$0 \to \operatorname{Syz}_2(L)/\mathfrak{m}\operatorname{Syz}_2(L) \to (R/\mathfrak{m}^2 R)^{\beta_1(L)} \to \operatorname{Syz}_1(L) \to 0,$$

where $L \simeq \operatorname{Hom}(K, D)$ is a suitable module, and tensor it by K. The sequence

$$0 \to (\operatorname{Syz}_2(L)/\mathfrak{m}\operatorname{Syz}_2(L)) \otimes K \to (K/\mathfrak{m}^2 K)^{\beta_1(L)} \to \operatorname{Syz}_1(L) \otimes K \to 0$$

is also exact by Remark 2.41. The computation of the lengths gives

$$\beta_1(L)(\mathrm{l}(K) - \mathrm{l}(\mathfrak{m}^2 K)) = \beta_2(L)\beta_0(K) + (\beta_0(L) - 1)\,\mathrm{l}(R), \tag{9}$$

where the equality

$$\mathrm{l}(\operatorname{Syz}_1(L) \otimes K) = (\beta_0(L) - 1)\,\mathrm{l}(R)$$

follows from the computation of the lengths in the exact sequence

$$0 \to \operatorname{Syz}_1(L) \otimes K \to K^{\beta_0(L)} \to D \to 0$$

and Proposition 2.52.

Using (9) and the formulas $\beta_0(L) = \mu^0(K)$, $\beta_1(L) = \mu^0(K)^2 - 1$, and $\beta_2(L) = (\mu^0(K)^2 - 1)\mu^0(K)$, which follow from Proposition 2.51 and the equality $\mathrm{l}(R) = (1 + \mu^0(K))(1 + \beta_0(K))$, we get the required equality. $\square$

Theorem 2.57. SD(2)*-complete rings are Koszul, i.e.,* $\operatorname{Ext}^i_R(k,k)_j = 0$ *for* $i \neq j$.

Proof. For $M = \operatorname{Syz}_1(K), \operatorname{Syz}_1(\operatorname{Hom}(K,D))$, we have $\operatorname{Ext}^i_R(M,k)_j = 0$ for $i \neq j$. It remains to prove that the modules $\operatorname{Syz}_1(K)$ and $\operatorname{Syz}_1(\operatorname{Hom}(K,D))$ are Tor-independent and their tensor product is annihilated by $\mathfrak{m}$. $\square$

3. PCI-Dimension

Definition 3.1 ([9])**.** A *quasi-deformation* of a ring R is a diagram of homomorphisms $R \to R' \leftarrow Q$, where $R \to R'$ is a flat extension and $R' \leftarrow Q$ is a *deformation*, i.e., the quotient homomorphism by the ideal I generated by a regular sequence.

Definition 3.2 ([9])**.** $\operatorname{CI-dim}_R M = \inf\{\operatorname{pd}_Q(M \otimes_R R') - \operatorname{pd}_Q R' \mid R \to R' \leftarrow Q \text{ is a quasi-deformation}\}$.

Definition 3.3. Consider the modules M over a ring R such that $\operatorname{G-dim}_R M = 0$ and their Betti numbers $\beta^R_n(M)$ are bounded by some polynomial in n. For such modules, we set $\operatorname{PCI-dim}_R M = 0$. For arbitrary modules, we set

$$\operatorname{PCI-dim}_R M = \inf\{n \mid \text{there exists an exact sequence } 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0, \text{ where } \operatorname{PCI-dim}_R P_i = 0\}.$$

The following statement is well known [26], but we provide a simpler proof of it.

Proposition 3.4. *If R is a complete intersection, then for any R-module M the numbers $\beta^R_n(M)$ are bounded by some polynomial in n.*

Proof. First, we reduce this assertion to the case where $\operatorname{depth} M = \operatorname{depth} R$. Let $n = \operatorname{depth} R - \operatorname{depth} M$. We denote $\operatorname{Syz}^R_n(M) = \operatorname{coker} \delta_{n+1}$, where $(\mathbf{F}, \delta)$ is the minimal free resolvent of M over R. For $i \gg 0$, we have $\beta^R_i(\operatorname{Syz}^R_n(M)) = \beta^R_{i+n}(M)$. On the other hand, $\operatorname{G-dim}_R M = \operatorname{depth} R - \operatorname{depth} M$, whence $\operatorname{G-dim}_R \operatorname{Syz}^R_n(M) = 0$, and therefore $\operatorname{depth} \operatorname{Syz}^R_n(M) = \operatorname{depth} R$. Now let $\operatorname{depth} M = \operatorname{depth} R$. Take an R- or M-regular sequence $(x) = (x_1, x_2, \ldots, x_{\operatorname{depth} R})$. Since in this case $\operatorname{Tor}^R_i(R/(x), M) = 0$, we have $\beta^R_i(M) = \beta^{R/(x)}_i(M/(x))$. Thus, the statement is reduced to the case of an Artinian ring. We prove it by induction on the length of the module M. For the quotient field k, this is a classical result [38]. The induction step easily follows from the exact sequence $0 \to k \to M \to M/k \to 0$. $\square$

Proposition 3.5. *If R is a complete intersection, then for any R-module M we have* $\operatorname{PCI-dim}_R M < \infty$. *Conversely, if* $\operatorname{PCI-dim}_R k < \infty$*, then R is a complete intersection.*

Proof. Let R be a complete intersection. We take any R-module M and construct its resolvent consisting of modules of nonzero PCI-dimension. Let $n = \operatorname{depth} R - \operatorname{depth} M$. We denote $\operatorname{Syz}^R_n(M) = \operatorname{coker} \delta_{n+1}$, where $(\mathbf{F}, \delta)$ is the minimal free resolvent of M over R. As $\operatorname{G-dim} M$ is finite because R is a Gorenstein ring, $\operatorname{G-dim}_R \operatorname{Syz}^R_n(M) = 0$. For $i \gg 0$, we have $\beta^R_i(\operatorname{Syz}^R_n(M)) = \beta^R_{i+n}(M)$. From this and because the Betti numbers of any module over a complete intersection are bounded by a polynomial (3.4), we obtain $\operatorname{PCI-dim}_R \operatorname{Syz}^R_n(M) = 0$.

If $\operatorname{PCI-dim}_R k < \infty$, then $\beta^R_i(k)$ are bounded by a polynomial and then the ring R is a complete intersection [27]. $\square$

Proposition 3.6. $\operatorname{PCI-dim}_R M \leq \operatorname{CI-dim}_R M$, *and if* $\operatorname{CI-dim}_R M$ *is finite, then we have the equality.*

Proof. If $\operatorname{CI-dim}_R M < \infty$, then by [9, Theorem 1.4] $\operatorname{G-dim}_R M < \infty$. Let $n = \operatorname{depth} R - \operatorname{depth} M$. We denote $\operatorname{Syz}^R_n(M) = \operatorname{coker} \delta_{n+1}$, where $(\mathbf{F}, \delta)$ is the minimal free resolvent of M over R. Since $\operatorname{G-dim} M$ is finite, it follows that $\operatorname{G-dim}_R \operatorname{Syz}^R_n(M) = 0$. For $i \gg 0$, we have $\beta^R_i(\operatorname{Syz}^R_n(M)) = \beta^R_{i+n}(M)$. From this and from the fact that the Betti numbers of any module of finite CI-dimension are bounded by a polynomial [9, Lemma 1.5], we obtain $\text{PCI-dim}_R \operatorname{Syz}_n^R(M) = 0$. $\square$

Proposition 3.7. *If* $\text{PCI-dim}\, M < \infty$, *then* $\text{PCI-dim}\, M + \operatorname{depth} M = \operatorname{depth} R$.

Proof. The statement is evident, since under these assumptions $\text{PCI-dim}\, M = \text{G-dim}\, M$, and for G-dimension the corresponding formula is true. $\square$

Using a similar method for PCI-dimension, one can show some other properties of CI-dimension. The key point here is that if the Betti numbers of two modules in a short exact sequence are bounded by a polynomial, then this is also true for the third module. Moreover, from the properties of G-dimension the following statement is obvious.

Proposition 3.8. *If two modules in a short exact sequence have finite* PCI-*dimension, then so does the third one.*

However, it is unknown whether CI-dimension has a similar property.

As Proposition 3.6 shows, the class of modules of finite PCI-dimension contains the class of modules of finite CI-dimension. There is a problem due to this: do these classes coincide? A negative answer was given by O. Veliche in [39]. The following was shown.

Proposition 3.9 ([39])**.** *Let Q be a ring containing a field and* $\operatorname{depth} Q \geq 4$. *Then there exists a perfect ideal $I \subset Q$ such that* $\operatorname{grade} R/I = 4$ *and there exists a module M over the ring $R = Q/I$ such that* $0 = \text{PCI-dim}_R M < \text{CI-dim}_R M = \infty$.

Now we prove that a localization of a module of finite PCI-dimension also has finite PCI-dimension.

Proposition 3.10. $\beta_i^{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \beta_i^R(M)$. *In particular, if the right-hand side is bounded by a polynomial in i, then so is the left-hand side.*

Proof. We take the minimal free resolvent of M over R and tensor it by the (R-flat) module $R_\mathfrak{p}$. The resulting complex is a complex of free $R_\mathfrak{p}$-modules that is a direct sum of the minimal resolvent of $M_\mathfrak{p}$ over $R_\mathfrak{p}$ and some complexes of the form $0 \to R_\mathfrak{p} \to R_\mathfrak{p} \to 0$. As the ith Betti number equals the rank of the ith free module in the free resolvent, we are done. $\square$

Proposition 3.11. $\text{PCI-dim}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \text{PCI-dim}_R M$.

Proof. The statement evidently follows from the corresponding property of G-dimension and Proposition 3.10. $\square$

Proposition 3.12 ([5])**.** *If R is a complete intersection, then $R_\mathfrak{p}$ is a complete intersection.*

Proof. Indeed, if R is a complete intersection, then, by Proposition 3.4, the $\beta_n^R(R/\mathfrak{p})$ are bounded by a polynomial. By Proposition 3.10, the $\beta_i^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ are also bounded by a polynomial. But $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the quotient field of the ring $R_\mathfrak{p}$, and from the fact that the Betti numbers are bounded by a polynomial we obtain (see [27]) that $R_\mathfrak{p}$ is a complete intersection. $\square$

4. CM-Dimension

Assumptions. Consider the following two groups of notions:

(1) regular ring; complete intersection; ideal generated by a regular sequence from $\mathfrak{m} \setminus \mathfrak{m}^2$; ideal generated by an arbitrary regular sequence; projective dimension;
(2) Gorenstein ring; Cohen–Macaulay ring; G-Gorenstein ideal; G-perfect ideal; G-dimension.

For a large variety of statements about the first group of notions there are analogues for the second group. We look at several examples.

The quotient-ring of a regular ring by some ideal is a regular ring if and only if the ideal is generated by a regular sequence from $\mathfrak{m}/\mathfrak{m}^2$, and, similarly, the quotient-ring of a Gorenstein ring by some ideal is a Gorenstein ring if and only if the ideal is G-Gorenstein.

The quotient-ring of a regular ring by some ideal is a complete intersection if and only if the ideal is generated by a regular sequence, and, similarly, the quotient-ring of a Gorenstein ring by some ideal is a Cohen–Macaulay ring if and only if the ideal is G-perfect.

Let S be a ring, R be its quotient-ring by the ideal I generated by a regular sequence from $\mathfrak{m}\backslash\mathfrak{m}^2$, and M be an R-module. Then $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{pd}_S M < \infty$, and if one of these conditions holds, then $\operatorname{pd}_R M + \operatorname{grade} R/I = \operatorname{pd}_S M$ (see [33]). The analogue of this statement is the following. Let S be a ring, R be its quotient-ring by a G-Gorenstein ideal, and M be an R-module. Then $\text{G-dim}_R M < \infty$ if and only if $\text{G-dim}_S M < \infty$, and if one of these conditions holds, then $\text{G-dim}_R M + \operatorname{grade} R/I = \text{G-dim}_S M$ (see [24]).

Now we give a definition of CM-dimension that would be an analogue for Definitions 3.1 and 3.2 for CI-dimension.

Definition 4.1. A G-*quasi-deformation* of a ring R is a diagram of local homomorphisms $R \to R' \leftarrow Q$, where $R \to R'$ is a flat extension and $R' \leftarrow Q$ is a G-*deformation*, i.e., the quotient homomorphism by the G-perfect ideal I.

Definition 4.2. $\text{CM-dim}_R M = \inf\{\text{G-dim}_Q(M \otimes_R R') - \text{G-dim}_Q R' \mid R \to R' \leftarrow Q$ is a G-quasi-deformation$\}$.

We prove that from the finiteness of the G_K-dimension of a module M w.r.t. suitable modules K it follows that its CM-dimension is finite. To do this, we give a new criterion for G-perfectness of an ideal.

Theorem 4.3. *The following condition for an ideal I is equivalent to conditions* (1) *and* (2) *from Proposition* 1.56:

(3) *there exists an ideal J such that the ideals I and J are directly* G*-connected,* $\operatorname{Ext}_R^{\operatorname{grade} R/I}(R/I, R)$ *is a suitable R/I-module, and* $\operatorname{Ext}_R^{\operatorname{grade} R/J}(R/J, R)$ *is a suitable R/J-module.*

Proof. We check that condition (3) follows from the validity of condition (1). As $\mathfrak{a}$ we can take the ideal generated by a maximal regular sequence in I. Let $J = (\mathfrak{a} : I)$. Then

$$\operatorname{grade} R/I = \operatorname{grade} R/J$$

and

$$\text{G-dim}_R R/J = \text{G-dim}_{R/\mathfrak{a}} R/J + \text{G-dim}_R R/\mathfrak{a} = \text{G-dim}_{R/\mathfrak{a}} R/I + \text{G-dim}_R R/\mathfrak{a} = \text{G-dim}_R R/I.$$

This yields the G-perfectness of the ideal J. Below, we use condition (2).

We check that condition (1) follows from condition (3). Let $\mathfrak{a}$ be the corresponding G-Gorenstein ideal. Consider the ideals $I/\mathfrak{a}$ and $J/\mathfrak{a}$ in the ring $R/\mathfrak{a}$. The G-perfectness of these ideals is equivalent to the G-perfectness of I and J. Condition (3) by Lemma 1.57 can be carried over to the quotient-ring $R/\mathfrak{a}$, which is why it suffices to consider the following case: $\operatorname{Ann} I = J$, $\operatorname{Ann} J = I$, I is a suitable R/J-module, and J is a suitable R/I-module. Under these conditions, the exact sequences $0 \to I \to R \to R/I \to 0$ and $0 \to J \to R \to R/J \to 0$ are duals of each other and we obtain the following isomorphisms for $i > 1$:

$$\operatorname{Ext}_R^i(R/I, R) \cong \operatorname{Ext}_R^{i-1}(I, R), \quad \operatorname{Ext}_R^i(R/J, R) \cong \operatorname{Ext}_R^{i-1}(J, R),$$

and also

$$\operatorname{Ext}_R^1(R/I, R) = 0, \quad \operatorname{Ext}_R^1(R/J, R) = 0.$$

Now it suffices to show that for $i > 0$

$$\operatorname{Ext}_R^i(R/I, R) = 0 = \operatorname{Ext}_R^i(R/J, R).$$

The induction basis is shown. Let $k \geq 1$ and the statement be true for $i \leq k$. Consider the spectral sequences of the ring change

$$\operatorname{Ext}^i_{R/I}(J, \operatorname{Ext}^j_R(R/I, R)) \Rightarrow \operatorname{Ext}^{i+j}_R(J, R)$$

and

$$\operatorname{Ext}^i_{R/J}(I, \operatorname{Ext}^j_R(R/J, R)) \Rightarrow \operatorname{Ext}^{i+j}_R(I, R).$$

For $i \geq 0$, we have

$$\operatorname{Ext}^i_{R/I}(J, \operatorname{Ext}^{k-i}_R(R/I, R)) = 0$$

by the induction assumption and the suitability of the R/I-module J. Hence,

$$\operatorname{Ext}^{k+1}_R(R/J, R) \simeq \operatorname{Ext}^k_R(J, R) = 0.$$

Similarly,

$$\operatorname{Ext}^{k+1}_R(R/I, R) = 0.$$

The theorem is proved. □

Remark 4.4. The case where $\mathfrak{a} = 0$ and the ideals I and $J = \operatorname{Ann} I$ are principal was first considered in [5] (see also [6]).

We list several immediate corollaries.

Corollary 4.5. *If I is a principal ideal, then $\text{G-dim}_R R/I = 0$ if and only if $\operatorname{Ann} I$ is a suitable R/I-module.*

Corollary 4.6. *If there exists a G-Gorenstein ideal $\mathfrak{a}$ such that $(\mathfrak{a} : I) = I$ and $\operatorname{Ext}^{\operatorname{grade} R/I}_R(R/I, R)$ is a suitable R/I-module, then I is G-perfect.*

The following construction was considered in [10] in a similar context for the case where K is a dualizing module. Let K be a suitable module over a ring R. We define multiplication on the R-module $S = R \oplus K$ by the formula

$$(a_1, r_1)(a_2, r_2) = (a_1 a_2, a_1 r_2 + a_2 r_1).$$

Obviously, we have introduced a ring structure on S. Note that there is a surjective ring homomorphism ϕ from S into R, i.e., R can be considered as an S-module. The kernel of this homomorphism is an ideal of K. This ideal is G-perfect by Corollary 4.6, where as $\mathfrak{a}$ we consider the zero ideal.

Theorem 4.7. *Let $\text{G}_K\text{-dim}\, M < \infty$ for some suitable module K. Then $\text{CM-dim}\, M < \infty$.*

Proof. Consider a G-quasi-deformation $R \to R \leftarrow S$, where as S we take the ring $R \oplus K$ considered above. By Theorem 1.54, $\text{G}_K\text{-dim}_R M = \text{G-dim}_S M$. □

Corollary 4.8. *If K is a suitable module over R, then there exists a ring S such that $R \simeq S/I$, $\text{G-dim}_S R = 0$, and $\operatorname{Hom}_S(R, S) \simeq K$.*

Now we can give one more definition of CM-dimension, equivalent to Definition 4.2 but technically more suitable.

Definition 4.2′. $\text{CM-dim}_R M = \inf\{\text{G}_K\text{-dim}_{R'}(M \otimes_R R') \mid R \to R' \text{ is a flat extension}, K \text{ is a suitable } R'\text{-module}\}$.

In particular, it is now evident that $\text{CM-dim}\, M \geq 0$. Using this definition, we prove that CM-dimension indeed characterizes Cohen–Macaulay rings.

Theorem 4.9. *If $\text{CM-dim}_R M < \infty$, then $\text{CM-dim}_R M + \operatorname{depth} M = \operatorname{depth} R$.*

Proof. The statement follows from the corresponding equality for G-dimension:

$$\begin{aligned}\text{CM-dim}_R M &= \text{G-dim}_Q M' - \text{G-dim}_Q R' = (\operatorname{depth} Q - \operatorname{depth}_Q M') - (\operatorname{depth} Q - \operatorname{depth}_Q R')\\ &= \operatorname{depth}_Q R' - \operatorname{depth}_Q M' = \operatorname{depth} R' - \operatorname{depth}_{R'}(M \otimes_R R') = \operatorname{depth} R - \operatorname{depth} M.\end{aligned}$$

The theorem is proved. □

Theorem 4.10. *If a ring R is a Cohen–Macaulay ring, then for any R-module M we have* $\text{CM-dim}_R M < \infty$*. Conversely, if* $\text{CM-dim}_R k < \infty$*, then R is a Cohen–Macaulay ring.*

Proof. If the ring R is a Cohen–Macaulay ring, then its completion R' is the quotient-ring of a regular ring S by a G-perfect ideal, and the statement follows from the fact that all modules over regular rings have finite G- (moreover, projective) dimension.

We prove the converse. Let $\text{CM-dim}_R k < \infty$, $R \to R'$ be the corresponding flat extension, and K be a suitable module. Let $(\boldsymbol{x}) = (x_1, \ldots, x_{\operatorname{depth} R})$ be a maximal R-regular sequence. Then $\text{CM-dim}_{R/(\boldsymbol{x})} k < \infty$. Indeed, since $R \to R'$ is a flat extension, we see that $(\boldsymbol{x})$ is an R'-regular sequence, $R/(\boldsymbol{x}) \to R'/(\boldsymbol{x})$ is also a flat extension, and the suitability of the $R'/(\boldsymbol{x})$-module $K/(\boldsymbol{x})K$ and the equality

$$\mathrm{G}_{K/(\boldsymbol{x})K}\text{-dim}_{R'/(\boldsymbol{x})}(k \otimes R'/(\boldsymbol{x})) = \mathrm{G}_K\text{-dim}_{R'}(k \otimes R') - \operatorname{depth} R$$

follow from Theorem 1.54. Thus, it suffices to consider the case $\operatorname{depth} R = 0$. We prove that in this case R is Artinian. If not, then for any n the ideal $\mathfrak{m}^n$ is nonzero. For any n, we have the embedding $0 \to \operatorname{Hom}(k, \mathfrak{m}^n) \to \operatorname{Hom}(k, R)$. Since $\bigcap \mathfrak{m}^n = 0$, for $n \gg 0$ we have $\operatorname{Hom}(k, \mathfrak{m}^n) = 0$, whence $\operatorname{depth} \mathfrak{m}^n \neq 0$. On the other hand, for R-modules M of finite length, using induction on the length with the help of Lemma 2.6 one can show that $\mathrm{G}_K\text{-dim}_{R'} M \otimes R' < \infty$. Consider the exact sequence

$$0 \to \mathfrak{m}^n \otimes_R R' \to R' \to R/\mathfrak{m}^n \otimes_R R' \to 0.$$

Since the length of the module $R/\mathfrak{m}^n$ is finite, we have $\mathrm{G}_K\text{-dim}_{R'} R/\mathfrak{m}^n \otimes R' < \infty$, and from Lemma 2.6 we obtain $\mathrm{G}_K\text{-dim}_{R'} \mathfrak{m}^n \otimes_R R' < \infty$, whence $\text{CM-dim}_R \mathfrak{m}^n < \infty$. We have a contradiction with Theorem 4.9:

$$0 < \operatorname{depth} \mathfrak{m}^n + \text{CM-dim}_R \mathfrak{m}^n = \operatorname{depth} R = 0.$$

The theorem is proved. □

Proposition 4.11. $\text{CM-dim}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \text{CM-dim}_R M$.

Proof. Obviously, we may assume that $\text{CM-dim}_R M$ is finite. Let $R \to R' \leftarrow Q$ be the corresponding G-quasi-deformation. Since $R \to R'$ is a flat extension, there exists an ideal $\mathfrak{p}' \subset R'$ such that $R \cap \mathfrak{p}' = \mathfrak{p}$. Let $\mathfrak{q} \subset Q$ be the preimage of $\mathfrak{p}'$. It is easy to see that the diagram $R_\mathfrak{p} \to R'_{\mathfrak{p}'} \leftarrow Q_\mathfrak{q}$ is a G-quasi-deformation. Then,

$$\begin{gathered}\text{G-dim}_Q M \otimes_R R' \geq \text{G-dim}_{Q_\mathfrak{q}} (M \otimes_R R')_\mathfrak{q} = \text{G-dim}_{Q_\mathfrak{q}} M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'},\\ \text{G-dim}_Q R' = \text{G-dim}_{Q_\mathfrak{q}} R'_{\mathfrak{p}'},\end{gathered}$$

as required. □

Remark 4.12. From Theorem 4.9 and Proposition 4.11 we obtain that for modules M of finite CM-dimension and for any prime ideal $\mathfrak{p}$ the following inequality holds:

$$\operatorname{depth} R - \operatorname{depth} M \geq \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p}.$$

The last condition for the module M was used in [12]; in particular, the authors point out (see [12, Remark 5]) that it holds for $\text{G-dim}\, M < \infty$. Thus, we have obtained some extension of the class of modules for which the conditions of [12, Corollary 4] hold in advance.

REFERENCES

1. T. Araya, R. Takahashi, and Y. Yoshino, *Homological Invariants Associated to Semi-Dualizing Bimodules*, preprint (2002).
2. T. Araya and Y. Yoshino, "Remarks on a depth formula, a grade inequality and a conjecture of Auslander," *Comm. Algebra*, **26**, 3793–3806 (1998).
3. M. Auslander and M. Bridger, "Stable module theory," *Mem. Amer. Math. Soc.*, **94** (1969).
4. M. Auslander, S. Ding, and Ø. Solberg, "Liftings and weak liftings of modules," *J. Algebra*, **156**, 273–397 (1993).
5. L. L. Avramov, "Flat morphisms of complete intersections," *Dokl. Akad. Nauk SSSR*, **225**, 11–14 (1975).
6. L. L. Avramov, "Modules of finite virtual projective dimension," *Invent. Math.*, **96**, 71–101 (1989).
7. L. L. Avramov and H.-B. Foxby, "Homological dimensions of unbounded complexes," *J. Pure Appl. Algebra*, **71**, 129–155 (1991).
8. L. L. Avramov and H.-B. Foxby, "Ring homomorphisms and finite Gorenstein dimension," *Proc. London Math. Soc.*, **307**, 241–270 (1997).
9. L. L. Avramov, V. N. Gasharov, and I. V. Peeva, "Complete intersection dimension," *Publ. Math. I.H.E.S.*, **86**, 67–114 (1997).
10. L. L. Avramov, S. S. Strogalov, and A. N. Todorov, "Gorenstein modules," *Usp. Mat. Nauk*, **27**, No. 4, 199–200 (1972).
11. H. Bass, "On the ubiquity of Gorenstein rings," *Math. Z.*, **82**, 8–28 (1963).
12. S. Choi and S. Iyengar, "On a depth formula for modules over local rings," *Comm. Algebra*, **29**, 3135–3143 (2001).
13. L. W. Christensen, *Gorenstein Dimensions*, Lecture Notes Math., Vol. 1747, Springer, Berlin (2000).
14. L. W. Christensen, "Semi-dualizing complexes and their Auslander categories," *Trans. Amer. Math. Soc.*, **353**, No. 5, 1839–1883 (2001).
15. H.-B. Foxby, "Gorenstein modules and related modules," *Math. Scand.*, **31**, 267–284 (1973).
16. H.-B. Foxby, unpublished notes (1987).
17. S. I. Gelfand and Yu. I. Manin, *Methods of Homological Algebra* [in Russian], Nauka, Moscow (1988).
18. A. A. Gerko, "On homological dimensions," in: *International Algebraic Seminar Devoted to the 70th Anniversary of the Research Seminar of Moscow State Univ. on Algebra, Founded by O. Yu. Shmidt in 1903 (November 13–16, 2000). Abstracts*, CPI Publisher at the Dept. of Mechanics and Mathematics of MSU, Moscow (2000), p. 13.
19. A. A. Gerko, "On homological dimensions," *Mat. Sb.*, **192**, No. 8, 79–94 (2001).
20. A. A. Gerko, "On suitable modules and G-perfect ideals," *Usp. Mat. Nauk*, **56**, No. 4, 141–142 (2001).
21. A. A. Gerko, "On the structure of the set of semidualizing complexes," *Illinois J. Math.*, **48**, No. 3, 965–976 (2004).
22. A. A. Gerko, "On the structure of the set of semidualizing complexes," *Usp. Mat. Nauk*, **59**, No. 5, 145–146 (2004).
23. A. A. Gerko, "On the structure of the set of semidualizing complexes," in: *International Algebraic Conference Devoted to the 250th Anniversary of Moscow University. Abstracts*, Dept. of Mechanics and Mathematics of MSU, Moscow (2004), pp. 36–38.
24. E. S. Golod, "G-dimension and generalized perfect ideals," *Tr. Mat. Inst. Steklov*, **165**, 62–66 (1984).
25. E. S. Golod, "Question," *Usp. Mat. Nauk*, **40**, No. 1, 234 (1985).
26. T. H. Gulliksen, "A change of the ring theorem, with applications to Poincaré series and intersection multiplicity," *Math. Scand.*, **34**, 167–183 (1974).
27. T. H. Gulliksen, "A homological characterization of local complete intersections," *Compositio Math.*, **23**, 251–255 (1973).
28. R. Hartshorne, *Residues and Duality*, Lecture Notes Math., Vol. 12, Springer, Berlin (1971).

29. C. Huneke, L. Sega, and A. Vraciu, “Vanishing of Ext and Tor over some Cohen–Macaulay local rings,” *Illinois J. Math.*, **48**, No. 1, 295–317 (2004).
30. S. Iyengar, “Depth for complexes, and intersection theorems,” *Math. Z.*, **230**, 545–567 (1999).
31. T. Kawasaki, “On Macaulayfication of Noetherian schemes,” *Trans. Amer. Math. Soc.*, **352**, No. 6, 2517–2552 (2000).
32. V. Maşek, *Gorenstein Dimension of Modules*, arXiv:math.AC/9809121.
33. M. Nagata, *Local Rings*, Wiley, New York (1962).
34. C. Peskine and L. Szpiro, “Dimension projective finié et cohomologie locale,” *Publ. Math. I.H.E.S.*, **42**, 47–119 (1972).
35. S. Sather-Wagstaff, *Semidualizing Modules and the Divisor Class Group*, preprint (2004).
36. R. Y. Sharp, “Dualizing complexes for commutative Noetherian rings,” *Math. Proc. Cambridge Philos. Soc.*, **75**, 369–386 (1975).
37. N. Spaltenstein, “Resolutions of unbounded complexes,” *Compositio Math.*, **65**, 121–154 (1988).
38. J. Tate, “Homology of Noetherian rings and local rings,” *Illinois J. Math.*, **1**, 14–25 (1957).
39. O. Veliche, “Construction of modules with finite homological dimensions,” *J. Algebra*, **250**, 427–449 (2002).
40. Y. Yoshino, “Modules of G-dimension zero over local rings with the cube of maximal ideal being zero,” in: *Commutative Algebra, Singularities and Computer Algebra (Sinaia, 2002)*, NATO Sci. Ser., II: Math., Phys. and Chem., Vol. 115, Kluwer, Dordrecht (2003), pp. 255–273.

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