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Parameter-dependent filtering of Gaussian processes in Hilbert spaces

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ABSTRACT

The filtering problem for non-Markovian Gaussian processes on rigged Hilbert spaces is considered. Continuous dependence of the filter and observation error on parameters which may be present both in the signal and observation processes is proved. The general results are applied to signals governed by stochastic heat equations driven by distributed or pointwise fractional noise. The observation process may be a noisy observation of the signal at given points in the domain, the position of which may depend on the parameter.

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Introduction

Signal processes defined by linear stochastic partial differential equations may be understood as Gaussian Hilbert space-valued processes, where the state space is an appropriate function space. The filtering problem for such signals in the case when observations are finite-dimensional and perturbed by noise white in time has been recently solved in [11, 12]. We are not aware of any other results on filtering of Gaussian processes that would be applicable to stochastic PDEs except for the standard case of Kalman–Bucy filter, in which the signal process is Markovian (cf. [5] for a pioneering result in this direction). Of course, much more is known about the "dual" LQ control problem which has been treated, for instance, in [3] and [2], and related statistical inference problems that were addressed in numerous papers, like [1, 13] or [10]. In finite-dimensional state spaces there are several papers by Kleptsyna et al. ([7, 9] or [8]) where the filtering problem for non-Markovian Gaussian signal is studied.

In the above-mentioned papers [11, 12], the signal is a general infinite-dimensional Gaussian process with a known covariance operator and the observation is finite-dimensional. The covariance of the observation error is shown to satisfy a Hilbert space-valued nonlinear integral equation and the filter itself satisfies a linear stochastic equation in infinite dimensions (containing the covariance of the observation error). A typical example of signal process covered by this approach is the one governed by stochastic heat equation driven by space-dependent Brownian motion fractional in time

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$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) = &\quad \Delta u(t,x) + \eta^h(t,x), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ u(t,\cdot)_{|\partial\mathcal{D}} = &\quad 0, \quad t \in [0,T], \end{aligned} \tag{0.1}$$

where $\mathcal{D} \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\mathcal{D}$, Δ is the Laplace operator, and the noise η^h is viewed as a fractional noise with the Hurst parameter h.

The observation is given by the stochastic differential

$$\begin{aligned} d\xi_t &= A(u(t,\cdot))\,dt + dW_t, \quad t \in [0,T], \\ \xi_0 &= 0, \end{aligned}$$

where A is the operator of pointwise observation at given points $z_1, ..., z_n$ in the domain $\mathcal{D}$,

$$A(\varphi) = (\varphi(z_1), ..., \varphi(z_n)), \tag{0.2}$$

defined on a suitable function space and W is an $\mathbb{R}^n$-valued Wiener process which is independent of η^h.

The filtering problem here and in similar cases is relatively delicate because the observations are well defined only if the signal process is sufficiently regular and so it is interesting to study the stability of the filter with respect to small changes of the observation points.

In this article, we solve a more general problem of continuous dependence of the filter on parameters which may be present both in signal and observation processes. This research is motivated by high level of uncertainty in models described by stochastic partial differential equations on one side and space-time irregularity of solutions on the other hand. The signal equations usually contain parameters that are not precisely known and often must be estimated in longer run of the process. Then it is important that the signal (as well as the filter) is sufficiently robust with respect to small changes of parameters. Also, the observations are noisy as well (nondegeneracy of the noise in observation equation is a crucial condition for a Kalman–Bucy type result) and it is of interest whether the filter depends continuously on the changes of the "parameters" of observations.

We consider the abstract setting as in [12]. So the signal is a Hilbert space-valued parameter-dependent Gaussian process and the observation is given by stochastic differential, the coefficients of which may also depend on the parameter. The above-mentioned lack of regularity is overcomed by posing the equation on a rigged Hilbert space.

The article is divided into five sections. In Section 2, the basic setting is explained in detail and the main result from [12] is recalled, where the forms of equations for filter and observation error are derived and existence and uniqueness of their solutions are proved. Section 3, which contains the heart of the proof of our main result, is devoted to continuous dependence of the covariance of observation error (Theorem 2.1). As mentioned above, this mapping satisfies a nonlinear integral equation with non-Lipschitz right-hand side. Hence it does not seem to be possible to proceed in a standard way by means of the Gronwall lemma. We use here a method based on compactness of the family of solutions (which is proved by Arzela–Ascoli theorem for mappings taking values in operator spaces, utilizing so-called collective compactness of solutions and their adjoints, cf. [14]). Section 4 contains the main result of the article (Theorem 3.1), the proof of continuous dependence of the filter (finite-dimensional analogue of this statement may be found in the unpublished work [16]). In Section 5, these results are applied in the case of signal defined by linear stochastic partial differential equation (SPDE) driven by cylindrical fractional Brownian motion. Two examples of signal are then elaborated in more detail: The heat equation perturbed either with distributed or by pointwise fractional noise. Observations are finite-dimensional and the case of pointwise observation at some points in the domain, that may depend on the parameter, is also considered.

The space of bounded linear operators mapping a Banach space X to a Banach space Y is denoted as $\mathcal{L}(X,Y)$, $\mathcal{L}(X) := \mathcal{L}(X,X)$. The space of Hilbert–Schmidt operators from a Hilbert space H into a Hilbert space V is denoted as $\mathcal{L}_2(H,V)$, $\mathcal{L}_2(H) := \mathcal{L}_2(H,H)$. Similarly, $\mathcal{L}_1(H)$ indicates the space of trace class operators on H.

1. Preliminaries

Let $H = (H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a separable Hilbert space. Consider a selfadjoint positive operator $\mathcal{B}$ on H with a compact resolvent. For $\alpha > 0$ consider a Hilbert space $V_\alpha = (V_\alpha, \langle\cdot,\cdot\rangle_{V_\alpha}, \|\cdot\|_{V_\alpha})$ defined by the fractional power of operator $\mathcal{B}$ as $V_\alpha = Dom(\mathcal{B}^\alpha)$ equipped with a graph norm $\|\cdot\|_{V_\alpha}$. Then (H, V_α) form together a rigged separable Hilbert space such that $V_\alpha \subset H$ and identifying H with the dual H^* the embeddings

$$V_\alpha \hookrightarrow H = H^* \hookrightarrow V_\alpha^*$$

are continuous and dense. The duality pairing between V_α and V_α^* is defined by the usual extension of the form $\langle u, v\rangle_{V_\alpha}, V_\alpha^* = \langle u, v\rangle_H$ for $u \in V_\alpha \subset H$ and $v \in H \subset V_\alpha^*$.

For arbitrary $x, y \in V_\alpha$ we define tensor product $x \circ y \in \mathcal{L}(V_\alpha^*, V_\alpha)$, $(x \circ y)v = x\langle y, v\rangle_{V_\alpha}, V_\alpha^*$, $v \in V_\alpha^*$.

Consider a stochastic basis $(\Omega, F, P, (F_t))$, a compact set of parameters Λ and a family of signals $\{\theta_t^\lambda, t \in [0,T], \lambda \in \Lambda\}$ that are (F_t) – progressively measurable centered Gaussian processes with paths P-a.s. in $L^2([0,T], V_{\alpha+\vartheta})$ for some $\alpha > \vartheta > 0$ such that

$$\sup_{\lambda\in\Lambda,\, t\in[0,T]} \mathbb{E}\|\theta_t^\lambda\|_{V_{\alpha+\vartheta}}^2 < \infty \tag{1.1}$$

and for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $\lambda \in \Lambda$ and for all $s, t \in [0,T]$, $|t - s| < \delta$

$$\mathbb{E}\|\theta_t^\lambda - \theta_s^\lambda\|_{V_\alpha}^2 < \epsilon. \tag{1.2}$$

The Equation (1.1) will be referred to as uniform boundedness of $\{\theta^\lambda\}$ and the property (1.2) as (mean - square) equicontinuity.

For every $\lambda \in \Lambda$ let $\xi^\lambda = \{\xi_t^\lambda, t \in [0,T]\}$ denotes an $\mathbb{R}^n$ -valued observation process given as

$$\xi_t^\lambda = \int_0^t A^\lambda(s)\theta_s^\lambda \mathrm{d}s + W_t, \quad t \in [0,T], \tag{1.3}$$

where $(A^\lambda(s))_{s\in[0,T]}$ is a family of linear operators $V_\alpha \to \mathbb{R}^n$ such that for every $\lambda \in \Lambda$ the mapping $t \to A^\lambda(t)$ is strongly measurable and uniformly bounded, that is,

$$\sup_{t\in[0,T],\,\lambda\in\Lambda} \|A^\lambda(t)\|_{\mathcal{L}(V_\alpha,\mathbb{R}^n)} < \infty.$$

Here $W = \{W_t, t \in [0, T]\}$ is a standard $\mathbb{R}^n$ -valued Wiener process independent of the family of signals $\{\theta^\lambda\}$.

Further, assume that for each $\lambda \in \Lambda$ and $t \in [0, T]$ the operator $A^\lambda(t)$ can be decomposed into functionals $A_1^\lambda(t), ..., A_n^\lambda(t) \in V_\alpha^*$ such that

$$A^\lambda(t)b = (\langle b, A_1^\lambda(t)\rangle_{V_\alpha}, V_\alpha^*, ..., \langle b, A_n^\lambda(t)\rangle_{V_\alpha}, V_\alpha^*)^T$$

for all $b \in V_\alpha$. The dual operator $(A^\lambda)^*(t) : \mathbb{R}^n \to V_\alpha^*$ then satisfies

$$(A^\lambda)^*(t)z = \sum_{i=1}^n z_i A_i^\lambda(t) \tag{1.4}$$

for all $z \in \mathbb{R}^n$.

We will study the optimal filter $\hat{\theta}_t^\lambda$, which is defined as

$$\hat{\theta}_t^\lambda = \mathbb{E}\left[\theta_t^\lambda | F_t^{\xi_\lambda}\right],$$

where $(F_t^{\xi_\lambda})_{t\in[0,T]}$ is the filtration generated by the observation process ξ^λ.

Set $K^\lambda(t, s) = \mathbb{E}[\theta_t^\lambda \circ \theta_s^\lambda]$, $t, s \in [0, T]$, $\lambda \in \Lambda$. In virtue of the uniform boundedness of the processes $\{\theta_t^\lambda, t \in [0, T], \lambda \in \Lambda\}$ the family of mappings $K^\lambda : [0, T]^2 \to \mathcal{L}(V_\alpha^*, V_\alpha)$, $\lambda \in \Lambda$ is uniformly bounded in $\mathcal{C}([0, T]^2, \mathcal{L}(V_\alpha^*, V_\alpha))$, i.e.

$$\sup_{\lambda\in\Lambda} \|K^\lambda\|_{\mathcal{C}\left([0,T]^2, \mathcal{L}(V_\alpha^*, V_\alpha)\right)} < \infty.$$

Then, for a fixed value of the parameter λ, we have following theorem (cf. Theorem 1.1 in [12]).

Theorem 1.1. *Let $\Delta = \{(t, s) \in [0, T]^2; 0 \le s \le t \le T\}$ and $\lambda \in \Lambda$. The filter $\hat{\theta}^\lambda$ satisfies the stochastic integral equation*

$$\hat{\theta}_t^\lambda = \int_0^t \Phi^\lambda(t, s)(A^\lambda)^*(s)\mathrm{d}\xi_s - \int_0^t \Phi^\lambda(t, s)(A^\lambda)^*(s)A^\lambda(s)\hat{\theta}_s^\lambda \mathrm{d}s, \ t \in [0, T], \tag{1.5}$$

where operator $\Phi^\lambda\colon \Delta \to \mathcal{L}(V_\alpha^*, V_\alpha)$ defined as $\Phi^\lambda(t, s) = \mathbb{E}[\theta_t^\lambda \circ (\theta_s^\lambda - \hat{\theta}_s^\lambda)]$ for all $(t, s) \in \Delta$ is strongly continuous and satisfies the integral equation

$$\Phi^\lambda(t, s) = K^\lambda(t, s) - \sum_{j=1}^n \int_0^s \left(\Phi^\lambda(t, r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s, r)A_j^\lambda(r)\right)\mathrm{d}r, \quad (t, s) \in \Delta. \tag{1.6}$$

Moreover, for all $t \in [0, T]$, $\Phi^\lambda(t, t)$ is the covariance of the estimation error at time $t \in [0, T]$, that is,

$$\Phi^\lambda(t, t) = \mathbb{E}\left[(\theta_t^\lambda - \hat{\theta}_t^\lambda) \circ (\theta_t^\lambda - \hat{\theta}_t^\lambda)\right] \tag{1.7}$$

holds.

Remark 1.2. According to Lemma 2.2 in [6] process $\{\tilde{W}_t, t \in [0, T]\}$ defined as

$$\tilde{W}_t^\lambda = \xi_t^\lambda - \int_0^t \mathbb{E}\left[A^\lambda(r)\theta_r^\lambda | F_t^{\xi_\lambda}\right]\mathrm{d}r = \xi_t^\lambda - \int_0^t A^\lambda(r)\hat{\theta}_r^\lambda \mathrm{d}r. \tag{1.8}$$

is $\mathbb{R}^n$ -valued $(F_t^{\xi_\lambda})$ – standard Wiener process called innovation process. Equation (1.5) can be rewritten as

$$\hat{\theta}_t^\lambda = \int_0^t \Phi^\lambda(t, s)(A^\lambda)^*(s)\mathrm{d}\tilde{W}_s^\lambda, \ t \in [0, T]. \tag{1.9}$$

2. Continuous dependence for the covariance

In this section, continuous dependence of the covariance operator Φ^λ on $\lambda \in \Lambda$ is shown. The main result of the section is stated below.

Theorem 2.1. *Under the assumptions in Section 2 if*

$$K^\lambda \to K^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda \tag{2.1}$$

in $\mathcal{C}([0,T]^2, \mathcal{L}(V_\alpha^*, V_\alpha))$ *and*

$$A^\lambda \to A^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda \tag{2.2}$$

in $\mathcal{C}([0,T], \mathcal{L}(V_\alpha, \mathbb{R}^n))$ *then*

$$\Phi^\lambda \to \Phi^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda \tag{2.3}$$

in $\mathcal{C}(\Delta, \mathcal{L}(V_\alpha^*, V_\alpha))$.

For the proof of Theorem 2.1, we need following lemma.

Lemma 2.2. *Under the assumptions in Section 2 the set of functions* $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ *is relatively compact in* $\mathcal{C}(\Delta, \mathcal{L}(V_\alpha^*, V_\alpha))$.

Proof. In virtue of the infinite-dimensional version of the Arzela–Ascoli theorem the statement of Lemma 2.2 holds if and only if the family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is uniformly bounded and equicontinuous in $\mathcal{C}(\Delta, \mathcal{L}(V_\alpha^*, V_\alpha))$ and $\{\Phi^\lambda(t,s)\}_{\lambda\in\Lambda}$ is relatively compact in $\mathcal{L}(V_\alpha^*, V_\alpha)$ for every $(t,s) \in \Delta$.

First, we show that mappings $(t,s,\lambda) \to \Phi^\lambda(t,s)$, $\lambda \in \Lambda$ are uniformly bounded on $\Delta \times \Lambda$. For arbitrary $x, y \in V_\alpha^*$, $(t,s) \in \Delta$ and for all $\lambda \in \Lambda$ using Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\left| \left\langle \left(\sum_{j=1}^n \int_0^s \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \mathrm{d}r \right) x, y \right\rangle_{V_\alpha}, V_\alpha^* \right| \\
&\quad \le \sum_{j=1}^n \int_0^s \left| \left\langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \right] x, y \right\rangle_{V_\alpha}, V_\alpha^* \right| \mathrm{d}r \\
&\quad = \sum_{j=1}^n \int_0^s |\langle \Phi^\lambda(t,r)A_j^\lambda(r) \langle \Phi^\lambda(s,r)A_j^\lambda(r), x\rangle_{V_\alpha}, V_\alpha^*, y\rangle_{V_\alpha}, V_\alpha^*| \mathrm{d}r \\
&\quad = \sum_{j=1}^n \int_0^s |\langle \Phi^\lambda(t,r)A_j^\lambda(r), y\rangle_{V_\alpha}, V_\alpha^*| \, |\langle \Phi^\lambda(s,r)A_j^\lambda(r), x\rangle_{V_\alpha}, V_\alpha^*| \mathrm{d}r \\
&\quad \le \sum_{j=1}^n \left(\int_0^s \left(\langle \Phi^\lambda(t,r)A_j^\lambda(r), y\rangle_{V_\alpha}, V_\alpha^* \right)^2 \mathrm{d}r \right)^{\frac12} \left(\int_0^s \left(\langle \Phi^\lambda(s,r)A_j^\lambda(r), x\rangle_{V_\alpha}, V_\alpha^* \right)^2 \mathrm{d}r \right)^{\frac12} \\
&\quad = \sum_{j=1}^n \left(\int_0^s \left\langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \right] y, y \right\rangle_{V_\alpha}, V_\alpha^* \mathrm{d}r \right)^{\frac12} \\
&\qquad \left(\int_0^s \left\langle \left[\left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \right] x, x \right\rangle_{V_\alpha}, V_\alpha^* \mathrm{d}r \right)^{\frac12}.
\end{aligned} \tag{2.4}$$

In the last inequality, we can increase the upper bound of the first integral from s to t because the integrand is nonnegative. Hence we only need to estimate the term

$$\int_0^t \left\langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right)\right]x, x\right\rangle_{V_\alpha}, V_\alpha^* \mathrm{d}r$$

for $j = 1, \ldots, n$ and $x \in V_\alpha^*$. By (1.6) and the uniform boundedness of the $\{K^\lambda\}_{\lambda\in\Lambda}$ we have

$$\begin{aligned}
0 &\le \int_0^t \left\langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right)\right]x, x\right\rangle_{V_\alpha}, V_\alpha^* \mathrm{d}r \\
&\le \sum_{j=1}^n \int_0^t \left\langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right)\right]x, x\right\rangle_{V_\alpha}, V_\alpha^* \mathrm{d}r \\
&= \langle K^\lambda(t,t)x, x\rangle_{V_\alpha}, V_\alpha^* - \langle \Phi^\lambda(t,t)x, x\rangle_{V_\alpha}, V_\alpha^* \\
&\le \langle K^\lambda(t,t)x, x\rangle_{V_\alpha}, V_\alpha^* \le C_1(T)\|x\|^2_{V_\alpha^*}, \quad t \in [0,T], \quad C_1(T) < \infty.
\end{aligned} \tag{2.5}$$

Now, by (1.6), (2.4), (2.5) and again by the uniform boundedness of the family $\{K^\lambda\}_{\lambda\in\Lambda}$ we obtain

$$\begin{aligned}
&|\langle \Phi^\lambda(t,s)x, y\rangle_{V_\alpha}, V_\alpha^*| \\
&\quad \le \|K^\lambda(t,s)\|_{\mathcal{L}(V_\alpha^*, V)}\|x\|_{V_\alpha^*}\|y\|_{V_\alpha^*} \\
&\qquad + \left|\left\langle \left(\sum_{j=1}^n \int_0^s \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s,r)A_j^\lambda(r)\right)\mathrm{d}r\right)x, y\right\rangle_{V_\alpha}, V_\alpha^*\right| \\
&\quad \le C_2(T)\|x\|_{V_\alpha^*}\|y\|_{V_\alpha^*}, \quad C_2(T) < \infty
\end{aligned}$$

for all $\lambda \in \Lambda$, $x, y \in V_\alpha^*$ and $(t,s) \in \Delta$, which proves that the family of operators $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is uniformly bounded in $C(\Delta, \mathcal{L}(V_\alpha^*, V_\alpha))$.

Further, we show that the family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is equicontinuous on Δ (as mappings with values in $\mathcal{L}(V_\alpha^*, V_\alpha)$). To this end, it is enough to show equicontinuity of $\{\Phi^\lambda(\cdot, s)\}_{\lambda\in\Lambda}$ on $[s,T]$ for all $s \in [0,T]$ and equicontinuity of $\{\Phi^\lambda(t,\cdot)\}_{\lambda\in\Lambda}$ on $[0,t]$ for all $t \in [0,T]$. By definition of Φ^λ and Cauchy - Schwarz inequality we have

$$\begin{aligned}
&\left|\langle\left(\Phi^\lambda(t_1,s) - \Phi^\lambda(t_2,s)\right)x, y\rangle_{V_\alpha, V_\alpha^*}\right| = \left|\mathbb{E}\left(\langle \theta_s^\lambda - \hat{\theta}_s^\lambda, x\rangle_{V_\alpha, V_\alpha^*}\langle \theta_{t_1}^\lambda - \theta_{t_2}^\lambda, y\rangle_{V_\alpha, V_\alpha^*}\right)\right| \\
&\quad \le \sqrt{\mathbb{E}\left|\langle \theta_s^\lambda - \hat{\theta}_s^\lambda, x\rangle_{V_\alpha, V_\alpha^*}\right|^2}\sqrt{\mathbb{E}\left|\langle \theta_{t_1}^\lambda - \theta_{t_2}^\lambda, y\rangle_{V_\alpha, V_\alpha^*}\right|^2} \\
&\quad \le \sqrt{\mathbb{E}\|\theta_s^\lambda\|^2_{V_\alpha}}\|x\|_{V_\alpha^*}\sqrt{\mathbb{E}\left|\langle \theta_{t_1}^\lambda - \theta_{t_2}^\lambda, y\rangle_{V_\alpha, V_\alpha^*}\right|^2}
\end{aligned}$$

for all $s \in [0,T]$, $t_1, t_2 \in [s,T]$, $\lambda \in \Lambda$ and all $x, y \in V_\alpha^*$. Therefore, using the mean-square equicontinuity and uniform boundedness of signals $\{\theta_t^\lambda, t \in [0,T], \lambda \in \Lambda\}$, for $\epsilon > 0$ we find $\delta > 0$ such that

$$\left|\langle\left(\Phi^\lambda(t_1,s) - \Phi^\lambda(t_2,s)\right)x, y\rangle_{V_\alpha, V_\alpha^*}\right| < \epsilon\|x\|_{V_\alpha^*}\|y\|_{V_\alpha^*} \tag{2.6}$$

for every $\lambda \in \Lambda$ and all $s \in [0,T]$, $t_1, t_2 \in [s,T]$, $|t_2 - t_1| < \delta$, which proves equicontinuity of $\{\Phi^\lambda(\cdot, s)\}_{\lambda\in\Lambda}$.

Furthermore, we have

$$\begin{aligned} &\|\big(\Phi^\lambda(t,s_1)-\Phi^\lambda(t,s_2)\big)x\|_{V_\alpha}\\ &\quad\le \|\mathbb{E}\Big(\theta^\lambda_t\langle\theta^\lambda_{s_1}-\theta^\lambda_{s_2},x\rangle_{V_\alpha,V_\alpha^*}\Big)\|_{V_\alpha}+\|\mathbb{E}\Big(\theta^\lambda_t\langle\hat\theta^\lambda_{s_2}-\hat\theta^\lambda_{s_1},x\rangle_{V_\alpha,V_\alpha^*}\Big)\|_{V_\alpha}\\ &\quad\le\sqrt{\mathbb{E}\|\theta^\lambda_s\|^2_{V_\alpha}}\left(\sqrt{\mathbb{E}\Big|\langle\theta^\lambda_{s_1}-\theta^\lambda_{s_2},x\rangle_{V_\alpha,V_\alpha^*}\Big|^2}+\sqrt{\mathbb{E}\Big|\langle\hat\theta^\lambda_{s_2}-\hat\theta^\lambda_{s_1},x\rangle_{V_\alpha,V_\alpha^*}\Big|^2}\right)\end{aligned} \tag{2.7}$$

for all $t\in[0,T]$, $s_1,s_2\in[0,t]$, $\lambda\in\Lambda$ and all $x\in V_\alpha^*$. Using (1.9), (1.4), Itô isometry and the uniform boundednes of $(A^\lambda)_{\lambda\in\Lambda}$ and $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ we have

$$\begin{aligned} &\mathbb{E}\Big|\langle\hat\theta^\lambda_{s_2}-\hat\theta^\lambda_{s_1},x\rangle_{V_\alpha,V_\alpha^*}\Big|^2\\ &\le 2\sum_{j=1}^n\mathbb{E}\left|\int_0^{s_1}\Big\langle\big(\Phi^\lambda(s_2,r)-\Phi^\lambda(s_1,r)\big)A^\lambda_j(s),x\Big\rangle_{V_\alpha,V_\alpha^*}\mathrm{d}\tilde W^\lambda_{j,r}\right|^2\\ &\quad+2\sum_{j=1}^n\mathbb{E}\left|\int_{s_1}^{s_2}\langle\Phi^\lambda(s_2,r)A^\lambda_j(s),x\rangle_{V_\alpha,V_\alpha^*}\mathrm{d}\tilde W^\lambda_{j,r}\right|^2\\ &=2\sum_{j=1}^n\int_0^{s_1}\left|\Big\langle\big(\Phi^\lambda(s_2,r)-\Phi^\lambda(s_1,r)\big)A^\lambda_j(s),x\Big\rangle_{V_\alpha,V_\alpha^*}\right|^2\mathrm{d}r\\ &\quad+2\sum_{j=1}^n\int_{s_1}^{s_2}\Big|\langle\Phi^\lambda(s_2,r)A^\lambda_j(s),x\rangle_{V_\alpha,V_\alpha^*}\Big|^2\mathrm{d}r\\ &\le C(T)\|x\|^2_{V_\alpha^*}\Big(\int_0^{s_1}\|\Phi^\lambda(s_2,r)-\Phi^\lambda(s_1,r)\|^2_{\mathcal{C}([r,T],\mathcal{L}(V_\alpha^*,V_\alpha))}\mathrm{d}r+|s_2-s_1|\Big),0\le s_1<s_2\le T,\end{aligned} \tag{2.8}$$

where $C(T)<\infty$.

Using (2.7) and (2.8), the equicontinuity of $\{\Phi^\lambda(\cdot,r)\}_{\lambda\in\Lambda}$ on $[r,T]$ for all $r\in[0,T]$ shown in (2.6) and the mean-square equicontinuity and uniform boundedness of signals $\{\theta^\lambda_t,t\in[0,T],\lambda\in\Lambda\}$, for all $\epsilon>0$ we can find $\delta>0$ such that

$$\|\big(\Phi^\lambda(t,s_1)-\Phi^\lambda(t,s_2)\big)x\|_{V_\alpha}<\epsilon\|x\|_{V_\alpha^*} \tag{2.9}$$

for every $\lambda\in\Lambda$ and all $t\in[0,T]$, $s_1,s_2\in[0,t]$, $|s_2-s_1|<\delta$ and $x\in V_\alpha^*$. This completes the proof of equicontinuity of $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ on Δ.

It remains to show that $\{\Phi^\lambda(t,s)\}_{\lambda\in\Lambda}$ is relatively compact in $\mathcal{L}(V_\alpha^*,V_\alpha)$ for every $(t,s)\in\Delta$. This property is equivalent to collective compactness of the family $\{[\Phi^\lambda(t,s),(\Phi^\lambda)^*(t,s)]\}_{\lambda\in\Lambda}$ in $\mathcal{L}(V_\alpha^*,V_\alpha)\times\mathcal{L}(V_\alpha,V_\alpha^*)$, cf. [14]. Employing the compactness of embeddings

$$V_{\alpha+\vartheta}\hookrightarrow V_\alpha,\qquad V^*_{\alpha-\vartheta}\hookrightarrow V^*_\alpha$$

(cf. (1.1) for the definition of ϑ) it is enough to show *Range*$(\Phi^\lambda(t,s))\subset V_{\alpha+\vartheta}$, *Range*$((\Phi^\lambda)^*(t,s))\subset V^*_{\alpha-\vartheta}$ and the uniform boundedness

$$\sup_{\lambda\in\Lambda}\|\Phi^\lambda(t,s)\|_{\mathcal{L}(V_\alpha^*,V_{\alpha+\vartheta})}<\infty \tag{2.10}$$

and

$$\sup_{\lambda\in\Lambda}\|(\Phi^\lambda)^*(t,s)\|_{\mathcal{L}(V_\alpha,V^*_{\alpha-\vartheta})}<\infty \tag{2.11}$$

holds for arbitrarily chosen $(t,s) \in \Delta$. We have that

$$\|\Phi^\lambda(t,s)\|_{\mathcal{L}(V_\alpha^*, V_{\alpha+\vartheta})} = \sup_{\|x\|_{V_\alpha^*}\leq 1} \|\Phi^\lambda(t,s)x\|_{V_{\alpha+\vartheta}} \leq \sup_{\|x\|_{V_\alpha^*}\leq 1} \mathbb{E}\|\theta_t^\lambda\|_{V_{\alpha+\vartheta}}\|\theta_s^\lambda - \hat{\theta}_s^\lambda\|_{V_\alpha}\|x\|_{V_\alpha^*}$$
$$\leq \tilde{C}_1(T)\mathbb{E}\|\theta_t^\lambda\|_{V_{\alpha+\vartheta}}\|\theta_s^\lambda\|_{V_\alpha} \leq C_1(T),$$

where the constant $C_1(T) < \infty$ does not depend on $\lambda \in \Lambda$ due to (1.1), which proves (2.10). Similarly, we have

$$\|(\Phi^\lambda(^*(t,s)\|_{\mathcal{L}(V_\alpha, V_{\alpha-\vartheta}^*)} = \sup_{\|y\|_{V_\alpha}\leq 1} \|(\Phi^\lambda)^*(t,s)y\|_{V_{\alpha-\vartheta}^*}$$
$$= \sup_{\|y\|_{V_\alpha}\leq 1} \sup\left\{\left|\langle \mathcal{B}^{2\vartheta}x, (\Phi^\lambda)^*(t,s)y\rangle_{V_\alpha^*}\right|; \|\mathcal{B}^\vartheta x\|_{V_\alpha^*} \leq 1,\ x \in V_{\alpha-2\vartheta}^*\right\}$$
$$= \sup_{\|y\|_{V_\alpha}\leq 1} \sup\left\{\left|\langle \Phi^\lambda(t,s)\mathcal{B}^\vartheta z, y\rangle_{V_\alpha}\right|; \|z\|_{V_\alpha^*} \leq 1,\ z \in V_{\alpha-\vartheta}^*\right\}$$
$$\leq \sup_{\|y\|_{V_\alpha}\leq 1} \sup\left\{\left|\langle \theta_t^\lambda, y\rangle_{V_\alpha}\langle \mathcal{B}^\alpha(\theta_s^\lambda - \hat{\theta}_s^\lambda), \mathcal{B}^{-\alpha}\mathcal{B}^\vartheta z\rangle_H\right|; \|z\|_{V_\alpha^*} \leq 1,\ z \in V_{\alpha-\vartheta}^*\right\}$$
$$\leq \sup_{\|y\|_{V_\alpha}\leq 1} \sup\left\{\left|\langle \theta_t^\lambda, y\rangle_{V_\alpha}\langle \mathcal{B}^{\alpha+\vartheta}(\theta_s^\lambda - \hat{\theta}_s^\lambda), \mathcal{B}^{-\alpha} z\rangle_H\right|; \|z\|_{V_\alpha^*} \leq 1,\ z \in V_{\alpha-\vartheta}^*\right\}$$
$$\leq \sup_{\|y\|_{V_\alpha}\leq 1} \sup\left\{\|y\|_{V_\alpha}\|z\|_{V_\alpha^*}\mathbb{E}\|\theta_t^\lambda\|_{V_\alpha}\|\theta_s^\lambda - \hat{\theta}_s^\lambda\|_{V_{\alpha+\vartheta}}; \|z\|_{V_\alpha^*} \leq 1,\ z \in V_{\alpha-\vartheta}^*\right\}$$
$$\leq C(T),$$

where the constant $C_2(T) < \infty$ does not depend on $\lambda \in \Lambda$. Therefore, Equation (2.11) holds true and the proof of Lemma 2.2 is complete. □

Now, we prove Theorem 2.1.

Proof. Set $\mathcal{C} = \mathcal{C}(\Delta, \mathcal{L}(V_\alpha^*, V_\alpha))$ and assume the converse, i.e. (2.1) and (2.2) hold and (2.3) do not. Then we can find $\epsilon_0 > 0$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \in \Lambda$ such that $\lambda_n \to \lambda_0$ and

$$\|\Phi^{\lambda_n} - \Phi^{\lambda_0}\|_{\mathcal{C}} > \epsilon_0, \quad n \in \mathbb{N}. \tag{2.12}$$

By the relative compactness of $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ proved in Lemma 2.2, we can find a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and a limit $\Psi \in \mathcal{C}$ such that

$$\|\Phi^{\lambda_{n_k}} - \Psi\|_{\mathcal{C}} \to 0, \quad k \to 0. \tag{2.13}$$

We show that $\Psi = \Phi^{\lambda_0}$ and, therefore, (2.12) contradicts (2.13).

Using (1.6) we have

$$\left\|\Psi(t,s) - K^{\lambda_0}(t,s) + \sum_{j=1}^n \int_0^s \left(\Psi(t,r)A_j^{\lambda_0}(r)\right) \circ \left(\Psi(s,r)A_j^{\lambda_0}(r)\right)\mathrm{d}r\right\|_{\mathcal{L}(V_\alpha^*, V_\alpha)}$$
$$\leq \|\Psi(t,s) - \Phi^{\lambda_{n_k}}(t,s)\|_{\mathcal{L}(V_\alpha^*, V_\alpha)} + \|K^{\lambda_{n_k}}(t,s) - K^{\lambda_0}(t,s)\|_{\mathcal{L}(V_\alpha^*, V_\alpha)}$$
$$+ \sum_{j=1}^n \Bigg\| \int_0^s \left(\Psi(t,r)A_j^{\lambda_0}(r)\right) \circ \left(\Psi(s,r)A_j^{\lambda_0}(r)\right)$$
$$- \left(\Phi^{\lambda_{n_k}}(t,r)A_j^{\lambda_{n_k}}(r)\right) \circ \left(\Phi^{\lambda_{n_k}}(s,r)A_j^{\lambda_{n_k}}(r)\right)\mathrm{d}r\Bigg\|_{\mathcal{L}(V_\alpha^*, V_\alpha)}$$

for all $(t,s)\in\Delta$. The first two terms tend to zero uniformly in Δ as $k\to\infty$ by (2.13) and (2.1). Using uniform boundedness of $\{\Phi^\lambda\}$ and $\{A^\lambda\}$ the third term can be estimated by

$$
\begin{aligned}
&s\sum_{j=1}^{n}(\|(\Psi A_j^{\lambda_0})\circ[(\Psi-\Phi^{\lambda_{n_k}})A_j^{\lambda_0}]\|_{\mathcal{C}}+\|(\Psi A_j^{\lambda_0})\circ[\Phi^{\lambda_{n_k}}(A_j^{\lambda_0}-A_j^{\lambda_{n_k}})]\|_{\mathcal{C}})\\
&\qquad+s\sum_{j=1}^{n}(\|[\Psi(A_j^{\lambda_0}-A_j^{\lambda_{n_k}})]\circ(\Phi^{\lambda_{n_k}}A_j^{\lambda_{n_k}})\|_{\mathcal{C}}\\
&\qquad+\|[(\Psi-\Phi^{\lambda_{n_k}})A_j^{\lambda_{n_k}}]\circ(\Phi^{\lambda_{n_k}}(s,r)A_j^{\lambda_{n_k}}(r))\|_{\mathcal{C}})\\
&\le C(T)(\|\Psi-\Phi^{\lambda_{n_k}}\|_{\mathcal{C}}+\|A^{\lambda_0}-A^{\lambda_{n_k}}\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))}),\quad C(T)<\infty
\end{aligned}
$$

which tends to zero uniformly in Δ as $k\to\infty$ by (2.1) and (2.2). It follows that

$$
\Psi(t,s)=K^{\lambda_0}(t,s)-\sum_{j=1}^{n}\int_0^s\Big(\Psi(t,r)A_j^{\lambda_0}(r)\Big)\circ\Big(\Psi(s,r)A_j^{\lambda_0}(r)\Big)\mathrm{d}r
$$

for all $(t,s)\in\Delta$, hence Ψ solves (1.6) with $\lambda=\lambda_0$. In virtue of Theorem 2.1 in [12] on uniqueness of solutions to (1.6) we conclude that $\Psi=\Phi^{\lambda_0}$, which completes the proof of Theorem 2.1. □

3. Continuous dependence for the filter

In this section, continuous dependence of the filter $\hat{\theta}^\lambda$ on $\lambda\in\Lambda$ is proved, which is the main result of the article.

First, note that in virtue of (1.9), (1.4), Itô isometry and the uniform boundednes of $(A^\lambda)_{\lambda\in\Lambda}$ and $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ we obtain

$$
\begin{aligned}
&\mathbb{E}\|\hat{\theta}^\lambda_{t_2}-\hat{\theta}^\lambda_{t_1}\|^2_{V_\alpha}\\
&\quad\le 2\mathbb{E}\left\|\int_0^{t_1}\big(\Phi^\lambda(t_2,r)-\Phi^\lambda(t_1,r)\big)(A^\lambda)^*(r)\mathrm{d}\tilde{W}^\lambda_r\|^2_{V_\alpha}+2\mathbb{E}\|\int_{t_1}^{t_2}\Phi^\lambda(t_2,r)(A^\lambda)^*(r)\mathrm{d}\tilde{W}^\lambda_r\right\|^2_{V_\alpha}\\
&\quad=2\int_0^{t_1}\|\big(\Phi^\lambda(t_2,r)-\Phi^\lambda(t_1,r)\big)A^\lambda_j(s)\|^2_{V_\alpha}\mathrm{d}r+2\int_{t_1}^{t_2}\|\Phi^\lambda(t_2,r)A^\lambda_j(s)\|^2_{V_\alpha}\mathrm{d}r\\
&\quad\le C(T)\left(\int_0^{t_1}\|\Phi^\lambda(t_2,r)-\Phi^\lambda(t_1,r)\|^2_{\mathcal{L}(V_\alpha^*,V_\alpha)}\mathrm{d}r+|t_2-t_1|\right)
\end{aligned}
$$

for some $C(T)<\infty$ and all $t_1,t_2\in[0,T]$, $t_1<t_2$.

Therefore, using the equicontinuity of $\{\Phi^\lambda(\cdot,s)\}_{\lambda\in\Lambda}$ on $[r,T]$ for all $r\in[0,T]$ shown in (2.6) it follows that

$$
\mathbb{E}\|\hat{\theta}^\lambda_{t_2}-\hat{\theta}^\lambda_{t_1}\|^2_{V_\alpha}\to 0,\quad |t_2-t_1|\to 0
$$

for every $\lambda\in\Lambda$, which yields

$$
\hat{\theta}^\lambda\in\mathcal{C}\big([0,T],L^2(\Omega,V_\alpha)\big).
$$

Theorem 3.1. *Under the assumptions stated in Section 2 if*

$$\theta^{\lambda} \to \theta^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda, \tag{3.1}$$

in $\mathcal{C}([0,T], L^2(\Omega, V_\alpha))$ *and*

$$A^{\lambda} \to A^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda, \tag{3.2}$$

in $\mathcal{C}([0,T], \mathcal{L}(V_\alpha, \mathbb{R}^n))$ *then*

$$\hat{\theta}^{\lambda} \to \hat{\theta}^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda, \tag{3.3}$$

in $\mathcal{C}([0,T], L^2(\Omega, V_\alpha))$.

Proof. Given $\lambda \in \Lambda$ set

$$u(r) = \sup_{t\in[0,r]} \|\hat{\theta}_t^{\lambda} - \hat{\theta}_t^{\lambda_0}\|_{L^2(\Omega, V_\alpha)}, \quad r \in [0,T].$$

Note that u is nondecreasing and measurable on $[0,T]$ which follows from the continuity of the filter.

Using (1.5) and (1.3) we have

$$u(r) \le 2(I_1 + I_2),$$

where

$$I_1 = \sup_{t\in[0,r]} \mathbb{E}\bigg\| \int_0^t \Phi^{\lambda}(t,s)(A^{\lambda})^*(s)A^{\lambda}(s)(\theta_s^{\lambda} - \hat{\theta}_s^{\lambda}) - \Phi^{\lambda_0}(t,s)(A^{\lambda_0})^*(s)A^{\lambda_0}(s)(\theta_s^{\lambda_0} - \hat{\theta}_s^{\lambda_0})\mathrm{d}s\bigg\|^2_{L^2(\Omega, V_\alpha)},$$

$$I_2 = \sup_{t\in[0,r]} \mathbb{E}\bigg\| \int_0^t \Phi^{\lambda}(t,s)(A^{\lambda})^*(s) - \Phi^{\lambda_0}(t,s)(A^{\lambda_0})^*(s)\mathrm{d}W_s\bigg\|^2_{L^2(\Omega, V_\alpha)}.$$

Furthermore, we can estimate

$$\begin{aligned}
I_1 \le\ & \sup_{t\in[0,r]} 2\mathbb{E}\bigg\| \int_0^t \left[\Phi^{\lambda}(t,s) - \Phi^{\lambda_0}(t,s)\right](A^{\lambda})^*(s)A^{\lambda}(s)\left(\theta_s^{\lambda} - \hat{\theta}_s^{\lambda}\right)\mathrm{d}s\bigg\|^2_{L^2(\Omega, V_\alpha)} \\
& + \sup_{t\in[0,r]} 2\mathbb{E}\bigg\| \int_0^t \Phi^{\lambda_0}(t,s)\left[(A^{\lambda})^*(s) - (A^{\lambda_0})^*(s)\right]A^{\lambda}(s)\left(\theta_s^{\lambda} - \hat{\theta}_s^{\lambda}\right)\mathrm{d}s\bigg\|^2_{L^2(\Omega, V_\alpha)} \\
& + \sup_{t\in[0,r]} 2\mathbb{E}\bigg\| \int_0^t \Phi^{\lambda_0}(t,s)(A^{\lambda_0})^*(s)\left[A^{\lambda}(s) - A^{\lambda_0}(s)\right]\left(\theta_s^{\lambda} - \hat{\theta}_s^{\lambda}\right)\mathrm{d}s\bigg\|^2_{L^2(\Omega, V_\alpha)} \\
& + \sup_{t\in[0,r]} 2\mathbb{E}\bigg\| \int_0^t \Phi^{\lambda_0}(t,s)(A^{\lambda_0})^*(s)A^{\lambda_0}(s)\left(\theta_s^{\lambda} - \theta_s^{\lambda_0} + \hat{\theta}_s^{\lambda_0} - \hat{\theta}_s^{\lambda}\right)\mathrm{d}s\bigg\|^2_{L^2(\Omega, V_\alpha)} \\
& \le C_1(T)\|\Phi^{\lambda} - \Phi^{\lambda_0}\|_{\mathcal{C}} + C_2(T)\|A^{\lambda} - A^{\lambda_0}\|_{\mathcal{C}\left([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n)\right)} \\
& + C_3(T)\|\theta^{\lambda} - \theta^{\lambda_0}\|_{\mathcal{C}\left([0,T],L^2(\Omega,V_\alpha)\right)} + C_4(T)\textstyle\int_0^r u(s)\mathrm{d}s
\end{aligned}$$

and

$$
\begin{aligned}
I_2 &\le \sup_{t\in[0,r]} \sum_{j=1}^{n} \mathbb{E}\left\| \int_0^t \Phi^{\lambda}(t,s)A_j^{\lambda}(s) - \Phi^{\lambda_0}(t,s)A_j^{\lambda_0}(s)\mathrm{d}W_s^j \right\|^2_{L^2(\Omega, V_\alpha)} \\
&= \sup_{t\in[0,r]} \sum_{j=1}^{n} \int_0^t \left\| \Phi^{\lambda}(t,s)A_j^{\lambda}(s) - \Phi^{\lambda_0}(t,s)A_j^{\lambda_0}(s) \right\|^2_{L^2(\Omega, V_\alpha)} \mathrm{d}s \\
&\le C_5(T)\|A^{\lambda} - A^{\lambda_0}\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))} + C_6(T)\|\Phi^{\lambda} - \Phi^{\lambda_0}\|_{\mathcal{C}},
\end{aligned}
$$

where $C_1 - C_6$ are finite constants dependent only on T. We used boundedness of $\{\Phi^{\lambda}\}$ and $\{A^{\lambda}\}$ and Itô isometry. Therefore, we have

$$
u(r) \le \alpha(T) + \int_0^r C_4(T)u(s)\mathrm{d}s, \quad r \in [0,T],
$$

where

$$
\alpha(T) = \bar{C}(T)\Big(\|\Phi^{\lambda} - \Phi^{\lambda_0}\|_{\mathcal{C}} + \|A^{\lambda} - A^{\lambda_0}\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))} + \|\theta^{\lambda} - \theta^{\lambda_0}\|_{\mathcal{C}([0,T],L^2(\Omega,V_\alpha))} \Big)
$$

and $\bar{C}(T) < \infty$. Using Gronwall's inequality we obtain

$$
\begin{aligned}
&u(T) \le \alpha(T)\exp\{TC_4(T)\} \\
&\le C(T)\Big(\|\Phi^{\lambda} - \Phi^{\lambda_0}\|_{\mathcal{C}} + \|A^{\lambda} - A^{\lambda_0}\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))} + \|\theta^{\lambda} - \theta^{\lambda_0}\|_{\mathcal{C}([0,T],L^2(\Omega,V_\alpha))} \Big)
\end{aligned}
$$

for a constant $C(T) < \infty$ independent of $\lambda \in \Lambda$.

Using Assumptions (3.1), (3.2) and Theorem 2.1 we obtain

$$
\sup_{t\in[0,r]} \|\hat{\theta}_t^{\lambda} - \hat{\theta}_t^{\lambda_0}\|_{L^2(\Omega,V_\alpha)} \to 0, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda,
$$

which completes the proof. □

4. Signal governed by stochastic evolution equation

Consider a stochastic basis $(\Omega, F, P, (F_t))$ and two-sided cylindrical fractional Brownian motion $\{B_t, t \in \mathbb{R}\}$ with Hurst parameter $h > 1/2$ on separable Hilbert space U defined by the formal series

$$
B_t = \sum_{n=1}^{\infty} \beta_n(t)e_n, \quad t \in \mathbb{R},
$$

where $\{e_n, n \in \mathbb{N}\}$ is an orthonormal basis in U and $\{\beta_n(t), t \in \mathbb{R}\}_{n\in\mathbb{N}}$ is a sequence of independent real-valued fractional Brownian motions (for the definitions see [4] and [5]).

Let H be a separable Hilbert space and for any $\lambda \in \Lambda$, Λ being a compact metric space, let the H-valued signal θ^{λ} satisfy the equation

$$
d\theta_t^{\lambda} = \mathcal{A}_{\lambda}\theta_t^{\lambda}dt + G_{\lambda}dB_t, \quad t \in [0,T], \tag{4.1}
$$

where the linear operator $\mathcal{A}_{\lambda} : Dom(\mathcal{A}_{\lambda}) \subset H \to H$ is strictily negative, selfadjoint and has compact resolvent, hence it generates a compact strongly continuous analytic semigroup $\{S_\lambda(t), t \geq 0\}$ in H. Finally, G_λ is an operator defined on U with values in an appropriate space.

By the analyticity of $S_\lambda, \lambda \in \Lambda,$ the Hilbert spaces

$$V_\delta^\lambda = Dom((-\mathcal{A}_\lambda)^\delta), \quad \delta \geq 0 \tag{4.2}$$

equipped with the graph norm topology are well defined. We assume that V_δ^λ do not depend on $\lambda \in \Lambda$ for every $\delta \geq 0$ (the graph norms $\|\cdot\|_{V_\delta^\lambda}, \lambda \in \Lambda$ are equivalent) and we set $V_\delta = V_\delta^{\lambda_0}, \delta \geq 0,$ for a fixed, arbitrarily chosen $\lambda_0 \in \Lambda$. Finally,

$$G_\lambda : U \to V_1',$$

where V_1' is the dual of V_1 with respect to topology of H.

In order to use results from previous sections impose the following assumptions:

- *Uniform exponential stability*: For some $c_1 > 0$ and $\rho_1 > 0$ we have

$$\|S_\lambda(t)\|_{\mathcal{L}(H)} \leq c_1 e^{-\rho_1 t}, \quad t > 0 \tag{A1}$$

  for $\lambda \in \Lambda$.

- *Uniform singularity at time $t = 0$*: For some $c_2 > 0$ and $0 \leq \gamma < h$ we have

$$\|S_\lambda(t) G_\lambda\|_{\mathcal{L}_2(U,H)} \leq c_2 t^{-\gamma}, t > 0 \tag{A2}$$

  for $\lambda \in \Lambda$.

- *Equicontinuity of the semigroups*: There exists $\alpha \geq 0$ such that $\gamma + \alpha < h$ and for any $x \in V_\alpha$ the mappings

$S_\lambda(\cdot)x : [0, T] \to V_\alpha$ are continuous uniformly in $\lambda \in \Lambda$.(A3)

- *Continuous dependence*: For $t > 0$ and $\lambda_0 \in \Lambda$ we have

$$S_\lambda(t) G_\lambda \overset{\mathcal{L}_2(U, V_\alpha)}{\longrightarrow} S_{\lambda_0}(t) G_{\lambda_0}, \quad \lambda \to \lambda_0. \tag{A4}$$

Note that the above conditions imply the uniform analyticity of the family of semigroups $(S_\lambda)_{\lambda \in \Lambda}$, hence by (A1) and (A2) we obtain

$$\|S_\lambda(t) G_\lambda\|_{\mathcal{L}_2(U, V_\delta)} \leq C_\delta e^{-\rho t} t^{-(\gamma+\delta)}, \quad t > 0 \tag{4.3}$$

for any $\delta \geq 0$ and a constant C_δ independent of $t > 0$ and $\lambda \in \Lambda$. Indeed, we may estimate

$$\|S_\lambda(t) G_\lambda\|_{\mathcal{L}_2(U, V_\delta)} \leq \|S_\lambda(t/2) G_\lambda\|_{\mathcal{L}_2(U,H)} \|S_\lambda(t/2)\|_{\mathcal{L}_2(H, V_\delta)}, \quad t > 0.$$

Also, notice that the inequality in (A2) may be verified only on a finite time interval $(0, T)$ for a $T > 0$ if we take into account (A1).

The above hypotheses imply existence of a strictly stationary solution to (4.1) with continuous paths in V_α, understood in the mild sense, which may be expressed as

$$\theta_t^\lambda = \int_{-\infty}^t S_\lambda(t-u) G_\lambda dB(u), \quad t \in [0, T], \tag{4.4}$$

(see [10] for details). Similar computations as in [4] yield a representation for the covariance $K^\lambda(t,s) = K^\lambda(t-s) = \mathbb{E}\left[\theta_t^\lambda \circ \theta_0^\lambda\right]$ for $(t,s) \in \Delta$:

$$K^\lambda(t) = \int_{-\infty}^{t} \int_{-\infty}^{0} S_\lambda(-u) G_\lambda G_\lambda^* S_\lambda(t-v) \gamma_h(u-v) du dv, \tag{4.5}$$

where $\gamma_h(u) = h(2h-1)|u|^{2h-2}, u \in \mathbb{R}$ (note that $S_\lambda(t)^* = S_\lambda(t)$). The integral (4.5) is correctly defined due to the estimate

$$\begin{aligned}
&\int_{-\infty}^{t} \int_{-\infty}^{0} \|S_\lambda(-r) G_\lambda G_\lambda^* S_\lambda(t-v)\|_{\mathcal{L}(H)} \gamma_h(r-v) dr dv \\
&\quad \le \int_{-\infty}^{t} \int_{-\infty}^{0} \|S_\lambda(-r) G_\lambda G_\lambda^* S_\lambda(t-v)\|_{\mathcal{L}_1(H)} \gamma_h(r-v) dr dv \\
&\quad \le \int_{-\infty}^{t} \int_{-\infty}^{0} \|S_\lambda(-r) G_\lambda\|_{\mathcal{L}_2(U,H} \|S_\lambda(t-v) G_\lambda\|_{\mathcal{L}_2(U,H)} \gamma_h(r-v) dr dv \\
&\quad \le c_0 \int_{-\infty}^{t} \int_{-\infty}^{0} e^{\rho(r+v)} (-r)^{-\gamma} (t-v)^{-\gamma} \gamma_h(r-v) dr dv
\end{aligned}$$

for $t \in [0,T]$, with some constant $c_0 < \infty$, which follows by (4.3) with $\delta = 0$ (so $V_\delta = H$). The right-hand side is finite since

$$\int_{-\infty}^{t} \int_{-\infty}^{0} e^{\rho(r+v)} (-r)^{-\eta} (t-v)^{-\eta} \gamma_h(r-v) dr dv < \infty, \quad t \in [0,T], \tag{4.6}$$

for any $\rho > 0$ and $0 \le \eta < h$ which we will also use in the sequel.

We are now ready to verify the boundedness condition (1.1) and equicontinuity condition (1.2) from Section 2. Take $\vartheta \in (0,\alpha)$ such that $\gamma + \alpha + \vartheta < h$ and any $t \in [0,T]$, $\lambda \in \Lambda_0$. Using (4.6), (4.3), and the strict stationarity of (4.4) we have

$$\begin{aligned}
\mathbb{E}\|\theta_t^\lambda\|_{V_{\alpha+\vartheta}}^2 &= \mathbb{E}\|\theta_0^\lambda\|_{V_{\alpha+\vartheta}}^2 \\
&= \left\| \int_{-\infty}^{0} \int_{-\infty}^{0} S_\lambda(-r) G_\lambda G_\lambda^* S_\lambda(-v) \gamma_h(r-v) dr dv \right\|_{\mathcal{L}_1(V_{\alpha+\vartheta})} \\
&\le \int_{-\infty}^{0} \int_{-\infty}^{0} \|S_\lambda(-r) G_\lambda\|_{\mathcal{L}_2(U, V_{\alpha+\vartheta})} \|S_\lambda(-v) G_\lambda\|_{\mathcal{L}_2(U, V_{\alpha+\vartheta})} \gamma_h(r-v) dr dv \\
&\le C_{\alpha+\vartheta}^2 \int_{-\infty}^{0} \int_{-\infty}^{0} e^{\rho(r+v)} (-r)^{-(\gamma+\alpha+\vartheta)} (-v)^{-(\gamma+\alpha+\vartheta)} \gamma_h(r-v) dr dv < \infty,
\end{aligned}$$

with the last integral being independent of t, λ and finite due to (4.6) with $\eta = \gamma + \alpha + \theta$. This proves the boundedness of $\{\theta^\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{C}([0,T], L^2(\Omega, V_{\alpha+\vartheta}))$.

By the strict stationarity, it is enough to verify the equicontinuity at zero from the right. For $t \in [0,T]$ and $\lambda \in \Lambda$ we obtain

$$\mathbb{E}\left\|\int_{-\infty}^{t} S_\lambda(t-u)G_\lambda dB_u - \int_{-\infty}^{0} S_\lambda(-u)G_\lambda dB_u\right\|_{V_\alpha}^2$$
$$= \mathbb{E}\left\|\int_0^t S_\lambda(t-u)G_\lambda dB_u - \int_{-\infty}^{0}(S_\lambda(t)-I)S_\lambda(-u)G_\lambda dB_u\right\|_{V_\alpha}^2$$
$$\leq 2\mathbb{E}\left\|\int_0^t S_\lambda(t-u)G_\lambda dB_u\right\|_{V_\alpha}^2 + 2\mathbb{E}\left\|\int_{-\infty}^{0}(S_\lambda(t)-I)S_\lambda(-u)G_\lambda dB_u\right\|_{V_\alpha}^2$$
$$\leq 2\left\|\int_0^t\int_0^t S_\lambda(t-u)G_\lambda G_\lambda^* S_\lambda(t-v)\gamma_h(u-v)dudv\right\|_{\mathcal{L}_1(V_\alpha)}$$
$$+2\left\|\int_{-\infty}^{0}\int_{-\infty}^{0}(S_\lambda(t)-I)S_\lambda(-u)G_\lambda G_\lambda^* S_\lambda(-v)(S_\lambda(t)-I)\gamma_h(u-v)dudv\right\|_{\mathcal{L}_1(V_\alpha)}$$
$$\leq 2\int_0^t\int_0^t \|S_\lambda(t-u)G_\lambda\|_{\mathcal{L}_2(U,V_\alpha)}\|S_\lambda(t-v)G_\lambda\|_{\mathcal{L}_2(U,V_\alpha)}\gamma_h(u-v)dudv$$
$$+2\int_{-\infty}^{0}\int_{-\infty}^{0}\|(S_\lambda(t)-I)S_\lambda(-u)G_\lambda\|_{\mathcal{L}_2(U,V_\alpha)}\|(S_\lambda(t)-I)S_\lambda(-v)G_\lambda\|_{\mathcal{L}_2(U,V_\alpha)}\gamma_h(u-v)dudv$$
$$=: 2I_1(t,\lambda) + 2I_2(t,\lambda)$$

Now $I_1(t,\lambda)$ can be estimated

$$I_1(t,\lambda) = \int_0^t\int_0^t \|S_\lambda(u)G_\lambda\|_{\mathcal{L}_2(U,V_\alpha)\mathcal{L}_2(U,V_\alpha)}\|S_\lambda(v)G_\lambda\|_{\mathcal{L}_2(U,V_\alpha)}\gamma_h(u-v)dudv$$
$$\leq C_\alpha^2\int_0^t\int_0^t e^{-\rho(u+v)}(uv)^{-(\gamma+\alpha)}\gamma_h(u-v)dudv.$$

The last term is independent of $\lambda \in \Lambda$ and tends to 0 as $t \to 0+$ when we take into account (4.6) with $\eta = \gamma + \alpha$.

For $I_2(t,\lambda)$ we construct an integrable majorant by (4.6) with $\eta = \gamma + \alpha$ and the equicontinuity of the semigroups in (A3) which implies that $|S_\lambda(\cdot) - I|_{\mathcal{L}(V_\alpha)}$ is bounded on $[0,T]$ by some $N_\alpha > 0$ depending only on α. Moreover,

$$(S_\lambda(t)-I)S_\lambda(u)G_\lambda \overset{\mathcal{L}_2(U,V_\alpha)}{\to} 0, \quad t \to 0+ \tag{4.7}$$

holds for any $u > 0$. The convergence (4.7) is obtained by the analyticity of S_λ and equicontinuity in (A3) again by Dominated Convergence Theorem. Indeed, if $\{f_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in U we have

$$\|(S_\lambda(t)-I)S_\lambda(u)G_\lambda f_n\|_{V_\alpha} \to 0, \quad t \to 0+,$$

for any $n \in \mathbb{N}$ and $u > 0$. Moreover,

$$\sum_{n=0}^{\infty}\|(S_\lambda(t)-I)S_\lambda(u)G_\lambda f_n\|_{V_\alpha}{}^2 \leq N_\alpha^2\sum_{n=0}^{\infty}\|S_\lambda(u)G_\lambda f_n\|_{V_\alpha}{}^2 = N_\alpha^2\|S_\lambda(u)G_\lambda\|_{\mathcal{L}_2(U,V_\alpha)},$$

which is finite by (4.3) with $\delta = \alpha$. Hence we have verified that the family $\{\theta^\lambda\}_{\lambda\in\Lambda} \subset \mathcal{C}([0,T], L^2(\Omega, V_\alpha))$ is equicontinuous.

Now, we verify the condition (3.1) on continuous dependence of θ^λ on λ. Assume $t \in [0,T]$ and $\lambda, \lambda_0 \in \Lambda$ then we have

$$\mathbb{E}\|\theta_t^\lambda - \theta_t^{\lambda_0}\|_{V_\alpha}^2 = \mathbb{E}\left\| \int_{-\infty}^t S_\lambda(t-u)G_\lambda dB_u - \int_{-\infty}^t S_{\lambda_0}(t-u)G_{\lambda_0} dB_u \right\|_{V_\alpha}^2$$
$$= \mathbb{E}\left\| \int_{-\infty}^t (S_\lambda(t-u)G_\lambda - S_{\lambda_0}(t-u)G_{\lambda_0}) dB_u \right\|_{V_\alpha}^2$$
$$= \left\| \int_{-\infty}^t \int_{-\infty}^t (S_\lambda(t-u)G_\lambda - S_{\lambda_0}(t-u)G_{\lambda_0})(S_\lambda(t-v)G_\lambda - S_{\lambda_0}(t-v)G_{\lambda_0})^* \gamma_h(u-v) du dv \right\|_{\mathcal{L}_1(V_\alpha)}$$
$$= \left\| \int_0^\infty \int_0^\infty (S_\lambda(u)G_\lambda - S_{\lambda_0}(u)G_{\lambda_0})(S_\lambda(v)G_\lambda - S_{\lambda_0}(v)G_{\lambda_0})^* \gamma_h(u-v) du dv \right\|_{\mathcal{L}_1(V_\alpha)}$$
$$\leq \int_0^\infty \int_0^\infty \|S_\lambda(u)G_\lambda - S_{\lambda_0}(u)G_{\lambda_0}\|_{\mathcal{L}_2(U,V_\alpha)} \|S_\lambda(v)G_\lambda - S_{\lambda_0}(v)G_{\lambda_0}\|_{\mathcal{L}_2(U,V_\alpha)} \gamma_h(u-v) du dv$$

This upper bound does not depend on t and the integrand converges pointwise to zero as $\lambda \to \lambda_0$ by (A4). The Dominated Convergence Theorem (we use an upper bound constructed using (4.6) with $\eta = \gamma + \alpha$) yields desired convergence. We have verified that $\theta^\lambda \to \theta^{\lambda_0}$ in $\mathcal{C}([0,T], L^2(\Omega, V_\alpha))$ as $\lambda \to \lambda_0$.

4.1. Distributed fractional noise in heat equation

Consider the stationary solution of the equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) = & \quad \lambda^1 \Delta u(t,x) + \eta_{\lambda^2}^h(t,x), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ u(t,\cdot)_{|\partial\mathcal{D}} = & \quad 0, \quad t \in [0,T], \end{aligned} \tag{4.8}$$

where $\mathcal{D} \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\mathcal{D}$, Δ is the Laplace operator and the parameter $\lambda = (\lambda^1, \lambda^2)$ takes values in a compact metric space $\Lambda = \Lambda^1 \times \Lambda^2, \Lambda^1 \subset (0, \infty)$. The noise $\eta_{\lambda^2}^h$ is viewed as a fractional noise with the Hurst parameter $h > 1/2$.

Equation (4.8) is treated rigorously as the Hilbert space-valued equation

$$d\theta_t^\lambda = \mathcal{A}_\lambda \theta_t^\lambda dt + G_\lambda dB_t^h, \quad t \in [0,T], \tag{4.9}$$

for $\lambda \in \Lambda$ as in (4.1), where we set

$$U = H = L^2(\mathcal{D}), \quad \mathcal{A}_\lambda = \lambda^1 \Delta, \quad Dom\mathcal{A}_\lambda = W^{2,2}(\mathcal{D}) \cap W_0^{1,2}(\mathcal{D}),$$

B^h is the cylindrical fractional Brownian motion in U and $G_\lambda = G_{\lambda^2} : \Lambda^2 \to \mathcal{L}_2(U, H)$ is continuous.

It is well known that $\mathcal{A}_1$ is strictly negative and generates strongly continuous compact semigroup on H which we denote by S (here we formally assume that $1 \in \Lambda^1$). For the semigroups S_λ generated by $\mathcal{A}_\lambda, \lambda \in \Lambda$ we have

$$S_\lambda(t) = S(\lambda^1 t), \quad t > 0, \lambda \in \Lambda. \tag{4.10}$$

To establish continuous dependence of the filter we verify (A1), (A2), (A3), and (A4). First, S is exponentially stable so the condition (A1) is satisfied.

Furthermore,

$$\|S_\lambda(t)G_\lambda\|_{\mathcal{L}_2(U,H)} \le \|S(\lambda^1 t)\|_{\mathcal{L}(H)}\|G_{\lambda^2}\|_{\mathcal{L}_2(U,H)} \le K$$

for some $K<\infty$ by the Resonance Theorem, compactness of Λ^1 and continuous dependence of G_{λ^2} on $\lambda^2 \in \Lambda^2$. Therefore, (A2) is verified with $\gamma = 0$.

Taking arbitrary $\alpha > 0$ so that

$$\gamma + \alpha = \alpha < h \tag{4.11}$$

we obtain (A3) by analyticity of S and (4.10).

Finally, the continuous dependence in (A4) is verified as follows: First, we observe that in our case (A4) follows from the weaker condition

$$S_\lambda(t)G_\lambda \xrightarrow{\mathcal{L}_2(U,H)} S_{\lambda_0}(t)G_{\lambda_0}, \quad \lambda \to \lambda_0,$$

for $t>0$. For $t>0$ fixed taking $c>0$ such that $\lambda^1 t > 0$ for every $\lambda^1 \in \Lambda^1$ we may write

$$\|S_\lambda(t)G_\lambda - S_{\tilde\lambda}(t)G_{\tilde\lambda}\|_{\mathcal{L}_2(U,V_\alpha)} \le \|S(c)\|_{\mathcal{L}(H,V_\alpha)}\|S(\lambda^1 t - c)G_\lambda - S(\tilde\lambda^1 t - c)G_{\tilde\lambda}\|_{\mathcal{L}_2(U,H)}$$

and we use analyticity of the semigroup S.

Now let $\{f_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in U, then we have

$$\begin{aligned}\|S_\lambda(t)G_\lambda - S(t)_{\tilde\lambda}G_{\tilde\lambda}\|_{\mathcal{L}_2(U,H)} &= \sum_{n=0}^{\infty}\|(S_\lambda(t)G_{\lambda^2} - S_{\tilde\lambda}(t)G_{\tilde\lambda^2})f_n\|_H^2 \\ &= \sum_{n=0}^{\infty}\|(S(\lambda^1 t)G_{\lambda^2} - S(\tilde\lambda^1 t)G_{\tilde\lambda^2})f_n\|_H^2 \\ &\le 2\sum_{n=0}^{\infty}\|S(\lambda^1 t)(G_{\tilde\lambda^2} - G_{\lambda^2})f_n\|_H^2 \\ &+2\sum_{n=0}^{\infty}\|(S(\lambda^1 t) - S(\tilde\lambda^1 t))G_{\tilde\lambda^2}f_n\|_H^2\end{aligned}$$

for $t>0, \lambda = (\lambda^1,\lambda^2) \in \Lambda$ and $\tilde\lambda = (\tilde\lambda^1, \tilde\lambda^2) \in \Lambda$. The right-hand side converges to 0 if $\lambda \to \tilde\lambda$ by the Resonance Theorem, compactness of Λ and continuous dependence of G_{λ^2} on $\lambda^2 \in \Lambda^2$.

We have verified the conditions for continuous dependence of the filter $\hat\theta^\lambda$ in Theorem 3.1 for arbitrary $0<\alpha<h$.

Assume moreover, that the condition

$$\alpha > \frac{d}{4} \tag{4.12}$$

is additionally satisfied. Then by the Sobolev embedding theorem and [15] we have

$$V_\alpha \hookrightarrow W^{2\alpha,2}(\mathcal{D}) \hookrightarrow \mathcal{C}^{0,\beta}(\mathcal{D}), \tag{4.13}$$

where $W^{2\alpha,2}(\mathcal{D})$ is the Sobolev space and $\mathcal{C}^{0,\beta}(\mathcal{D})$ is the space of uniformly β-Hölder continuous functions on $\mathcal{D}$, $\beta = (4\alpha - d)/2$. Hence, for arbitrary chosen set of points $z_1^\lambda, \ldots, z_n^\lambda \in \mathcal{D}$ (possibly depending on $\lambda \in \Lambda$) the evaluation map

$$A^\lambda\varphi = (\varphi(z_1^\lambda), \ldots, \varphi(z_n^\lambda)), \quad \varphi \in V_\alpha \tag{4.14}$$

is well defined for $\lambda \in \Lambda$. Suppose that the mapping $\lambda \mapsto (z_1^\lambda, \ldots z_n^\lambda)$ is continuous. Then $A^\lambda : \Lambda \to \mathcal{L}(V_\alpha, \mathbb{R}^n)$ is continuous as well, since by (4.13) for a constant $c_0 > 0$ we have

$$\sup_{\varphi \in V_\alpha} \{|\varphi(z_i^\lambda) - \varphi(z_{i\tilde{\lambda}})|, \|\varphi\|_{V_\alpha} \leq 1\} \leq c_0 \sup_{\varphi \in C^{0,\beta}} \{|\varphi(z_i^\lambda) - \varphi(z_{i\tilde{\lambda}})|, \|\varphi\|_{C^{0,\beta}} \leq 1\} \to 0$$

whenever $\lambda \to \tilde{\lambda}$ for $i = 1, \ldots, n$. This verifies the condition (3.2) and we may conclude that Theorem 3.1 with V_α defined as above holds for signal defined by (4.9) and the observation process

$$\begin{aligned} d\xi_t^\lambda &= A^\lambda \theta_t^\lambda dt + dW_t, \quad t \in [0, T], \\ \xi_0^\lambda &= 0, \end{aligned}$$

with arbitrary $\mathbb{R}^n$-valued Wiener process W which is independent of B^h and pointwise observation A^λ given by (4.14). In this case the equations for the filter (1.5) and (1.6) in Theorem 1.1 can be simplified in the same way as in the Corollary 3.1. in [12].

Note that since we assume $h > 1/2$, both (4.11) and (4.12) are satisfied if either $d = 1$, 2 or $d = 3$ and $h > 3/4$.

4.2. Pointwise fractional noise in heat equation

Consider the signal given as a stationary solution to the parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= \lambda^1 \Delta u(t, x) + \delta_{\lambda^2} \eta^h(t), \quad (t, x) \in [0, T] \times \mathcal{D}, \\ u(t, \cdot)_{|\partial \mathcal{D}} &= 0, \quad t \in [0, T]. \end{aligned} \tag{4.15}$$

The setup is similar to the previous example except that the noise η^h is not distributed on the whole domain $\mathcal{D}$, but is scalar and acting at the point $\lambda^2 \in \mathcal{D}$. Here, $\mathcal{D} \subset \mathbb{R}^d$ is again a bounded domain with smooth boundary $\partial \mathcal{D}$, Δ is the Laplace operator, δ_y stands for the Dirac distribution at $y \in \mathcal{D}$ and the parameter $\lambda = (\lambda^1, \lambda^2)$ takes values in a compact metric space $\Lambda = \Lambda^1 \times \Lambda^2$, where $\Lambda^1 \subset (0, \infty)$ and $\Lambda^2 \subset \mathcal{D}$. The noise η^h is an one-dimensional fractional Brownian motion with the Hurst parameter $h > 1/2$.

We treat (4.15) as the equation

$$d\theta_t^\lambda = \mathcal{A}_\lambda \theta_t^\lambda dt + G_\lambda dB_t, \quad t \in [0, T],$$

for $\lambda \in \Lambda$ as in (4.1), where

$$U = \mathbb{R}, \quad H = L^2(\mathcal{D}), \quad \mathcal{A}_\lambda = \lambda^1 \Delta, \quad Dom \mathcal{A}_\lambda = W^{2,2}(\mathcal{D}) \cap W_0^{1,2}(\mathcal{D}), \quad G_\lambda = \delta_{\lambda^2}$$

with a real fractional Brownian motion B^h. The semigroups S and S_λ are the same as in the previous example, it is therefore sufficient to verify (A2), (A3) and (A4). Note that as $U = \mathbb{R}$, the Hilbert-Schmidt and operator norms are equal for operators defined on U.

To verify (A2) we estimate

$$\begin{aligned} \|S_\lambda(t) G_\lambda\|_{\mathcal{L}_2(U,H)} &= \|S(\lambda^1 t) \delta_{\lambda^2}\|_{\mathcal{L}(U,H)} \\ &\leq \|S(\lambda^1 t)\|_{\mathcal{L}(V_\gamma^*, H)} \|\delta_{\lambda^2}\|_{\mathcal{L}(U, V_\gamma^*)} \\ &\leq c_0 t^{-\gamma}, \quad t > 0, \end{aligned} \tag{4.16}$$

for some $c_0 > 0$ whenever $d/4 < \gamma < 1$. We used the analyticity of S, isomorphism $V_\gamma^* \cong Dom(-\mathcal{A})^{-\gamma}$, compactness of Λ and continuous dependence of δ_{λ^2} on $\lambda^2 \in \Lambda^2$ in

$(\mathcal{C}^{0,\beta})^* \hookrightarrow V_\gamma^*$, where $\beta = (4\gamma - d)/2$. Assuming that

$$\frac{d}{4} < h, \tag{4.17}$$

we have verified (A2). Fix γ such that $d/4 < \gamma < h$. Then (A3) is satisfied for any $\alpha \geq 0$ with

$$\gamma + \alpha < h$$

by the analyticity of S. Finally, to verify (A4) we examine the norm in $\mathcal{L}_2(U,H)$ as in the previous example and estimate

$$\begin{aligned}\|S_\lambda(t)G_\lambda - S_{\tilde{\lambda}}(t)G_{\tilde{\lambda}}\|_{\mathcal{L}_2(U,H)} &= \|S_\lambda(t)G_\lambda - S_{\tilde{\lambda}}(t)G_{\tilde{\lambda}}\|_{\mathcal{L}(U,H)} \\ &\leq \|S_\lambda(t)(G_\lambda - G_{\tilde{\lambda}})\|_{\mathcal{L}(U,H)} =: H_1 \\ &+ \|(S_\lambda(t) - S_{\tilde{\lambda}}(t))G_{\tilde{\lambda}}\|_{\mathcal{L}(U,H)} =: H_2\end{aligned}$$

for $\lambda = (\lambda^1, \lambda^2) \in \Lambda$ and $\tilde{\lambda} = (\tilde{\lambda}^1, \tilde{\lambda}^2) \in \Lambda$. The term H_1 is estimated similarly as in (4.16) as

$$H_1 \leq c_0 t^{-\gamma} \|\delta_{\lambda^2} - \delta_{\tilde{\lambda}^2}\|_{\mathcal{L}(U,V_\gamma^*)}$$

by analyticity of S. We see that H_1 tends to 0 as $\lambda \to \tilde{\lambda}$ by continuous dependence of δ_{λ^2} on λ^2 in $(\mathcal{C}^{0,\beta})^* \hookrightarrow V_\gamma^*$. For H_2 we have

$$H_2 = \|(S(\lambda^1 t) - S(\tilde{\lambda}^1 t))\delta_{\tilde{\lambda}^2}\|_{\mathcal{L}(U,H)} = \|(S(\lambda^1 t) - S(\tilde{\lambda}^1 t))(-A)^\gamma(-A)^{-\gamma}\delta_{\tilde{\lambda}^2}\|_{\mathcal{L}(U,H)}.$$

Now $\delta_{\tilde{\lambda}^2} \in \mathcal{L}(U, V_\gamma^*)$ and it easily follows that $H_2 \to 0$ as $\lambda \to \tilde{\lambda}$, which verifies (A4). As in the previous example we may also examine the conditions under which we shall consider pointwise observation of the signal as defined in (4.14). Similarly, we obtain the condition $d/4 < \alpha$ which can be satisfied only when $d = 1$.

Remark 4.1. The Hurst parameter h of the driving fractional Brownian motion in the signal equation in Section 5 is supposed to be greater than 1/2 and there is an interesting open question what happens if $h < 1/2$. We conjecture that analogous continuous dependence results would hold, however, under more stringent conditions, especially in (A4), where convergence in an appropriate Hölder norm would be needed. The reason is that the kernel in the formula for covariance (an analog of (4.5)) has a different (and more singular) form. In our main examples, the irregularity of the noise leads to the lost of space regularity which would be a serious limitation in our filtering problem. For instance, in Subsection 5.1 (the distributed noise) the pointwise observation is possible only for $h > \frac{d}{4}$, so the singular fractional Brownian motion may be considered only in one-dimensional case and only if $h > 1/4$. In Subsection 5.2 (the pointwise noise) the signal is not regular enough to allow pointwise observation if $h < 1/2$ (the corresponding regularity results in these cases have been proved in [4]).

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