
PAC-Bayes Bounds for Multivariate Linear Regression and Linear Autoencoders

Ruixin Guo

Kent State University
rguo5@kent.edu

Ruoming Jin

Kent State University
rjin1@kent.edu

Xinyu Li

Kent State University
xli74@kent.edu

Yang Zhou

Auburn University
yangzhou@auburn.edu

Abstract

Linear Autoencoders (LAEs) have shown strong performance in state-of-the-art recommender systems. However, this success remains largely empirical, with limited theoretical understanding. In this paper, we investigate the generalizability – a theoretical measure of model performance in statistical learning – of multivariate linear regression and LAEs. We first propose a PAC-Bayes bound for multivariate linear regression, extending the earlier bound for single-output linear regression by Shalaeva et al. [45], and establish sufficient conditions for its convergence. We then show that LAEs, when evaluated under a relaxed mean squared error, can be interpreted as constrained multivariate linear regression models on bounded data, to which our bound adapts. Furthermore, we develop theoretical methods to improve the computational efficiency of optimizing the LAE bound, enabling its practical evaluation on large models and real-world datasets. Experimental results demonstrate that our bound is tight and correlates well with practical ranking metrics such as Recall@K and NDCG@K.

1 Introduction

In recent years, simple linear recommendation models have consistently demonstrated impressive performance, often rivaling deep learning models [12, 24, 35]. In particular, linear autoencoders (LAEs) such as EASE [48] and EDLAE [49] have shown a surprising edge over classical linear methods like ALS [23]. Despite their empirical success and widespread adoption, the theoretical understanding of why LAEs perform so well remains limited. Moreover, much of recommender system research has focused heavily on empirical comparisons, where weak baselines and unreliable sampled metrics often render evaluations biased and difficult to reproduce [12, 11]. A solid theoretical foundation is therefore urgently needed to explain and justify the true performance of recommendation models beyond purely empirical assessments.

Statistical learning theory [51] provides such a foundation by estimating a model’s theoretical performance over the underlying data distribution. A classic result in this area is the uniform convergence PAC bound established by Vapnik and Chervonenkis [52]. Although this bound guarantees convergence, it characterizes the *worst-case* generalization gap and is typically vacuous for large models such as neural networks, thus failing to reflect true model performance in practice [39, 13]. To address this limitation, a variant PAC framework incorporates information-theoretic techniques to bound the *expected* generalization gap, which often yields tighter results [20]. One notable example is the PAC-Bayes bound, first introduced by McAllester [37]. Recently, Dziugaite and Roy [13] empirically demonstrated that PAC-Bayes bounds can remain non-vacuous even for large neural networks, suggesting that PAC-Bayes theory provides a more accurate and practically meaningful characterization of model performance.

While statistical learning has been extensively developed for a wide range of machine learning and deep learning models [51, 6, 53], its application to recommendation systems remains largely

underexplored, with only a few exceptions [47, 46, 16]. To the best of our knowledge, it has not yet been directly applied to LAEs. In this work, we aim to advance the theoretical understanding of LAE performance through PAC-Bayes theory.

When evaluated under mean squared error, LAEs are closely related to multivariate linear regression models. Prior works have developed several PAC-Bayes bounds for linear regression. Notably, Alquier et al. [5] proposed a PAC-Bayes bound for general machine learning models with unbounded loss; Germain et al. [17] adapted this bound to linear regression models with mean squared loss, but their result does not converge; and Shalaeva et al. [45] improved Germain’s bound deriving a strictly tighter and convergent version. We aim to extend existing PAC-Bayes bounds for linear regression to the setting of LAEs. Several challenges arise in doing so:

Multivariate Data: Existing PAC-Bayes bounds for linear regression primarily focus on the single-output setting. Extending them to the multivariate (multi-output) case is nontrivial, as the outputs can exhibit statistical dependencies. To derive a valid bound in this setting, more general assumptions on the data distribution must first be established to capture potential output dependence.

LAE-specific Characteristics: LAEs differ from multivariate linear regression in several important aspects. They typically operate on bounded data, so the standard Gaussian data assumption used in linear regression does not apply. Moreover, LAEs impose unique structural constraints, such as the zero-diagonal constraint on the weight matrix and the hold-out constraint on data. These characteristics must be rigorously defined and integrated into the theoretical framework.

Computational Inefficiency: Optimizing PAC-Bayes bounds is typically computationally expensive, as estimating the distance between prior and posterior in high-dimensional spaces is complex [41]. Since LAE models often contain hundreds of millions of parameters and are computed on large real-world datasets (Table 2), developing computationally efficient methods is crucial for practical bound evaluation.

This paper addresses the aforementioned challenges and makes the following key contributions:

- (Section 3) We generalize Shalaeva’s Gaussian data assumption [45] to the multivariate setting (Assumption 3.1) and propose a corresponding PAC-Bayes bound for multivariate linear regression (Theorem 3.2), extending Shalaeva’s single-output bound [45]. We further establish sufficient conditions (Theorem 3.3) that guarantee convergence for both bounds.
- (Section 4) We propose a relaxed mean squared error for evaluating LAE models and show that, under this loss, LAEs can be viewed as constrained multivariate linear regression models on bounded data. Building on this, we adapt our PAC-Bayes bound to LAEs by replacing the Gaussian data assumption with a bounded one (Assumption 4.1) and incorporating LAE-specific constraints: the zero-diagonal constraint on weights and the hold-out constraint on data (Section 4.2).
- (Section 5) We develop theoretical methods to improve the computational efficiency of optimizing the bound. Following Dziugaite and Roy [13], we restrict the prior and posterior distributions to be Gaussian (Assumption 5.1), leading to a closed-form expression for the tightest bound (Theorem 5.2). We further establish a practical upper bound with reduced complexity to mitigate the computational cost introduced by the zero-diagonal constraint (Theorem 5.4).
- (Section 6) We evaluate the bound for LAEs on real-world datasets. Experimental results demonstrate that our bound is tight and correlates well with practical evaluation metrics such as Recall@K and NDCG@K, suggesting that it effectively reflects the actual model performance of LAEs.

All proofs of the theorems, lemmas and propositions in the main paper are provided in Appendix A. Related Works are in Appendix D. Conclusions and Discussions are in Appendix E.

2 Preliminaries

Notation: We denote $S = \{(x_i, y_i)\}_{i=1}^m$ as the dataset, where $x_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}^p$ for all i , and m is the number of samples. For each sample (x_i, y_i) , x_i is the input and y_i is the target. Let $f_W : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be the linear regression model parameterized by $W \in \mathbb{R}^{p \times n}$. The model prediction is $f_W(x_i) = Wx_i$, and the mismatch between target and prediction is measured by the squared Frobenius norm loss $\|y_i - Wx_i\|_F^2$.

Denote $X = [x_1, x_2, \dots, x_m] \in \mathbb{R}^{n \times m}$ as the input matrix and $Y = [y_1, y_2, \dots, y_m] \in \mathbb{R}^{p \times m}$ as the target matrix. The *empirical risk*, representing the average loss on the observed dataset S , is then defined as $R^{\text{emp}}(W) = \frac{1}{m} \sum_{i=1}^m \|y_i - Wx_i\|_F^2 = \frac{1}{m} \|Y - WX\|_F^2$. To evaluate performance on

unseen data, we assume that each (x_i, y_i) is i.i.d. sampled from an unknown distribution \mathcal{D} , and define the *true risk* as $R^{\text{true}}(W) = \mathbb{E}_{(x,y) \sim \mathcal{D}} [\|y - Wx\|_F^2]$.

To construct a PAC-Bayes bound, we treat f_W as a stochastic model by considering W a random variable. Denote π as the prior distribution over W and ρ as the posterior distribution. π represents our initial belief about W before observing any data, whereas ρ represents our updated belief after incorporating information from the dataset S [28].

Multivariate Linear Regression [25]: From the definition above, the linear regression equation can be written as $Y = WX + E$, where $E = [e_1, e_2, \dots, e_m] \in \mathbb{R}^{p \times m}$ is the error matrix.

Usually the first dimension of every x_i is set 1 (i.e., X_{1*} is a vector of all 1s) to represent the bias term. We say the linear regression is *multivariate* (or *multi-output*) if $p > 1$.

Multivariate linear regression typically assumes that the error vectors e_i and e_j are independent for $i \neq j$, but allows dependencies among the elements within each e_i . This leads to our statistical assumption stated in Assumption 3.1.

Alquier's Bound [5] adapted to Linear Regression: Alquier's bound is a general bound that can be applied to any model with unbounded loss. When adapted to the linear regression model using our notation, Alquier's bound can be stated as follows: Given π , for any $\lambda > 0$ and $\delta > 0$,

$$P \left(\forall \rho, \mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)] < \mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)] + \frac{1}{\lambda} \left[D(\rho \parallel \pi) + \ln \frac{1}{\delta} + \Psi_{\pi, \mathcal{D}}(\lambda, m) \right] \right) \geq 1 - \delta \quad (1)$$

where $\Psi_{\pi, \mathcal{D}}(\lambda, m) = \ln \mathbb{E}_{W \sim \pi} \mathbb{E}_{S \sim \mathcal{D}^m} [e^{\lambda(R^{\text{true}}(W) - R^{\text{emp}}(W))}]$, and $D(\rho \parallel \pi) = \mathbb{E}_{W \sim \rho} \left[\ln \frac{\rho(dW)}{\pi(dW)} \right]$ denotes the Kullback-Leibler (KL) Divergence where the measure ρ is absolutely continuous with respect to π . The loss is typically assumed following a light-tailed distribution, such as sub-Gaussian or sub-exponential, to ensure that $\Psi_{\pi, \mathcal{D}}(\lambda, m)$ is bounded [17, 18, 19].

Note that Alquier's bound holds simultaneously for all posteriors ρ . If replacing $\forall \rho$ with any single ρ such as a Gaussian posterior or a Gibbs posterior, the bound still holds.

Shalaeva's Bound [45]: Shalaeva's bound is an application of Alquier's bound to the *single-output* linear regression model, i.e., the case $p = 1$. It further assumes \mathcal{D} is Gaussian: Given constants σ_x and σ_e , for any draw $(x, y) \sim \mathcal{D}$, 1. $x \sim \mathcal{N}(0, \sigma_x^2 I)$, and 2. there exists $W^* \in \mathbb{R}^{1 \times n}$ such that $y = W^*x + e$, where $e \sim \mathcal{N}(0, \sigma_e^2)$ is Gaussian noise. Under this assumption, \mathcal{D} is fixed in $\Psi_{\pi, \mathcal{D}}(\lambda, m)$, while π remains unspecified. Germain et al. [17] showed that, if π is Gaussian, $\Psi_{\pi, \mathcal{D}}(\lambda, m)$ is bounded, since the true risk $R^{\text{true}}(W)$ with $W \sim \pi$ is sub-gamma; however, their bound is independent of m and does not guarantee convergence. Shalaeva et al. [45] improve this by showing that $\Psi_{\pi, \mathcal{D}}(\lambda, m)$ in fact has a strictly tighter upper bound: For any π ,

$$\Psi_{\pi, \mathcal{D}}(\lambda, m) = \ln \mathbb{E}_{W \sim \pi} \frac{\exp(\lambda v_W)}{(1 + \frac{\lambda v_W}{m/2})^{m/2}} \leq \ln \mathbb{E}_{W \sim \pi} \exp \left(\frac{2\lambda^2 v_W^2}{m} \right) \quad (2)$$

where $v_W = \sigma_x^2 \|W - W^*\|_2^2 + \sigma_e^2$. This bound depends on m and can be used to establish convergence.

Convergence of Shalaeva's Bound: The convergence analysis in Shalaeva et al.'s paper [45] is presented informally. Here we formally state their results as follows: Since $\lim_{m \rightarrow \infty} (1 + \frac{\lambda v_W}{m/2})^{m/2} = \exp(\lambda v_W)$, for any $\lambda > 0$, the convergence of $\Psi_{\pi, \mathcal{D}}(\lambda, m)$ follows from

$$\lim_{m \rightarrow \infty} \Psi_{\pi, \mathcal{D}}(\lambda, m) = \lim_{m \rightarrow \infty} \ln \mathbb{E}_{W \sim \pi} \frac{\exp(\lambda v_W)}{(1 + \frac{\lambda v_W}{m/2})^{m/2}} = \ln \mathbb{E}_{W \sim \pi} \lim_{m \rightarrow \infty} \frac{\exp(\lambda v_W)}{(1 + \frac{\lambda v_W}{m/2})^{m/2}} = 0 \quad (3)$$

Upon careful examination of their analysis, we found that additional conditions are required to guarantee (3), which were not discussed in their original paper. Specifically, swapping \lim and \mathbb{E} is valid only under certain conditions. For example, by the *dominated convergence theorem* [44, 15], the condition can be $\mathbb{E}_{W \sim \pi} [\exp(\lambda v_W)] < \infty$. If the choice of (λ, π) does not satisfy this condition, convergence is not guaranteed. These issues are discussed in Section 3.2 and Appendix B.

Collaborative Filtering Recommenders for Implicit Feedback: In collaborative filtering, an implicit feedback dataset is typically represented as a binary user-item interaction matrix $H \in \{0, 1\}^{n \times m}$, with n items and m users (Section 1.3.1.1, [1]). Each H_{ij} denotes an interaction: $H_{ij} = 1$ means that user j has interacted with item i , while $H_{ij} = 0$ means no observed interaction.

Suppose H is a test set. To evaluate a model, we typically *hold out* a fraction $1 - p$ ($p \in (0, 1)$) of 1s in H (Section 7.4.2, [1]; also [32, 48, 38]). Formally, we use a binary mask matrix $\Delta \in \{0, 1\}^{n \times m}$ to perform the hold-out operation. Δ denotes a random matrix, where each Δ_{ij} is independently drawn from a Bernoulli distribution conditioned on H_{ij} : $P(\Delta_{ij} = 1 | H_{ij} = 1) = p$, $P(\Delta_{ij} = 0 | H_{ij} = 1) = 1 - p$ and $P(\Delta_{ij} = 0 | H_{ij} = 0) = 1$.

Let Δ be a realization of Δ . Define the input matrix as $H^{\text{input}} = \Delta \odot H$ and the target matrix as $H^{\text{target}} = (\mathbf{1} - \Delta) \odot H$, where \odot denotes the Hadamard (element-wise) product and $\mathbf{1} \in \{1\}^{n \times m}$ ¹. For any i, j such that $H_{ij} = 1$, we say H_{ij} is held out if $\Delta_{ij} = 0$, which yields $H_{ij}^{\text{input}} = 0$ and $H_{ij}^{\text{target}} = 1$. Consequently, H^{input} retains a p fraction of the 1s in H , while the remaining $1 - p$ fraction are held-out and moved to H^{target} .

A collaborative filtering model takes H^{input} as input and generates a prediction matrix $H^{\text{pred}} \in \mathbb{R}^{n \times m}$. Model performance is typically evaluated by how well $(\mathbf{1} - \Delta) \odot H^{\text{pred}}$ approximates H^{target} . The masked prediction $(\mathbf{1} - \Delta) \odot H^{\text{pred}}$ indicates that only the entries in H^{pred} that coincide with the held-out interactions contribute to the evaluation. This approximation quality is typically measured using ranking-based metrics such as Recall@K or NDCG@K (Section 7.5.3, 7.5.4, [1]).

LAE Models and EASE [48]: LAE models are a class of collaborative filtering models. They are typically represented by a square matrix $W \in \mathbb{R}^{n \times n}$ and trained by solving $\arg \min_W \|H - WH\|$, where H denotes the training set. The model takes H as input and generates a prediction WH , which aims to reconstruct H itself. W is commonly constrained by a zero diagonal (i.e., $\text{diag}(W) = 0$ ²) to prevent overfitting towards the identity matrix I [48, 49, 50]. Some studies relax this constraint by allowing a diagonal with bounded norm instead [38].

EASE is one of the most popular LAE models, obtained by solving

$$\arg \min_W \|H - WH\|_F^2 + \gamma \|W\|_F^2 \quad \text{s.t. } \text{diag}(W) = 0 \quad (4)$$

where γ is the regularization parameter. Let W_0 be the solution of (4), then W_0 has a closed form: Let $P = (HH^T + \gamma I)^{-1}$, then $(W_0)_{ij} = 0$ if $i = j$ and $(W_0)_{ji} = -P_{ij}/P_{jj}$ if $i \neq j$.

3 PAC-Bayes Bound for Multivariate Linear Regression

3.1 The Statistical Assumption and the Bound

We first generalize Shalaeva et al.'s Gaussian data assumption [45] to the multivariate data with dependent outputs and potentially degenerate covariance. Based on this assumption, we derive our bound and show that Shalaeva's bound is a special case of ours.

Assumption 3.1. Let $\mu_x \in \mathbb{R}^n$, $\Sigma_x \in \mathbb{R}^{n \times n}$ be positive semi-definite, and $\Sigma_e \in \mathbb{R}^{p \times p}$ be positive-definite. Suppose $(x, y) \sim \mathcal{D}$ satisfies: 1. $x \sim \mathcal{N}(\mu_x, \Sigma_x)$; 2. there exists $W^* \in \mathbb{R}^{p \times n}$ such that $y = W^*x + e$, where $e \sim \mathcal{N}(0, \Sigma_e)$; in other words, $y|x \sim \mathcal{N}(W^*x, \Sigma_e)$.

The positive semi-definite assumption of Σ_x allows it to be singular, implying a degenerate Gaussian distribution whose support lies on a lower-dimensional manifold embedded in \mathbb{R}^n . This includes the standard multivariate linear regression setting in which the first element of x is 1 and the remaining $n - 1$ elements are Gaussian. In this case, the first row and first column of Σ_x are 0.

Under Assumption 3.1, for any model $W \in \mathbb{R}^{p \times n}$, the prediction error $y - Wx = (W^* - W)x + e$ follows the Gaussian distribution $\mathcal{N}(\mu_w, \Sigma_w)$, where

$$\mu_w = \mathbb{E}[(W^* - W)x + e] = (W^* - W)\mathbb{E}[x] + \mathbb{E}[e] = (W^* - W)\mu_x$$

$$\Sigma_w = \mathbb{E}[(W^* - W)(x - \mu_x) + e][(W^* - W)(x - \mu_x) + e]^T = (W^* - W)\Sigma_x(W^* - W)^T + \Sigma_e$$

Note that Σ_w is positive definite due to the positive definiteness of Σ_e . Let $\Sigma_w = S^T \Lambda S$ be its eigenvalue decomposition where S is orthogonal and $\Lambda = \text{diag}(\eta_1, \eta_2, \dots, \eta_p)$ with $\eta_i > 0$ for all i . Both S and Λ depend on W . The **PAC-Bayes bound for multivariate linear regression** is then stated as follows:

¹The same symbol $\mathbf{1}$ will be used elsewhere in this paper to represent all-ones matrices of different sizes.

²The notation diag is defined as follows: If $W \in \mathbb{R}^{n \times n}$, then $\text{diag}(W) \in \mathbb{R}^n$ denotes the vector consisting of the diagonal elements of W . If $w \in \mathbb{R}^n$, then $\text{diag}(w) \in \mathbb{R}^{n \times n}$ denotes the diagonal matrix whose diagonal entries are the elements of w .

Theorem 3.2. Denote $b = S\Sigma_w^{-1/2}\mu_w$. Given π , for any $\lambda > 0$ and $\delta > 0$,

$$P \left(\forall \rho, \mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)] < \mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)] + \frac{1}{\lambda} \left[D(\rho \parallel \pi) + \ln \frac{1}{\delta} + \Psi_{\pi, \mathcal{D}}(\lambda, m) \right] \right) \geq 1 - \delta \quad (5)$$

where

$$\Psi_{\pi, \mathcal{D}}(\lambda, m) = \ln \mathbb{E}_{W \sim \pi} \left[\exp \left(\lambda \left(\text{tr}(\Sigma_w) + \mu_w^T \mu_w \right) \right) \frac{\exp \left(\sum_{i=1}^p \frac{-\lambda m b_i^2 \eta_i}{m + 2\lambda \eta_i} \right)}{\prod_{i=1}^p (1 + 2\lambda \eta_i/m)^{m/2}} \right] \leq \ln \mathbb{E}_{W \sim \pi} \exp \left(\frac{2\lambda^2 \|\Sigma_w\|_F^2}{m} \right)$$

The bound of Theorem 3.2 is a general case of Shalaeva's bound. It can be reduced to Shalaeva's bound by taking $p = 1$, $\mu_x = 0$, $\Sigma_x = \sigma_x^2 I$ and $\Sigma_e = \sigma_e^2$ for some constants σ_x, σ_e .

3.2 Convergence Analysis

This section presents the convergence analysis of Theorem 3.2. We provide a sufficient condition based on the *dominated convergence theorem* that guarantees convergence, thereby completing and rigorously formalizing the convergence analysis of Shalaeva's bound [45]. This condition is stated as follows:

Theorem 3.3. If λ and π satisfies $\mathbb{E}_{W \sim \pi} [\exp (\lambda \|(\Sigma_x + \mu_x \mu_x^T)^{1/2}(W^* - W)\|_F^2)] < \infty$, then $\lim_{m \rightarrow \infty} \Psi_{\pi, \mathcal{D}}(\lambda, m) = 0$.

Here are some examples of the combinations (λ, π) that satisfy the condition of Theorem 3.3:

Example 3.4. Let π be a distribution with bounded support, then for any $\lambda > 0$, the condition holds, because there exists a constant $G > 0$ with $\|W\|_F < G$ such that

$$\begin{aligned} \mathbb{E}_{W \sim \pi} \left[\exp \left(\lambda \|(\Sigma_x + \mu_x \mu_x^T)^{1/2}(W^* - W)\|_F^2 \right) \right] &\leq \mathbb{E}_{W \sim \pi} \left[\exp \left(\lambda \|(\Sigma_x + \mu_x \mu_x^T)^{1/2}\|_F^2 \|W^* - W\|_F^2 \right) \right] \\ &\leq \mathbb{E}_{W \sim \pi} \left[\exp \left(\lambda \|(\Sigma_x + \mu_x \mu_x^T)^{1/2}\|_F^2 (\|W^*\|_F + \|W\|_F)^2 \right) \right] < \exp \left(\lambda \|(\Sigma_x + \mu_x \mu_x^T)^{1/2}\|_F^2 (\|W^*\|_F + G)^2 \right) < \infty \end{aligned}$$

Example 3.5. Let π be a Gaussian distribution parameterized by $\mathcal{U}_0 \in \mathbb{R}^{n \times n}$ and $\sigma > 0$, such that each W_{ij} is independently drawn from $\mathcal{N}((\mathcal{U}_0)_{ij}, \sigma^2)$. Let $\Sigma_x + \mu_x \mu_x^T = Q^T \Lambda Q$ be its eigenvalue decomposition, where $\Lambda = \text{diag}(\nu_1, \nu_2, \dots, \nu_n)$ and ν_1 is the largest eigenvalue, then

$$\mathbb{E}_{W \sim \pi} \left[\exp \left(\lambda \|(\Sigma_x + \mu_x \mu_x^T)^{1/2}(W^* - W)\|_F^2 \right) \right] = \prod_{i=1}^p \prod_{j=1}^p \frac{\exp \left(\frac{\lambda \nu_j (Q_{j*}(W^* - \mathcal{U}_0)_{*i})^2}{1 - 2\lambda \sigma^2 \nu_j} \right)}{(1 - 2\lambda \sigma^2 \nu_j)^{1/2}}$$

In this case, the sufficient condition holds for any $\lambda \in (0, \frac{1}{2\nu_1 \sigma^2})$.

Applying Theorem 3.3 to Shalaeva's bound, then (3) is guaranteed if λ and π satisfies $\mathbb{E}_{W \sim \pi} [\exp (\lambda \sigma_x^2 \|W^* - W\|_2^2)] < \infty$. Moreover, since

$$\mathbb{E}_{W \sim \pi} [\exp (\lambda \sigma_x^2 \|W^* - W\|_2^2)] < \mathbb{E}_{W \sim \pi} [\exp (\lambda \sigma_x^2 \|W^* - W\|_2^2 + \lambda \sigma_e^2)] = \mathbb{E}_{W \sim \pi} [\exp (\lambda v_w)]$$

a sufficient condition is therefore $\mathbb{E}_{W \sim \pi} [\exp (\lambda v_w)] < \infty$.

4 PAC-Bayes Bound for LAEs

This section presents a PAC-Bayes bound for LAEs. The model W can be obtained using any training method, such as EASE, EDLAE or ELSA; our bound only focuses on analyzing its test performance and is independent of the training procedure.

4.1 Adapting to Bounded Data Assumption

Most real-world recommendation datasets are bounded rather than Gaussian. For example, the user-item interaction matrix H introduced in Section 2 is binary. If we assume that each user vector H_{*i} is i.i.d. sampled from an underlying distribution, this distribution must have bounded support. Therefore, to adapt the PAC-Bayes bound for multivariate linear regression to recommendation datasets, we replace the Gaussian assumption on \mathcal{D} (Assumption 3.1) with the following bounded-support assumption:

Assumption 4.1. Suppose \mathcal{D} is characterized by three finite cross-correlation matrices $\Sigma_{xx} = \mathbb{E}_{(x,y) \sim \mathcal{D}}[xx^T]$, $\Sigma_{xy} = \mathbb{E}_{(x,y) \sim \mathcal{D}}[xy^T]$ and $\Sigma_{yy} = \mathbb{E}_{(x,y) \sim \mathcal{D}}[yy^T]$, where Σ_{xx} is positive definite.

This assumption is indeed general. It holds for all \mathcal{D} with bounded support, and also holds for certain \mathcal{D} with unbounded support such as Gaussian, since Assumption 3.1 implies Assumption 4.1. Consequently, it enables the derivation of a more general bound, although in this work we focus on its application to bounded-data settings.

We first note that the true risk under Assumption 4.1 can be expressed in an explicit form:

Lemma 4.2. *Given any W , the true risk can be expressed as*

$$R^{\text{true}}(W) = \|W\Sigma_{xx}^{1/2} - \Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2 - \|\Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2 + \text{tr}(\Sigma_{yy}) \quad (6)$$

Then, in (1), we have the upper bound $\Psi_{\pi, \mathcal{D}}(\lambda, m) \leq \ln \mathbb{E}_\pi \left[e^{\lambda R^{\text{true}}(W)} \right]$, which is obtained by removing $-R^{\text{emp}}(W)$ due to its non-positivity. This upper bound, originally used by Germain et al. (Appendix A.4, [17]), does not ensure convergence since it is independent of m , but it simplifies computation. By plugging in (6), we get

Proposition 4.3. *Denote $B = -\Sigma_{xy}^T \Sigma_{xx}^{-1/2}$ and $C = e^{\lambda(\text{tr}(\Sigma_{yy}) - \|\Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2)}$. Then (1) holds, with $\Psi_{\pi, \mathcal{D}}(\lambda, m)$ upper-bounded by*

$$\Psi_{\pi, \mathcal{D}}(\lambda, m) \leq \ln \mathbb{E}_\pi \left[e^{\lambda R^{\text{true}}(W)} \right] = \ln C \mathbb{E}_\pi \left[e^{\lambda \|W\Sigma_{xx}^{1/2} + B\|_F^2} \right] \quad (7)$$

4.2 Applying the PAC-Bayes Bound to LAEs

Recall from Section 2 that the LAE model is defined by a squared matrix $W \in \mathbb{R}^{n \times n}$, and evaluated by comparing the closeness between the masked prediction $(\mathbf{1} - \Delta) \odot (W(\Delta \odot H))$ and the target $(\mathbf{1} - \Delta) \odot H$. This closeness is typically measured by ranking-based metrics such as Recall@K and NDCG@K, which are discrete and difficult to analyze statistically. To simplify the analysis, we use *mean squared error* (MSE; see Section 7.5.1, [1]) instead. The classic MSE is defined as the mean squared Frobenius norm of all held-out interactions:

$$\frac{1}{m} \|(\mathbf{1} - \Delta) \odot H - (\mathbf{1} - \Delta) \odot (W(\Delta \odot H))\|_F^2$$

where the mask $\mathbf{1} - \Delta$ on prediction distinguishes it from a multivariate linear regression. We relax the MSE by removing this mask, allowing all predicted interactions to participate in the evaluation rather than only the held-out ones:

$$\frac{1}{m} \|(\mathbf{1} - \Delta) \odot H - W(\Delta \odot H)\|_F^2 \quad (8)$$

Let $X = \Delta \odot H$ be the input and $Y = (\mathbf{1} - \Delta) \odot H$ be the target. The 1s in X_{*j} represent items observed by user j , while the 1s in Y_{*j} represent items that are hidden but potentially of interest to the user. The classic MSE only evaluates on held-out items, i.e., those with $(X_{ij} = 0, Y_{ij} = 1)$. In contrast, our relaxed MSE (8) also accounts for items with $(X_{ij} = 0, Y_{ij} = 0)$, indicating that items unobserved and unlikely to interest the user should not be recommended; and those with $(X_{ij} = 1, Y_{ij} = 0)$, indicating that the model should avoid recommending items already observed by the user (see Section 7.3.4, [1]). Consequently, (8) can be viewed as a special case of the empirical risk of linear regression $R^{\text{emp}}(W) = \frac{1}{m} \|Y - WX\|_F^2$, with output dimension $p = n$, and under the following **LAE-specific constraints**:

1. Hold-out constraint on X and Y : For any i, j , X_{ij} and Y_{ij} are either 0 or 1, but cannot both be 1.
2. Zero-diagonal constraint on W : $\text{diag}(W) = 0$ (Optional).

Since both X and Y are derived from H and Δ , the true risk can be defined by introducing statistical assumptions on H and Δ respectively. For H , we assume that each user vector H_{*j} is i.i.d. sampled from a multivariate Bernoulli distribution \mathcal{M} , and denote $\Sigma_{hh} = \mathbb{E}_{h \sim \mathcal{M}}[hh^T]$ as the cross-correlation matrix of \mathcal{M} . For Δ , denote \mathcal{B} as the distribution from which each column Δ_{*j} is independently drawn. Note that \mathcal{B} depends on \mathcal{M} , and this dependence encodes the hold-out mechanism: for $\delta \sim \mathcal{B}$ and $h \sim \mathcal{M}$, $P(\delta_i = 1 | h_i = 1) = p$, $P(\delta_i = 0 | h_i = 1) = 1 - p$ and $P(\delta_i = 0 | h_i = 0) = 1$. In the true risk $R^{\text{true}}(W) = \mathbb{E}_{(x,y) \sim \mathcal{D}} [\|y - Wx\|_F^2]$, by plugging in $x = \delta \odot h$, $y = (\mathbf{1} - \delta) \odot h$, we get

$$R^{\text{true}}(W) = \mathbb{E}_{\delta \sim \mathcal{B}, h \sim \mathcal{M}} [\|\mathbf{1} - \delta \odot h - W(\delta \odot h)\|_F^2] \quad (9)$$

By Lemma 4.2, we further obtain the following result:

Lemma 4.4. (9) can be written in the same form as (6) by plugging in

$$\Sigma_{xx} = p^2 \Sigma_{hh} + p(1-p)(I \odot \Sigma_{hh}), \quad \Sigma_{yy} = (1-p)^2 \Sigma_{hh} + p(1-p)(I \odot \Sigma_{hh}), \quad \Sigma_{xy} = p(1-p)(\Sigma_{hh} - I \odot \Sigma_{hh})$$

Furthermore, Σ_{xx} is positive definite if Σ_{hh} is positive definite.

If W is subject to a zero-diagonal constraint, such as models trained from EASE, EDLAE or ELSA, then an additional condition $\text{diag}(W) = 0$ is applied to both the empirical risk (8) and the true risk (9). The **PAC-Bayes bound for LAEs** is formed by plugging (8) and (9) into (1). Since Assumption 4.1 holds for this bound, it directly leads to (7).

5 Practical Methods for Computing the PAC-Bayes Bound for LAEs

In the PAC-Bayes bound for LAEs proposed in Section 4, the choice of π and ρ is so far unspecified, and the computation of the bound has not yet been addressed. This section develops theoretical methods to improve the computational efficiency, enabling the bound to be evaluated on large models and datasets.

Our goal is to optimize the right hand side of (1): Given δ , find π, ρ, λ that minimize

$$\underbrace{\mathbb{E}_{W \sim \rho}[R^{\text{emp}}(W)] + \frac{1}{\lambda} D(\rho \parallel \pi)}_{\text{part 1}} + \underbrace{\frac{1}{\lambda} \ln \frac{1}{\delta} + \frac{1}{\lambda} \Psi_{\pi, \mathcal{D}}(\lambda, m)}_{\text{part 2}} \quad (10)$$

with $R^{\text{emp}}(W)$ given by (8) and $R^{\text{true}}(W)$ given by (9). However, solving for λ, π, ρ simultaneously is generally intractable [4]. We therefore consider a weaker problem: Given λ and π , we optimize (10) with respect to ρ . We compute the optimal ρ for different choices of λ and π , and select the combination yielding the tightest bound.

We discuss the computation of part 1 of (10) in Section 5.1 and part 2 in Section 5.2.

5.1 Closed-form Solution for the Optimal ρ under Gaussian Constraint

Given π and λ , we search for the optimal ρ by

$$\arg \min_{\rho} \mathbb{E}_{W \sim \rho}[R^{\text{emp}}(W)] + \frac{1}{\lambda} D(\rho \parallel \pi) \quad (11)$$

Not all choices of π and ρ make (11) easy to solve. For example, given any π , the optimal ρ is the Gibbs posterior, defined as $\rho(dW) = \frac{e^{-\lambda R^{\text{emp}}(W)} \pi(dW)}{\mathbb{E}_{\pi}[e^{-\lambda R^{\text{emp}}(W)}]}$ [4]. However, the Gibbs posterior is generally a complex distribution without a closed-form density function or parameterization, making it analytically intractable in practice.

To obtain a tractable and efficient solution, we instead optimize (11) under the constraint that π, ρ are restricted to specific distribution families. Notably, Dziugaite and Roy [13] proposed a practical way to compute PAC-Bayes bounds for deep neural networks by assuming π and ρ to be entry-wise Gaussian distributions, which allows the KL-Divergence $D(\rho \parallel \pi)$ to be computed analytically. We follow Dziugaite and Roy's assumption [13] and formally state it as follows:

Assumption 5.1. Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n \times n}$ with $\mathcal{B} \geq 0$ (entry-wise non-negative), and denote $\bar{\mathcal{N}}(\mathcal{A}, \mathcal{B})$ as the entry-wise Gaussian distribution such that $W \sim \bar{\mathcal{N}}(\mathcal{A}, \mathcal{B})$ means each W_{ij} is independently drawn from $\mathcal{N}(\mathcal{A}_{ij}, \mathcal{B}_{ij})$. Assume $\rho = \bar{\mathcal{N}}(\mathcal{U}, \mathcal{S})$ and $\pi = \bar{\mathcal{N}}(\mathcal{U}_0, \sigma^2 \mathcal{J})$, where $\mathcal{U}, \mathcal{U}_0, \mathcal{S} \in \mathbb{R}^{n \times n}$ with $\mathcal{S} > 0$ (entry-wise positive), $\mathcal{J} \in \{1\}^{n \times n}$ is the all-ones matrix, and $\sigma > 0$.

Applying the constraint $\text{diag}(W) = 0$ to ρ and π implies setting $\text{diag}(\mathcal{U}) = 0, \text{diag}(\mathcal{S}) = 0, \text{diag}(\mathcal{U}_0) = 0$ and $\text{diag}(\sigma^2 \mathcal{J}) = 0$, since a deterministically zero random variable has zero mean and zero variance.

Since neural network models are typically non-linear and do not admit closed-form solutions for the optimal ρ , Dziugaite and Roy [13] solved for the optimal ρ using stochastic gradient descent

with gradients estimated via Monte Carlo sampling, which requires a trade-off between sample size and computational cost. Surprisingly, due to the linearity of to LAE models, we find that their assumption adapted to LAEs admits a closed-form solution for the optimal ρ , as shown in Theorem 5.2. This closed-form solution enables efficient, direct computation of ρ , avoiding sampling, iteration or trade-off procedures.

Theorem 5.2. (a) Under Assumption 5.1, the closed-form solution of the optimal ρ of (11) is given by

$$\mathcal{U} = \left(\frac{1}{m} Y X^T + \frac{1}{2\lambda\sigma^2} \mathcal{U}_0 \right) \left(\frac{1}{m} X X^T + \frac{1}{2\lambda\sigma^2} I \right)^{-1}, \quad \mathcal{S}_{ij} = \frac{1}{\frac{2\lambda}{m} X_{j*} X_{j*}^T + \frac{1}{\sigma^2}} \quad \text{for } i, j \in \{1, 2, \dots, n\}$$

(b) If we add the constraint $\text{diag}(W) = 0$ to both ρ and π , then the optimal ρ becomes

$$\begin{aligned} \mathcal{S}_{ij} &= \frac{1}{\frac{2\lambda}{m} X_{j*} X_{j*}^T + \frac{1}{\sigma^2}}, \quad \mathcal{S}_{ii} = 0 \quad \text{for } i, j \in \{1, 2, \dots, n\} \text{ and } i \neq j \\ \mathcal{U} &= \left(\frac{1}{m} Y X^T + \frac{1}{2\lambda\sigma^2} \mathcal{U}_0 - \frac{1}{2} \text{diag}(x) \right) \left(\frac{1}{m} X X^T + \frac{1}{2\lambda\sigma^2} I \right)^{-1} \end{aligned}$$

where

$$x = 2 \cdot \text{diag} \left[\left(\frac{1}{m} Y X^T + \frac{1}{2\lambda\sigma^2} \mathcal{U}_0 \right) \left(\frac{1}{m} X X^T + \frac{1}{2\lambda\sigma^2} I \right)^{-1} \right] \oslash \text{diag} \left[\left(\frac{1}{m} X X^T + \frac{1}{2\lambda\sigma^2} I \right)^{-1} \right]$$

and \oslash denotes element-wise division.

5.2 Reducing the Complexity of $\Psi_{\pi, \mathcal{D}}(\lambda, m)$ under the Zero-Diagonal Constraint

Under Assumptions 4.1 and 5.1, the closed-form expression of $\Psi_{\pi, \mathcal{D}}(\lambda, m)$ is too complex to derive explicitly, making direct computation infeasible. We therefore compute the upper bound given in (7) instead. This section shows that, under the zero-diagonal constraint, this computation has a high complexity of $O(n^4)$, which we reduce to $O(n^3)$ by establishing a simpler upper bound.

We first consider the case without the constraint $\text{diag}(W) = 0$. Since we assume $\pi = \mathcal{N}(\mathcal{U}_0, \sigma^2 J)$ in Assumption 5.1, $W_{i*}^T \sim \mathcal{N}((\mathcal{U}_0)_{i*}^T, \sigma^2 I)$, thus $(W_{i*} \Sigma_{xx}^{1/2} + B_{i*})^T = \Sigma_{xx}^{1/2} W_{i*}^T + B_{i*}^T \sim \mathcal{N}(\Sigma_{xx}^{1/2} (\mathcal{U}_0)_{i*}^T + B_{i*}^T, \sigma^2 \Sigma_{xx})$. In this case, the $\mathbb{E}_\pi [e^{\lambda R^{\text{true}}(W)}]$ term in (7) can be further expressed as follows:

Proposition 5.3. Let $A = \sigma^2 \Sigma_{xx}$, and $A = S^T \Lambda S$ be the eigenvalue decomposition where S is orthogonal and $\Lambda = \text{diag}(\eta_1, \eta_2, \dots, \eta_n)$ with $\eta_1 \geq \dots \geq \eta_n \geq 0$ ³. Denote $\mu^i = \Sigma_{xx}^{1/2} (\mathcal{U}_0)_{i*}^T + B_{i*}^T$.

$$\mathbb{E}_\pi [e^{\lambda R^{\text{true}}(W)}] = C \prod_{i=1}^n \prod_{j=1}^n \frac{\exp \left(\frac{\lambda(\bar{b}_j^i)^2 \eta_j}{1-2\lambda\eta_j} \right)}{(1-2\lambda\eta_j)^{1/2}}, \quad \text{where } \bar{b}^i = S A^{-1/2} \mu^i \quad (12)$$

The computational complexity of (12) is $O(n^3)$, mainly due to the eigenvalue decomposition of A .

Now we discuss the case that $\text{diag}(W) = 0$ is applied. Denote π' as the distribution π with the constraint $\text{diag}(W) = 0$, that is, for $W \sim \pi'$, $W_{ii} = 0$ for all i . Then $\pi' = \mathcal{N}(\mathcal{U}_0, \sigma^2 (J - I))$ where $\text{diag}(\mathcal{U}_0) = 0$, and $W_{i*}^T \sim \mathcal{N}((\mathcal{U}_0)_{i*}^T, \sigma^2 (I - I^T))$ where I^i is a matrix with $I_{ii}^i = 1$ and other entries being 0. Therefore, $(W_{i*} \Sigma_{xx}^{1/2} + B_{i*})^T \sim \mathcal{N}(\Sigma_{xx}^{1/2} (\mathcal{U}_0)_{i*}^T + B_{i*}^T, \sigma^2 (\Sigma_{xx} - (\Sigma_{xx}^{1/2})_{*i} (\Sigma_{xx}^{1/2})_{*i}^T))$.

Denote $A^{(i)} = \sigma^2 (\Sigma_{xx} - (\Sigma_{xx}^{1/2})_{*i} (\Sigma_{xx}^{1/2})_{*i}^T)$, then $A^{(i)}$ is singular and positive semi-definite. Let $A^{(i)} = S^{(i)T} \Lambda^{(i)} S^{(i)}$ be the eigenvalue decomposition where $S^{(i)}$ is orthogonal and $\Lambda^{(i)} = \text{diag}(\eta_1^{(i)}, \eta_2^{(i)}, \dots, \eta_n^{(i)})$ with $\eta_1^{(i)} \geq \dots \geq \eta_n^{(i)} \geq 0$. Then

$$\mathbb{E}_{\pi'} [e^{\lambda R^{\text{true}}(W)}] = C \prod_{i=1}^n \prod_{j=1}^n \frac{\exp \left(\frac{\lambda(b_j^{(i)})^2 \eta_j^{(i)}}{1-2\lambda\eta_j^{(i)}} \right)}{(1-2\lambda\eta_j^{(i)})^{1/2}}, \quad \text{where } b^{(i)} = S^{(i)} (A^{(i)})^{-1/2} \mu^i \quad (13)$$

³We slightly abuse the notation of S and Λ . In Theorem 3.2, they denote the decomposition of Σ_W , whereas here they denote the decomposition of A .

The issue with (13) is its high computational complexity: We need to compute the eigenvalue decomposition for each $A^{(i)}$ in order to obtain $S^{(i)}$ and $\Lambda^{(i)}$. Since each eigenvalue decomposition costs $O(n^3)$, the computation of (13) costs $O(n^4)$, which is impractical for large n .

Since the direct computation of $\mathbb{E}_{\pi'} \left[e^{\lambda R^{\text{true}}(W)} \right]$ is difficult, we instead compute an upper bound, as established by the following theorem:

Theorem 5.4. *Given (12) and (13), for any $\lambda \in \left(0, \frac{1}{2\eta_1}\right)$ where η_1 is the largest eigenvalue of A ,*

$$\mathbb{E}_{\pi'} \left[e^{\lambda R^{\text{true}}(W)} \right] \leq \mathbb{E}_{\pi} \left[e^{\lambda R^{\text{true}}(W)} \right]$$

Theorem 5.4 holds for any \mathcal{U}_0 , including the special case where $\text{diag}(\mathcal{U}_0) = 0$ for both π' and π . This theorem allows us to compute $\mathbb{E}_{\pi} \left[e^{\lambda R^{\text{true}}(W)} \right]$ instead of $\mathbb{E}_{\pi'} \left[e^{\lambda R^{\text{true}}(W)} \right]$, thereby reducing the complexity from $O(n^4)$ to $O(n^3)$.

5.3 The Final Bound and the Algorithm for its Computation

The final step in computing the bound is to select λ that yields the tightest bound. Following [4], we search over a finite grid $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_L\}$ where L is the number of elements in Λ and $\lambda_i > 0$ for $i \in \{1, 2, \dots, L\}$; details are provided in Appendix C. Applying this grid search to (1), we obtain the final bound: Given π , for any $\lambda \in \Lambda$ and $\delta > 0$, with probability at least $1 - \delta$, the following bound holds for any ρ :

$$\mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)] \leq \mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)] + \frac{1}{\lambda} \left[D(\rho \parallel \pi) + \ln \frac{L}{\delta} + \ln \mathbb{E}_{\pi} \left[e^{\lambda R^{\text{true}}(W)} \right] \right] \quad (14)$$

We now summarize computation of (14) under the LAE setting in Algorithm 1. By default, the algorithm assumes that the zero-diagonal constraint on W is applied. For unconstrained W , the algorithm can be adapted by switching the solution for ρ from Theorem 5.2 (b) to Theorem 5.2 (a), and by computing $D(\rho \parallel \pi)$ using the unconstrained case (25) instead of (42).

Note that this algorithm requires $\Sigma_{hh} = \mathbb{E}_{h \sim \mathcal{M}} [hh^T]$ as input, which depends on \mathcal{M} . In practice, \mathcal{M} may be an oracle distribution, making Σ_{hh} unknown and inaccessible. However, there are practical scenarios where a non-oracle \mathcal{M} is available. For example, if there exists a larger and fixed dataset $H^{\text{whole}} \in \{0, 1\}^{n \times m'} (m' > m)$ such that the columns of H are sampled without replacement from the columns of H^{whole} , we can take \mathcal{M} as the *population distribution* over the columns of H^{whole} . This is similar to the matrix completion setting of Srebro et al. [47], where all entries of a fixed matrix are treated as underground truth and observed entries are sampled from them. Under this assumption, $\Sigma_{hh} = \frac{1}{m'} H^{\text{whole}} (H^{\text{whole}})^T$ is known and accessible. We provide further discussion in Appendix E.1.

Algorithm 1 Computing the PAC-Bayes bound for LAEs

Input: $\Sigma_{hh}, p, \delta, \sigma, \Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_L\}, X, Y$, and an LAE model W (with $\text{diag}(W) = 0$).

Compute $\Sigma_{xx}, \Sigma_{xy}, \Sigma_{yy}$ with Σ_{hh}, p by Lemma 4.4.

Set $\pi = \bar{\mathcal{N}}(W, \sigma^2 I)$ (i.e., let $\mathcal{U}_0 = W$ such that W is the mean prior of π).

Let $G = \{\}$ be a set to store the results.

for each λ_i **in** Λ :

 Compute $\rho = \bar{\mathcal{N}}(\mathcal{U}, \mathcal{S})$ with π, λ_i by Theorem 5.2 (b).

 Compute $D(\rho \parallel \pi)$ with ρ, π by (42) in Appendix F.

 Compute $\mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)]$ with ρ, X, Y by (40) in Appendix F.

 Compute $\mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)]$ with $\rho, \Sigma_{xx}, \Sigma_{xy}, \Sigma_{yy}$ by (41) in Appendix F, and let it be the left hand side of (14), denoted as LH_i .

 Compute $\mathbb{E}_{\pi} \left[e^{\lambda R^{\text{true}}(W)} \right]$ with $\pi, \Sigma_{xx}, \Sigma_{xy}, \Sigma_{yy}, \lambda_i$ by (12).

 Compute the right hand side of (14), denoted as RH_i , with $\mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)], D(\rho \parallel \pi), \mathbb{E}_{\pi} \left[e^{\lambda R^{\text{true}}(W)} \right]$.

 Append $(\text{LH}_i, \text{RH}_i)$ to G .

Output: the pair $(\text{LH}^*, \text{RH}^*)$ in G , where $\text{RH}^* = \min_{1 \leq i \leq L} \{\text{RH}_i\}$.

6 Experiments

In this section, we conduct experiments to compute the PAC-Bayes bound for LAEs using Algorithm 1 on real-world datasets, evaluate the tightness of the bound, and empirically assess its correlation with practical ranking metrics such as Recall@K and NDCG@K.

We adopt the *strong generalization* evaluation setting, which divides the entire dataset into a training set and a test set with disjoint users [48, 38]. Let the entire dataset be $H^{\text{whole}} \in \{0, 1\}^{n \times m'}$. We split it into a training set $H^{\text{train}} \in \{0, 1\}^{n \times (m' - m)}$ and a test set $H^{\text{test}} \in \{0, 1\}^{n \times m}$ by setting $m = 0.3m'$. The test set H^{test} is further split into an input matrix X and a target matrix Y , with a hold-out fraction $1 - p = \frac{1}{2}$. The LAE model W is obtained by solving the EASE objective (4) on the training set H^{train} (The EASE model can also be replaced by other LAE models such as EDLAE or ELSA. Our bound only focuses on evaluation and is independent of the training method.). We set γ in (4) to values of 50, 100, 200, 500, 1000, 2000 and 5000 to generate seven different LAE models and evaluate them accordingly. Other inputs of the algorithm are set as follows: $\delta = 0.01$, $\sigma = 0.001$, $\Lambda = \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512\}$.

Our experiments run on a machine with 500 GB RAM and an Nvidia A100 GPU. The GPU has 80 GB RAM. We use three datasets: MovieLens 20M (ML 20M), Netflix and MSD, with their details shown in Table 2 in Appendix F.

The results are presented in Table 1. On the left side, each pair (LH, RH) is the output of Algorithm 1, where LH and RH represent the left-hand side and right-hand side of (14), respectively. Detailed values of the components of RH are provided in Table 3 in Appendix F. The results demonstrate that our bound is tight: in all cases, RH is within 3 times LH, in contrast to Dziugaite and Roy’s non-vacuous bound for deep neural networks [13], where RH can reach up to 10 times LH in their experiments.

The right side of Table 1 reports Recall@50 and NDCG@100 for each model on test sets, with both metrics referenced from the EASE paper [48]. Across all datasets, models with smaller LH and RH generally achieve higher Recall@50 and NDCG@100. This negative correlation reflects the expected relationship: LH and RH are loss-based, where lower values are better; Recall@50 and NDCG@100 are ranking metrics, where higher values are better. Although minor deviations exist – for example, on MSD, the best LH/RH occur at $\gamma = 1000$, while the best Recall@50/NDCG@100 occur at $\gamma = 500$ – the overall trend demonstrates a strong alignment between LH/RH and Recall@50/NDCG@100, suggesting that our bound effectively reflects the practical performance of LAE models. Further discussion on this alignment is provided in Appendix E.4.

Table 1: Experiment results

Models	PAC-Bayes Bound for LAEs			Ranking Performance		
	ML 20M	Netflix	MSD	ML 20M	Netflix	MSD
$\gamma = 50$	LH	61.66	87.22	15.96	Recall@50	0.3434
	RH	128.66	178.11	32.60	NDCG@100	0.4342
$\gamma = 100$	LH	60.75	86.54	15.85	Recall@50	0.3453
	RH	125.90	176.25	32.26	NDCG@100	0.4373
$\gamma = 200$	LH	60.06	85.96	15.76	Recall@50	0.3471
	RH	123.67	174.55	31.94	NDCG@100	0.4402
$\gamma = 500$	LH	59.46	85.35	15.66	Recall@50	0.3489
	RH	121.41	172.64	31.62	NDCG@100	0.4439
$\gamma = 1000$	LH	59.19	85.00	15.64	Recall@50	0.3502
	RH	120.17	171.44	31.50	NDCG@100	0.4464
$\gamma = 2000$	LH	59.09	84.72	15.68	Recall@50	0.3510
	RH	119.34	170.45	31.52	NDCG@100	0.4487
$\gamma = 5000$	LH	59.19	84.48	15.83	Recall@50	0.3506
	RH	118.91	169.47	31.77	NDCG@100	0.4509

Acknowledgments and Disclosure of Funding

This research was partially supported by NSF grant IIS 2142675. RG acknowledges the NeurIPS 2025 Financial Aid Award.

We thank all anonymous reviewers of the current and previous submissions of this manuscript for their valuable comments. In particular, the use of the relaxed MSE as a testing metric for LAEs was suggested by Reviewer nWHo, and the experimental comparison between our bound and practical ranking metrics was suggested by Reviewer TXNp.

References

- [1] Charu C Aggarwal. *Recommender systems: the textbook*. Springer, 2016.
- [2] Pierre Alquier. Bayesian methods for low-rank matrix estimation: short survey and theoretical study. In *International Conference on Algorithmic Learning Theory*, pages 309–323. Springer, 2013.
- [3] Pierre Alquier and Gérard Biau. Sparse single-index model. *Journal of Machine Learning Research*, 14(1):243–280, 2013.
- [4] Pierre Alquier et al. User-friendly introduction to PAC-Bayes bounds. *Foundations and Trends in Machine Learning*, 17(2):174–303, 2024.
- [5] Pierre Alquier, James Ridgway, and Nicolas Chopin. On the properties of variational approximations of Gibbs posteriors. *Journal of Machine Learning Research*, 17(236):1–41, 2016.
- [6] Peter L Bartlett, Andrea Montanari, and Alexander Rakhlin. Deep learning: a statistical viewpoint. *Acta numerica*, 30:87–201, 2021.
- [7] Emmanuel J Candès and Terence Tao. The power of convex relaxation: Near-optimal matrix completion. *IEEE transactions on Information Theory*, 56(5):2053–2080, 2010.
- [8] Ioar Casado, Luis A Ortega, Aritz Pérez, and Andrés R Masegosa. PAC-Bayes-Chernoff bounds for unbounded losses. In *Advances in Neural Information Processing Systems*, volume 37, pages 24350–24374, 2024.
- [9] Olivier Catoni. A PAC-Bayesian approach to adaptive classification. *preprint LPMA*, 840(2):6, 2003.
- [10] Badr-Eddine Chérif-Abdellatif, Yuyang Shi, Arnaud Doucet, and Benjamin Guedj. On PAC-Bayesian reconstruction guarantees for VAEs. In *International Conference on Artificial Intelligence and Statistics*, pages 3066–3079. PMLR, 2022.
- [11] Paolo Cremonesi and Dietmar Jannach. Progress in recommender systems research: Crisis? what crisis? *AI Magazine*, 42(3):43–54, 2021.
- [12] Maurizio Ferrari Dacrema, Paolo Cremonesi, and Dietmar Jannach. Are we really making much progress? a worrying analysis of recent neural recommendation approaches. In *Proceedings of the 13th ACM Conference on Recommender Systems*, pages 101–109, 2019.
- [13] Gintare Karolina Dziugaite and Daniel M Roy. Computing nonvacuous generalization bounds for deep (stochastic) neural networks with many more parameters than training data. In *Proceedings of the Conference on Uncertainty in Artificial Intelligence*, 2017.
- [14] Xinlong Feng and Zhinan Zhang. The rank of a random matrix. *Applied mathematics and computation*, 185(1):689–694, 2007.
- [15] Gerald B Folland. *Real analysis: modern techniques and their applications*. John Wiley & Sons, 1999.
- [16] Rina Foygel, Ohad Shamir, Nati Srebro, and Russ R Salakhutdinov. Learning with the weighted trace-norm under arbitrary sampling distributions. *Advances in Neural Information Processing Systems*, 24, 2011.
- [17] Pascal Germain, Francis Bach, Alexandre Lacoste, and Simon Lacoste-Julien. PAC-Bayesian theory meets Bayesian inference. In *Advances in Neural Information Processing Systems*, volume 29, 2016.
- [18] Maxime Haddouche and Benjamin Guedj. PAC-Bayes generalisation bounds for heavy-tailed losses through supermartingales. *Transactions on Machine Learning Research*, 2023.

[19] Maxime Haddouche, Benjamin Guedj, Omar Rivasplata, and John Shawe-Taylor. PAC-Bayes unleashed: Generalisation bounds with unbounded losses. *Entropy*, 23(10):1330, 2021.

[20] Fredrik Hellström, Giuseppe Durisi, Benjamin Guedj, Maxim Raginsky, et al. Generalization bounds: Perspectives from information theory and PAC-Bayes. *Foundations and Trends in Machine Learning*, 18(1):1–223, 2025.

[21] Roger A Horn and Charles R Johnson. *Topics in matrix analysis*. Cambridge university press, 1991.

[22] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

[23] Yifan Hu, Yehuda Koren, and Chris Volinsky. Collaborative filtering for implicit feedback datasets. In *IEEE International Conference on Data Mining*, pages 263–272. IEEE, 2008.

[24] Ruoming Jin, Dong Li, Jing Gao, Zhi Liu, Li Chen, and Yang Zhou. Towards a better understanding of linear models for recommendation. In *Proceedings of the 27th ACM SIGKDD Conference on Knowledge Discovery & Data Mining*, pages 776–785, 2021.

[25] Richard A Johnson and Dean W Wichern. *Applied multivariate statistical analysis*. Prentice Hall Upper Saddle River, NJ, 2007.

[26] Yehuda Koren. Factorization meets the neighborhood: a multifaceted collaborative filtering model. In *Proceedings of the 14th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 426–434, 2008.

[27] Yehuda Koren, Robert Bell, and Chris Volinsky. Matrix factorization techniques for recommender systems. *Computer*, 42(8):30–37, 2009.

[28] John Langford and Robert Schapire. Tutorial on practical prediction theory for classification. *Journal of Machine Learning Research*, 6(3), 2005.

[29] John Langford and Matthias Seeger. *Bounds for averaging classifiers*. School of Computer Science, Carnegie Mellon University, 2001.

[30] Antoine Ledent and Rodrigo Alves. Generalization analysis of deep nonlinear matrix completion. In *International Conference on Machine Learning*. PMLR, 2024.

[31] Antoine Ledent, Rodrigo Alves, Yunwen Lei, and Marius Kloft. Fine-grained generalization analysis of inductive matrix completion. In *Advances in Neural Information Processing Systems*, volume 34, pages 25540–25552, 2021.

[32] Dawen Liang, Rahul G Krishnan, Matthew D Hoffman, and Tony Jebara. Variational autoencoders for collaborative filtering. In *Proceedings of the World Wide Web Conference*, pages 689–698, 2018.

[33] David G Luenberger and Yinyu Ye. *Linear and nonlinear programming, 3rd Edition*. Springer, 2008.

[34] The Tien Mai. From bilinear regression to inductive matrix completion: a quasi-Bayesian analysis. *Entropy*, 25(2):333, 2023.

[35] Kelong Mao, Jieming Zhu, Jinpeng Wang, Quanyu Dai, Zhenhua Dong, Xi Xiao, and Xiuqiang He. Simplex: A simple and strong baseline for collaborative filtering. In *Proceedings of the 30th ACM International Conference on Information & Knowledge Management*, pages 1243–1252, 2021.

[36] Arakaparampil M Mathai and Serge B Provost. *Quadratic forms in random variables: theory and applications*. Marcel Dekker, 1992.

[37] David A McAllester. Some PAC-Bayesian theorems. In *Proceedings of the 11th Annual Conference on Computational Learning Theory*, pages 230–234, 1998.

[38] Jaewan Moon, Hye-young Kim, and Jongwuk Lee. It's enough: Relaxing diagonal constraints in linear autoencoders for recommendation. In *Proceedings of the 46th International ACM SIGIR Conference on Research and Development in Information Retrieval*, pages 1639–1648, 2023.

[39] Vaishnavh Nagarajan and J Zico Kolter. Uniform convergence may be unable to explain generalization in deep learning. In *Advances in Neural Information Processing Systems*, volume 32, 2019.

[40] Xia Ning and George Karypis. SLIM: Sparse linear methods for top-N recommender systems. In *International Conference on Data Mining*, pages 497–506. IEEE, 2011.

- [41] Luca Oneto, Sandro Ridella, and Davide Anguita. Do we really need a new theory to understand over-parameterization? *Neurocomputing*, 543:126227, 2023.
- [42] Benjamin Recht. A simpler approach to matrix completion. *Journal of Machine Learning Research*, 12(12), 2011.
- [43] Borja Rodríguez-Gálvez, Ragnar Thobaben, and Mikael Skoglund. More PAC-Bayes bounds: From bounded losses, to losses with general tail behaviors, to anytime validity. *Journal of Machine Learning Research*, 25(110):1–43, 2024.
- [44] Walter Rudin. *Principles of mathematical analysis*. McGraw-hill New York, 1976.
- [45] Vera Shalaeva, Alireza Fakhrizadeh Esfahani, Pascal Germain, and Mihaly Petreczky. Improved PAC-Bayesian bounds for linear regression. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 5660–5667, 2020.
- [46] Ohad Shamir and Shai Shalev-Shwartz. Matrix completion with the trace norm: Learning, bounding, and transducing. *Journal of Machine Learning Research*, 15(1):3401–3423, 2014.
- [47] Nathan Srebro, Noga Alon, and Tommi Jaakkola. Generalization error bounds for collaborative prediction with low-rank matrices. In *Advances in Neural Information Processing Systems*, volume 17, 2004.
- [48] Harald Steck. Embarrassingly shallow autoencoders for sparse data. In *The World Wide Web Conference*, pages 3251–3257, 2019.
- [49] Harald Steck. Autoencoders that don’t overfit towards the identity. In *Advances in Neural Information Processing Systems*, volume 33, pages 19598–19608, 2020.
- [50] Vojtěch Vančura, Rodrigo Alves, Petr Kasalický, and Pavel Kordík. Scalable linear shallow autoencoder for collaborative filtering. In *Proceedings of the 16th ACM Conference on Recommender Systems*, pages 604–609, 2022.
- [51] Vladimir Vapnik. *The nature of statistical learning theory*. Springer science & business media, 1999.
- [52] Vladimir Vapnik and Alexey Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability & Its Applications*, 16(2):264–280, 1971.
- [53] Chiyuan Zhang, Samy Bengio, Moritz Hardt, Benjamin Recht, and Oriol Vinyals. Understanding deep learning requires rethinking generalization. In *International Conference on Learning Representations*, 2017.

A Proofs of Theorems, Lemmas, and Propositions

Proof of Theorem 3.2:

Given W and $(x, y) \sim \mathcal{D}$, denote $v = y - Wx$, then $v \sim \mathcal{N}(\mu_w, \Sigma_w)$. Let $Q \in \mathbb{R}^{p \times p}$ such that $\Sigma_w = QQ^T$. Such Q exists since we can take $Q = \Sigma_w^{1/2} = S^T \Lambda^{1/2} S$, but we do not assume it to be unique. Let $\epsilon \sim \mathcal{N}(0, I)$, then we can write $v = Q\epsilon + \mu_w$. Thus,

$$\begin{aligned} R^{\text{true}}(W) &= \mathbb{E}_{(x,y) \sim \mathcal{D}} [\|y - Wx\|_F^2] = \mathbb{E}_\epsilon [\|Q\epsilon + \mu_w\|_F^2] = \mathbb{E}_\epsilon [(Q\epsilon + \mu_w)^T (Q\epsilon + \mu_w)] \\ &= \mathbb{E}_\epsilon [\epsilon^T Q^T Q\epsilon + \mu_w^T Q\epsilon + \epsilon^T Q^T \mu_w + \mu_w^T \mu_w] = \text{tr}(Q^T Q) + \mu_w^T \mu_w \\ &= \text{tr}(QQ^T) + \mu_w^T \mu_w = \text{tr}(\Sigma_w) + \mu_w^T \mu_w \end{aligned} \quad (15)$$

Also, we can express the random variable $\|v\|_F^2$ in quadratic form (Representation 3.1a.1, [36]):

$$\begin{aligned} \|v\|_F^2 &= v^T v = (Q\epsilon + \mu_w)^T (Q\epsilon + \mu_w) \\ &= (Q\epsilon + \mu_w)^T \Sigma_w^{-1/2} \Sigma_w \Sigma_w^{-1/2} (Q\epsilon + \mu_w) \\ &= (\Sigma_w^{-1/2} Q\epsilon + \Sigma_w^{-1/2} \mu_w)^T \Sigma_w (\Sigma_w^{-1/2} Q\epsilon + \Sigma_w^{-1/2} \mu_w) \\ &= (\Sigma_w^{-1/2} Q\epsilon + \Sigma_w^{-1/2} \mu_w)^T S^T \Lambda S (\Sigma_w^{-1/2} Q\epsilon + \Sigma_w^{-1/2} \mu_w) \\ &= (S \Sigma_w^{-1/2} Q\epsilon + S \Sigma_w^{-1/2} \mu_w)^T \Lambda (S \Sigma_w^{-1/2} Q\epsilon + S \Sigma_w^{-1/2} \mu_w) \end{aligned} \quad (16)$$

Denote $\epsilon' = S \Sigma_w^{-1/2} Q\epsilon$, then $\epsilon' \sim \mathcal{N}(0, I)$, because $\mathbb{E}[\epsilon'] = S \Sigma_w^{-1/2} Q \mathbb{E}[\epsilon] = 0$ and

$$\text{Cov}[\epsilon'] = \mathbb{E}[\epsilon' \epsilon'^T] = S \Sigma_w^{-1/2} Q \mathbb{E}[\epsilon \epsilon^T] Q^T \Sigma_w^{-1/2} S^T = I$$

As $b = S \Sigma_w^{-1/2} \mu_w$, we can write

$$\|v\|_F^2 = (\epsilon' + b)^T \Lambda (\epsilon' + b) = \sum_{i=1}^p \eta_i (\epsilon'_i + b_i)^2$$

Hence, each $\epsilon'_i + b_i$ is independently from $\mathcal{N}(b_i, 1)$, and each $(\epsilon'_i + b_i)^2$ is independently from the non-central chi-squared distribution of noncentrality parameter b_i^2 and with degree 1 of freedom. Thus the MGF of $(\epsilon'_i + b_i)^2$ is

$$M_{(\epsilon'_i + b_i)^2}(t) = \mathbb{E}_{(\epsilon'_i + b_i)^2} [e^{t(\epsilon'_i + b_i)^2}] = \frac{\exp\left(\frac{b_i^2 t}{1-2t}\right)}{(1-2t)^{1/2}} \quad (17)$$

Given i.i.d. samples $\{(x_j, y_j)\}_{j=1}^m$ from \mathcal{D} , let $v_j = y_j - Wx_j$. Then v_1, v_2, \dots, v_m are i.i.d. from $\mathcal{N}(\mu_w, \Sigma_w)$, and

$$R^{\text{emp}}(W) = \frac{1}{m} \sum_{j=1}^m \|y_j - Wx_j\|_F^2 = \frac{1}{m} \sum_{j=1}^m \|v_j\|_F^2$$

Hence the MGF of $R^{\text{emp}}(W)$ is

$$\begin{aligned} M_{R^{\text{emp}}(W)}(t) &= \mathbb{E}_{S \sim \mathcal{D}^m} [e^{t R^{\text{emp}}(W)}] = \mathbb{E}_{S \sim \mathcal{D}^m} \left[\exp\left(\frac{t}{m} \sum_{j=1}^m \|v_j\|_F^2\right) \right] \\ &= \left(\mathbb{E}_{S \sim \mathcal{D}^m} \left[\exp\left(\frac{t}{m} \|v\|_F^2\right) \right] \right)^m = \left(\mathbb{E}_{S \sim \mathcal{D}^m} \left[\exp\left(\frac{t}{m} \sum_{i=1}^p \eta_i (\epsilon'_i + b_i)^2\right) \right] \right)^m \\ &= \left(\prod_{i=1}^p \mathbb{E}_{(\epsilon'_i + b_i)^2} \left[\exp\left(\frac{t \eta_i}{m} (\epsilon'_i + b_i)^2\right) \right] \right)^m = \left(\prod_{i=1}^p \frac{\exp\left(\frac{t b_i^2 \eta_i}{m-2t\eta_i}\right)}{(1-2t\eta_i/m)^{1/2}} \right)^m \\ &= \frac{\exp\left(\sum_{i=1}^p \frac{t m b_i^2 \eta_i}{m-2t\eta_i}\right)}{\prod_{i=1}^p (1-2t\eta_i/m)^{m/2}} \end{aligned} \quad (18)$$

By (15) and (18), we can expand $\Psi_{\pi, \mathcal{D}}(\lambda, m)$ as

$$\begin{aligned}
\Psi_{\pi, \mathcal{D}}(\lambda, m) &= \ln \mathbb{E}_{W \sim \pi} \mathbb{E}_{S \sim \mathcal{D}^m} [e^{\lambda(R^{\text{true}}(W) - R^{\text{emp}}(W))}] \\
&= \ln \mathbb{E}_{W \sim \pi} \left[e^{\lambda R^{\text{true}}(W)} \mathbb{E}_{S \sim \mathcal{D}^m} [e^{-\lambda R^{\text{emp}}(W)}] \right] \\
&= \ln \mathbb{E}_{W \sim \pi} \left[\exp(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W)) \frac{\exp\left(\sum_{i=1}^p \frac{-\lambda m b_i^2 \eta_i}{m+2\lambda\eta_i}\right)}{\prod_{i=1}^p (1+2\lambda\eta_i/m)^{m/2}} \right] \tag{19}
\end{aligned}$$

Use the inequality that for any $x > 0$ and $k > 0$, $e^{\frac{xk}{x+k}} < (\frac{x}{k} + 1)^k$ ⁴, and the fact $\text{tr}(\Sigma_W) = \sum_{i=1}^p \eta_i$, we have

$$\begin{aligned}
&\ln \mathbb{E}_{W \sim \pi} \left[\exp(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W)) \frac{\exp\left(\sum_{i=1}^p \frac{-\lambda m b_i^2 \eta_i}{m+2\lambda\eta_i}\right)}{\prod_{i=1}^p (1+2\lambda\eta_i/m)^{m/2}} \right] \\
&\leq \ln \mathbb{E}_{W \sim \pi} \left[\exp(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W)) \frac{\exp\left(\sum_{i=1}^p \frac{-\lambda m b_i^2 \eta_i}{m+2\lambda\eta_i}\right)}{\prod_{i=1}^p \exp\left(\frac{m\lambda\eta_i}{m+2\lambda\eta_i}\right)} \right] \\
&= \ln \mathbb{E}_{W \sim \pi} \exp\left(\lambda \mu_W^T \mu_W + \sum_{i=1}^p \lambda(\eta_i - \frac{mb_i^2 \eta_i}{m+2\lambda\eta_i}) - \sum_{i=1}^p \frac{m\lambda\eta_i}{m+2\lambda\eta_i}\right) \\
&= \ln \mathbb{E}_{W \sim \pi} \exp\left(\lambda \mu_W^T \mu_W + \sum_{i=1}^p \frac{2\lambda^2 \eta_i^2 - \lambda m b_i^2 \eta_i}{m+2\lambda\eta_i}\right) \\
&\leq \ln \mathbb{E}_{W \sim \pi} \exp\left(\lambda(\mu_W^T \mu_W - \sum_{i=1}^p b_i^2 \eta_i) + \frac{2\lambda^2(\sum_{i=1}^p \eta_i^2)}{m}\right) = \ln \mathbb{E}_{W \sim \pi} \exp\left(\frac{2\lambda^2(\sum_{i=1}^p \eta_i^2)}{m}\right)
\end{aligned}$$

The last equality above is because

$$\sum_{i=1}^p b_i^2 \eta_i = b^T \Lambda b = \mu_W^T \Sigma_W^{-1/2} S^T \Lambda S \Sigma_W^{-1/2} \mu_W = \mu_W^T \mu_W$$

Since

$$\sum_{i=1}^p \eta_i^2 = \text{tr}(S^T \Lambda^2 S) = \text{tr}(\Sigma_W^2) = \text{tr}(\Sigma_W \Sigma_W^T) = \|\Sigma_W\|_F^2$$

we have

$$\ln \mathbb{E}_{W \sim \pi} \exp\left(\frac{2\lambda^2(\sum_{i=1}^p \eta_i^2)}{m}\right) = \ln \mathbb{E}_{W \sim \pi} \exp\left(\frac{2\lambda^2 \|\Sigma_W\|_F^2}{m}\right)$$

□

Proof of Theorem 3.3:

By (19), given any $\lambda > 0$, we let $\{f_m\}_{m \in \mathbb{N}}$ be a sequence of functions where

$$f_m(W) = \exp(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W)) \frac{\exp\left(\sum_{i=1}^p \frac{-\lambda m b_i^2 \eta_i}{m+2\lambda\eta_i}\right)}{\prod_{i=1}^p (1+2\lambda\eta_i/m)^{m/2}}$$

for $m > 0$, and

$$f_0(W) = \exp(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W))$$

Note that each f_i is a non-negative function.

⁴Since $\frac{x}{x+1} < \ln(x+1)$ for any $x > -1$, replacing x with $\frac{x}{k}$, and taking exponential on both sides, we get $e^{\frac{xk}{x+k}} < (\frac{x}{k} + 1)^k$.

Since $W \sim \pi$ and $W \in \mathbb{R}^{p \times p}$, let $E = \mathbb{R}^{p \times p}$, then f_m is a measurable function on E , π is a Borel probability measure on E , and we can express $\mathbb{E}_\pi[f_m]$ as a Lebesgue integral (Section 10.1, [15]):

$$\mathbb{E}_\pi[f_m] = \int_E f_m d\pi$$

Now we prove the following three conditions:

(a) $f_m(W) \leq f_0(W)$ for any $m \geq 0$ and $W \in E$.

For any given W , η_i and b_i are fixed for all i , where $\eta_i > 0$ and $b_i^2 \geq 0$. Thus for any $m \geq 0$, the numerator of f_m satisfies $\exp\left(\sum_{i=1}^p \frac{-\lambda m b_i^2 \eta_i}{m+2\lambda\eta_i}\right) \leq 1$, and the denominator of f_m satisfies $\prod_{i=1}^p (1 + 2\lambda\eta_i/m)^{m/2} \geq 1$. Note that the numerator is monotonically decreasing with m and the denominator is monotonically increasing with m .

(b) $f_m \rightarrow 1$ pointwisely as $m \rightarrow \infty$.

For any W ,

$$\begin{aligned} \lim_{m \rightarrow \infty} f_m(W) &= \exp\left(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W)\right) \lim_{m \rightarrow \infty} \frac{\exp\left(\sum_{i=1}^p \frac{-\lambda m b_i^2 \eta_i}{m+2\lambda\eta_i}\right)}{\prod_{i=1}^p (1 + 2\lambda\eta_i/m)^{m/2}} \\ &= \exp\left(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W)\right) \frac{\exp\left(\sum_{i=1}^p \lim_{m \rightarrow \infty} \frac{-\lambda m b_i^2 \eta_i}{m+2\lambda\eta_i}\right)}{\prod_{i=1}^p \lim_{m \rightarrow \infty} (1 + 2\lambda\eta_i/m)^{m/2}} \\ &= \exp\left(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W)\right) \frac{\exp\left(-\lambda \sum_{i=1}^p b_i^2 \eta_i\right)}{\prod_{i=1}^p \exp(\lambda \eta_i)} = 1 \end{aligned}$$

The last inequality uses the facts that $\sum_{i=1}^p b_i^2 \eta_i = \mu_W^T \mu_W$ and $\sum_{i=1}^p \eta_i = \text{tr}(\Sigma_W)$.

(c) $\int_E f_0 d\pi = \mathbb{E}_\pi[f_0] < \infty$.

$$\begin{aligned} \mathbb{E}_\pi[f_0] &= \mathbb{E}_\pi \exp\left(\lambda(\text{tr}(\Sigma_W) + \mu_W^T \mu_W)\right) \\ &= \mathbb{E}_\pi \exp\left(\lambda[\text{tr}((W^* - W)\Sigma_x(W^* - W)^T + \Sigma_e) + \|(W^* - W)\mu_x\|_F^2]\right) \\ &= \mathbb{E}_\pi \exp\left(\lambda \left[\sum_{i=1}^p (W^* - W)_{i*} \Sigma_x (W^* - W)_{i*}^T + \text{tr}(\Sigma_e) + \sum_{i=1}^p (W^* - W)_{i*} \mu_x \mu_x^T (W^* - W)_{i*}^T \right]\right) \\ &= \mathbb{E}_\pi \exp\left(\lambda \left[\sum_{i=1}^p (W^* - W)_{i*} [\Sigma_x + \mu_x \mu_x^T] (W^* - W)_{i*}^T + \text{tr}(\Sigma_e) \right]\right) \\ &= \mathbb{E}_\pi \exp\left(\lambda \left[\left\| (\Sigma_x + \mu_x \mu_x^T)^{1/2} (W^* - W) \right\|_F^2 + \text{tr}(\Sigma_e) \right]\right) \\ &= \exp(\lambda \text{tr}(\Sigma_e)) \mathbb{E}_\pi \exp\left(\lambda \left[\left\| (\Sigma_x + \mu_x \mu_x^T)^{1/2} (W^* - W) \right\|_F^2 \right]\right) < \infty \end{aligned}$$

The last inequality holds because $\mathbb{E}_\pi \exp\left(\lambda \left[\left\| (\Sigma_x + \mu_x \mu_x^T)^{1/2} (W^* - W) \right\|_F^2 \right]\right) < \infty$ is our assumption and $\exp(\lambda \text{tr}(\Sigma_e))$ is a constant.

Since the conditions (a), (b) and (c) hold, by the *dominated convergence theorem* (Theorem 11.32, [44]), we have

$$\lim_{m \rightarrow \infty} \mathbb{E}_\pi[f_m] = \lim_{m \rightarrow \infty} \int_E f_m d\pi = \int_E \lim_{m \rightarrow \infty} f_m d\pi = \int_E 1 d\pi = \mathbb{E}_\pi[1] = 1$$

Since \ln is continuous on $(0, \infty)$, we can interchange \lim and \ln . Therefore,

$$\lim_{m \rightarrow \infty} \Psi_{\pi, \mathcal{D}}(\lambda, m) = \lim_{m \rightarrow \infty} \ln \mathbb{E}_\pi[f_m] = \ln \lim_{m \rightarrow \infty} \mathbb{E}_\pi[f_m] = \ln 1 = 0$$

□

Proof of Lemma 4.2:

$$\begin{aligned}
R^{\text{true}}(W) &= \mathbb{E} [\|y - Wx\|_F^2] = \sum_{i=1}^n \mathbb{E} [\|y_i - W_{i*}x\|_F^2] = \sum_{i=1}^n W_{i*} \mathbb{E} [xx^T] W_{i*}^T - 2W_{i*} \mathbb{E} [y_i x] + \mathbb{E} [y_i^2] \\
&= \sum_{i=1}^n W_{i*} \Sigma_{xx} W_{i*}^T - 2W_{i*} (\Sigma_{xy})_{*i} + (\Sigma_{yy})_{ii} \\
&= \sum_{i=1}^n (W_{i*} \Sigma_{xx}^{1/2}) (W_{i*} \Sigma_{xx}^{1/2})^T - 2(W_{i*} \Sigma_{xx}^{1/2}) \Sigma_{xx}^{-1/2} (\Sigma_{xy})_{*i} + (\Sigma_{yy})_{ii} \\
&= \sum_{i=1}^n (W_{i*} \Sigma_{xx}^{1/2} - (\Sigma_{xy})_{*i}^T \Sigma_{xx}^{-1/2}) (W_{i*} \Sigma_{xx}^{1/2} - (\Sigma_{xy})_{*i}^T \Sigma_{xx}^{-1/2})^T - (\Sigma_{xy})_{*i}^T \Sigma_{xx}^{-1} (\Sigma_{xy})_{*i} + (\Sigma_{yy})_{ii} \\
&= \sum_{i=1}^n \|W_{i*} \Sigma_{xx}^{1/2} - (\Sigma_{xy})_{*i}^T \Sigma_{xx}^{-1/2}\|_F^2 - \|\Sigma_{xx}^{-1/2} (\Sigma_{xy})_{*i}\|_F^2 + (\Sigma_{yy})_{ii} \\
&= \|W \Sigma_{xx}^{1/2} - \Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2 - \|\Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2 + \text{tr}(\Sigma_{yy})
\end{aligned}$$

Since we assume Σ_{xx} is positive definite, $\Sigma_{xx}^{-1/2}$ exists.

□

Proof of Proposition 4.3:

By Lemma 4.2,

$$\begin{aligned}
\mathbb{E}_\pi \left[e^{\lambda R^{\text{true}}(W)} \right] &= \mathbb{E}_\pi \left[e^{\lambda (\|W \Sigma_{xx}^{1/2} - \Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2 - \|\Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2 + \text{tr}(\Sigma_{yy}))} \right] \\
&= \mathbb{E}_\pi \left[e^{\lambda (\|W \Sigma_{xx}^{1/2} - \Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2)} \right] e^{\lambda (\text{tr}(\Sigma_{yy}) - \|\Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2)} = C \mathbb{E}_\pi \left[e^{\lambda \|W \Sigma_{xx}^{1/2} + B\|_F^2} \right]
\end{aligned} \tag{20}$$

□

Proof of Lemma 4.4:

Since $x = \delta \odot h$ and $y = (\mathbf{1} - \delta) \odot h$, we have

$$\begin{aligned}
\Sigma_{xx} &= \mathbb{E} [xx^T] = \mathbb{E} [(\delta \odot h)(\delta \odot h)^T] \\
\Sigma_{xy} &= \mathbb{E} [xy^T] = \mathbb{E} [(\delta \odot h)((\mathbf{1} - \delta) \odot h)^T] \\
\Sigma_{yy} &= \mathbb{E} [yy^T] = \mathbb{E} [((\mathbf{1} - \delta) \odot h)((\mathbf{1} - \delta) \odot h)^T]
\end{aligned}$$

We first prove Σ_{xx} . For $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$, $(\Sigma_{xx})_{ij} = \mathbb{E}[\delta_i \delta_j h_i h_j]$. Note that $\delta_i \delta_j h_i h_j$ is a Bernoulli random variable (as its value can either be 0 or 1), δ_i depends on h_i , and δ_j depends on h_j . We have

$$\begin{aligned}
\mathbb{E}[\delta_i \delta_j h_i h_j] &= P(\delta_i \delta_j h_i h_j = 1) = P(\delta_i = 1, \delta_j = 1, h_i = 1, h_j = 1) \\
&= P(\delta_i = 1 | \delta_j = 1, h_i = 1, h_j = 1) P(\delta_j = 1 | h_i = 1, h_j = 1) P(h_i = 1, h_j = 1) \\
&= P(\delta_i = 1 | h_i = 1) P(\delta_j = 1 | h_j = 1) P(h_i = 1, h_j = 1) \\
&= p^2 \mathbb{E}[h_i h_j] = p^2 (\Sigma_{hh})_{ij}
\end{aligned} \tag{21}$$

For any i , $(\Sigma_{xx})_{ii} = \mathbb{E}[(\delta_i h_i)^2]$. Using the property that a Bernoulli random variable X has $\mathbb{E}[X^2] = \mathbb{E}[X]$,

$$\begin{aligned}
\mathbb{E}[(\delta_i h_i)^2] &= P(\delta_i h_i = 1) = P(\delta_i = 1, h_i = 1) = P(\delta_i = 1 | h_i = 1) P(h_i = 1) = p \mathbb{E}[h_i] \\
&= p \mathbb{E}[h_i^2] = p (\Sigma_{hh})_{ii}
\end{aligned} \tag{22}$$

Combining (21) and (22), we get

$$\Sigma_{xx} = p^2 \Sigma_{hh} + p(1-p)(I \odot \Sigma_{hh}) \tag{23}$$

Since $(\Sigma_{yy})_{ij} = \mathbb{E}[(1 - \delta_i)(1 - \delta_j)h_i h_j]$ and $(\Sigma_{yy})_{ii} = \mathbb{E}[(1 - \delta_i)h_i]^2$, replacing p with $1 - p$ in (23), we get $\Sigma_{yy} = (1 - p)^2 \Sigma_{hh} + p(1 - p)(I \odot \Sigma_{hh})$.

Since $(\Sigma_{xy})_{ij} = \mathbb{E}[\delta_i(1 - \delta_j)h_i h_j] = p(1 - p)\Sigma_{hh}$ and $(\Sigma_{xy})_{ii} = \mathbb{E}[\delta_i(1 - \delta_i)h_i^2] = 0$ (Note that $\delta_i(1 - \delta_i)h_i^2 = 0$ regardless of whether δ_i is 0 or 1.), we have $\Sigma_{xy} = p(1 - p)(\Sigma_{hh} - I \odot \Sigma_{hh})$.

Note that in (23), if Σ_{hh} is positive definite, then $I \odot \Sigma_{hh}$ is also positive definite (Theorem 7.5.3 (b), [22]), which implies that Σ_{xx} is positive definite.

□

Proof of Theorem 5.2:

(a) Since $\mathbb{E}_\rho[W] = \mathcal{U}$ and $\mathbb{E}_\rho[W^T W] = \mathcal{U}^T \mathcal{U} + \text{diag}(\sum_{k=1}^n \mathcal{S}_{k1}, \sum_{k=1}^n \mathcal{S}_{k2}, \dots, \sum_{k=1}^n \mathcal{S}_{kn})$,

$$\begin{aligned} \mathbb{E}_\rho[R^{\text{emp}}(W)] &= \frac{1}{m} \mathbb{E}_\rho[\|Y - WX\|_F^2] = \frac{1}{m} \sum_{l=1}^m \mathbb{E}_\rho[\|Y_{*l} - WX_{*l}\|_F^2] \\ &= \frac{1}{m} \sum_{l=1}^m \mathbb{E}_\rho[(Y_{*l} - WX_{*l})^T (Y_{*l} - WX_{*l})] = \frac{1}{m} \sum_{l=1}^m Y_{*l}^T Y_{*l} - 2Y_{*l}^T \mathbb{E}_\rho[W] X_{*l} + X_{*l}^T \mathbb{E}_\rho[W^T W] X_{*l} \\ &= \frac{1}{m} \sum_{l=1}^m Y_{*l}^T Y_{*l} - 2Y_{*l}^T \mathcal{U} X_{*l} + X_{*l}^T \mathcal{U}^T \mathcal{U} X_{*l} + X_{*l}^T \text{diag} \left(\sum_{k=1}^n \mathcal{S}_{k1}, \sum_{k=1}^n \mathcal{S}_{k2}, \dots, \sum_{k=1}^n \mathcal{S}_{kn} \right) X_{*l} \end{aligned} \quad (24)$$

And $D(\rho \parallel \pi)$ can be expressed as

$$D(\rho \parallel \pi) = \frac{1}{2} \left[n^2(2 \ln \sigma - 1) - \sum_{k=1}^n \sum_{l=1}^n (\ln \mathcal{S}_{kl} - \frac{\mathcal{S}_{kl}}{\sigma^2}) + \frac{\|\mathcal{U} - \mathcal{U}_0\|_F^2}{\sigma^2} \right] \quad (25)$$

Denote $f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) = \mathbb{E}_\rho[R^{\text{emp}}(W)] + \frac{1}{\lambda} D(\rho \parallel \pi)$, our optimization problem becomes

$$\min_{\mathcal{U}, \mathcal{S}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) \quad (26)$$

The optimal \mathcal{U} and \mathcal{S} are obtained by solving $\frac{\partial}{\partial \mathcal{U}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) = 0$ and $\frac{\partial}{\partial \mathcal{S}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) = 0$.

First we show the partial derivatives of the $\frac{1}{\lambda} D(\rho \parallel \pi)$ term:

$$\frac{\partial}{\partial \mathcal{U}_{ij}} \frac{1}{\lambda} D(\rho \parallel \pi) = \frac{(\mathcal{U}_{ij} - (\mathcal{U}_0)_{ij})}{\lambda \sigma^2}, \quad \frac{\partial}{\partial \mathcal{S}_{ij}} \frac{1}{\lambda} D(\rho \parallel \pi) = -\frac{1}{2\lambda} \left(\frac{1}{\mathcal{S}_{ij}} - \frac{1}{\sigma^2} \right)$$

Then we show the partial derivatives of the $\mathbb{E}_\rho[R^{\text{emp}}(W)]$ term. By (24), for any i, j ,

$$\begin{aligned} \frac{\partial}{\partial \mathcal{S}_{ij}} \mathbb{E}_\rho[R^{\text{emp}}(W)] &= \frac{\partial}{\partial \mathcal{S}_{ij}} \frac{1}{m} \sum_{l=1}^m X_{*l}^T \text{diag} \left(\sum_{k=1}^n \mathcal{S}_{k1}, \sum_{k=1}^n \mathcal{S}_{k2}, \dots, \sum_{k=1}^n \mathcal{S}_{kn} \right) X_{*l} \\ &= \frac{\partial}{\partial \mathcal{S}_{ij}} \frac{1}{m} \sum_{l=1}^m X_{jl} \mathcal{S}_{ij} X_{jl} = \frac{1}{m} \sum_{l=1}^m X_{jl}^2 = \frac{1}{m} X_{j*} X_{j*}^T \\ \frac{\partial}{\partial \mathcal{U}_{ij}} \mathbb{E}_\rho[R^{\text{emp}}(W)] &= \frac{\partial}{\partial \mathcal{U}_{ij}} \frac{1}{m} \sum_{l=1}^m -2Y_{*l}^T \mathcal{U} X_{*l} + X_{*l}^T \mathcal{U}^T \mathcal{U} X_{*l} = \frac{1}{m} \sum_{l=1}^m \left(-2Y_{il} X_{jl} + \frac{\partial}{\partial \mathcal{U}_{ij}} \sum_{k=1}^n (\mathcal{U}_{k*} X_{*l})^2 \right) \\ &= \frac{1}{m} \sum_{l=1}^m \left(-2Y_{il} X_{jl} + \frac{\partial}{\partial \mathcal{U}_{ij}} (\mathcal{U}_{i*} X_{*l})^2 \right) = \frac{1}{m} \sum_{l=1}^m (-2Y_{il} X_{jl} + 2(\mathcal{U}_{i*} X_{*l}) X_{jl}) \\ &= \frac{2}{m} (-Y_{i*} X_{j*}^T + \mathcal{U}_{i*} X X_{j*}^T) \end{aligned}$$

Wrap up the above results, we get

$$\frac{\partial}{\partial \mathcal{S}_{ij}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) = \frac{1}{m} X_{j*} X_{j*}^T - \frac{1}{2\lambda} \left(\frac{1}{\mathcal{S}_{ij}} - \frac{1}{\sigma^2} \right) \quad (27)$$

$$\frac{\partial}{\partial \mathcal{U}_{ij}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) = \frac{2}{m} (-Y_{i*} X_{j*}^T + \mathcal{U}_{i*} X X_{j*}^T) + \frac{(\mathcal{U}_{ij} - (\mathcal{U}_0)_{ij})}{\lambda \sigma^2} \quad (28)$$

Therefore, by (27), the solution of $\frac{\partial}{\partial \mathcal{S}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) = 0$ is that

$$\mathcal{S}_{ij} = \frac{1}{\frac{2\lambda}{m} X_{j*} X_{j*}^T + \frac{1}{\sigma^2}} \quad \text{for any } i, j \in \{1, 2, \dots, n\} \quad (29)$$

By (28) we have

$$\frac{\partial}{\partial \mathcal{U}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) = \left[\frac{2}{m} (-Y X^T + \mathcal{U} X X^T) + \frac{1}{\lambda \sigma^2} (\mathcal{U} - \mathcal{U}^0) \right]^T \quad (30)$$

Thus the solution of $\frac{\partial}{\partial \mathcal{U}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) = 0$ is

$$\mathcal{U} = \left(\frac{1}{m} Y X^T + \frac{1}{2\lambda \sigma^2} \mathcal{U}_0 \right) \left(\frac{1}{m} X X^T + \frac{1}{2\lambda \sigma^2} I \right)^{-1} \quad (31)$$

Now we show that $f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda)$ is a convex function, such that the solutions of \mathcal{S} in (29) and \mathcal{U} in (31) are the global minimizer of (26). By (27) and (28) we have

$$\frac{\partial^2 f}{\partial \mathcal{S}_{ij} \partial \mathcal{S}_{kl}} = \begin{cases} \frac{1}{2\lambda (\mathcal{S}_{ij})^2} & \text{if } i = k, j = l \\ 0 & \text{otherwise} \end{cases}, \quad \frac{\partial^2 f}{\partial \mathcal{U}_{ij} \partial \mathcal{U}_{kl}} = \begin{cases} \frac{2}{m} X_{j*} X_{l*}^T + \frac{1}{\lambda \sigma^2} & \text{if } i = k, j = l \\ \frac{2}{m} X_{j*} X_{l*}^T & \text{if } i = k, j \neq l \\ 0 & \text{otherwise} \end{cases}$$

Denote $\nu \in \mathbb{R}^{2n^2}$ where for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$, $\nu_{(i-1)n+j} = \mathcal{U}_{ij}$ and $\nu_{n^2+(i-1)n+j} = \mathcal{S}_{ij}$. Let $H_f \in \mathbb{R}^{2n^2 \times 2n^2}$ be the Hessian matrix where $(H_f)_{ij} = \frac{\partial^2 f}{\partial \nu_i \partial \nu_j}$. Then we can write $H_f = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where $A = \frac{2}{m} (X X^T) \otimes I_n + \frac{1}{\lambda \sigma^2} I_{n^2}$ and B is a $n^2 \times n^2$ diagonal matrix with $B_{(i-1)n+j, (i-1)n+j} = \frac{1}{2\lambda (\mathcal{S}_{ij})^2}$. Here \otimes means Kronecker product.

The Kronecker product has a property that, let $\{\lambda_i | i = 1, \dots, m\}$ be the eigenvalues of $P \in \mathbb{R}^{m \times m}$ and $\{\mu_j | j = 1, \dots, n\}$ be the eigenvalues of $Q \in \mathbb{R}^{n \times n}$, then $\{\lambda_i \mu_j | i = 1, \dots, m, j = 1, \dots, n\}$ are the eigenvalues of $P \otimes Q$ (Theorem 4.2.12, [21]). Since $X X^T$ is positive semi-definite and I_n is positive definite, $(X X^T) \otimes I_n$ is positive semi-definite. Thus A is positive definite. Since all elements of \mathcal{S} are positive, B is positive definite. Therefore, H_f is a positive definite matrix for any \mathcal{U} and \mathcal{S} , which means $f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda)$ is a convex function. Thus, the solutions of \mathcal{S} in (29) and \mathcal{U} in (31) give the global minimum.

(b) Applying the constraint $\text{diag}(W) = 0$ to ρ and π implies taking $\text{diag}(\mathcal{U}) = 0$, $\text{diag}(\mathcal{S}) = 0$, $\text{diag}(\mathcal{U}_0) = 0$, and $\text{diag}(\sigma^2 J) = 0$. In this case, (26) becomes a constrained optimization problem.

$$\min_{\mathcal{U}, \mathcal{S}} f(\mathcal{U}, \mathcal{S} | \mathcal{U}_0, \sigma, \lambda) \quad \text{s.t. } \text{diag}(\mathcal{U}) = 0, \text{diag}(\mathcal{S}) = 0 \quad (32)$$

Then we remove the constraint $\text{diag}(\mathcal{S}) = 0$ by defining f as a function of only the off-diagonal elements of \mathcal{S} . Let $\mathcal{S}^- = \{\mathcal{S}_{ij} : i, j \in \{1, \dots, n\}, i \neq j\}$, then (32) is equivalent to

$$\min_{\mathcal{U}, \mathcal{S}^-} f(\mathcal{U}, \mathcal{S}^- | \mathcal{U}_0, \sigma, \lambda) \quad \text{s.t. } \text{diag}(\mathcal{U}) = 0 \quad (33)$$

To solve (33), we construct the Lagrangian function

$$L(\mathcal{U}, \mathcal{S}^-, x | \mathcal{U}_0, \sigma, \lambda) = f(\mathcal{U}, \mathcal{S}^- | \mathcal{U}_0, \sigma, \lambda) + x^T \text{diag}(\mathcal{U})$$

where $x \in \mathbb{R}^n$, and solve

$$\frac{\partial L}{\partial x} = [\text{diag}(\mathcal{U})]^T = 0 \quad (34)$$

$$\frac{\partial L}{\partial \mathcal{U}} = \frac{\partial}{\partial \mathcal{U}} f(\mathcal{U}, \mathcal{S}^- | \mathcal{U}_0, \sigma, \lambda) + \text{diag}(x) = 0 \quad (35)$$

$$\frac{\partial L}{\partial \mathcal{S}_{ij}} = \frac{\partial}{\partial \mathcal{S}_{ij}} f(\mathcal{U}, \mathcal{S}^- | \mathcal{U}_0, \sigma, \lambda) = 0 \quad \text{for } i, j \in \{1, 2, \dots, n\}, i \neq j \quad (36)$$

Since (36) is the $i \neq j$ case of (27), the optimal \mathcal{S}^- is obtained by (29) with $i \neq j$.

The optimal \mathcal{U} is obtained by solving (35) and (34). By (35),

$$\begin{aligned} & \frac{2}{m}(-YX^T + \mathcal{U}XX^T) + \frac{1}{\lambda\sigma^2}(\mathcal{U} - \mathcal{U}^0) + \text{diag}(x) = 0 \\ \iff & \mathcal{U} = \left(\frac{1}{m}YX^T + \frac{1}{2\lambda\sigma^2}\mathcal{U}_0 - \frac{1}{2}\text{diag}(x) \right) \left(\frac{1}{m}XX^T + \frac{1}{2\lambda\sigma^2}I \right)^{-1} \end{aligned} \quad (37)$$

Then we solve x to satisfy (34),

$$\begin{aligned} \text{diag}(\mathcal{U}) &= \text{diag} \left[\left(\frac{1}{m}YX^T + \frac{1}{2\lambda\sigma^2}\mathcal{U}_0 \right) \left(\frac{1}{m}XX^T + \frac{1}{2\lambda\sigma^2}I \right)^{-1} \right] - \text{diag} \left[\frac{1}{2}\text{diag}(x) \left(\frac{1}{m}XX^T + \frac{1}{2\lambda\sigma^2}I \right)^{-1} \right] \\ &= \text{diag} \left[\left(\frac{1}{m}YX^T + \frac{1}{2\lambda\sigma^2}\mathcal{U}_0 \right) \left(\frac{1}{m}XX^T + \frac{1}{2\lambda\sigma^2}I \right)^{-1} \right] - \frac{1}{2}x \odot \text{diag} \left[\left(\frac{1}{m}XX^T + \frac{1}{2\lambda\sigma^2}I \right)^{-1} \right] = 0 \end{aligned}$$

we get

$$x = 2 \cdot \text{diag} \left[\left(\frac{1}{m}YX^T + \frac{1}{2\lambda\sigma^2}\mathcal{U}_0 \right) \left(\frac{1}{m}XX^T + \frac{1}{2\lambda\sigma^2}I \right)^{-1} \right] \odot \text{diag} \left[\left(\frac{1}{m}XX^T + \frac{1}{2\lambda\sigma^2}I \right)^{-1} \right]$$

Now we show that the solution of (34), (35) and (36) gives the global minimum of the problem (33). Let H_L be the Hessian matrix of the Lagrangian L . It is easy to verify that, by removing the rows and columns of H_f corresponding to $\mathcal{S}_{11}, \mathcal{S}_{22}, \dots, \mathcal{S}_{nn}$ to obtain $H'_f \in \mathbb{R}^{(2n^2-n) \times (2n^2-n)}$, we have $H_L = H'_f$. This shows that H_L is positive definite for any \mathcal{U} and \mathcal{S}^- . Hence, by the second-order sufficiency conditions (Section 11.5, [33]), any solution $(\mathcal{U}, \mathcal{S}^-, x)$ satisfying (34), (35) and (36) is a strict local minimum. Since the solution is unique, it is also a strict global minimum.

□

Proof of Proposition 5.3:

Denote $v_i = (W_{i*}\Sigma_{xx}^{1/2} + B_{i*})^T$, and write $v_i = A^{1/2}\epsilon + \mu^i$ where $\epsilon \sim \mathcal{N}(0, I)$. Since $A = S^T\Lambda S$, using the quadratic form shown in (16), we have

$$\begin{aligned} \|v_i\|_F^2 &= (A^{1/2}\epsilon + \mu^i)^T(A^{1/2}\epsilon + \mu^i) = (A^{1/2}\epsilon + \mu^i)^T A^{-1/2} S^T \Lambda S A^{-1/2} (A^{1/2}\epsilon + \mu^i) \\ &= (S\epsilon + SA^{-1/2}\mu^i)^T \Lambda (S\epsilon + SA^{-1/2}\mu^i) = (S\epsilon + \bar{b}^i)^T \Lambda (S\epsilon + \bar{b}^i) = \sum_{j=1}^n \eta_j (S_{j*}\epsilon + \bar{b}_j^i)^2 \end{aligned}$$

It is easy to show that each $S_{j*}\epsilon$ are i.i.d. from $\mathcal{N}(0, 1)$ for all j , thus each $S_{j*}\epsilon + \bar{b}_j^i$ is independently from $\mathcal{N}(\bar{b}_j^i, 1)$. Since each v_i is independent, by (20) we have

$$\begin{aligned} \mathbb{E}_\pi \left[e^{\lambda R^{\text{true}}(W)} \right] &= C \mathbb{E}_\pi \left[e^{\lambda \sum_{i=1}^n \|W_{i*}\Sigma_{xx}^{1/2} + B_{i*}\|_F^2} \right] = C \prod_{i=1}^n \mathbb{E}_\pi \left[e^{\lambda \|v_i\|_F^2} \right] = C \prod_{i=1}^n \prod_{j=1}^n \mathbb{E}_\pi \left[e^{\lambda \eta_j (S_{j*}\epsilon + \bar{b}_j^i)^2} \right] \\ &= C \prod_{i=1}^n \prod_{j=1}^n \frac{\exp \left(\frac{\lambda (\bar{b}_j^i)^2 \eta_j}{1 - 2\lambda \eta_j} \right)}{(1 - 2\lambda \eta_j)^{1/2}} \end{aligned}$$

The last equality above follows from (17).

□

Proof of Theorem 5.4:

Let $P, Q \in \mathbb{R}^{n \times n}$ be two symmetric matrices, we write $P \succeq Q$ if $P - Q$ is positive semi-definite, and write $P \succ Q$ if $P - Q$ is positive definite.

By Corollary 7.7.4 (c) of [22], if $P \succeq Q$, then $\eta_j(P) \geq \eta_j(Q)$ for any j , where $\eta_j(P)$ and $\eta_j(Q)$ denote the j th largest eigenvalues of P and Q , respectively. Since $A - A^{(i)} = \sigma^2(\Sigma_{xx}^{1/2})_{*i}(\Sigma_{xx}^{1/2})_{*i}^T \succeq 0$ for any i , we have $\eta_j \geq \eta_j^{(i)}$ for any i, j .

Since $b^{(i)} = S^{(i)}(A^{(i)})^{-1/2}\mu^i$, we have

$$\begin{aligned}(b_j^{(i)})^2\eta_j^{(i)} &= \eta_j^{(i)}(\mu^i)^T(A^{(i)})^{-1/2}(S_{j*}^{(i)})^T S_{j*}^{(i)}(A^{(i)})^{-1/2}\mu^i \\ &= \eta_j^{(i)}(\mu^i)^T(S^{(i)})^T(\Lambda^{(i)})^{-1/2}[S^{(i)}(S_{j*}^{(i)})^T][S_{j*}^{(i)}(S^{(i)})^T](\Lambda^{(i)})^{-1/2}(S^{(i)})\mu^i \\ &= (\mu^i)^T(S_{j*}^{(i)})^T(S_{j*}^{(i)})\mu^i\end{aligned}$$

Therefore, (13) can be expressed as

$$\begin{aligned}\frac{1}{C}\mathbb{E}_{\pi'}\left[e^{\lambda R^{\text{true}}(W)}\right] &= \prod_{i=1}^n \prod_{j=1}^n \frac{\exp\left(\frac{\lambda(b_j^{(i)})^2\eta_j^{(i)}}{1-2\lambda\eta_j^{(i)}}\right)}{\left(1-2\lambda\eta_j^{(i)}\right)^{1/2}} = \prod_{i=1}^n \prod_{j=1}^n \frac{\exp\left(\frac{\lambda(\mu^i)^T(S_{j*}^{(i)})^T(S_{j*}^{(i)})\mu^i}{1-2\lambda\eta_j^{(i)}}\right)}{\left(1-2\lambda\eta_j^{(i)}\right)^{1/2}} \\ &= \prod_{i=1}^n \frac{\exp\left(\lambda(\mu^i)^T\left(\sum_{j=1}^n \frac{(S_{j*}^{(i)})^T(S_{j*}^{(i)})}{1-2\lambda\eta_j^{(i)}}\right)\mu^i\right)}{\prod_{j=1}^n \left(1-2\lambda\eta_j^{(i)}\right)^{1/2}} = \prod_{i=1}^n \frac{\exp\left(\lambda(\mu^i)^T(S^{(i)})^T\bar{\Lambda}^{(i)}S^{(i)}\mu^i\right)}{\prod_{j=1}^n \left(1-2\lambda\eta_j^{(i)}\right)^{1/2}}\end{aligned}$$

where $\bar{\Lambda}^{(i)} = \text{diag}\left(\frac{1}{1-2\lambda\eta_1^{(i)}}, \frac{1}{1-2\lambda\eta_2^{(i)}}, \dots, \frac{1}{1-2\lambda\eta_n^{(i)}}\right)$.

Similarly, (12) can be expressed as

$$\frac{1}{C}\mathbb{E}_{\pi}\left[e^{\lambda R^{\text{true}}(W)}\right] = \prod_{i=1}^n \frac{\exp\left(\lambda(\mu^i)^T S^T \bar{\Lambda} S \mu^i\right)}{\prod_{j=1}^n \left(1-2\lambda\eta_j\right)^{1/2}}$$

where $\bar{\Lambda} = \text{diag}\left(\frac{1}{1-2\lambda\eta_1}, \frac{1}{1-2\lambda\eta_2}, \dots, \frac{1}{1-2\lambda\eta_n}\right)$.

Now we show that $S^T \bar{\Lambda} S \succeq (S^{(i)})^T \bar{\Lambda}^{(i)} S^{(i)}$ for any i . By Corollary 7.7.4 (a) of [22], if $P \succ 0$ and $Q \succ 0$, then $P \succeq Q$ if and only if $Q^{-1} \succeq P^{-1}$. Since we assume $0 < \lambda < \frac{1}{2\eta_1}$, it follows that $1-2\lambda\eta_j^{(i)} > 0$ and $1-2\lambda\eta_j > 0$ for any i, j . Thus, all diagonal elements of $\bar{\Lambda}^{(i)}$ and $\bar{\Lambda}$ are positive, implying that $(S^{(i)})^T \bar{\Lambda}^{(i)} S^{(i)} \succ 0$ and $S^T \bar{\Lambda} S \succ 0$.

Since $((S^{(i)})^T \bar{\Lambda}^{(i)} S^{(i)})^{-1} = (S^{(i)})^T (I - 2\lambda\Lambda^{(i)}) S^{(i)} = I - 2\lambda A^{(i)}$ and $(S^T \bar{\Lambda} S)^{-1} = I - 2\lambda A$, we have

$$\left((S^{(i)})^T \bar{\Lambda}^{(i)} S^{(i)}\right)^{-1} \succeq (S^T \bar{\Lambda} S)^{-1} \iff I - 2\lambda A^{(i)} \succeq I - 2\lambda A \iff A \succeq A^{(i)}$$

Thus, $S^T \bar{\Lambda} S \succeq (S^{(i)})^T \bar{\Lambda}^{(i)} S^{(i)}$ holds, implying that $(\mu^i)^T S^T \bar{\Lambda} S \mu^i \geq (\mu^i)^T (S^{(i)})^T \bar{\Lambda}^{(i)} S^{(i)} \mu^i$ for any μ^i . Therefore,

$$\frac{1}{C}\mathbb{E}_{\pi'}\left[e^{\lambda R^{\text{true}}(W)}\right] = \prod_{i=1}^n \frac{\exp\left(\lambda(\mu^i)^T(S^{(i)})^T\bar{\Lambda}^{(i)}S^{(i)}\mu^i\right)}{\prod_{j=1}^n \left(1-2\lambda\eta_j^{(i)}\right)^{1/2}} \leq \prod_{i=1}^n \frac{\exp\left(\lambda(\mu^i)^T S^T \bar{\Lambda} S \mu^i\right)}{\prod_{j=1}^n \left(1-2\lambda\eta_j\right)^{1/2}} = \frac{1}{C}\mathbb{E}_{\pi}\left[e^{\lambda R^{\text{true}}(W)}\right]$$

□

B Further Discussion on the Convergence of Shalaeva's Bound

Another convergence result by Shalaeva et al [45] is that: For any fixed π, ρ, δ such that $D(\rho || \pi) < \infty$, take $\lambda = m^{1/d}$ for some constant $d > 2$, then the right hand side of Shalaeva's bound (See Section 2) converges to the left hand side as $m \rightarrow \infty$.

This result is based on the convergence of the upper bound $\mathbb{E}_{W \sim \pi} \exp\left(\frac{2\lambda^2 v_W^2}{m}\right)$: By taking $\lambda = m^{1/d}$, we have $\mathbb{E}_{W \sim \pi} \exp\left(\frac{2\lambda^2 v_W^2}{m}\right) = \mathbb{E}_{W \sim \pi} \exp(2m^{2/d-1}v_W^2)$. In this case, the following convergence statement

$$\lim_{m \rightarrow \infty} \frac{1}{\lambda} \left[D(\rho \parallel \pi) + \ln \frac{1}{\delta} + \Psi_{\pi, \mathcal{D}}(\lambda, m) \right] \leq \lim_{m \rightarrow \infty} m^{-1/d} \left[D(\rho \parallel \pi) + \ln \frac{1}{\delta} \right] + \lim_{m \rightarrow \infty} m^{-1/d} \ln \mathbb{E}_{W \sim \pi} \exp \left(2m^{2/d-1} v_w^2 \right) = 0$$

holds if

$$\lim_{m \rightarrow \infty} m^{-1/d} \ln \mathbb{E}_{W \sim \pi} \exp \left(2m^{2/d-1} v_w^2 \right) = 0 \quad (38)$$

Shalaeva et al. [45] provided only one condition $d > 2$ to ensure (38), and they did not discuss the choice of π . However, we find that $d > 2$ alone is not sufficient to guarantee convergence, since (38) does not hold for all π . A few examples are given below:

Example B.1. If π is a distribution with bounded support, then (38) holds. This is because there exists a constant $G > 0$ such that $\|W\|_2 < G$. We can show that for any $\lambda > 0$, $\mathbb{E}_{W \sim \pi} \exp(\lambda v_w^2) < \infty$:

$$\begin{aligned} \mathbb{E}_{W \sim \pi} \exp(\lambda v_w^2) &= \mathbb{E}_{W \sim \pi} [\exp(\lambda(\sigma_x^2 \|W^* - W\|_2^2 + \sigma_e^2)^2)] \leq \mathbb{E}_{W \sim \pi} [\exp(\lambda(\sigma_x^2 (\|W^*\|_2 + \|W\|_2)^2 + \sigma_e^2)^2)] \\ &< \mathbb{E}_{W \sim \pi} [\exp(\lambda(\sigma_x^2 (\|W^*\|_2 + G)^2 + \sigma_e^2)^2)] = \exp(\lambda(\sigma_x^2 (\|W^*\|_2 + G)^2 + \sigma_e^2)^2) < \infty \end{aligned}$$

Thus, when $d > 2$,

$$\lim_{m \rightarrow \infty} m^{-1/d} \ln \mathbb{E}_{W \sim \pi} \exp \left(2m^{2/d-1} v_w^2 \right) \leq \lim_{m \rightarrow \infty} m^{-1/d} \ln \mathbb{E}_{W \sim \pi} \exp(2v_w^2) = 0$$

Example B.2. Let π be a Gaussian distribution. We show that (38) does not hold.

We first show that $\mathbb{E}_{W \sim \pi} \exp(\lambda v_w^2) = \infty$ for any $\lambda > 0$. Denote $w = W^* - W \in \mathbb{R}^{1 \times n}$ where $W \sim \pi$, then w is a Gaussian random vector. Thus

$$\begin{aligned} \mathbb{E}_{W \sim \pi} \exp(\lambda v_w^2) &= \mathbb{E}_{W \sim \pi} [\exp(\lambda(\sigma_x^2 \|W^* - W\|_2^2 + \sigma_e^2)^2)] \geq \mathbb{E}_{W \sim \pi} [\exp(\lambda(\sigma_x^2 \|W^* - W\|_2^2)^2)] \\ &= \mathbb{E}_w [\exp(\lambda \sigma_x^4 \|w\|_2^4)] = \mathbb{E}_w \left[\exp \left(\lambda \sigma_x^4 \left(\sum_{i=1}^n w_i^2 \right)^2 \right) \right] \geq \mathbb{E}_w [\exp(\lambda \sigma_x^4 w_1^4)] \end{aligned}$$

Here $w_1 \in \mathbb{R}$ is the first element of w , which is a Gaussian random variable. Suppose $w_1 \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\begin{aligned} \mathbb{E}_w [\exp(\lambda \sigma_x^4 w_1^4)] &= \int \exp(\lambda \sigma_x^4 w_1^4) \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp \left(-\frac{(w_1 - \mu)^2}{2\sigma^2} \right) dw_1 \\ &= \int \frac{1}{\sqrt{2\pi}\sigma} \exp \left(\lambda \sigma_x^4 w_1^4 - \frac{(w_1 - \mu)^2}{2\sigma^2} \right) dw_1 = \infty \end{aligned}$$

because $\lambda \sigma_x^4 w_1^4 - \frac{(w_1 - \mu)^2}{2\sigma^2} \rightarrow \infty$ as $w_1 \rightarrow \infty$. Hence, $\mathbb{E}_{W \sim \pi} \exp(\lambda v_w^2) = \infty$ for any $\lambda > 0$.

If (38) holds, then by the definition of limit, for any $\epsilon > 0$, there exists a finite integer M such that for any $m > M$, $m^{-1/d} \ln \mathbb{E}_{W \sim \pi} \exp(2m^{2/d-1} v_w^2) < \epsilon$. The negation of this statement is that, there exists $\epsilon > 0$ such that for any finite integer M , there exists $m > M$ satisfying $m^{-1/d} \ln \mathbb{E}_{W \sim \pi} \exp(2m^{2/d-1} v_w^2) \geq \epsilon$. Let $\epsilon = 1$, M be any finite integer, and $m = M + 1$. Then $\mathbb{E}_{W \sim \pi} \exp(2m^{2/d-1} v_w^2) = \infty$ and $1 = \epsilon \leq m^{-1/d} \ln \mathbb{E}_{W \sim \pi} \exp(2m^{2/d-1} v_w^2) = \infty$. So the negation is true, and (38) does not hold.

In summary, the validity of (38) depends on π , and a sufficient condition for convergence requires that π be appropriately specified.

C Allowing Multiple Trials on λ

Finding the optimal λ that yields the tightest bound is nontrivial. As suggested in Section 2.1.4 of [4], we approximate the optimal λ by searching over a finite grid $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_L\}$, where each $\lambda_i > 0$ and L denotes the cardinality of Λ .

$$P \left(\forall \lambda \in \Lambda, \forall \rho, \mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)] < \mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)] + \frac{1}{\lambda} \left[D(\rho \parallel \pi) + \ln \frac{L}{\delta} + \Psi_{\pi, \mathcal{D}}(\lambda, m) \right] \right) \geq 1 - \delta$$

This is because

$$\begin{aligned}
& P \left(\forall \lambda \in \Lambda, \forall \rho, \mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)] < \mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)] + \frac{1}{\lambda} \left[D(\rho \parallel \pi) + \ln \frac{L}{\delta} + \Psi_{\pi, \mathcal{D}}(\lambda, m) \right] \right) \\
& = 1 - P \left(\exists \lambda \in \Lambda, \exists \rho, \mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)] \geq \mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)] + \frac{1}{\lambda} \left[D(\rho \parallel \pi) + \ln \frac{L}{\delta} + \Psi_{\pi, \mathcal{D}}(\lambda, m) \right] \right) \\
& = 1 - P \left(\bigcup_{i=1}^L \left\{ \exists \rho, \mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)] \geq \mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)] + \frac{1}{\lambda_i} \left[D(\rho \parallel \pi) + \ln \frac{L}{\delta} + \Psi_{\pi, \mathcal{D}}(\lambda_i, m) \right] \right\} \right) \\
& \geq 1 - \sum_{i=1}^L P \left(\exists \rho, \mathbb{E}_{W \sim \rho} [R^{\text{true}}(W)] \geq \mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)] + \frac{1}{\lambda_i} \left[D(\rho \parallel \pi) + \ln \frac{L}{\delta} + \Psi_{\pi, \mathcal{D}}(\lambda_i, m) \right] \right) \\
& \geq 1 - \sum_{i=1}^L \frac{\delta}{L} = 1 - \delta
\end{aligned}$$

The grid search is a standard method for optimizing λ and is also used by Dziugaite and Roy [13]. Note that λ cannot be directly optimized via gradient descent while keeping ρ and π , as this would make the optimal λ depend on the dataset S , which is a random variable [4, 43]. Consequently, the optimal λ would itself become a random variable, contradicting the assumption that it is fixed.

D Related Works

In statistical learning, generalization bounds commonly provide an upper bound on the generalization gap, $R^{\text{true}}(W) - R^{\text{emp}}(W)$ (or, in the two-sided case, $|R^{\text{true}}(W) - R^{\text{emp}}(W)|$), thereby estimating the true risk $R^{\text{true}}(W)$ for any given model W . A generalization bound is called a Probably Approximately Correct (PAC) bound if it guarantees, with probability at least $1 - \delta$, that the gap does not exceed a small ϵ . It is PAC-Bayesian if, in addition, the model W is treated as a random variable and a KL-divergence term, $D(\rho \parallel \pi)$, is introduced to allow Bayesian inference between the prior π and posterior ρ of W . Classic PAC bounds typically estimate the worst-case generalization gap, $\sup_W \{R^{\text{true}}(W) - R^{\text{emp}}(W)\}$ [51], which often becomes loose as the number of parameters in W increases [39, 41]. In contrast, PAC-Bayes bounds estimate the expected generalization gap, $\mathbb{E}_W [R^{\text{true}}(W) - R^{\text{emp}}(W)]$, which is typically much tighter than the worst-case bound [13].

Early PAC-Bayes bounds, including those by McAllester [37], Catoni [9], Langford and Seeger [29], primarily focus on binary or bounded losses. In recent years, PAC-Bayes bounds have been extended to unbounded losses, as in Alquier et al. [5], Haddouche et al. [19], Haddouche and Guedj [18], Rodríguez-Gálvez et al. [43], Casado et al. [8]. These works typically assume that the loss follows a light-tailed or heavy-tailed distribution. However, such assumptions usually rely on oracle knowledge of the underlying data distribution; consequently, these bounds are often oracle bounds and can only be computed when the true data distribution is known.

Linear regression typically employs the squared loss, which is unbounded. Several PAC-Bayes bounds targeting the generalization gap have been proposed for linear regression. One major line of work adapts the unbounded squared loss to the framework of Alquier et al. [5], such as Germain et al. [17] and Shalaeva et al. [45], which we follow here. Other studies, such as Haddouche et al. [19], consider the ℓ_1 loss for linear regression instead of the standard ℓ_2 (squared) loss. To the best of our knowledge, all these bounds focus on the single-output setting and have not yet been extended to the multivariate case.

Another class of generalization bounds for linear regression targets the excess risk $R^{\text{true}}(W) - R^{\text{true}}(W^*)$, where W^* denotes the minimizer of the true risk. The advantage of this formulation is that, when the tail distribution of the loss satisfies the Bernstein assumption (Definition 4.1, [4]), such bounds (including both PAC and PAC-Bayes types) typically achieve a $1/m$ convergence rate, which is faster than the $1/\sqrt{m}$ rate commonly observed for PAC-Bayes bounds targeting the generalization gap (Section 4.2, [4]). Several bounds for single-output and multivariate linear regression follow this setting, including those by Alquier and Bieu [3], Alquier [2], and Mai [34]. However, these bounds do not depend on the empirical risk, making them less practical for real-world applications.

Collaborative filtering can be viewed as a special case of matrix completion problem, i.e., predicting missing values in a matrix (Section 1.3.1.2, [1]). In recent years, LAEs have become a popular

model for collaborative filtering due to their simplicity and effectiveness. Unlike other models, LAEs have the distinctive property that their training objective, such as in (4), resembles a constrained linear regression problem where the input and target matrices are the same, highlighting their close relationship with linear regression. The earliest LAE model can be traced back to SLIM [40], followed by models such as EASE [48], EDLAE [49] and ELSA [50]. All of these models introduce a zero-diagonal constraint on the weight matrix W , preventing items from learning themselves and thereby avoiding overfitting toward the identity. A recent study by Moon et al. [38] shows that this zero-diagonal constraint can be slightly relaxed to a diagonal with small bounded norm, potentially improving performance.

While the generalizability of LAE models remains unexplored, related work has investigated generalizability in matrix completion for collaborative filtering, including Srebro et al. [47], Shamir et al. [46], and Foygel et al. [16]. Other studies focus on the generalizability of general matrix completion, such as Candès and Tao [7], Recht [42], Ledent et al. [31], and Ledent and Alves [30]. Our work differs from these studies in that we analyze LAEs mainly from the perspective of linear regression rather than matrix completion. In addition, Variational Autoencoders (VAEs) are another type of collaborative filtering model [32], and PAC-Bayes bounds have been developed for VAEs [10]. It should be noted that VAEs and LAEs are fundamentally different models, so the PAC-Bayes bounds for VAEs are not directly comparable to our bounds, as discussed in Appendix E.3.

Therefore, to the best of our knowledge, we propose the first PAC-Bayes bound for multivariate linear regression targeting the generalization gap, and the first PAC-Bayes bound for LAEs.

E Conclusions and Discussions

This paper studies the generalizability of multivariate linear regression and LAEs. We first propose a PAC-Bayes bound for multivariate linear regression under a Gaussian data assumption, extending Shalaeva’s bound for single-output linear regression, and establish a sufficient condition that guarantees convergence. Next, we build the connection between multivariate linear regression and LAE models by introducing a relaxed MSE as an evaluation metric, under which LAE models can be interpreted as multivariate linear regression on bounded data, subject to a zero-diagonal constraint on weights and a hold-out constraint on the input and target data. This connection allows us to adapt our bound for multivariate linear regression to LAEs.

In practice, LAEs are typically large models evaluated on large datasets, which makes computing the tightest bound inefficient. To address this, we develop theoretical methods to improve computational efficiency. Specifically, by restricting both the prior and posterior to be Gaussian, we obtain an efficient sub-optimal bound in closed-form. We then address the computational cost imposed by the zero-diagonal constraint by establishing and computing an upper bound with reduced complexity. Experimental results demonstrate that our bound is tight and correlates strongly with practical metrics such as Recall@K and NDCG@K, suggesting that it effectively reflects the real-world performance of LAE models.

Below are the discussions of our work.

E.1 Limitations

One limitation of our work lies in Algorithm 1, which takes $\Sigma_{hh} = \mathbb{E}_{h \sim \mathcal{M}}[hh^T]$ as input and requires it to be known, which is only possible when \mathcal{M} is non-oracle. Whether \mathcal{M} is non-oracle depends on how the dataset is modeled statistically. If the dataset H is drawn from a meta-dataset H^{whole} that is known and of fixed size, we may assume \mathcal{M} to be the population distribution of this meta-dataset, thereby making \mathcal{M} non-oracle. However, this assumption is rather restrictive: In most real-world scenarios, such a meta-dataset may not exist, as data are continuously collected or expand over time. In this case, \mathcal{M} is typically modeled as an unknown oracle distribution to account for unseen data. Therefore, whether \mathcal{M} is non-oracle ultimately depends on how the dataset is modeled, and our work shows that the bound is at least computable in restricted scenarios where datasets can be represented by a non-oracle \mathcal{M} .

To avoid introducing an oracle \mathcal{M} , one may also consider reconstructing the bound by applying linear regression to empirical PAC-Bayes bounds such as those of McAllester [37], and Langford and Seeger [29], rather than to the Alquier’s oracle bound [5]. However, this approach is not feasible

because empirical PAC-Bayes bounds are typically derived under the assumption of bounded loss, whereas the loss in linear regression is unbounded since W itself is unbounded in $\|y_i - Wx_i\|_F^2$.

Moreover, PAC-Bayes bounds for unbounded losses are inherently oracle, as they are derived from tail-distribution assumptions that are themselves oracle in nature. These distributions are usually characterized by a bounded exponential moment $\mathbb{E}_{X \sim \mathcal{M}} [e^{\lambda(\mathbb{E}_{X \sim \mathcal{M}}[X] - X)}]$ for some $\lambda \in \mathbb{R}$, as in the Hoeffding assumption [5] and the bounded Cumulant Generating Function (CGF) assumption [43, 19]. Assuming these quantities are bounded implicitly presumes access to oracle information about \mathcal{M} , which is rarely realistic in practice.

Another limitation is that the computational methods introduced in Section 5 are primarily designed for full rank or nearly full-rank LAEs. By ‘nearly-full rank’, we refer to matrices of rank $n - 1$ formed by applying the zero-diagonal constraint to a full-rank $n \times n$ matrix. Some LAE models, however, impose low-rank constraints [49] on W , where a $W^{n \times n}$ of rank k ($k < n$) can be decomposed as $W = UV$ with $U^{n \times k}$ and $V^{k \times n}$. Assumption 5.1 does not hold in this case, since it requires W to be a random Gaussian matrix, implying full rank [14]. Consequently, results that rely on this assumption, including Theorems 5.2 and 5.4, are not applicable to low-rank LAEs. One possible approach to adapting our bound for low-rank W is to impose distributional assumptions on U and V so that any realization of $W = UV$ is always low-rank; however, this contradicts Assumption 5.1.

E.2 Comparison with Mai’s Excess Risk Bounds [34]

Mai proposed generalization bounds for bilinear regression (Theorem 2, [34]) and matrix completion (Theorem 3, [34]) based on excess risk. Their bilinear regression setting is defined as follows: Given two input matrices $Z \in \mathbb{R}^{p \times r}$ and $X \in \mathbb{R}^{n \times m}$, let $Y \in \mathbb{R}^{p \times m}$ be the target matrix following a distribution conditioned on Z and X , then there exists $W^* \in \mathbb{R}^{r \times n}$ such that $Y = ZW^*X + E$, where $E \in \mathbb{R}^{p \times m}$ is a random noise matrix whose entries E_{ij} has zero mean and finite variance. If taking $r = p$ and $Z = I$, it reduces to the multivariate linear regression presented in Section 2.

Given a model $W \in \mathbb{R}^{p \times m}$, let the true risk be defined as $R^{\text{true}}(W) = \frac{1}{pm} \mathbb{E}_{Y|Z,X} [\|Y - ZWX\|_F^2]$, then the excess risk can be expressed as

$$R^{\text{true}}(W) - R^{\text{true}}(W^*) = \frac{1}{pm} \|ZWX - ZW^*X\|_F^2$$

If we further suppose that W is a random variable following a distribution $\hat{\rho}$, and define the expected excess risk as $\mathbb{E}_{W \sim \hat{\rho}} [R^{\text{true}}(W)] - R^{\text{true}}(W^*)$, then by Jensen’s inequality we have

$$\mathbb{E}_{W \sim \hat{\rho}} [R^{\text{true}}(W)] - R^{\text{true}}(W^*) \geq \frac{1}{pm} \|\mathbb{E}_{W \sim \hat{\rho}} [ZWX] - ZW^*X\|_F^2 \quad (39)$$

Mai first derived a generalization bound for bilinear regression by establishing an upper bound on $\|\mathbb{E}_{W \sim \hat{\rho}} [ZWX] - ZW^*X\|_F^2$. They then adapted this bound to the matrix completion setting, which assumes that k ($k < pm$) pairs in $\{(ZW^*X)_{ij}, Y_{ij}\}$ are observed, where the trained model W is used to recover the remaining pairs. Further, their bound converges linearly with respect to k .

It is well known that if the loss satisfies Bernstein assumption (Definition 4.1, [4]), one can construct an upper bound on the excess risk (or the expected excess risk) that converges at a linear rate of $1/m$. By (39), Mai’s bound is indeed an upper bound on a *lower bound* of the excess risk, which makes its linear convergence rate reasonable.

In contrast, our bounds are based on the generalization gap $R^{\text{true}}(W) - R^{\text{emp}}(W)$, rather than the excess risk. PAC-Bayes bounds on the generalization gap typically converge at a slower rate of $1/\sqrt{m}$ (Section 4.2, [4]). The Bernstein assumption, which enables linear convergence for excess risk bounds, is not applicable in the setting of the generalization gap.

E.3 Comparison with PAC-Bayes Bounds for Variational Autoencoders [10]

Like LAEs, Variational Autoencoders (VAEs) are another class of autoencoders that have been applied to collaborative filtering [32]. Recently, PAC-Bayes bounds for VAEs were proposed by Chérief-Abdellatif et al. [10].

Although LAEs and VAEs are both autoencoders, they differ fundamentally in model architecture, learning objectives, and loss formulations, making our bound for LAEs not directly comparable to the PAC-Bayes bound for VAEs.

In detail, the architecture of VAEs [10] is:

$$\text{input } \mathbf{x} \xrightarrow{\text{encoder } q_\phi(\mathbf{z}|\mathbf{x})} \text{latent code } \mathbf{z} \xrightarrow{\text{decoder } p_\theta(\mathbf{x}|\mathbf{z})} \text{prediction} \approx \text{target } \mathbf{x}$$

with the loss defined as $-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})}[\log p_\theta(\mathbf{x}|\mathbf{z})]$.

And the architecture of LAEs is

$$\text{input } \mathbf{x} \xrightarrow{\text{encoder and decoder } W} \text{prediction } W\mathbf{x} \approx \text{target } \mathbf{y}$$

with the loss defined as the relaxed MSE $\|\mathbf{y} - W\mathbf{x}\|_F^2$ in our framework, see Section 4.2.

The differences between VAEs and LAEs are summarized as follows:

Difference in Architecture: A VAE consists of an encoder and a decoder and explicitly involves the latent code \mathbf{z} . In an LAE, the model W serves as both the encoder and the decoder, with the latent code being implicit. While one could decompose $W = AB$ and treat B as the encoder and A as the decoder, such a decomposition is uncommon in recommender systems. Most LAE recommender models do not decompose W [40, 48, 49, 50]; instead, they focus on studying W directly by introducing constraints such as a zero diagonal. As a result, LAEs and VAEs are generally regarded as two distinct model classes.

Difference in Input and Target: VAEs require the input and target to be identical. In contrast, for LAEs, the input \mathbf{x} and target \mathbf{y} are the same during training but differ during evaluation due to the hold-out constraint. Our bound is defined for evaluation, where \mathbf{x} and \mathbf{y} are indeed different. This fundamental difference in input-target configuration makes LAEs incompatible with the VAE framework.

Difference in Loss: The VAE loss $-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})}[\log p_\theta(\mathbf{x}|\mathbf{z})]$ aims to minimize the mismatch between the encoder $q_\phi(\mathbf{z}|\mathbf{x})$ and the decoder $p_\theta(\mathbf{x}|\mathbf{z})$, while the LAE loss $\|\mathbf{y} - W\mathbf{x}\|_F^2$ aims to minimize the mismatch between the target \mathbf{y} and the prediction $W\mathbf{x}$. These losses reflect fundamentally different learning objectives, so the two models cannot share the same loss function. Moreover, the LAE loss aligns more closely with a multivariate linear regression loss than with the VAE loss, which is why our bound for multivariate linear regression can be naturally extended to LAE models.

Therefore, due to the fundamental differences between VAEs and LAEs, our bound for LAEs is not directly comparable to the PAC-Bayes bound for VAEs proposed by Chérif-Abdellatif et al. [10].

E.4 Relationship between MSE and Ranking Metrics

While minimizing MSE is not theoretically consistent with achieving optimal ranking, empirical studies have observed a strong correlation between reductions in MSE and improvements in ranking metrics [26]. Moreover, ranking metrics are typically set-based, discrete, and non-differentiable, making them difficult to optimize directly using gradient-based methods. Consequently, it has become standard practice in collaborative filtering to employ regression-style or reconstruction-based surrogate losses – most commonly the squared error – as training objectives (Section 2.6, [1]), which resemble the MSE used during evaluation. This approach underlies many successful algorithms, including SLIM [40], EASE [48], EDLAE [49], ELSA [50], and matrix factorization models [23, 27], which consistently demonstrate strong empirical performance on ranking tasks despite optimizing a non-ranking loss.

Since LAEs typically use a linear regression loss for training and ranking metrics for evaluation – and model performance is ultimately measured by the latter – we derive our generalization bound solely with respect to the evaluation metric. As discussed in Section 4.2, because ranking metrics are difficult to analyze statistically, we instead adopt MSE as the evaluation metric, which follows the form of a linear regression loss but differs from the training loss in both definition and purpose.

E.5 Broader Impacts

This work advances the theoretical foundations of machine learning by introducing the first PAC-Bayes bound for multivariate linear regression targeting generalization gap, extending beyond single-output regression to handle multiple dependent variables simultaneously. This establishes new generalization guarantees for structured prediction, multi-task learning, and recommendation systems.

Additionally, we identify and correct a limitation in an existing PAC-Bayes proof for single-output linear regression, further strengthening the theoretical foundation of regression analysis.

Building on this, we apply our bound to LAEs in recommendation systems, delivering their first rigorous generalization analysis. Our approach accounts for key structural constraints, such as the zero-diagonal weight requirement, ensuring applicability to models like EASE and EDLAE.

Beyond theory, our work has direct practical implications for model evaluation and selection. Our bound provides a post-training diagnostic tool for assessing the generalization of any LAE model, regardless of its training process. While not directly guiding training or hyperparameter tuning, a smaller PAC-Bayes bound suggests better generalization on unseen data. Empirical results confirm that our bound remains within a reasonable multiple of the test error, offering reliable probabilistic estimates of true risk independent of training error.

Our work focuses on theoretical generalization analysis and poses no immediate ethical risks. However, recommendation systems shape content exposure and user behavior in domains like e-commerce and social media. Strengthening generalization theory alongside other recommendation criteria may help mitigate bias, enhance fairness, and improve trust in AI-driven systems.

F Other Supplemental Materials

F.1 Details of Algorithm 1

Here we provide details on the computation of $\mathbb{E}_{W \sim \rho}[R^{\text{emp}}(W)]$, $\mathbb{E}_{W \sim \rho}[R^{\text{true}}(W)]$ and $D(\rho \parallel \pi)$ in Algorithm 1, which are not fully described in the main paper.

Given λ and $\pi = \mathcal{N}(\mathcal{U}_0, \sigma^2 I)$, the optimal $\rho = \mathcal{N}(\mathcal{U}, \mathcal{S})$ that minimizes the right hand side of (14) is obtained by Theorem 5.2. Once ρ is obtained, we can compute $\mathbb{E}_{W \sim \rho}[R^{\text{emp}}(W)]$ by (24), which can be simplified as

$$\mathbb{E}_{W \sim \rho}[R^{\text{emp}}(W)] = \frac{1}{m} \|Y - \mathcal{U}X\|_F^2 + \frac{n-1}{m} \|\text{diag}(\mathcal{S}_{1*})^{1/2}X\|_F^2 \quad (40)$$

The $n-1$ term in (40) is due to the zero-diagonal constraint, which enforces $\text{diag}(\mathcal{S}) = 0$. Without this constraint, the term becomes n instead of $n-1$.

Similarly, by (6), $\mathbb{E}_{W \sim \rho}[R^{\text{true}}(W)]$ can be expressed as

$$\begin{aligned} \mathbb{E}_{W \sim \rho}[R^{\text{true}}(W)] &= \|\Sigma_{xy}^T \Sigma_{xx}^{-1/2} - \mathcal{U} \Sigma_{xx}^{1/2}\|_F^2 + (n-1) \|\text{diag}(\mathcal{S}_{1*})^{1/2} \Sigma_{xx}^{1/2}\|_F^2 \\ &\quad + \text{tr}(\Sigma_{yy}) - \|\Sigma_{xy}^T \Sigma_{xx}^{-1/2}\|_F^2 \end{aligned} \quad (41)$$

The derivation of (41) is analogous to (24) by substituting Y with $\Sigma_{xy}^T \Sigma_{xx}^{-1/2}$ and X with $\Sigma_{xx}^{-1/2}$.

The $D(\rho \parallel \pi)$ term under zero-diagonal constraint is obtained from (25) by removing the diagonal elements of \mathcal{S} :

$$D(\rho \parallel \pi) = \frac{1}{2} \left[(n^2 - n)(2 \ln \sigma - 1) - \sum_{k=1}^n \sum_{l=1, l \neq k}^n (\ln \mathcal{S}_{kl} - \frac{\mathcal{S}_{kl}}{\sigma^2}) + \frac{\|\mathcal{U} - \mathcal{U}_0\|_F^2}{\sigma^2} \right] \quad (42)$$

F.2 Dataset Description

The following table shows the details of the datasets used in the experiments.

Table 2: Dataset description

Dataset	ML 20M	Netflix	MSD
#users (m)	138493	480189	1017982
#items (n)	26744	17770	40000
#interactions	2000263	100480507	33687193

F.3 Details of the Results in Table 1

The following table presents the detailed values of the components of the RH terms in Table 1, illustrating how the results were obtained. This information may be helpful for reproducing the experiments.

Table 3: Details of the terms of each RH in Table 1

Models	PAC-Bayes Bound for LAEs			
	ML 20M	Netflix	MSD	
$\gamma = 50$	λ	512	512	512
	$\mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)]$	66.99	90.87	16.58
	$D(\rho \parallel \pi)$	0.28	0.18	0.0019
$\gamma = 100$	$\ln \mathbb{E}_{\pi} [e^{\lambda R^{\text{true}}(W)}]$	31571.14	44659.37	8196.30
	λ	512	512	512
	$\mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)]$	65.14	89.68	16.34
$\gamma = 200$	$D(\rho \parallel \pi)$	0.27	0.17	0.0018
	$\ln \mathbb{E}_{\pi} [e^{\lambda R^{\text{true}}(W)}]$	31102.53	44313.39	8141.72
	λ	512	512	512
$\gamma = 500$	$\mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)]$	63.59	88.57	16.12
	$D(\rho \parallel \pi)$	0.26	0.17	0.0018
	$\ln \mathbb{E}_{\pi} [e^{\lambda R^{\text{true}}(W)}]$	30753.19	44014.86	8092.62
$\gamma = 1000$	λ	512	512	512
	$\mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)]$	61.93	87.26	15.89
	$D(\rho \parallel \pi)$	0.23	0.17	0.0016
$\gamma = 2000$	$\ln \mathbb{E}_{\pi} [e^{\lambda R^{\text{true}}(W)}]$	30444.39	43703.53	8044.64
	λ	512	512	512
	$\mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)]$	60.96	86.42	15.79
$\gamma = 5000$	$D(\rho \parallel \pi)$	0.23	0.16	0.0016
	$\ln \mathbb{E}_{\pi} [e^{\lambda R^{\text{true}}(W)}]$	30310.47	43522.96	8033.10
	λ	512	512	512
$\gamma = 5000$	$\mathbb{E}_{W \sim \rho} [R^{\text{emp}}(W)]$	60.23	85.71	15.78
	$D(\rho \parallel \pi)$	0.22	0.15	0.0015
	$\ln \mathbb{E}_{\pi} [e^{\lambda R^{\text{true}}(W)}]$	30255.43	43382.46	8052.90

NeurIPS Paper Checklist

1. Claims

Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

Answer: [\[Yes\]](#)

Justification: We have accurately stated our contributions in Abstract and Introduction.

Guidelines:

- The answer NA means that the abstract and introduction do not include the claims made in the paper.
- The abstract and/or introduction should clearly state the claims made, including the contributions made in the paper and important assumptions and limitations. A No or NA answer to this question will not be perceived well by the reviewers.
- The claims made should match theoretical and experimental results, and reflect how much the results can be expected to generalize to other settings.
- It is fine to include aspirational goals as motivation as long as it is clear that these goals are not attained by the paper.

2. Limitations

Question: Does the paper discuss the limitations of the work performed by the authors?

Answer: [\[Yes\]](#)

Justification: We have discussed the limitations of our work in Appendix E.

Guidelines:

- The answer NA means that the paper has no limitation while the answer No means that the paper has limitations, but those are not discussed in the paper.
- The authors are encouraged to create a separate "Limitations" section in their paper.
- The paper should point out any strong assumptions and how robust the results are to violations of these assumptions (e.g., independence assumptions, noiseless settings, model well-specification, asymptotic approximations only holding locally). The authors should reflect on how these assumptions might be violated in practice and what the implications would be.
- The authors should reflect on the scope of the claims made, e.g., if the approach was only tested on a few datasets or with a few runs. In general, empirical results often depend on implicit assumptions, which should be articulated.
- The authors should reflect on the factors that influence the performance of the approach. For example, a facial recognition algorithm may perform poorly when image resolution is low or images are taken in low lighting. Or a speech-to-text system might not be used reliably to provide closed captions for online lectures because it fails to handle technical jargon.
- The authors should discuss the computational efficiency of the proposed algorithms and how they scale with dataset size.
- If applicable, the authors should discuss possible limitations of their approach to address problems of privacy and fairness.
- While the authors might fear that complete honesty about limitations might be used by reviewers as grounds for rejection, a worse outcome might be that reviewers discover limitations that aren't acknowledged in the paper. The authors should use their best judgment and recognize that individual actions in favor of transparency play an important role in developing norms that preserve the integrity of the community. Reviewers will be specifically instructed to not penalize honesty concerning limitations.

3. Theory assumptions and proofs

Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

Answer: [\[Yes\]](#)

Justification: We provide full set of assumptions (Assumption 3.1, 4.1 and 5.1), and all proofs are presented in Appendix A. We have aimed to make the proofs both complete and easy to follow.

Guidelines:

- The answer NA means that the paper does not include theoretical results.
- All the theorems, formulas, and proofs in the paper should be numbered and cross-referenced.
- All assumptions should be clearly stated or referenced in the statement of any theorems.
- The proofs can either appear in the main paper or the supplemental material, but if they appear in the supplemental material, the authors are encouraged to provide a short proof sketch to provide intuition.
- Inversely, any informal proof provided in the core of the paper should be complemented by formal proofs provided in appendix or supplemental material.
- Theorems and Lemmas that the proof relies upon should be properly referenced.

4. Experimental result reproducibility

Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

Answer: [Yes]

Justification: We have disclosed all the information needed to reproduce the main experimental results in Section 5.3 and Section 6.

Guidelines:

- The answer NA means that the paper does not include experiments.
- If the paper includes experiments, a No answer to this question will not be perceived well by the reviewers: Making the paper reproducible is important, regardless of whether the code and data are provided or not.
- If the contribution is a dataset and/or model, the authors should describe the steps taken to make their results reproducible or verifiable.
- Depending on the contribution, reproducibility can be accomplished in various ways. For example, if the contribution is a novel architecture, describing the architecture fully might suffice, or if the contribution is a specific model and empirical evaluation, it may be necessary to either make it possible for others to replicate the model with the same dataset, or provide access to the model. In general, releasing code and data is often one good way to accomplish this, but reproducibility can also be provided via detailed instructions for how to replicate the results, access to a hosted model (e.g., in the case of a large language model), releasing of a model checkpoint, or other means that are appropriate to the research performed.
- While NeurIPS does not require releasing code, the conference does require all submissions to provide some reasonable avenue for reproducibility, which may depend on the nature of the contribution. For example
 - (a) If the contribution is primarily a new algorithm, the paper should make it clear how to reproduce that algorithm.
 - (b) If the contribution is primarily a new model architecture, the paper should describe the architecture clearly and fully.
 - (c) If the contribution is a new model (e.g., a large language model), then there should either be a way to access this model for reproducing the results or a way to reproduce the model (e.g., with an open-source dataset or instructions for how to construct the dataset).
 - (d) We recognize that reproducibility may be tricky in some cases, in which case authors are welcome to describe the particular way they provide for reproducibility. In the case of closed-source models, it may be that access to the model is limited in some way (e.g., to registered users), but it should be possible for other researchers to have some path to reproducing or verifying the results.

5. Open access to data and code

Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

Answer: [\[Yes\]](#)

Justification: We have provided the code and the links to the open datasets (MovieLens 20M, Netflix, MSD) in the supplemental material.

Guidelines:

- The answer NA means that paper does not include experiments requiring code.
- Please see the NeurIPS code and data submission guidelines (<https://nips.cc/public/guides/CodeSubmissionPolicy>) for more details.
- While we encourage the release of code and data, we understand that this might not be possible, so “No” is an acceptable answer. Papers cannot be rejected simply for not including code, unless this is central to the contribution (e.g., for a new open-source benchmark).
- The instructions should contain the exact command and environment needed to run to reproduce the results. See the NeurIPS code and data submission guidelines (<https://nips.cc/public/guides/CodeSubmissionPolicy>) for more details.
- The authors should provide instructions on data access and preparation, including how to access the raw data, preprocessed data, intermediate data, and generated data, etc.
- The authors should provide scripts to reproduce all experimental results for the new proposed method and baselines. If only a subset of experiments are reproducible, they should state which ones are omitted from the script and why.
- At submission time, to preserve anonymity, the authors should release anonymized versions (if applicable).
- Providing as much information as possible in supplemental material (appended to the paper) is recommended, but including URLs to data and code is permitted.

6. Experimental setting/details

Question: Does the paper specify all the training and test details (e.g., data splits, hyper-parameters, how they were chosen, type of optimizer, etc.) necessary to understand the results?

Answer: [\[Yes\]](#)

Justification: We have specified all experimental settings and details in Section 6.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The experimental setting should be presented in the core of the paper to a level of detail that is necessary to appreciate the results and make sense of them.
- The full details can be provided either with the code, in appendix, or as supplemental material.

7. Experiment statistical significance

Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

Answer: [\[No\]](#)

Justification: Our models are obtained from closed-form solution, which is deterministic and does not involve random errors.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The authors should answer “Yes” if the results are accompanied by error bars, confidence intervals, or statistical significance tests, at least for the experiments that support the main claims of the paper.
- The factors of variability that the error bars are capturing should be clearly stated (for example, train/test split, initialization, random drawing of some parameter, or overall run with given experimental conditions).

- The method for calculating the error bars should be explained (closed form formula, call to a library function, bootstrap, etc.)
- The assumptions made should be given (e.g., Normally distributed errors).
- It should be clear whether the error bar is the standard deviation or the standard error of the mean.
- It is OK to report 1-sigma error bars, but one should state it. The authors should preferably report a 2-sigma error bar than state that they have a 96% CI, if the hypothesis of Normality of errors is not verified.
- For asymmetric distributions, the authors should be careful not to show in tables or figures symmetric error bars that would yield results that are out of range (e.g. negative error rates).
- If error bars are reported in tables or plots, The authors should explain in the text how they were calculated and reference the corresponding figures or tables in the text.

8. Experiments compute resources

Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

Answer: [\[Yes\]](#)

Justification: We have provided information on the computer resources in Section 6 and supplemental material.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The paper should indicate the type of compute workers CPU or GPU, internal cluster, or cloud provider, including relevant memory and storage.
- The paper should provide the amount of compute required for each of the individual experimental runs as well as estimate the total compute.
- The paper should disclose whether the full research project required more compute than the experiments reported in the paper (e.g., preliminary or failed experiments that didn't make it into the paper).

9. Code of ethics

Question: Does the research conducted in the paper conform, in every respect, with the NeurIPS Code of Ethics <https://neurips.cc/public/EthicsGuidelines>?

Answer: [\[Yes\]](#)

Justification: We have reviewed NeurIPS Code of Ethics and ensured that our research conforms to it in every respect.

Guidelines:

- The answer NA means that the authors have not reviewed the NeurIPS Code of Ethics.
- If the authors answer No, they should explain the special circumstances that require a deviation from the Code of Ethics.
- The authors should make sure to preserve anonymity (e.g., if there is a special consideration due to laws or regulations in their jurisdiction).

10. Broader impacts

Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

Answer: [\[Yes\]](#)

Justification: We have discussed the broader impacts in Appendix E.

Guidelines:

- The answer NA means that there is no societal impact of the work performed.
- If the authors answer NA or No, they should explain why their work has no societal impact or why the paper does not address societal impact.

- Examples of negative societal impacts include potential malicious or unintended uses (e.g., disinformation, generating fake profiles, surveillance), fairness considerations (e.g., deployment of technologies that could make decisions that unfairly impact specific groups), privacy considerations, and security considerations.
- The conference expects that many papers will be foundational research and not tied to particular applications, let alone deployments. However, if there is a direct path to any negative applications, the authors should point it out. For example, it is legitimate to point out that an improvement in the quality of generative models could be used to generate deepfakes for disinformation. On the other hand, it is not needed to point out that a generic algorithm for optimizing neural networks could enable people to train models that generate Deepfakes faster.
- The authors should consider possible harms that could arise when the technology is being used as intended and functioning correctly, harms that could arise when the technology is being used as intended but gives incorrect results, and harms following from (intentional or unintentional) misuse of the technology.
- If there are negative societal impacts, the authors could also discuss possible mitigation strategies (e.g., gated release of models, providing defenses in addition to attacks, mechanisms for monitoring misuse, mechanisms to monitor how a system learns from feedback over time, improving the efficiency and accessibility of ML).

11. Safeguards

Question: Does the paper describe safeguards that have been put in place for responsible release of data or models that have a high risk for misuse (e.g., pretrained language models, image generators, or scraped datasets)?

Answer: [NA]

Justification: Our work is mainly theoretical and is unlikely to involve risks or potential misuse in application.

Guidelines:

- The answer NA means that the paper poses no such risks.
- Released models that have a high risk for misuse or dual-use should be released with necessary safeguards to allow for controlled use of the model, for example by requiring that users adhere to usage guidelines or restrictions to access the model or implementing safety filters.
- Datasets that have been scraped from the Internet could pose safety risks. The authors should describe how they avoided releasing unsafe images.
- We recognize that providing effective safeguards is challenging, and many papers do not require this, but we encourage authors to take this into account and make a best faith effort.

12. Licenses for existing assets

Question: Are the creators or original owners of assets (e.g., code, data, models), used in the paper, properly credited and are the license and terms of use explicitly mentioned and properly respected?

Answer: [Yes]

Justification: We use open datasets (MovieLens 20M, Netflix, MSD) in accordance with their licenses, and we do not use existing code or models.

Guidelines:

- The answer NA means that the paper does not use existing assets.
- The authors should cite the original paper that produced the code package or dataset.
- The authors should state which version of the asset is used and, if possible, include a URL.
- The name of the license (e.g., CC-BY 4.0) should be included for each asset.
- For scraped data from a particular source (e.g., website), the copyright and terms of service of that source should be provided.

- If assets are released, the license, copyright information, and terms of use in the package should be provided. For popular datasets, paperswithcode.com/datasets has curated licenses for some datasets. Their licensing guide can help determine the license of a dataset.
- For existing datasets that are re-packaged, both the original license and the license of the derived asset (if it has changed) should be provided.
- If this information is not available online, the authors are encouraged to reach out to the asset's creators.

13. New assets

Question: Are new assets introduced in the paper well documented and is the documentation provided alongside the assets?

Answer: [\[Yes\]](#)

Justification: We have included a README file as documentation along with the code.

Guidelines:

- The answer NA means that the paper does not release new assets.
- Researchers should communicate the details of the dataset/code/model as part of their submissions via structured templates. This includes details about training, license, limitations, etc.
- The paper should discuss whether and how consent was obtained from people whose asset is used.
- At submission time, remember to anonymize your assets (if applicable). You can either create an anonymized URL or include an anonymized zip file.

14. Crowdsourcing and research with human subjects

Question: For crowdsourcing experiments and research with human subjects, does the paper include the full text of instructions given to participants and screenshots, if applicable, as well as details about compensation (if any)?

Answer: [\[NA\]](#)

Justification: Our paper does not involve crowdsourcing nor research with human subjects.

Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Including this information in the supplemental material is fine, but if the main contribution of the paper involves human subjects, then as much detail as possible should be included in the main paper.
- According to the NeurIPS Code of Ethics, workers involved in data collection, curation, or other labor should be paid at least the minimum wage in the country of the data collector.

15. Institutional review board (IRB) approvals or equivalent for research with human subjects

Question: Does the paper describe potential risks incurred by study participants, whether such risks were disclosed to the subjects, and whether Institutional Review Board (IRB) approvals (or an equivalent approval/review based on the requirements of your country or institution) were obtained?

Answer: [\[NA\]](#)

Justification: Our paper does not involve crowdsourcing nor research with human subjects.

Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Depending on the country in which research is conducted, IRB approval (or equivalent) may be required for any human subjects research. If you obtained IRB approval, you should clearly state this in the paper.

- We recognize that the procedures for this may vary significantly between institutions and locations, and we expect authors to adhere to the NeurIPS Code of Ethics and the guidelines for their institution.
- For initial submissions, do not include any information that would break anonymity (if applicable), such as the institution conducting the review.

16. Declaration of LLM usage

Question: Does the paper describe the usage of LLMs if it is an important, original, or non-standard component of the core methods in this research? Note that if the LLM is used only for writing, editing, or formatting purposes and does not impact the core methodology, scientific rigorousness, or originality of the research, declaration is not required.

Answer: [NA]

Justification: We use LLMs for writing purposes and for generating code unrelated to the core methodology. All code related to the core methodology is written by the authors.

Guidelines:

- The answer NA means that the core method development in this research does not involve LLMs as any important, original, or non-standard components.
- Please refer to our LLM policy (<https://neurips.cc/Conferences/2025/LLM>) for what should or should not be described.