Low-Degree Implied Equalities for Strengthening Semidefinite Relaxations

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***Abstract*—Moment/SOS relaxations of polynomial optimization problems (POPs) are becoming increasingly popular in robotics, but in practice only low-order relaxations are tractable, and these are often too weak unless the formulation is augmented with redundant constraints. Many successful applications add such implied equalities by hand, with limited explanation of why they strengthen the relaxation. In this work, we give a simple degree-based necessary condition for when an implied equality can strengthen a fixed-order Moment/SOS relaxation. The condition depends on the gap between the degree of the equality and the minimum degree of an ideal-membership certificate proving it from the original constraints, and it can be checked by solving a linear system. This leads to a linear-algebraic procedure for generating candidate implied equalities that can strengthen the relaxation. We illustrate the construction in the quadratic case, where useful candidate equalities can be generated by a nullspace computation followed by a linear projection, turning their design into a lightweight preprocessing step.**

I. INTRODUCTION

Moment/Sums-of-Squares (SOS) relaxations of polynomial optimization problems are becoming a practical tool in robotics and vision [1–6]. However, to obtain tight (or useful) relaxations, one typically has to resort to higher-order relaxations, whose computational cost grows quickly with the hierarchy degree and limits their practical application in robotics. For many problems of interest, a common approach has therefore been to stay at the first order of the hierarchy (often referred to as just "the semidefinite relaxation" or "Shor's relaxation"), but instead identify and add additional tightening constraints to the relaxation. This pattern appears repeatedly in certifiably correct estimation, registration, and motion planning, where carefully chosen implied constraints substantially tighten first-order relaxations without having to resort to higher-order relaxations [2–5, 7]. In the literature, the additional equalities are often added by hand and constitute a significant part of the paper's contribution, but with limited explanation of why they help or how they were discovered [4, 6, 8, 9].

Our contribution is twofold: First, we give a simple, degree-based necessary condition for when a given equality constraint can strengthen the relaxation at any fixed order of the hierarchy, allowing one to cheaply disqualify candidate equalities by solving a linear system. Second, we specialize this condition to homogenized QCQPs and the first-order relaxation that dominates most current robotics applications of SOS programming. In this setting, we show how to generate all tightening equality constraints through a cheap nullspace computation followed by a linear projection. This turns the search for useful implied equalities into a lightweight preprocessing step, eliminating the need to derive tightening constraints by hand, and also provides a principled justification for why many hand-designed constraints from the literature are effective. The proposed method is adapted from well-established tools from algebraic geometry related to computing Gröbner bases of ideals [10–12].

A. Related Work

Strengthening fixed-degree SOS relaxations has been studied in many communities in different forms. Adding strengthening inequalities is central to many well-known hierarchies [13–15]. Problem-specific implied inequalities can be found [16], but automating their search is involved and less well-understood [17–19].

On the other hand, the set of all redundant equalities that one can add to a SOS relaxation is exactly understood in algebraic geometry. In principle, one can eliminate the equalities entirely by working with a Gröbner basis and a quotient-ring representation [10, 20, 21], but these approaches are often computationally prohibitive and can destroy sparsity [22–24]. Cheaper alternatives include basis-selection methods [25] and sampling-based constructions [26], though sampling may be difficult for robotics problems where even finding feasible points is hard [6, 27]. Most closely related to the current work is [5], which also uses nullspace computations to generate implied equalities, but relies on sampling, which can be difficult to perform efficiently, if at all. Moreover, the method can return constraints that are weak or difficult to interpret. By contrast, our method avoids solution sampling and directly characterizes the implied equalities that can strengthen a SOS relaxation.

II. NOTATION AND BACKGROUND

We denote $[N] = \{1, 2, \dots, N\}$. Let $\mathbb{R}[x]$ denote the polynomial ring in the decision variables x, and let $\mathbb{R}[x]_{\leq d}$ denote the polynomials of total degree at most d. The SOS polynomials up to degree-$2d$ are denoted $\boldsymbol{\Sigma}[x]_{\leq 2d} := \left\{p \in \mathbb{R}[x]_{\leq 2d} \mid p = \sum_i q_i^2(x),\ q_i \in \mathbb{R}[x]_{\leq d}\right\}$. For a polyno-

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mial f, we write $\deg(f)$ for its total degree, i.e. the highest degree of any monomial in its terms.

We consider POPs of the form

$$\min\ p(x) \tag{1a}$$

$$\text{s.t. } h_i(x) = 0,\ i \in [M] \tag{1b}$$

In general, (1) can contain inequalities of the form $g_j(x) \geq 0,\ j \in [N]$, but these will play no role in our development, so we drop them for simplicity.

The degree-$2D$ SOS relaxation of (1) is given by

$$\max\ \gamma \tag{2a}$$

$$\text{s.t. } p - \gamma = \sigma + \sum_{i=1}^{M} \lambda_i h_i \tag{2b}$$

$$\lambda_i \in \mathbb{R}[x]_{\leq 2D - \deg(h_i)},\ \sigma \in \mathbf{\Sigma}[x]_{\leq 2D} \tag{2c}$$

and is a semidefinite program [28, 29].

We call a constraint $\hat{h}(x) = 0$ redundant or an implied equality for (1) if it does not change the feasible set. Though redundant equalities $\hat{h}_k(x) = 0$ do not change (1), they do change (2) as their inclusion changes (2b) to

$$p - \gamma = \sigma + \sum_{j=1}^{M} \lambda_j h_j + \sum_{k} \hat{\lambda}_k \hat{h}_k.$$

We give an example in Section III where including $\hat{h}$ strengthens the relaxation.

The set of all redundant equalities for the system $h_1(x) = 0, \ldots, h_M(x) = 0$ is fundamentally tied to the ideal generated by $h_1, \ldots, h_M$:

$$\langle h_1, \ldots, h_M \rangle := \left\{ f \;\middle|\; f = \sum_{i=1}^{M} \lambda_i h_i,\ \lambda_i \in \mathbb{R}[x] \right\}. \tag{3}$$

Notice that if $h_1(x) = 0, \ldots, h_M(x) = 0$ and $\hat{h} \in \langle h_1, \ldots, h_M \rangle$, then $\hat{h}(x) = 0$ and so every element of the ideal is an implied equality for (1). If $\hat{h} = \sum_i \lambda_i h_i$, we call $\{\lambda_i\}$ an *ideal-membership certificate* for $\hat{h}$, and we define the degree of that certificate as $D := \max_i \deg(\lambda_i h_i)$. Alternatively, we say that $\{\lambda_i\}$ proves membership of $\hat{h}$ in the ideal $\langle h_1, \ldots, h_M \rangle$ in degree D.

Remark 1. *Since we are interested in real solutions of* (1), $\hat{h} \in \langle h_1, \ldots, h_M \rangle$ *is sufficient, but not necessary, to be redundant. The necessary and sufficient condition is given by membership in the so-called real-radical ideal [10, 29]. Extending the results of this paper to the radical case is straightforward, but more involved, so we defer this to future work.*

III. When Does a Redundant Equality Help?

In this section, we begin by showing how adding redundant equalities to (1) can change the strength of the relaxation (2). Then we provide a simple condition for testing whether a redundant equality can actually add any strength.

Example 1. *Consider the POP*

$$\min\ x \quad \text{s.t.} \quad x^2 y - x = 0,\ x^d = 0, \tag{4}$$

for a given $d \in \mathbb{N}$. *The degree-*$2D$ *SOS relaxation of* (4) *attains the globally optimal value of* 0 *if* $2D \geq 2d - 1$ *since*

$$x - 0 = -\left(\sum_{i=0}^{d-2} x^i y^i \right) (x^2 y - x) + y^{d-1}(x^d).$$

If $2D < d$, *the relaxation is infeasible and gives the trivial bound* $-\infty$.

By contrast, the SOS relaxation of the POP

$$\min\ x \quad \text{s.t.} \quad x^2 y - x = 0,\ x^d = 0,\ x^2 = 0 \tag{5}$$

attains value 0 *for every* $2D > 3$ *since*

$$x - 0 = -(x^2 y - x) + 0(x^d) + y(x^2).$$

The only difference between (4) *and* (5) *is the redundant equality* $x^2 = 0$. *This extra equality is useful because it lowers the certificate degree, even though it does not change the feasible set.*

The preceding example shows that adding a redundant constraint $\hat{h}_k(x) = 0$ can lower the degree needed to prove the lower bound γ. The next lemma quantifies how a redundant constraint $\hat{h}(x) = 0$ affects the relaxation.

Lemma 1 (Simulation Lemma)**.** *Fix* $2D$. *Let* $\hat{h}$ *have degree* $\hat{d}$, *and suppose* $\hat{h}$ *has an ideal-membership certificate of degree* $\hat{D}$,

$$\hat{h} = \sum_{i=1}^{M} \mu_i h_i, \qquad \deg(\mu_i h_i) \leq \hat{D}. \tag{6}$$

If there is a degree-$2D$ *certificate of the form*

$$p - \gamma = \sigma + \sum_{i=1}^{M} \lambda_i h_i + \hat{\lambda} \hat{h}, \tag{7}$$

then there is also a certificate using only $h_1, \ldots, h_M$ *of degree at most* $2D + \hat{D} - \hat{d}$.

Proof. Substituting (6) into (7) gives

$$p - \gamma = \sigma + \sum_{i=1}^{M} \left(\lambda_i + \hat{\lambda} \mu_i \right) h_i. \tag{8}$$

Since $\deg(\hat{\lambda}) \leq 2D - \hat{d}$ and $\deg(\mu_i h_i) \leq \hat{D}$, the new multiplier terms have degree at most $2D + \hat{D} - \hat{d}$. ☐

Lemma 1 shows that the effect of adding a redundant $\hat{h}$ of degree $\hat{d}$ to the relaxation can be simulated by simply increasing the degree of the relaxation. However, increasing the degree of the relaxation is frequently intractable, so it is usually computationally preferred to include the $\hat{h}$. An immediate corollary of Lemma 1 characterizes which implied constraints *do not* help strengthen the relaxation.

Corollary 1. *If* $\hat{h}$ *has degree* $\hat{d}$ *and admits an ideal-membership certificate of the same degree* $\hat{D} = \hat{d}$, *then adding* $\hat{h}$ *cannot strengthen a degree-*$2D$ *SOS relaxation.*

The corollary says that only a *degree gap* can matter. An implied equality is potentially useful only when proving its membership in the original ideal requires higher degree than the equality itself.

Notice that (6) is linear in $\hat{h}$ and μ_i for fixed h_i. Therefore, testing the condition (6) requires checking the feasibility of a system of linear equations of size $\binom{n+\hat{D}}{n}$. The condition can be used to rigorously explain why specific implied equalities help certain QCQPs found in the literature.

Example 2 (Contact Planning [6]). *In [6], a QCQP model of planning through contact is presented that includes quadratic equalities of the form $Ru = w$, where R is a rotation matrix. The authors report that including the constraint $u = R^T w$ strengthens their semidefinite relaxation. We can prove that $u = R^T w$ from $Ru = w$ and $R^T R = I$ in degree 3:*

$$R^T w - u = -R^T(Ru - w) + (R^T R - I)u.$$

No lower-degree proof exists, explaining the empirical finding that $R^T w = u$ can strengthen their SOS relaxation.

Example 3 (Pose Estimation [4, 8]). *In [4, Eq. 23], a QCQP model of robust pose estimation is presented that includes the constraints $x_0^T x_0 = 1$ and $x_i x_i^T = x_0 x_0^T$ for $i \in [N]$. These imply that $x_i x_j^T = x_j x_i^T$ and a degree-4 certificate of this fact is given in Eq. 24 of the same paper. It can be shown that no lower-degree certificate can exist, again supporting the empirical finding that the proposed constraint strengthens the relaxation.*

IV. Generating New Quadratic Equalities for QCQPs

In this section, we consider homogenized quadratically constrained quadratic programs (QCQPs) of the form

$$\begin{aligned} &\min\ x^T Q_0 x \\ &\text{s.t. } e_0^T x = 1,\ x^T P_i x = 0,\ i \in [M], \end{aligned} \tag{9}$$

which are common in robotics. This form can be studied without loss of generality as nonhomogeneous equalities can always be homogenized into this form.

We show how to combine the result of Corollary 1 with simple linear algebra to *systematically generate* new redundant quadratic equalities $x^T \hat{P} x = 0$ that help a SOS relaxation. This method makes the search for useful redundant equalities a simple linear algebraic preprocessing step that is fast enough to run before solving a SOS program. The extension of the method to general POPs will be shown in follow-on work.

A. Generating New Equalities

To find useful redundant equalities, we fix a proof degree $\hat{D} > 2$. Any degree-$\hat{D}$ proof that $x^T \hat{P} x$ lies in the ideal $\langle x^T P_1 x, \dots, x^T P_M x, e_0^T x - 1\rangle$ has the form

$$x^T \hat{P} x = \sum_{i=1}^{M} \mu_i(x)\left(x^T P_i x\right) + \mu_0(x)\left(e_0^T x - 1\right), \tag{10}$$

with $\mu_i \in \mathbb{R}[x]_{\leq \hat{D}-2}$, $\mu_0 \in \mathbb{R}[x]_{\leq \hat{D}-1}$. Similar to (14a), matching the terms of the same degree on the left- and right-hand sides of Equation (10) gives a linear equation between the coefficients of $\mu_i(x)$ and $\hat{P}$. It also requires all terms which are not homogeneous of degree 2 to vanish on the right-hand side, forcing the coefficients of $\mu_i(x)$ to satisfy certain linear equalities.

Because (10) is linear, we can encode it as a linear map:

$$H_2^{\langle P_i\rangle}(u) = \hat{P} \tag{11a}$$

$$N_{\hat{D}}^{\langle P_i\rangle}(u) = 0 \tag{11b}$$

In (11), u stacks all the coefficients of the multiplier μ_i, $H_2^{\langle P_i\rangle}$ is a linear map keeping the degree 2 part of the right-hand side of (10), and $N_{\hat{D}}^{\langle P_i\rangle}$ is the linear map to all other terms. The maps $H_2^{\langle P_i\rangle}$ and $N_{\hat{D}}^{\langle P_i\rangle}$ depend on the coefficients of the matrices P_i and the vector e_0. $N_{\hat{D}}^{\langle P_i\rangle}$ also depends on the chosen $\hat{D}$. The size of the linear operator in (11) depends on the choice of the proof degree $\hat{D}$ and is at most $\binom{n+\hat{D}}{n} \times \left(M\binom{n+\hat{D}-2}{n} + \binom{n+\hat{D}-1}{n}\right)$.

This conversion from a polynomial equality to a system of linear equations is standard and is how a SOS program is converted into a standard form SDP. Mapping a basis for (11b) through (11a) constructs a basis for all redundant equalities that could be added to (9). We denote the space of implied quadratic equalities proven in degree $\hat{D}$ as

$$E_{\hat{D}}^{\langle P_i\rangle} := H_2^{\langle P_i\rangle}\left(\ker(N_{\hat{D}}^{\langle P_i\rangle})\right). \tag{12}$$

B. Removing Useless Redundant Equalities

As written, (11) parametrizes *all* redundant equalities that can be added to (9), including those that provably add no strength by Corollary 1. In this section, we show how these useless equalities can be removed.

From Corollary 1, the redundant equalities which admit degree-2 proofs do not help our relaxation. These can be parameterized by scalar multipliers $\mu_i \in \mathbb{R}$ for the quadratic constraints and a linear multiplier $\zeta_0^T x - \mu_0$ for the constraint $e_0^T x = 1$:

$$x^T \hat{P} x = \sum_{i=1}^{M} x^T(\mu_i P_i)x + (\zeta_0^T x - \mu_0)(e_0^T x - 1). \tag{13}$$

By matching terms with the same degree, we see that (13) gives the following linear equations

$$\hat{P} = \sum_{i=1}^{M} \mu_i P_i + (\zeta_0 e_0^T + e_0 \zeta_0^T), \quad \zeta_0 = 0, \quad \mu_0 = 0 \tag{14a}$$

$$\hat{P} \in \textbf{span}\{P_1, \dots, P_M\}, \tag{14b}$$

The last equation (14b) characterizes which new quadratic equalities add no strength to a SOS relaxation of (9) according to Corollary 1. Note that this procedure is identical to the one from Section IV-A, but made concrete for the degree-2 case.

To compute only the useful equalities, we compute a basis for the space $N_{\hat{D}}^{\langle P_i\rangle}(u) = 0$ and then project away those u

such that $H_2^{\langle P_i \rangle}(u) \in \textbf{span}\{P_1, \ldots, P_M\}$. Alternatively, we complete $\textbf{span}\{P_1, \ldots, P_M\}$ to a basis of $E_{\hat{D}}^{\langle P_i \rangle}$. The useful implied equalities are the new basis elements. Formally, this can be written as the quotient

$$U_{\hat{D}}^{\langle P_i \rangle} = E_{\hat{D}}^{\langle P_i \rangle} / \textbf{span}\{P_1, \ldots, P_M\} \tag{15}$$

C. Summary

Equation (15) is the main computational takeaway. It characterizes all quadratic equalities that strengthen the relaxation of (9).

1) Compute the nullspace of $N_{\hat{D}}^{\langle P_i \rangle}$.
2) Map it through $H_2^{\langle P_i \rangle}(\ker N_{\hat{D}}^{\langle P_i \rangle})$ to compute $E_{\hat{D}}^{\langle P_i \rangle}$.
3) Complete $\textbf{span}\{P_1, \ldots, P_M\}$ to a basis of $E_{\hat{D}}^{\langle P_i \rangle}$. The new basis elements generate $U_{\hat{D}}^{\langle P_i \rangle}$.

These computations are standard linear algebra computations and can be carried out using algorithms such as QR, SVD, or Gaussian elimination.

V. COMPUTATIONAL EXAMPLES

We consider the computation of implied equalities for various ideals. All experiments were run on an 11th Gen Intel Core i9-11980HK CPU with 64 GiB RAM using Julia. For all examples, we report the size of the nullspace in (11b), the size of the space (15), the time required to compute the new candidate equalities, and which new constraints are found by the procedure at a given degree.

A. Ideal of Rotated Vectors

Proof Deg.	dim (11b)	dim (15)	New constraints	t (s)
3	60	3	$R^T w = u$	0.002
4	457	4	$\|w\|^2 = \|u\|^2$	0.013
5	3040	4	Same as degree 4	0.087
6	16512	9	$RR^T = I$	0.647

TABLE I: Quadratic implied equalities recovered for $Ru = w$ and $R^T R = I$. This encodes $R \in O(3)$.

Proof Deg.	dim (11b)	dim (15)	New constraints	time (s)
3	68	8	$R^T w = u$ $\tilde{r}_1^T \tilde{r}_2 = 0,\ \ \tilde{r}_2^T \tilde{r}_3 = 0$ $\tilde{r}_1 \times u = w_2 \tilde{r}_3 - w_3 \tilde{r}_2$	0.003
4	520	16	$\|w\|^2 = \|u\|^2$ $\tilde{r}_1^T r_1 = 1$ $r_2 = r_3 \times r_1,\ \ r_3 = r_1 \times r_2$	0.026
5	3805	24	$\tilde{r}_2^T \tilde{r}_2 = 1,\ \tilde{r}_2^T \tilde{r}_3 = 0$ $\tilde{r}_2 \times u = w_3 \tilde{r}_1 - w_1 \tilde{r}_3$ $\tilde{r}_3 \times u = w_1 \tilde{r}_2 - w_2 \tilde{r}_1$	0.16
6	22151	24	Same as degree 5	1.337

TABLE II: Quadratic implied equalities recovered for $Ru = w$, $R^T R = I$, and $r_1 = r_2 \times r_3$. This encodes $R \in SO(3)$ using a quadratic cross-product representation.

In this section, we consider the ideal from Example 2 in dimension 3, where vectors u and w are related by a matrix R. We consider two cases.

1) We only constrain $R \in O(3)$ by using the constraint $R^T R = I$. The results are presented in Table I.
2) We constrain R to be a rotation $R \in SO(3)$. We use the quadratic parametrization $R^T R = I$ and $r_1 = r_2 \times r_3$, where r_i is a column of R and $\times$ is the cross product. The results are presented in Table II.

We denote the columns of R as r_i and the rows as $\tilde{r}_i$.

We note that in both cases, the computation times are negligible. Moreover, it is interesting to track the evolution of proving that $RR^T = I$ in the case of $O(3)$ versus $SO(3)$. In the $O(3)$ case, the quadratic equality $RR^T = I$ is not proven until degree 6. However, in the $SO(3)$ case, we are able to prove that $\tilde{r}_i^T \tilde{r}_j = 0$ for $i \neq j$ at degree 3 and the full relation $RR^T = I$ is proven at degree 4.

Additionally, we note that at degree 6 for $O(3)$ and degree 5 for $SO(3)$, our procedure has already found a basis for all possible implied constraints provable in any degree. Though this cannot be proven by the proposed method, it can be certified by computing a Gröbner basis of the considered ideals.

B. Cloned-Quaternion Pose Estimation

Proof Deg.	dim (11b)	dim (15)	New constraints	time (s)
3	21	0	None	0.001
4	394	18	$x_0 x_1^T = x_1 x_0^T$ $x_0 x_2^T = x_2 x_0^T$ $x_1 x_2^T = x_2 x_1^T$	0.009
5	3770	18	Same as degree 4	0.084

TABLE III: Quadratic implied equalities recovered for Example 3.

We consider the ideal from Example 3, namely

$$x_0^T x_0 = 1, \quad x_i x_i^T = x_0 x_0^T$$

with $i \in [2]$, where each $x_i \in \mathbb{R}^4$. We report the results of running our procedure in Table III.

Our procedure recovers the implied constraints of [4, 8] at degree 4. A Gröbner basis computation confirms that these are all quadratic equalities obtainable in this example.

VI. CONCLUSION

We presented a simple necessary condition for testing whether a redundant equality can strengthen a fixed-order SOS relaxation: the equality must exhibit a degree gap between the polynomial itself and the lowest-degree certificate proving that it lies in the original ideal. This condition is linear and can be used to parametrize the set of all redundant equalities that add no strength. Since the set of all redundant equalities of a fixed degree can also be posed as a linear condition, the search for all redundant equalities that can strengthen a fixed-order SOS relaxation is a linear algebra problem. We demonstrated how to use this procedure to compute new implied quadratic equalities for homogenized QCQPs.

The construction in Section IV extends naturally to general POPs, higher-degree relaxations, and higher-degree implied equalities, but becomes more involved as the degree of the relaxation and the degree of the new equalities need to be tracked separately. The construction will be laid out in future work.

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