LEARNABILITY OF LEARNED NEURAL NETWORKS

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ABSTRACT

This paper explores the simplicity of learned neural networks under various settings: learned on real vs random data, varying size/architecture and using large minibatch size vs small minibatch size. The notion of simplicity used here is that of learnability i.e., how accurately can the prediction function of a neural network be learned from labeled samples from it. While learnability is different from (in fact often higher than) test accuracy, the results herein suggest that there is a strong correlation between small generalization errors and high learnability. This work also shows that there exist significant qualitative differences in shallow networks as compared to popular deep networks. More broadly, this paper extends in a new direction, previous work on understanding the properties of learned neural networks. Our hope is that such an empirical study of understanding learned neural networks might shed light on the right assumptions that can be made for a theoretical study of deep learning.

1 INTRODUCTION

Over the last few years neural networks have significantly advanced state of the art on several tasks such as image classification, machine translation, structured prediction,on, and have transformed the areas of computer vision and natural language processing (see e.g. LeCun et al. (2015)). Despite the success of neural networks in making these advances, understanding the reasons for their success remains largely an open problem. While there are many facets of this problem, one of the main ones is *generalization*, i.e. how is it that the training error and test error are close to each other for large neural networks where the number of parameters in the network is much larger than the number of training examples (highly overparametrized case). This paper explores three aspects of generalization in neural networks. The first aspect is the performance of neural networks on random training labels.
(Q1): Do neural networks learn simple patterns on random training data?
A second, very curious, aspect about the generalization of neural networks is the observation that increasing the size of a neural network helps in achieving better test error even if zero training error has already been achieved (see, e.g., Neyshabur et al. (2014)) i.e., larger neural networks have better generalization error. This is contrary to traditional wisdom in statistical learning theory which holds that larger models give better training error but at the cost of higher generalization error. A recent line of work proposes that the reason for better generalization of larger neural networks is *implicit regularization*, or in other words larger learned models are *simpler* than smaller learned models (e.g. Neyshabur (2017)). The second question we ask is
(Q2): Do larger neural networks learn simpler patterns compared to smaller neural networks when trained on real data?
A third aspect of generalization we consider is the widely observed phenomenon that using large minibatches for SGD leads to poor generalization LeCun et al.
(Q3): Are neural networks learned with small minibatch sizes simpler compared to those learned with large minibatch sizes? Please see the experimental results in Section D.4.
We look at the above questions through the lens of *learnability*, which we suggest as a concrete measure of the informal notion of simplicity used above.

2 EXPERIMENTS

All our experiments were performed on CIFAR-10 Krizhevsky et al. (2009). The $60,000$ training examples were divided into three subsets D_1, D_2 and D_3 with D_1 and D_2 having 25000 samples each and D_3 having 10000 samples. For more experimental details please refer Section D.1. The experimental setup is as follows.

Step 1 Train a network N_1 on (labeled) D_1.
Step 2 Use N_1 to predict labels for (unlabeled) D_2, denoted by $N_1(D_2)$.
Step 3 Train another network N_2 on the data $(D_2, N_1(D_2))$.

Learnability of a network is computed as $\frac{1}{|D_3|}\sum_{i=1}^{|D_3|}\mathbb{1}_{\{N_1(D_3)=N_2(D_3)\}} \times 100\%$. All the numbers reported here were averaged over 5 independent runs. We now present experimental results aimed at answering **(Q1)**, **(Q2)** and **(Q3)** we raised in Section 1.

Effect of Data The first set of experiments are aimed at understanding the effect of data on the simplicity of learned neural networks. We work with three different kinds of data. In this section we vary the data in three ways

— *True data*: Use *labeled* images from CIFAR-10 for D_1 in **Step 1**.
— *Random labels*: Use *unlabeled* images from CIFAR-10 for D_1 in **Step 1** and assign them random labels uniformly between 1 and 10.
— *Random images*: Use random images and labels in **Step 1**, where each pixel in the image is drawn uniformly from $[-1, 1]$.
For this set of experiments architecture of N_1 was the same as that of N_2. The networks N_1 and N_2 were varied over different architectures: VGG Simonyan & Zisserman (2014), GoogleNet Szegedy et al. (2015), ResNet He et al. (2016a), PreActResnet He et al. (2016b) , DPN Chen et al. (2017) and DenseNet Huang et al. (2016). We also do the same experiment on shallow convolutional neural networks with one convolutional layer and one fully connected layer. For the shallow networks, we vary the number of filters in N_1 and N_2 from $\{16, 32, 64, 128, 256, 512, 1024\}$. We start with 16 filters since that is the minimum number of filters where the training zero one error goes below 1%. The learnability values for various networks for true data, random labels and random images are presented in Table 1 for popular deep convolutional networks, Table 4 for shallow convolutional networks and Table 5 for MLPs. We see from the results that the learnability values of neural networks learned using true data are much larger compared to the values for those learned using random labels or random images. This clearly demonstrates that the complexity of a learned neural network heavily depends on the training data. Given that complexity of the learned model is closely related to its generalizability, this further supports the view that generalization in neural networks heavily depends on training data. Similar results can be observed for shallow convolutional networks on CIFAR-100 in Table 6.

Network	Random Labels	Random Images	True Data	True Data Acc.
VGG11	17.99±0.34	11.30±0.15	73.47±0.63	72.93±0.36
VGG13	16.82±0.54	12.51±0.15	75.21±1.25	75.01±0.56
VGG16	17.88±0.49	11.97±0.34	75.41±0.77	75.78±0.60
VGG19	17.95±0.62	11.78±0.46	75.84±0.49	76.10±0.28
ResNet18	14.96±0.20	13.56±0.58	69.93±0.74	69.98±0.55
Resnet34	16.14±0.54	13.96±0.07	72.22±0.85	71.88±0.41
PreActResnet18	16.33 ±0.35	14.61±0.80	70.83±0.75	68.35±1.97
DPN26	17.03±0.30	12.61±0.05	70.36±0.70	69.84±0.27
DenseNet121	19.32±0.26	13.35±0.35	79.46±0.08	79.47±0.67
GoogleNet	16.10±0.55	13.77±0.02	78.55±1.26	78.58±0.31

Table 1: Learnability comparison of popular neural network architectures on CIFAR 10 dataset with a batchsize of 64 averaged across five independent runs.

Effect of Network Size/Architecture The second set of experiments are aimed at understanding the effect of network size and architecture on the learnability of the learned neural network. First, we work with shallow convolutional neural networks (CNN) with 1 convolutional layer and 1 fully connected layer.

The results are presented in Table 2. Even though training accuracy is always greater than 99%, test accuracy increases with increase in the size of N_1 – Neyshabur et al. (2014) reports similar results for 2-layer multilevel perceptrons (MLP). It is clear that for any fixed N_2, the learnability of the learned network increases as the number of filters in N_1 increases. This suggests that the larger learned networks are indeed simpler than the smaller learned networks. Note also that for every N_1, its learnability values are always larger than its test accuracy when N_2 has 16 or more filters. This suggests that N_2 learns information about N_1 that is not contained in the data.

N_1 \ N_2	# Params	Test Acc.	16	32	64	128	256	512	1024
16	42186	53.91±0.56	58.81±0.40	61.61±0.49	63.78±0.54	65.09±0.24	65.08±0.21	65.12±0.19	63.94±0.73
32	84362	58.67±0.41	63.86±0.06	68.32±0.30	69.25±0.88	70.00±1.23	70.03±0.62	71.14±0.22	70.44±0.66
64	168714	61.37±0.36	67.60±0.04	71.37±0.58	73.04±0.25	74.48±0.05	73.80±0.19	74.92±0.36	74.04±0.01
128	337418	62.90±0.35	68.80±0.11	73.52±0.38	75.04±0.10	76.12±0.34	76.73±0.27	76.60±0.14	76.89±0.09
256	674826	63.80±0.32	70.63±0.08	74.53±0.22	77.53±0.10	77.61±0.04	78.05±0.29	78.10±0.01	77.56±0.68
512	1349642	64.43±0.11	71.69±0.05	76.23±0.28	77.15±0.05	78.07±0.06	79.43±0.06	79.56±0.50	78.96±0.02
1024	2701910	64.88±0.33	71.85±0.43	76.45±0.37	78.01±0.32	79.19±0.38	79.36±0.31	79.90±0.19	80.41±0.09

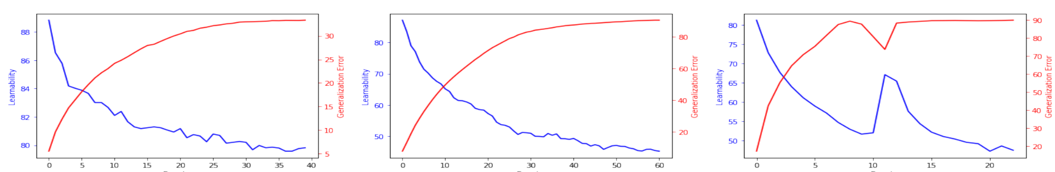
Table 2: Learnability values for shallow 2-layer CNNs of various sizes. Values in first column represent the number of filters in N_1 and values in header row represent the number of filters in N_2.

We performed the same experiment for some popular architectures as in Section 2. The results are presented in Table 3. Note that the accuracies reported here are significantly lower than those reported in published literature for the corresponding models; the reason for this is that our data size is essentially cut by half (see Section 2). Except for the case where N_2 is ResNet18 and N_1 is either a VGG or ResNet, there is a positive correlation between test accuracy and learnability i.e., a network with higher test accuracy is more learnable. We do not know the reason for the exception mentioned above. Furthermore, the pattern observed for shallow networks, that learnability is larger than accuracy, does not seem to always hold for these larger networks.

N_1 \ N_2	# of Layers, Params	Test Accuracy	VGG11	ResNet18	GoogLeNet	DenseNet121
DPN26	89-11574842	69.84±0.27	68.54±0.97	69.84±0.89	72.33±0.52	72.47±0.15
ResNet18	62-11173962	69.98±0.55	68.79±0.68	69.94±0.55	71.76±0.24	73.33±0.29
ResNet34	110-21282122	71.88±0.41	71.51±0.41	71.14±0.01	72.22±0.69	73.86±0.09
VGG11	34-9231114	72.93±0.36	73.47±0.39	69.46±0.57	72.39±0.11	73.42±0.03
VGG13	42-9416010	75.01±0.56	74.87±0.67	70.83±0.08	73.84±0.83	74.65±0.01
VGG16	54-14728266	75.78±0.60	74.23±1.04	72.28±0.14	74.40±0.66	74.76±0.39
VGG19	66-20040522	76.10±0.28	74.66±0.10	71.84±0.15	74.29±0.93	76.75±0.12
GoogLeNet	258-6166250	78.58±0.31	70.75±0.18	71.24±1.55	78.55±1.58	77.74±0.05
DenseNet121	362-6956298	79.47±0.67	73.41±0.37	73.95±0.56	78.61±0.01	79.46±0.01

Table 3: Learnability values for various popular architectures. The first column gives the architecture of N_1 and the header row shows the architecture of N_2. See text for discussion.

3 Why correlation between learnability and generalization?



(a) True Data (b) Random Labels (c) Random Images

Figure 1: Plot of learnability and generalization error vs epochs for shallow 2-layer CNNs

The experimental results so far show a clear correlation between learnability and generalizability of learned neural networks. This naturally leads to the question of why this is the case. We hypothesize that learnability captures the inductive bias of SGD training of neural networks. More precisely, when we start training, intuitively, the initial random network generalizes well (i.e., both train and test errors are high) and is also simple (learnability is high). As SGD changes the network to reduce the training error, it becomes more complex (learnability decreases) and generalization error increases. Figure 1 which shows the plots of learnability and generalizability of shallow 2-layer CNNs supports this hypothesis.

Conclusion We proposed learnability as a quantitative measure of the simplicity of (the function computed by) a network, and our experiments showed that it's intimately connected to generalization performance. Further study of this concept may lead to better understanding of the generalization phenomenon. We suggest one specific direction for future work: One way in which learnability may be useful in achieving better generalization is by regularizing training so as to guide it towards more learnable networks. Since learnability of a network can be estimated (but this is not very cheap) this is a reasonably concrete approach.

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A Appendix

B Introduction

Let us say we are considering a multi-class classification problem with c classes and let $\mathcal{D}$ denote a distribution over the inputs $x \sim \mathbb{R}^d$. Given a neural network $\mathcal{N}$, draw n independent samples $x_1^{\mathrm{tr}}, \cdots, x_n^{\mathrm{tr}}$ from $\mathcal{D}$ and train a neural network $\widehat{\mathcal{N}}$ on training data $(x_1^{\mathrm{tr}}, \mathcal{N}(x_1^{\mathrm{tr}})), \cdots, (x_n^{\mathrm{tr}}, \mathcal{N}(x_n^{\mathrm{tr}}))$, where $N(x) \in [c]$ denotes the prediction of $\mathcal{N}$ on x. The learnability of a neural network $\mathcal{N}$ is defined to be

$$L(\mathcal{N}) \stackrel{\mathrm{def}}{=} \mathop{\mathbb{E}}_{x^{\mathrm{tr}}, x \sim \mathcal{D}} \left[\mathbb{1}_{\left\{\mathcal{N}(x) = \widehat{\mathcal{N}}(x)\right\}} \right] \times 100\%. \tag{1}$$

Note that $L(\mathcal{N})$ implicitly depends on $\mathcal{D}$, the architecture and learning algorithm used to learn $\widehat{\mathcal{N}}$ as well as n. This dependence is suppressed in the notation but will be clear from context. Intuitively, larger the $L(\mathcal{N})$, easier it is to learn $\mathcal{N}$ from data. This notion of learnability is not new and is very closely related to probably approximately correct (PAC) learnability Valiant (1984); Kearns & Vazirani (1994). In the context of neural networks, learnability has been well studied from a theoretical point as we discuss briefly in Sec.C. There we also discuss some related empirical results but to the best of our knowledge there has been no work investigating the learnability of neural networks that are encountered in practice.

This paper empirically investigates the learnability of neural networks of varying sizes/architectures and minibatch sizes, learned on real/random data in order to answer (Q1) and (Q2) and (Q3).

Contributions: The results in this paper suggest that there is a strong correlation between generalizability and learnability of neural networks i.e., neural networks that generalize well are more learnable compared to those that do not generalize well. Our experiments suggest that

- Neural networks do not learn simple patterns on random data.
- Learned neural networks of large size/architectures that achieve higher accuracies are more learnable.
- Neural networks learned using small minibatch sizes are more learnable compared to those learned using large minibatch sizes.

Experiments also suggest that there are qualitative differences between learned shallow networks and deep networks and further investigation is warranted.

C Related work

The questions we raised in the introduction have been looked at from the point of view of flat/sharp minimizers (Hochreiter & Schmidhuber (1997) and the follow up work). This paper takes a complementary approach: it looks at the above questions through the lens of learnability.

Learnability of the concept class of neural networks has been addressed from a theoretical point of view in two recent lines of work. The first line of work shows hardness of learning by exhibiting a distribution and neural net that is hard to learn by certain type of algorithms. We will mention one of the recent results, further information can be obtained from references therein. Song et al. (2017) (see also Shamir (2016); Shalev-Shwartz et al. (2017)) show that there exist families of single hidden layer neural networks of small size that is hard to learn for *statistical query algorithms* (statistical query algorithms Kearns (1998) capture a large class of learning algorithms, in particular, many deep learning algorithms such as SGD). The result holds for log-concave distributions on the input and for a wide class of activation functions. If each sample is used only ones, then the hardness in their result means that the number of samples required is exponentially large. These results do not seem directly applicable to input distributions and networks encountered in practice.

The second line of work shows, under various assumptions on $\mathcal{D}$ and/or $\mathcal{N}$, that the learnability of neural networks is close to 1 Arora et al. (2014); Janzamin et al. (2015); Giryes et al. (2016); Zhong et al. (2017). Recently, Goel & Klivans (2017) give a provably efficient algorithm for learning one hidden layer neural networks consisting of sigmoids. However, their algorithm, which uses the kernel method, is different from the ones used in practice and the output hypothesis is not in the form of a neural network.

Using one neural net to train another has also been used in practice, e.g. Ba & Caurana (2013); Hinton et al. (2015); Urban et al. (2016). The goal in these works is to train a *small* neural net to the data with high accuracy by a process often called *distillation*. To this end, first a large network is trained to high accuracy. Then a smaller network is trained on the original data, but instead of class labels, the training now uses the classification probabilities or related quantities of the large network. Thus the goal of this line of research, while related, is different from our goal.

D EXPERIMENTS

D.1 EXPERIMENT SETUP

For all the experiments, we use vanilla stochastic gradient descent (SGD) i.e., no momentum parameter, with an initial learning rate of 0.01. We decrease the learning rate by a factor of $\frac{3}{4}$ if there is no decrease in train error for the last 10 epochs. Learning proceeds for 500 epochs or when the training zero-one error becomes smaller than 1%, whichever is earlier. Unless mentioned otherwise, minibatch size of 64 is used and the final training zero-one error is smaller than 1%. For training, we minimize logloss and do not use weight decay.

D.2 EFFECT OF DATA

More results for comparing learnability for shallow convolutional networks and MLP's are presented in Table 4 and Table 5

It is perhaps surprising that the learnability of networks trained on random data is substantially higher than 10% for shallow networks, on the other hand it's close to 10% for deeper networks. Some of this is due to class imbalance: in the case of true data, class imbalance is minimal for all architectures. While, when trained on random labels or random images output of N_1 on D_2 was skewed. This happened both for shallow networks and deeper networks but was slightly higher for shallow networks. Table 7 presents the percentage of each class in the labels of $N1$ on $D2$. However, we do not have a quantification of how much of learnability in the case of shallow networks arises due to class imbalance and a compelling reason for high learnability of shallow networks.

Network	Random Labels	Random Images	True Data	True Data Acc.
16	21.57±0.66	14.19±1.01	58.81±0.4	53.91±0.56
32	22.07±0.34	16.13±0.44	68.32±0.3	58.67±0.41
64	28.98±0.38	22.91±0.65	73.04±.25	61.37±0.36
128	35.34±0.55	31.05±1.75	76.12±0.34	62.90±0.35
256	40.93±0.62	40.5±0.84	78.05±0.29	63.80±0.32
512	43.51±0.91	49.43±2.78	79.56±0.5	64.43±0.11
1024	46.49±1.15	52.06±0.29	80.41±.09	64.88±0.33

Table 4: Learnability comparison of shallow networks on CIFAR-10 dataset with a batchsize of 64 averaged across five independent runs.

Network Depth	Random Labels	Random Images	True Data	True Data Acc.
1	21.39±0.35	45.13±0.83	50.18±0.72	40.88±0.65
2	17.65±0.43	34.89±1.29	48.58±0.57	42.62±0.53
3	15.99±0.21	24.78±0.91	48.62±1.01	42.83±0.63
4	13.74±0.01	20.33±0.99	48.12±0.83	42.74±0.83
5	12.60±0.60	16.55±0.12	46.12±0.81	42.76±0.52

Table 5: Learnability comparison of MLPs (Multi Layer Perceptrons) of fixed hidden unit size 64 and varying depth on CIFAR-10 dataset with a batchsize of 64 averaged across five independent runs.

Network	Random Labels	Random Images	True Data	True Data Acc.
16	13.52±0.54	8.98±0.59	38.56±1.10	26.13±0.36
32	18.92±0.43	16.93±0.90	44.05±0.67	28.89±0.43
64	22.61±1.13	25.74±0.72	47.18±0.73	30.32±0.18
128	25.88±0.30	34.34±2.27	48.87±0.47	30.71±0.18
256	28.52±0.88	43.27±2.15	49.75±0.32	31.39±0.13
512	29.62±1.07	47.45±2.04	50.71±0.29	31.68±0.19
1024	30.57±1.05	48.08±2.41	51.06±0.26	32.10±0.19

Table 6: Learnability comparison of shallow CNNs on CIFAR-100 dataset with a batchsize of 64 averaged across five independent runs.

Network Arch. \ Class	0	1	2	3	4	5	6	7	8	9
True Data										
GoogleNet	10.89	8.95	9.73	8.88	12.29	9.04	9.13	11.48	8.85	10.75
DenseNet	9.19	9.27	8.51	11.18	12.14	8.90	6.35	16.54	9.28	8.65
Shallow conv 1024	10.45	9.45	9.14	9.08	9.72	12.44	10.15	10.41	9.50	9.65
Shallow conv 16	11.78	9.67	9.44	9.48	9.83	10.35	9.90	10.37	9.16	10.03
Random Labels										
DenseNet	10.17	8.24	11.54	6.84	11.11	10.81	10.70	9.17	11.01	10.42
GoogleNet	9.33	11.51	6.76	9.71	12.82	9.20	10.32	7.08	10.70	12.57
Shallow conv 1024	11.86	10.05	13.93	10.07	9.79	12.69	14.60	4.88	4.94	7.19
Shallow conv 16	8.86	9.78	11.89	10.40	10.14	7.39	10.69	10.99	12.37	7.50
Random Images										
DenseNet	4.05	8.07	2.02	9.02	0.49	17.59	0.75	17.85	37.46	2.70
GoogleNet	28.78	7.29	4.56	4.50	1.13	12.12	21.79	8.45	0.93	10.46
Shallow conv 1024	3.42	22.07	19.03	6.79	5.00	3.46	8.99	7.88	4.19	19.17
Shallow conv 16	9.68	10.13	9.33	8.70	8.52	11.76	12.01	11.91	8.69	9.28

Table 7: Class wise Percentage distribution for N_1 predictions on D_2 for CIFAR-10 Dataset. Shallow 16 refers to a single layer ConvNet with 16 number of filters.

For any given example, let us denote TLP(x) $\stackrel{\text{def}}{=} \mathbb{1}_{\{N_1(x)=y(x)\}}$, where $y(x)$ denotes the true label of x and PLP(x) $\stackrel{\text{def}}{=} \mathbb{1}_{\{N_1(x)=N_2(x)\}}$. Tables 8 and 9 present the percentage of examples for the four

TLP \ PLP	0	1
0	11.68.	23.38
1	7.93	57.01

Table 8: N_1: Shallow net with 1024 filters, N_2: Shallow net with 1024 filters; in percentage

TLP \ PLP	0	1
0	25.84	19.83
1	15.29	39.04

Table 9: N_1: Shallow net with 16 filters, N_2: Shallow net with 16 filters; in percentage

TLP \ PLP	0	1
0	14.58.	12.93
1	11.94	60.55

Table 10: N_1: VGG11 and N_2: VGG11; in percentage

TLP \ PLP	0	1
0	11.21	9.97
1	10.23	68.59

Table 11: N_1: GoogleNet, N_2: GoogleNet; in percentage

different possibilities of TLP and PLP for shallow networks while Tables 10 and 11 present these results for VGG-11 and GoogleNet. The key point we would like to point out from these tables is that if we focus on those examples where N_1 does not predict the true label correctly i.e., TLP $= 0$

or the first row in the tables, we see that approximately half of these examples are still learned correctly by N_2. Contrast this with the learnability values of N_1 learned with random data which are all less than 20%. This suggests that networks learned on true data make simpler predictions even on examples which they misclassify.

No. of Hidden Units	# Params	Learnability	Test Accuracy
1	805	97.25 ± 0.34	37.43 ± 0.98
2	1600	96.06 ± 0.51	73.50 ± 1.49
3	2395	96.65 ± 0.36	84.58 ± 0.21
4	3190	95.69 ± 0.07	89.04 ± 0.54
5	3985	92.80 ± 0.74	92.09 ± 0.26
6	4780	90.27 ± 2.49	93.16 ± 0.13

Table 12: Learnability and Accuracy comparison of single layer MLP with varying hidden unit size for N_1 on MNIST dataset averaged across five independent runs. For all of the above results we fixed N_2 to a single layer MLP with hidden unit size of 4.

We also present in Table 12, the learnability and test accuracy values of single layer MLPs with different number of hidden units trained on MNIST. While we still observe correlation between learnability and test accuracy, the learnability values are much higher than the test accuracy values. This clearly demonstrates that high learnability does not necessarily require high test accuracy but can occur even when test accuracies are low.

D.3 Effect of Network Size/Architecture

N_1 \ N_2	# Params	Test Accuracy	1	2	3	4	5
1	197322	40.88±0.65	50.18±0.72	48.85±1.02	48.38±0.32	48.04±0.39	48.04±0.04
2	201482	42.62±0.53	48.48±0.10	48.58±0.57	50.03±0.43	48.96±0.33	47.80±1.11
3	205642	42.83±0.63	48.06±0.30	48.09±0.19	48.62±1.01	48.39±1.14	47.45±0.24
4	209802	42.74±0.83	46.35±0.41	46.89±0.76	48.32±0.95	48.12±0.83	46.40±0.12
5	213962	42.76±0.52	44.96±0.92	46.14±0.92	46.62±0.57	46.25±0.61	46.12±0.81

Table 13: Learnability values for shallow MLPs of various sizes. Values in the first column represent the depth ofN_1 and values in the header row represent the depth of N_2. Each MLP layer had a hidden unit size of 64 followed by a ReLU

D.4 Effect of Batch Size

The third set of experiments are aimed at this question. For this set of experiments, N_1 and N_2 are again varied over different architectures while keeping the architectures of N_1 and N_2 same. The minibatch size for training of N_2 (**Step 3**) is fixed to 64 while the minibatch size for training of N_1 (**Step 1**) is varied over $\{32, 64, 128, 256\}$. Table 14 presents these results.

N_1/N_2 \ Batch size	32	64	128	256
VGG11	74.56 ±0.71	73.75 ±0.07	72.05 ±0.83	69.10 ±0.53
VGG13	74.85 ±0.43	73.95 ±1.13	73.32 ±0.23	69.84 ±0.29
VGG16	74.87 ±0.60	74.73 ±0.44	73.14 ±0.25	69.88 ±0.32
VGG19	74.44 ±0.40	74.38 ±0.03	73.03 ±0.15	70.74 ±0.52
ResNet18	72.51 ±0.87	70.06 ±0.45	64.98 ±0.17	61.72 ±0.64
DenseNet121	73.79 ±0.25	73.33 ±0.06	-	-
GoogleNet	72.01 ±0.25	71.55 ±0.54	-	-

Table 14: Learnability comparison of network architectures on CIFAR-10 dataset with varying batch sizes. For this experiment we fixed N2 to be VGG11 with batch size of 64. GoogleNet and DenseNet architectures ran out of memory for batch size of 128 and 256.

It is clear from these results that for any architecture, increasing the minibatch size leads to a reduction in learnability. This suggests that using a larger minibatch size in SGD leads to a more complex neural network as compared to using a smaller minibatch size. Previous work on understanding effect of batch size on deep neural networks by Keskar et al. (2016) demonstrates that larger batch sizes tend to converge to sharp minimizers of the training function. Although more recent paper by Bengio et al. have demonstrated that the notion of sharpness can be easily modified or manipulated without actually changing the learned network. So we compare the learnability of popular neural networks with batch size

D.5 Variability of predictions

In this section, we will explore a slightly orthogonal question of whether neural networks learned with different random initializations converge to the same neural network, as functions. While there are some existing works e.g., Goodfellow et al. (2014), which explore linear interpolation between the parameters of two learned neural networks with different initializations, we are interested here in understanding if different SGD solutions still correspond to the same function. In order to do this, we compute the confusion matrix for different SGD solutions. If SGD is run k times ($k = 5$ in this case), recall that the (i, j) entry of the confusion matrix, where $1 \leq i, j \leq k$ gives the fraction of examples on which the i^{th} and j^{th} SGD solutions agree. The following are the confusion matrices for different SGD solutions, left for a shallow network with 1024 filters and right for VGG-11.

$$\begin{bmatrix} 1. & 0.93 & 0.93 & 0.93 & 0.93 \\ 0.93 & 1. & 0.93 & 0.94 & 0.94 \\ 0.93 & 0.93 & 1. & 0.93 & 0.94 \\ 0.93 & 0.94 & 0.93 & 1. & 0.93 \\ 0.93 & 0.94 & 0.94 & 0.93 & 1. \end{bmatrix} \qquad \begin{bmatrix} 1. & 0.73 & 0.73 & 0.73 & 0.72 \\ 0.73 & 1. & 0.74 & 0.73 & 0.74 \\ 0.73 & 0.74 & 1. & 0.74 & 0.75 \\ 0.73 & 0.73 & 0.74 & 1. & 0.74 \\ 0.72 & 0.74 & 0.75 & 0.74 & 1. \end{bmatrix}$$

For both the networks, we see that the off-diagonal entries are quite close to each other. This seems to suggest that while the different SGD solutions are not same as functions, they agree on a common subset (93% for shallow network and 73% for VGG-11) of examples. Furthermore, for VGG-11, the off-diagonal entries are very close to the test accuracy – this behavior of VGG-11 seems common to other popular architectures as well. This seems to suggest that different SGD solutions agree on precisely those examples which they predict correctly, which in turn means that the subset of examples on which different SGD solutions agree with each other are precisely the correctly predicted examples. However this does not seem to be the case. Figures 2 and 3 show the histograms of the number of distinct predictions for shallow network and VGG-11 respectively. For each number $i \in [k]$, it shows the fraction of examples for which the k SGD solutions make exactly i distinct predictions. The number of examples for which there is exactly 1 prediction, or equivalently all the SGD solutions agree is significantly smaller than the test accuracies reported above.

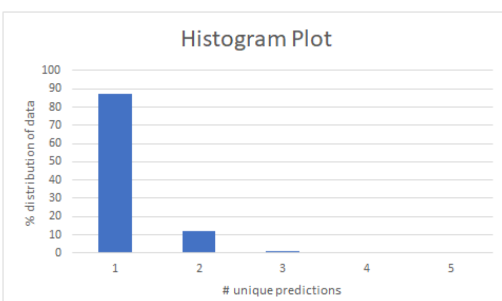

Figure 2: Shallow network with 1024 filters

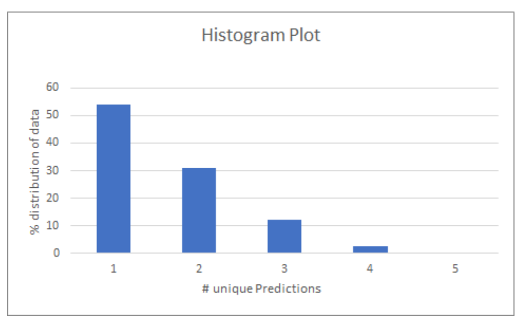

Figure 3: VGG11