QUANTIFYING CROSS-DOMAIN KNOWLEDGE DISTILLATION IN THE PRESENCE OF DOMAIN SHIFT

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ABSTRACT

Cross-domain knowledge distillation often suffers from domain shift. Although domain adaptation methods have shown strong empirical success in addressing this issue, their theoretical foundations remain underdeveloped. In this paper, we study knowledge distillation in a teacher–student framework for regularized linear regression and derive high-dimensional asymptotic excess risk for the student estimator, accounting for both covariate shift and model shift. This asymptotic analysis enables a precise characterization of the performance gain in cross-domain knowledge distillation. Our results demonstrate that, even under substantial shifts between the source and target domains, it remains feasible to identify an imitation parameter for which the student model outperforms the student-only baseline. Moreover, we show that the student's generalization performance exhibits the double descent phenomenon.

1 INTRODUCTION

The success of modern machine learning tasks typically requires the availability of large-scale labeled datasets. However, collecting labeled data for a new target task is often challenging and expensive. When data in the target domain is scarce, it is possible to leverage labeled data from related source domains. Knowledge distillation (KD) (Hinton et al., 2015), originally proposed for model compression, is a popular technique that transfers knowledge from a capable teacher model trained on a source domain to a smaller student model. This is achieved by guiding the student model to mimic the teacher model's outputs. The extra information in the teacher's predictions often improves the student model's performance when target domain data is limited. KD has recently achieved remarkable success across several fields including image classification (Radford et al., 2021; Li et al., 2024), speech recognition (Mingote et al., 2020), and language models (Gu et al., 2023; Agarwal et al., 2024).

We denote the source domain data and target domain data as $(\mathbf{X}_1, \mathbf{y}_1)$ and $(\mathbf{X}_2, \mathbf{y}_2)$, respectively. This work focuses on the following cross-domain KD process: a teacher model is first trained on the source domain data, and its predicted labels for the target domain inputs are then used to supervise the training of the student model by minimizing the per-sample objective function,

$$\mathcal{L}(\xi) = \xi \ell(y_2^{\mathsf{t}}, y_2^{\mathsf{s}}) + (1-\xi)\ell(y_2, y_2^{\mathsf{s}}), \tag{1}$$

where ℓ denotes the loss function, y_2 is the ground-truth label, y_2^{t} is the teacher's predicted label, and y_2^{s} denotes the student's prediction. The weight parameter ξ, known as the imitation parameter (Lopez-Paz et al., 2015), balances the contributions of the teacher's predictions and the observed labels during training.

Cross-domain KD often suffers from a *shift* between the source and target domains. For instance, the source domain may consist of standard American English speech, while a region-specific voice assistant must handle local dialects. Another example is a face detection model trained on images of light-skinned individuals (source domain) being applied to images of dark-skinned individuals (target domain). Learning a discriminative predictor under such domain shifts between source and target domains is known as domain adaptation (Ganin et al., 2016). While much of the literature on domain adaptation has focused on improving the performance of KD, relatively little is understood about when – and how effectively – the student model can learn from the teacher in the presence of domain shift.

Recently, Emrullah Ildiz et al. (2025) analyzed the weak-to-strong (W2S) generalization of linear models in a cross-domain setting, and identified the form of the optimal surrogate model. However, their analysis relies on the condition that the covariance matrices of the source and target domains are jointly diagonalizable, which limits its ability to capture the influence of eigenvectors. Moreover, their results are restricted to the setting $\xi = 1$ (i.e., pure teacher supervision), leaving the trade-off between distillation and learning from observed student data unexplored. Motivated by these limitations, we take a step toward a more complete understanding of the performance gains of cross-domain KD for linear regression.

In this paper, we present a theoretical analysis of cross-domain KD in the context of linear models, leveraging tools from random matrix theory. For ridge regression, we study two cases: (i) a deterministic-parameter setting, in which the teacher and student parameter vectors are non-random; and (ii) a random-parameter setting, in which a shared parameter vector is drawn from a prior distribution. We also analyze ridgeless regression in the under-parameterized regime (see the first inequality in equation 10). All proofs of the theoretical results are provided in the appendix. We summarize our contributions as follows:

- **High-dimensional risk characterization.** We derive precise high-dimensional asymptotics for the risk of cross-domain KD via a bias–variance decomposition. Our results reveal how the excess risk depends on the parameter vectors and the input distributions in both domains, generalizing the student-only setting of Hastie et al. (2022).
- **Utility of cross-domain KD.** (*"Stones from other mountains can polish jade"*). Intuitively, large domain shifts between the teacher's and student's training data might limit – or even harm – the value of teacher supervision for the student's generalization. Surprisingly, our analysis shows that even under substantial domain discrepancies, it is still possible to find an $\xi \in \mathbb{R}$ such that the student model can outperform the student-only baseline. The existence of such ξ depends on the geometry of the models and the covariance matrices of both domains.
- **Double descent phenomenon.** We observe that the excess risk, as a function of the dimension-to-sample-size ratio, exhibits the double-descent phenomenon in KD for teacher-student model – previously documented by Hastie et al. (2022); Nakkiran et al. (2021) in student-only models, and by Moniri & Hassani (2025) for $\xi = 1$ under no domain shift with isotropic covariance.

1.1 Related Works

Theory of KD. In recent years, a growing body of work has sought to understand the effects of KD. The theoretical understanding of distillation began with Phuong & Lampert (2019), who initially investigated linear student networks. Wei et al. (2021); Borup & Andersen (2021); Das & Sanghavi (2023); Pareek et al. (2024); Jeong & Chung (2025) theoretically studied self-distillation, a variant of KD in which the student model has the same architecture as the teacher and is trained on the same data. Menon et al. (2021) showed that a "Bayes teacher" providing true class probabilities can reduce the variance of the student's objective, leading to improved performance. Harutyunyan et al. (2023) proposed a framework that highlighted a delicate interplay among the teacher's accuracy, the student's margin with respect to the teacher predictions, and the complexity of the teacher predictions. From an information-theoretic perspective, Dissanayake et al. (2025) quantified and explained the transferred knowledge and knowledge left to distill for a downstream task.

Cross-domain KD and domain adaptation. Many studies have explored various methods to address the domain shift problem in the field of KD. Empirical works include Su & Maji (2016); Kundu et al. (2019); Asami et al. (2017); Li et al. (2023); Xu et al. (2024); Tang et al. (2025). The emergence of large language models (LLMs) has brought new advancements, such as distillation across vastly different architectures and scalable cross-domain transfer. For more details, readers may refer to Fedus et al. (2022); Ouyang et al. (2022); Yang et al. (2024). From a theoretical perspective, Emrullah Ildiz et al. (2025) focused on the setting where the student is trained using only the teacher's predictions, and analyzed the conditions under which the student can outperform the teacher in cross-domain KD.

Weak-to-strong generalization. Weak-to-strong (W2S) generalization (Burns et al., 2024), which concerns using predictions generated by a weaker teacher model to train a more powerful student

model, is closely connected to KD. Emrullah Ildiz et al. (2025) provided an analysis of ridgeless regression and proved that when using a weak model as the surrogate (teacher), W2S training can provably outperform training with true labels. Charikar et al. (2024) assumed that the models are selected over a convex set, and quantified the gain of the weak-label trained strong model over the weak model. Wu & Sahai (2025) explored W2S generalization for classification in a spiked covariance model. Medvedev et al. (2025) explained how W2S generalization can arise in random feature models described by two-layer networks. Theoretical research in this area has continued to grow, see Dong et al. (2025); Shin et al. (2025); Moniri & Hassani (2025); Oh et al. (2025), for example.

1.2 Notations

We use $\|\cdot\|$ to denote the spectral norm for matrices and the Euclidean norm for vectors, and $\|\cdot\|_{\mathsf{F}}$ for the Frobenius norm of a matrix. Standard big-O and small-o notations are employed. Moreover, we denote $x_n = o_{a.s.}(a_n)$, if $x_n/a_n \to 0$ almost surely. For any sequences $a_n \geq 0$ and $b_n \geq 0$, we write $a_n \lesssim b_n$ if $a_n = O(b_n)$, and $a_n \sim b_n$ if both $a_n \lesssim b_n$ and $b_n \lesssim a_n$. We use $\delta(\cdot)$ to denote the indicator function, which takes the value 1 if the condition $\cdot$ holds, and 0 otherwise. Throughout the paper, c and C denote constants that may vary from line to line. For a random variable x, we use $x \sim D$ to indicate that x follows the distribution D.

2 Preliminaries

2.1 Problem setup

Suppose there are N_1 covariates $\{\mathbf{x}_j^{(1)}\}_{j=1}^{N_1}$ drawn i.i.d. from an M-dimensional source distribution D_1 and N_2 covariates $\{\mathbf{x}_j^{(2)}\}_{j=1}^{N_2}$ drawn i.i.d. from an M-dimensional target distribution D_2. We consider a linear regression task specified by an unknown parameter vector $\boldsymbol{\beta}_i \in \mathbb{R}^M$:

$$y_j^{(i)} = \boldsymbol{\beta}_i^{\mathsf{T}} \mathbf{x}_j^{(i)} + \varepsilon_j^{(i)},\ i = 1, 2,\ 1 \leq j \leq N_i,$$

where $\varepsilon_j^{(i)} \in \mathbb{R}$ is a zero-mean random noise term with variance σ^2. For $i = 1, 2$ and $z \in \mathbb{C}\backslash\mathbb{R}^+$, define

$$\mathbf{X}_i = (\mathbf{x}_1^{(i)}, ..., \mathbf{x}_{N_i}^{(i)}) \in \mathbb{R}^{M \times N_i}, \quad \mathbf{y}_i = (y_1^{(i)}, ..., y_{N_i}^{(i)})^{\mathsf{T}} \in \mathbb{R}^{N_i},$$

$$\mathbf{Q}_i(z) = \left(\frac{1}{N_i}\mathbf{X}_i\mathbf{X}_i^{\mathsf{T}} - z\mathbf{I}_M\right)^{-1}, \quad \boldsymbol{\varepsilon}_i = (\varepsilon_1^{(i)}, ..., \varepsilon_{N_i}^{(i)})^{\mathsf{T}} \in \mathbb{R}^{N_i}.$$

We refer to the case where $D_1 \neq D_2$ as a ***covariate shift***, and the case where $\boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_2$ as a ***model shift***.

Teacher Model: The teacher model is finetuned on $\left\{(\mathbf{x}_j^{(1)}, y_j^{(1)})\right\}_{j=1}^{N_1}$:

$$\begin{aligned}\boldsymbol{\beta}_{\mathsf{t}} &= \arg\min_{\boldsymbol{\beta}} \left(\frac{1}{N_1}\|\mathbf{y}_1 - \mathbf{X}_1^{\mathsf{T}}\boldsymbol{\beta}\|^2 + \lambda_{\mathsf{t}}\|\boldsymbol{\beta}\|^2\right) \\ &= \left(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}} + N_1\lambda_{\mathsf{t}}\mathbf{I}_M\right)^{-1}\mathbf{X}_1\mathbf{y}_1 = \frac{1}{N_1}\mathbf{Q}_1(-\lambda_{\mathsf{t}})\mathbf{X}_1\mathbf{y}_1,\end{aligned} \tag{2}$$

where $\lambda_{\mathsf{t}} > 0$ is the teacher regularization parameter. The risk of $\boldsymbol{\beta}_{\mathsf{t}}$ when $M \sim N_1$ in the high-dimensional setting has been studied extensively in the literature such as Dobriban & Wager (2018); Hastie et al. (2022).

Student Model Trained with Cross-Domain KD: We use the pre-trained teacher model together with covariates $\{\mathbf{x}_j^{(2)}\}_{j=1}^{N_2}$ to generate predictions:

$$\mathbf{y}_2^{\mathsf{t}} = (y_1^{\mathsf{t}}, ..., y_{N_2}^{\mathsf{t}})^{\mathsf{T}} = \left(\mathbf{x}_1^{(2)}, ..., \mathbf{x}_{N_2}^{(2)}\right)^{\mathsf{T}}\boldsymbol{\beta}_{\mathsf{t}}.$$

The student model is finetuned on the target domain data $\{(\mathbf{x}_j^{(2)}, y_j^{(2)})\}_{j=1}^{N_2}$ and the teacher's predictions $\{(\mathbf{x}_j^{(2)}, y_j^{\mathsf{t}})\}_{j=1}^{N_2}$, using the per-sample objective function defined in equation 1 with an imitation

parameter ξ, as follows:

$$\begin{aligned}\boldsymbol{\beta}_{\mathsf{s}} = \arg\min_{\boldsymbol{\beta}} \mathcal{L}(\xi) &= \arg\min_{\boldsymbol{\beta}} \xi\left(\frac{1}{N_2}\|\mathbf{y}_2^{\mathsf{t}} - \mathbf{X}_2^{\mathsf{T}}\boldsymbol{\beta}\|^2\right) + (1-\xi)\left(\frac{1}{N_2}\|\mathbf{y}_2 - \mathbf{X}_2^{\mathsf{T}}\boldsymbol{\beta}\|^2\right) + \lambda_{\mathsf{s}}\|\boldsymbol{\beta}\|^2 \\ &= (\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}} + N_2\lambda_{\mathsf{s}}\mathbf{I}_M)^{-1}(\xi\mathbf{X}_2\mathbf{y}_2^{\mathsf{t}} + (1-\xi)\mathbf{X}_2\mathbf{y}_2),\end{aligned} \tag{3}$$

where λ_{s} is the student regularization parameter. While it is common to restrict $\xi \in [0,1]$ (Lopez-Paz et al., 2015), we do not impose this constraint, in line with Das & Sanghavi (2023); Pareek et al. (2024). For the covariates $\mathbf{x}_j^{(i)}$ and the noise terms $\varepsilon_j^{(i)}, i = 1,2,\ 1 \le j \le N_i$, we make the following assumptions, which are standard in the random matrix theory literature (see, e.g., Bai & Silverstein (2010)).

Assumption 1. *Suppose $\mathbf{X}_1, \mathbf{X}_2, \boldsymbol{\varepsilon}_1$, and $\boldsymbol{\varepsilon}_2$ are mutually independent. Moreover, we assume*

(a) the covariates are generated according to

$$\mathbf{X}_i = (\boldsymbol{\Sigma}_i)^{1/2}\mathbf{Z}_i, \ \text{ for } i = 1,2,$$

where $\mathbf{Z}_i = (z_{jk}^{(i)})$ is an $M \times N_i$ random matrix with i.i.d. entries of zero mean and unit variance, and $\boldsymbol{\Sigma}_i$ is a positive semi-definite matrix. Furthermore, we assume for all $p \in \mathbb{N}$, there is a constant C_p such that

$$\max_{i=1,2} \mathbb{E}|z_{11}^{(i)}|^p \le C_p. \tag{4}$$

(b) $M \sim N_1 \sim N_2$.

(c) $\boldsymbol{\varepsilon}_i \in \mathbb{R}^{N_i}$ is a random vector consisting of i.i.d. entries of zero mean, variance σ^2, and for all $p \in \mathbb{N}$, there is a constant c_p such that

$$\max_{i=1,2} \mathbb{E}|\varepsilon_1^{(i)}|^p \le c_p.$$

While we allow $z_{11}^{(1)}$ and $z_{11}^{(2)}$ to follow different distributions – a form of covariate shift – our theoretical results do not depend on their specific distributions, provided that the moment conditions in Assumption 1(a) are satisfied. The requirement that all moments of $z_{11}^{(i)}$ exist can be relaxed to the existence of finitely many moments, with minor modifications to our proof; however, we do not pursue this generalization here. The following assumption on the structure of the covariance matrices is imposed to facilitate theoretical analysis and rule out degenerate cases.

Assumption 2. *Let τ be a small constant. Denote the eigenvalues of $\boldsymbol{\Sigma}_i$ by $\sigma_1^i \ge \sigma_2^i \cdots \ge \sigma_M^i \ge 0$.*

(a) (Boundedness of $\boldsymbol{\Sigma}_i$). We assume that $\max_{i=1,2}\|\boldsymbol{\Sigma}_i\| = \sigma_1^i < \tau^{-1}$.

(b) (Anti-concentration at 0). For $i = 1,2$, the empirical spectral distribution of $\boldsymbol{\Sigma}_i$ satisfies

$$\frac{1}{M}\sum_{j=1}^{M} \delta(\sigma_j^i \le \tau) \le 1 - \tau.$$

Let $(\mathbf{x}, y)$ be an unseen sample of the target task, that is $y = \boldsymbol{\beta}_2^{\mathsf{T}}\mathbf{x} + \varepsilon$, where $\mathbf{x} \sim D_2$ and ε follows the same distribution with $\varepsilon_1^{(2)}$. Under the mean squared loss, the generalization ability is quantified by the risk of the estimator $\boldsymbol{\beta}_{\mathsf{s}}$:

$$\mathbf{R}(\boldsymbol{\beta}_{\mathsf{s}}) = \mathbb{E}_{\mathbf{x},y}|y - \boldsymbol{\beta}_{\mathsf{s}}^{\mathsf{T}}\mathbf{x}|^2 = \mathbb{E}_{\mathbf{x},y}|(\boldsymbol{\beta}_2 - \boldsymbol{\beta}_{\mathsf{s}})^{\mathsf{T}}\mathbf{x} + \varepsilon|^2 = \|\boldsymbol{\Sigma}_2^{1/2}(\boldsymbol{\beta}_2 - \boldsymbol{\beta}_{\mathsf{s}})\|^2 + \sigma^2,$$

where $\mathbb{E}_{\mathbf{x},y}$ denotes the expectation taken with respect to (w.r.t.) the pair $(\mathbf{x}, y)$. The excess test risk is defined as follows:

$$\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}}) = \mathbf{R}(\boldsymbol{\beta}_{\mathsf{s}}) - \sigma^2 = \|\boldsymbol{\Sigma}_2^{1/2}(\boldsymbol{\beta}_2 - \boldsymbol{\beta}_{\mathsf{s}})\|^2. \tag{5}$$

When $\xi = 0$, $\boldsymbol{\beta}_{\mathsf{s}}$ reduces to the ridge regression estimator for the student only model, and we denote the corresponding excess risk by $\mathbf{ER}_0$. Note that $\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}})$ can be decomposed into bias and variance as $\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}}) = \mathbf{Bias} + \mathbf{Var}$, where

$$\mathbf{Bias} = \|\boldsymbol{\Sigma}_2^{1/2}(\boldsymbol{\beta}_2 - \mathbb{E}_{\mathbf{x},y}\boldsymbol{\beta}_{\mathsf{s}})\|^2,$$
$$\mathbf{Var} = \|\boldsymbol{\Sigma}_2^{1/2}(\boldsymbol{\beta}_{\mathsf{s}} - \mathbb{E}_{\mathbf{x},y}\boldsymbol{\beta}_{\mathsf{s}})\|^2.$$

In the remainder of this paper, we derive asymptotic expressions for the bias and variance terms to analyze the generalization performance of the student model using tools from random matrix theory.

2.2 RANDOM MATRIX THEORY

Before proceeding to the theoretical analysis, we introduce several key quantities from random matrix theory that will appear in our main results. For any distribution G supported on $\mathbb{R}^+ = [0, \infty)$, its Stieltjes transform is defined as

$$m_G(z) = \int \frac{1}{x - z} \mathrm{d}G(x), \ z \notin \mathrm{supp}(G).$$

Next, we define the asymptotic eigenvalue density of random matrices via its Stieltjes transform.

Lemma 1. *Let $\mathbf{X} = \mathbf{\Sigma}^{1/2}\mathbf{Z}$ be a random matrix, where $\mathbf{Z} = (z_{jk}) \in \mathbb{R}^{M \times N}$, $M \sim N$ satisfies Assumption 1(a), and $\mathbf{\Sigma}$ satisfies Assumption 2. For each $z \in \mathbb{C}\backslash\mathbb{R}^+$, there exists a unique $m \equiv m_M(z) \in \mathbb{C}$ satisfying the equation*

$$z = -\frac{1}{m} + \frac{1}{N}\mathrm{Tr}\frac{\mathbf{\Sigma}}{1 + m\mathbf{\Sigma}} = -\frac{1}{m} - \frac{z}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}, \ \ \textit{with } \Im z \Im m(z) \geq 0, \tag{6}$$

where

$$\mathbf{\Pi}(z) = -(z + zm\mathbf{\Sigma})^{-1}.$$

This lemma is a well-known result in the random matrix theory literature (see, e.g., Bai & Silverstein (2010)).

3 THEORETICAL ANALYSIS

In this section, we analyze the excess risk $\mathbf{ER}(\boldsymbol{\beta}_\mathsf{s})$ defined in 5 under three distinct settings. In Section 3.1, we consider the case where $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are deterministic, with their difference being arbitrary. In Section 3.2, we study the scenario in which $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ and the common parameter vector is drawn from a prior distribution. Finally, in Section 3.3, we analyze ridgeless regression under the regime where $M < N_1, N_2$ and the covariance matrices $\mathbf{\Sigma}_1, \mathbf{\Sigma}_2$ are invertible.

Before presenting the main results, we first introduce some necessary notation. For $M, N_i, \mathbf{\Sigma}_i$ and $z < 0$, the Stieltjes transform determined by Lemma 1 is denoted by $m_i(z)$. Let

$$\mathbf{\Pi}_i(z) = -(z + zm_i(z)\mathbf{\Sigma}_i)^{-1}, \ i = 1, 2.$$

For notational simplicity, we write

$$\mathbf{Q}_1 = \mathbf{Q}_1(-\lambda_\mathsf{t}), \quad \mathbf{Q}_2 = \mathbf{Q}_2(-\lambda_\mathsf{s}), \quad \mathbf{\Pi}_1 = \mathbf{\Pi}_1(-\lambda_\mathsf{t}), \ \text{and} \ \mathbf{\Pi}_2 = \mathbf{\Pi}_2(-\lambda_\mathsf{s}).$$

For any deterministic matrix $\mathbf{A}$ with bounded spectral norm, we define

$$\mathcal{S}_i(\mathbf{A}) = \mathbf{A} + \frac{\frac{1}{N_i}\mathrm{Tr}\mathbf{\Sigma}_i\mathbf{\Pi}_i\mathbf{A}\mathbf{\Pi}_i}{(1 + \frac{1}{N_i}\mathrm{Tr}\mathbf{\Sigma}_i\mathbf{\Pi}_i)^2 - \frac{1}{N_i}\mathrm{Tr}(\mathbf{\Sigma}_i\mathbf{\Pi}_i)^2}\mathbf{\Sigma}_i, \ i = 1, 2.$$

Moreover, when $\mathbf{A} = \mathbf{I}_M$, we denote

$$\mathbf{\Pi}_1' = \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{\Pi}_1(z)\big|_{z=-\lambda_\mathsf{t}} = \mathbf{\Pi}_1\mathcal{S}_1(\mathbf{I}_M)\mathbf{\Pi}_1, \quad \mathbf{\Pi}_2' = \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{\Pi}_2(z)\big|_{z=-\lambda_\mathsf{s}} = \mathbf{\Pi}_2\mathcal{S}_2(\mathbf{I}_M)\mathbf{\Pi}_2.$$

3.1 DETERMINISTIC REGRESSION PARAMETERS

We now state our first main result.

Theorem 1. *Let $\boldsymbol{\gamma} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2$. For the deterministic vectors $\|\boldsymbol{\beta}_1\|$ and $\|\boldsymbol{\beta}_2\|$, assume that $\|\boldsymbol{\beta}_1\|, \|\boldsymbol{\beta}_2\| \leq c$ for some constant c. Under Assumptions 1-2, the following results hold:*

$$\begin{aligned}
\mathbf{Bias} = {} & \xi^2\boldsymbol{\beta}_1^\mathsf{T}\big[\lambda_\mathsf{t}^2\mathbf{\Pi}_1\mathcal{S}_1(\mathbf{\Sigma}_2)\mathbf{\Pi}_1 + \lambda_\mathsf{s}^2\lambda_\mathsf{t}^2\mathbf{\Pi}_1\mathcal{S}_1(\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{\Sigma}_2)\mathbf{\Pi}_2)\mathbf{\Pi}_1 - 2\lambda_\mathsf{t}^2\lambda_\mathsf{s}\mathbf{\Pi}_1\mathcal{S}_1(\mathbf{\Sigma}_2\mathbf{\Pi}_2)\mathbf{\Pi}_1\big]\boldsymbol{\beta}_1 \\
& + \lambda_\mathsf{s}^2\boldsymbol{\beta}_2^\mathsf{T}\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{\Sigma}_2)\mathbf{\Pi}_2\boldsymbol{\beta}_2 + 2\xi\boldsymbol{\beta}_1^\mathsf{T}\big[\lambda_\mathsf{t}\lambda_\mathsf{s}\mathbf{\Pi}_1\mathbf{\Sigma}_2\mathbf{\Pi}_2 - \lambda_\mathsf{t}\lambda_\mathsf{s}^2\mathbf{\Pi}_1\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{\Sigma}_2)\mathbf{\Pi}_2\big]\boldsymbol{\beta}_2 \\
& + 2\xi^2\boldsymbol{\gamma}^\mathsf{T}\big[\lambda_\mathsf{s}\lambda_\mathsf{t}\mathbf{\Pi}_2\mathbf{\Sigma}_2\mathbf{\Pi}_1 - \lambda_\mathsf{t}\lambda_\mathsf{s}^2\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{\Sigma}_2)\mathbf{\Pi}_2\mathbf{\Pi}_1 - \lambda_\mathsf{t}\mathbf{\Sigma}_2\mathbf{\Pi}_1 + \lambda_\mathsf{t}\lambda_\mathsf{s}\mathbf{\Sigma}_2\mathbf{\Pi}_2\mathbf{\Pi}_1\big]\boldsymbol{\beta}_1 \\
& + 2\xi\boldsymbol{\beta}_2^\mathsf{T}\big[\lambda_\mathsf{s}^2\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{\Sigma}_2)\mathbf{\Pi}_2 - \lambda_\mathsf{s}\mathbf{\Pi}_2\mathbf{\Sigma}_2\big]\boldsymbol{\gamma} + \xi^2\boldsymbol{\gamma}^\mathsf{T}\big[-2\lambda_\mathsf{s}\mathbf{\Pi}_2\mathbf{\Sigma}_2 + \lambda_\mathsf{s}^2\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{\Sigma}_2)\mathbf{\Pi}_2\big]\boldsymbol{\gamma} \\
& + o_{a.s.}(1),
\end{aligned} \tag{7}$$

and

$$\mathbf{Var} = \frac{\xi^2\sigma^2}{N_1}\mathrm{Tr}\big[\big(\mathbf{\Sigma}_2 - 2\lambda_s\mathbf{\Sigma}_2\mathbf{\Pi}_2 + \lambda_s^2\mathbf{\Pi}_2\mathcal{S}_1(\mathbf{\Sigma}_2)\mathbf{\Pi}_2\big)\big(\mathbf{\Pi}_1 - \lambda_t\mathbf{\Pi}_1'\big)\big] + \frac{(1-\xi)^2\sigma^2}{N_2}\mathrm{Tr}[\mathbf{\Sigma}_2(\mathbf{\Pi}_2 - \lambda_s\mathbf{\Pi}_2')] + o_{a.s.}(1).$$

This theorem characterizes the dependence of $\mathbf{ER}(\boldsymbol{\beta}_s)$ on the geometry of $\mathbf{\Sigma}_1, \boldsymbol{\beta}_1, \mathbf{\Sigma}_2$, and $\boldsymbol{\beta}_2$. For instance, in the absence of model shift, i.e., $\boldsymbol{\gamma} = 0$, the excess risk depends on how the common vector $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ aligns with the eigenvectors of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$, as well as on the alignment between the eigenvectors of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$. This observation extends the results of Hastie et al. (2022), which considers high-dimensional least squares regression within a single domain (corresponding to $\xi = 0$ in equation 3).

3.2 Random Regression Parameters

In this section, we assume that the vector $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \boldsymbol{\beta}$ is random, and consider the excess risk under two population covariance matrices, $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$, which may be equal or distinct. Before presenting the main result, we introduce the following assumption, commonly used in the literature (Dobriban & Wager, 2018; Moniri & Hassani, 2025).

Assumption 3. *The regression parameter vector $\boldsymbol{\beta} = (\beta_1, ..., \beta_M)^\mathsf{T} \in \mathbb{R}^M$ is random, with each entry i.i.d., and β_1 satisfies*

$$\mathbb{E}\beta_1 = 0,\ \mathbb{E}\beta_1^2 = \frac{\widetilde{\sigma}^2}{M},\ \textit{and}\ \mathbb{E}|\sqrt{M}\beta_1|^p \le C_p,$$

for any $p \in \mathbb{N}$, where C_p is a constant depending only on p.

Theorem 2. *Suppose Assumptions 1-3 hold. Then the following asymptotic expressions hold:*

$$\mathbf{Bias} = \frac{\tilde{\sigma}^2}{M}\Big[\xi^2\lambda_t^2\mathrm{Tr}\mathbf{\Sigma}_2\mathbf{\Pi}_1' + 2\xi\lambda_t\lambda_s\mathrm{Tr}\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{\Sigma}_2 + \lambda_s^2\mathrm{Tr}\mathbf{\Sigma}_2\mathbf{\Pi}_2' - 2\xi^2\lambda_t^2\lambda_s\mathrm{Tr}\mathbf{\Sigma}_2\mathbf{\Pi}_2\mathbf{\Pi}_1' + \xi\lambda_t\lambda_s^2\mathrm{Tr}\big[(\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{\Sigma}_2)\mathbf{\Pi}_2)(-2\mathbf{\Pi}_1 + \xi\lambda_t\mathbf{\Pi}_1')\big]\Big] + o_{a.s.}(1),$$

and $\mathbf{Var}$ *is the same as that in Theorem 1.*

This theorem extends the result of Moniri & Hassani (2025), which considers the case of no covariate shift, with inputs drawn i.i.d. from $\mathcal{N}(0, \mathbf{I}_M)$ in the context of W2S generalization (i.e., when $\xi = 1$). Our framework generalizes this analysis by allowing $\xi \in \mathbb{R}$, thereby providing a more comprehensive understanding of the trade-off between learning from the teacher and from the observed labels.

Let $\underline{m}_1(z), \underline{m}_2(z)$ be the Stieltjes transforms of the standard Marchenko-Pastur law with parameters $M/N_1, M/N_2$, respectively:

$$\underline{m}_i(z) = \int \frac{\mathrm{d}\varrho_{\mathrm{MP},i}(x)}{x - z} = \frac{\big(1 - \frac{M}{N_i}\big) - z - \sqrt{\big(z - 1 - \frac{M}{N_i}\big)^2 - 4\frac{M}{N_i}}}{2\frac{M}{N_i}z}. \tag{8}$$

The following corollary follows immediately from Theorem 2 and the fact that $\mathbf{\Pi}_1 = \underline{m}_1\mathbf{I}_M, \mathbf{\Pi}_2 = \underline{m}_2\mathbf{I}_M$ (see, e.g., Alex et al. (2014)).

Corollary 1. *Suppose $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \mathbf{I}_M$. Write $\underline{m}_1 = \underline{m}_1(-\lambda_t), \underline{m}_2 = \underline{m}_2(-\lambda_s)$. Under Assumption 1 and Assumption 3, we have the following expressions:*

$$\mathbf{Bias} = \tilde{\sigma}^2[\xi^2\lambda_t^2\underline{m}_1' + 2\xi\lambda_t\lambda_s\underline{m}_1\underline{m}_2 + \lambda_s^2\underline{m}_2' - 2\xi^2\lambda_t^2\lambda_s\underline{m}_2\underline{m}_1' - 2\xi\lambda_t\lambda_s^2\underline{m}_2'\underline{m}_1 + \xi^2\lambda_t^2\lambda_s^2\underline{m}_1'\underline{m}_2'] + o_{a.s.}(1),$$

and

$$\mathbf{Var} = \xi^2\sigma^2\frac{M}{N_1}\big[\underline{m}_1 - 2\lambda_s\underline{m}_1\underline{m}_2 + \lambda_s^2\underline{m}_1\underline{m}_2' - \lambda_t\underline{m}_1' + 2\lambda_t\lambda_s\underline{m}_2\underline{m}_1' - \lambda_t\lambda_s^2\underline{m}_1'\underline{m}_2'\big] + (1-\xi)^2\sigma^2\frac{M}{N_2}\big[\underline{m}_2 - \lambda_s\underline{m}_2'\big] + o_{a.s.}(1),$$

where $\underline{m}_1', \underline{m}_2'$ *denote the derivatives evaluated at* $z = -\lambda_t$ *and* $z = -\lambda_s$*, respectively:*

$$\underline{m}_1' = \frac{\mathrm{d}}{\mathrm{d}z}\underline{m}_1(z)\big|_{z=-\lambda_t}, \underline{m}_2' = \frac{\mathrm{d}}{\mathrm{d}z}\underline{m}_2(z)\big|_{z=-\lambda_s}.$$

As previously noted, we do not restrict ξ to the interval $[0, 1]$. It has been shown in Das & Sanghavi (2023) that the optimal value of ξ may exceed 1. In Corollary 2 below, we present a toy example demonstrating that even when the input data across domains are i.i.d. and in the absence of model shift – i.e., with no domain shift – the limiting optimal value of ξ can be negative.

Corollary 2. *Suppose the conditions in Corollary 1 hold. The limiting optimal value of* ξ *is negative if*

$$\lambda_s \lambda_t \underline{m}_1 \mathrm{SNR} - \frac{M}{N_2} > 0, \tag{9}$$

where $\mathrm{SNR} = \frac{\tilde{\sigma}^2}{\sigma^2} = \frac{\|\boldsymbol{\beta}\|^2}{\sigma^2} + o_{a.s.}(1)$.

Remark 1. *We call the case* $\xi < 0$ ***anti-learning against the teacher's supervision****, in contrast to* $\xi > 1$*, which Das & Sanghavi (2023) termed anti-learning the observed (possibly noisy) labels. This corollary provides insight into the selection of* ξ*: the sign of the limiting optimal value of* ξ *depends not only on parameters* (λ_t, λ_s) *but also on data-related factors (SNR, data dimension, and sample sizes of both domains).*

3.3 Ridgeless Regression

In this section, we consider the ridgeless regression for cross-domain KD in the under-parameterized setting, i.e.,

$$\frac{M}{N_1}, \frac{M}{N_2} < (1+\tau)^{-1}, \text{ and } \lambda_t = \lambda_s = 0. \tag{10}$$

Adopting the notation $\boldsymbol{\gamma} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2$ in Theorem 1, the high-dimensional asymptotic excess risk is given by the following result.

Theorem 3. *Suppose equation 10 holds and assume that* $\tau \le \sigma_{\min}(\boldsymbol{\Sigma}_i) \le \cdots \le \sigma_{\max}(\boldsymbol{\Sigma}_i) \le \tau^{-1}$ *for* $i = 1, 2$*. The estimator for student model obtained by equation 1 is the averaging estimator:*

$$\boldsymbol{\beta}_s = \xi \boldsymbol{\beta}_1^{\mathsf{OLS}} + (1-\xi)\boldsymbol{\beta}_2^{\mathsf{OLS}}, \tag{11}$$

where

$$\boldsymbol{\beta}_i^{\mathsf{OLS}} = (\mathbf{X}_i \mathbf{X}_i^{\mathsf{T}})^{-1} \mathbf{X}_i \mathbf{y}_i, \ i = 1, 2.$$

Under Assumptions 1-2, we have

$$\mathbf{ER}(\boldsymbol{\beta}_s) = [\widehat{\mathbf{Bias}} + \widehat{\mathbf{Var}}]\big(1 + o_{a.s.}(1)\big),$$

where

$$\widehat{\mathbf{Bias}} = \xi^2 \boldsymbol{\gamma}^{\mathsf{T}} \boldsymbol{\Sigma}_2 \boldsymbol{\gamma},$$

and

$$\widehat{\mathbf{Var}} = (1-\xi)^2 \sigma^2 \frac{M}{N_2 - M} + \xi^2 \sigma^2 \frac{1}{N_1 - M} \mathrm{Tr} \boldsymbol{\Sigma}_2 \boldsymbol{\Sigma}_1^{-1}.$$

Solving $\frac{\partial}{\partial \xi}(\widehat{\mathbf{Bias}} + \widehat{\mathbf{Var}}) = 0$, we obtain

$$\xi = \left(\boldsymbol{\gamma}^{\mathsf{T}} \boldsymbol{\Sigma}_2 \boldsymbol{\gamma} + \sigma^2 \frac{M}{N_2 - M} + \frac{\sigma^2}{N_1 - M} \mathrm{Tr} \boldsymbol{\Sigma}_2 \boldsymbol{\Sigma}_1^{-1}\right)^{-1} \frac{\sigma^2 M}{N_2 - M} \in (0, 1).$$

The following corollary follows immediately from this result and Theorem 3.

Corollary 3. *Suppose Assumption 3 and the conditions in Theorem 3 hold. If there is no model shift, i.e.,* $\boldsymbol{\gamma} = 0$*, we have*

$$\mathbf{ER}(\boldsymbol{\beta}_s) = \widehat{\mathbf{Var}}\big(1 + o_{a.s.}(1)\big). \tag{12}$$

Moreover, the limiting optimal value of ξ *lies in* $(0, 1)$.

Based on the conclusions of Theorems 1-3, the high-dimensional asymptotic excess risk, regarded as a function of ξ, is a quadratic function. Given that the excess risk is non-negative, the quadratic function opens upwards. This observation is consistent with Pareek et al. (2024), where self-distillation is considered. Given a $\xi \in \mathbb{R}$, the gain of cross-domain KD is characterized by the reduction in excess risk, $\mathbf{ER}_0 - \mathbf{ER}(\boldsymbol{\beta}_s)$.

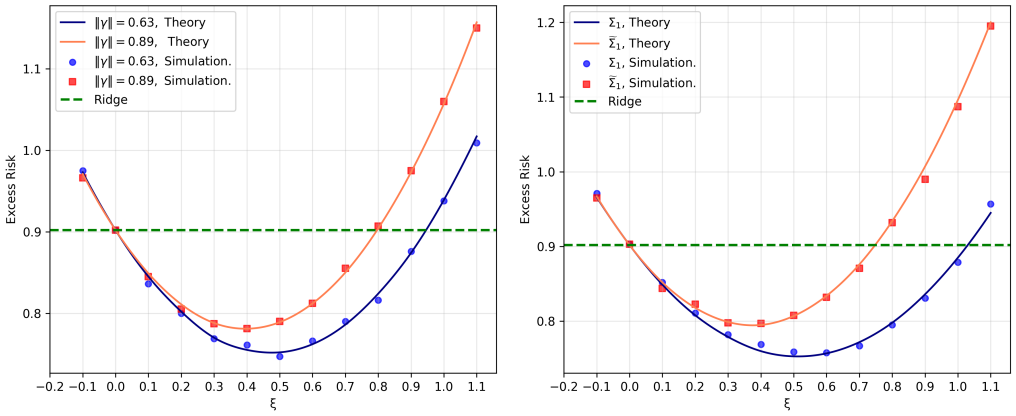
Proposition 1. *Under the conditions of Theorem 1 and Assumption A.1 for the deterministic case in Appendix B.6, there exists a value of $\xi \in \mathbb{R}$ such that*

$$\min_{\xi \in \mathbb{R}} \left(\mathbf{ER}(\boldsymbol{\beta}_s) - \mathbf{ER}_0 \right) < 0. \ a.s. \tag{13}$$

Moreover, under the conditions of Theorem 1 and Assumption A.2 for the random case in Appendix B.6, the inequality 13 also holds.

Remark 2. *This proposition shows that, even in the presence of a significant domain discrepancy, it is possible to find a value of $\xi \in \mathbb{R}$ such that the student model outperforms the student-only baseline (i.e., training on the observed labels only). We provide further details in Appendix B.6, where we demonstrate that covariate shift can, in some cases, be beneficial for KD.*

3.4 Numerical Simulations

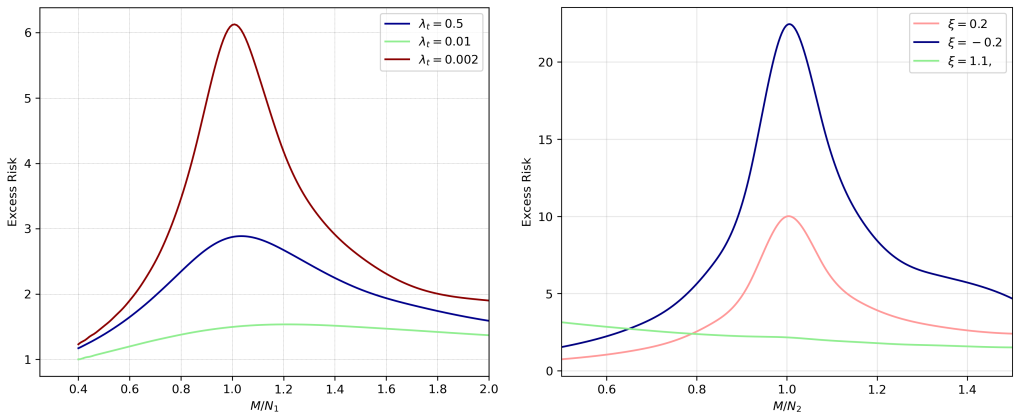


(a) Model shift (b) Covariate shift

Figure 1: Student's excess risk in the presence of domain shift. Solid lines represent theoretical values, while scattered points denote simulation results (averaged over 100 trials). The dashed green line indicates the theoretical performance for student-only baseline, corresponding to ridge regression trained solely on the target domain data. (a) Settings: $(\lambda_t, \lambda_s) = (0.1, 0.5), (M, N_1, N_2) = (400, 600, 200), \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \mathbf{I}_M$. The vectors $\boldsymbol{\beta}_2 = \frac{1}{\sqrt{M}}(1, ..., 1)^\mathsf{T}$, $\sigma^2 = 1$. We label the case $\|\boldsymbol{\gamma}\| = 0.63$ as $\boldsymbol{\gamma} = -\frac{2}{\sqrt{M}}(1, \ldots, 1, 0, \ldots, 0)^\mathsf{T}$ with the first $\frac{M}{10}$ entries equal to $-\frac{2}{\sqrt{M}}$, and the case $\|\boldsymbol{\gamma}\| = 0.89$ with the first $\frac{M}{5}$ entries equal to $-\frac{2}{\sqrt{M}}$. (b) Settings: $(\lambda_t, \lambda_s) = (0.1, 0.5)$, $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 \sim \mathcal{N}(0, M^{-1}\mathbf{I}_M)$, $(M, N_1, N_2) = (600, 200, 300)$, $\boldsymbol{\Sigma}_1 = 4\mathbf{I}_M$, $\widetilde{\boldsymbol{\Sigma}}_1 = \mathrm{diag}(d_1, \ldots, d_M)$ with $d_i = 0.64\delta(i \leq M/2) + 0.25\delta(M/2 < i \leq M)$, $\sigma^2 = 1$.

We plot the excess risk of the student model: (a) under model shift with identical covariate distributions, and (b) under covariate shift with identical parameter vectors, in Figure 1. All theoretical values of the Stieltjes transform presented in this paper are obtained by solving equation 6. Due to space limitations, the numerical validation of Theorem 3 is provided in Appendix C.1. Simulation results, averaged over 100 independent trials, show good agreement with the theoretical predictions.

Furthermore, we present numerical simulations of $\mathbf{ER}(\boldsymbol{\beta}_s)$ as a function of λ_s and λ_t for various values of ξ; these results are included in Appendix C.2.



(a) Excess risk as a function of $\frac{M}{N_1}$ for varying λ_t (b) Excess risk as a function of $\frac{M}{N_2}$ for varying ξ

Figure 2: Non-monotone student excess risk curves. We set $\mathbf{\Sigma}_2 = \mathbf{I}_M, \mathbf{\Sigma}_1 = \text{diag}(d_1, \cdots, d_M)$ where $d_i = 0.64\delta(i \leq \frac{M}{2}) + 0.25\delta(\frac{M}{2} < i \leq M)$. (a) Results are shown for fixed $M = 600$ and $\lambda_s = 0.05$ with different N_1. (b) Results are shown for fixed N_2 and $(\lambda_t, \lambda_s) = (0.05, 0.001)$, with varying M.

4 Double Descent of the Excess Risk

In this section, fixing ξ, λ_t and λ_s, we examine the excess risk as a function of the dimension M and the sample sizes N_1 and N_2. We find that the student model exhibits the double descent phenomenon, characterized by a non-monotonic behavior of the excess risk as a function of the ratio of dimension-to-sample-size. This phenomenon is consistent with findings in various linear regression settings (Hastie et al., 2022; Nakkiran et al., 2021; Belkin et al., 2020; Moniri & Hassani, 2024), and has been previously observed by Moniri & Hassani (2025) in the special case of pure teacher supervision without domain shift, where the risk was studied as a function of $\frac{M}{N_1}$.

Using our theoretical predictions from Theorem 2, we plot the excess risk of the student model, $\mathbf{ER} = \mathbf{ER}(\frac{M}{N_1})$, as a function of $\frac{M}{N_1}$ in Figure 2(a). The double descent phenomenon is evident for all three values of λ_t. As λ_t decreases, the peak of the risk curve shifts towards $\frac{M}{N_1} = 1$. In Figure 2(b), we plot $\mathbf{ER} = \mathbf{ER}(\frac{M}{N_2})$ against $\frac{M}{N_2}$, while allowing $\frac{M}{N_1}$ to vary simultaneously. We consider different values of ξ and observe that the double descent phenomenon is most pronounced in the regime of *anti-learning against the teacher's supervision* ($\xi < 0$). In contrast, when $\xi = 1.1$, no double descent occurs within the ratio range $[0.5, 1.5]$.

5 Conclusion

In this paper, we present a theoretical analysis of cross-domain KD for linear models using random matrix theory. Through the bias-variance decomposition, we precisely characterize the asymptotic expressions of excess risk for the student model in the high-dimensional setting. A surprising finding is that when the imitation parameter ξ is allowed to take any real value, cross-domain KD may outperform training solely on the target domain – even in the presence of significant discrepancies between source and target domains. This highlights the potential of distillation to effectively transfer knowledge across highly heterogeneous domains.

Our work also points to several promising directions for future research. Our theoretical analysis is currently limited to linear models; extending it to more complex architectures, particularly nonlinear models, would significantly broaden its applicability. Furthermore, while we observe the double descent phenomenon using the established theoretical limits; a rigorous theoretical characterization of this behavior in nonlinear models remains an important avenue for future investigation.

REFERENCES

Rishabh Agarwal, Nino Vieillard, Yongchao Zhou, Piotr Stanczyk, Sabela Ramos Garea, Matthieu Geist, and Olivier Bachem. On-policy distillation of language models: Learning from self-generated mistakes. In *The twelfth international conference on learning representations*, 2024.

Bloemendal Alex, László Erdős, Antti Knowles, Horng-Tzer Yau, and Jun Yin. Isotropic local laws for sample covariance and generalized Wigner matrices. *Electronic Journal of Probability*, 19 (none):1 – 53, 2014. doi: 10.1214/EJP.v19-3054.

Taichi Asami, Ryo Masumura, Yoshikazu Yamaguchi, Hirokazu Masataki, and Yushi Aono. Domain adaptation of dnn acoustic models using knowledge distillation. In *2017 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 5185–5189, 2017. doi: 10.1109/ICASSP.2017.7953145.

Zhidong Bai and Jack W Silverstein. *Spectral analysis of large dimensional random matrices*, volume 20. Springer, 2010.

Mikhail Belkin, Daniel Hsu, and Ji Xu. Two models of double descent for weak features. *SIAM Journal on Mathematics of Data Science*, 2(4):1167–1180, 2020. doi: 10.1137/20M1336072.

Kenneth Borup and Lars N Andersen. Even your teacher needs guidance: Ground-truth targets dampen regularization imposed by self-distillation. *Advances in Neural Information Processing Systems*, 34:5316–5327, 2021.

D. L. Burkholder. Distribution Function Inequalities for Martingales. *The Annals of Probability*, 1 (1):19 – 42, 1973. doi: 10.1214/aop/1176997023.

Collin Burns, Pavel Izmailov, Jan Hendrik Kirchner, Bowen Baker, Leo Gao, Leopold Aschenbrenner, Yining Chen, Adrien Ecoffet, Manas Joglekar, Jan Leike, Ilya Sutskever, and Jeffrey Wu. Weak-to-strong generalization: Eliciting strong capabilities with weak supervision. In *Forty-first International Conference on Machine Learning*, 2024.

Moses Charikar, Chirag Pabbaraju, and Kirankumar Shiragur. Quantifying the gain in weak-to-strong generalization. In *The Thirty-eighth Annual Conference on Neural Information Processing Systems*, 2024.

Rudrajit Das and Sujay Sanghavi. Understanding self-distillation in the presence of label noise. In *International Conference on Machine Learning*, pp. 7102–7140. PMLR, 2023.

Pasan Dissanayake, Faisal Hamman, Barproda Halder, Ilia Sucholutsky, Qiuyi Zhang, and Sanghamitra Dutta. Quantifying knowledge distillation using partial information decomposition. In *International Conference on Artificial Intelligence and Statistics*, pp. 4474–4482. PMLR, 2025.

Edgar Dobriban and Stefan Wager. High-dimensional asymptotics of prediction: Ridge regression and classification. *The Annals of Statistics*, 46(1):247 – 279, 2018. doi: 10.1214/17-AOS1549.

Yijun Dong, Yicheng Li, Yunai Li, Jason D. Lee, and Qi Lei. Discrepancies are virtue: Weak-to-strong generalization through lens of intrinsic dimension. In *Forty-second International Conference on Machine Learning*, 2025.

M Emrullah Ildiz, Halil Alperen Gozeten, Ege Onur Taga, Marco Mondelli, and Samet Oymak. High-dimensional analysis of knowledge distillation: Weak-to-strong generalization and scaling laws. In *13th International Conference on Learning Representations*, 2025.

William Fedus, Barret Zoph, and Noam Shazeer. Switch transformers: Scaling to trillion parameter models with simple and efficient sparsity. *Journal of Machine Learning Research*, 23(120):1–39, 2022.

Yaroslav Ganin, Evgeniya Ustinova, Hana Ajakan, Pascal Germain, Hugo Larochelle, François Laviolette, Mario March, and Victor Lempitsky. Domain-adversarial training of neural networks. *Journal of machine learning research*, 17(59):1–35, 2016.

Yuxian Gu, Li Dong, Furu Wei, and Minlie Huang. Minillm: Knowledge distillation of large language models. *arXiv preprint arXiv:2306.08543*, 2023.

Hrayr Harutyunyan, Ankit Singh Rawat, Aditya Krishna Menon, Seungyeon Kim, and Sanjiv Kumar. Supervision complexity and its role in knowledge distillation. In *The Eleventh International Conference on Learning Representations*, 2023.

Trevor Hastie, Andrea Montanari, Saharon Rosset, and Ryan J. Tibshirani. Surprises in high-dimensional ridgeless least squares interpolation. *The Annals of Statistics*, 50(2):949 – 986, 2022. doi: 10.1214/21-AOS2133.

Geoffrey Hinton, Oriol Vinyals, and Jeff Dean. Distilling the knowledge in a neural network. *arXiv preprint arXiv:1503.02531*, 2015.

Hyeonsu Jeong and Hye Won Chung. Rethinking self-distillation: Label averaging and enhanced soft label refinement with partial labels. In *The Thirteenth International Conference on Learning Representations*, 2025.

Antti Knowles and Jun Yin. Anisotropic local laws for random matrices. *Probability Theory and Related Fields*, 169(1):257–352, 2017.

Jogendra Nath Kundu, Nishank Lakkakula, and R Venkatesh Babu. Um-adapt: Unsupervised multi-task adaptation using adversarial cross-task distillation. In *Proceedings of the IEEE/CVF international conference on computer vision*, pp. 1436–1445, 2019.

Lujun Li, Yufan Bao, Peijie Dong, Chuanguang Yang, Anggeng Li, Wenhan Luo, Qifeng Liu, Wei Xue, and Yike Guo. Detkds: Knowledge distillation search for object detectors. In *Forty-first International Conference on Machine Learning*, 2024.

Wei Li, Kefeng Fan, and Huihua Yang. Teacher–student mutual learning for efficient source-free unsupervised domain adaptation. *Knowledge-Based Systems*, 261:110204, 2023. ISSN 0950-7051. doi: https://doi.org/10.1016/j.knosys.2022.110204.

Zeqin Lin and Guangming Pan. Eigenvector overlaps in large sample covariance matrices and non-linear shrinkage estimators. *arXiv preprint arXiv:2404.18173*, 2024.

David Lopez-Paz, Léon Bottou, Bernhard Schölkopf, and Vladimir Vapnik. Unifying distillation and privileged information. *arXiv preprint arXiv:1511.03643*, 2015.

Marko Medvedev, Kaifeng Lyu, Dingli Yu, Sanjeev Arora, Zhiyuan Li, and Nathan Srebro. Weak-to-strong generalization even in random feature networks, provably. In *Forty-second International Conference on Machine Learning*, 2025.

Aditya K Menon, Ankit Singh Rawat, Sashank Reddi, Seungyeon Kim, and Sanjiv Kumar. A statistical perspective on distillation. In Marina Meila and Tong Zhang (eds.), *Proceedings of the 38th International Conference on Machine Learning*, volume 139 of *Proceedings of Machine Learning Research*, pp. 7632–7642. PMLR, 18–24 Jul 2021.

Victoria Mingote, Antonio Miguel, Dayana Ribas, Alfonso Ortega, and Eduardo Lleida. Knowledge distillation and random erasing data augmentation for text-dependent speaker verification. In *ICASSP 2020-2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 6824–6828. IEEE, 2020.

Behrad Moniri and Hamed Hassani. Asymptotics of linear regression with linearly dependent data. *arXiv preprint arXiv:2412.03702*, 2024.

Behrad Moniri and Hamed Hassani. On the mechanisms of weak-to-strong generalization: A theoretical perspective. *arXiv preprint arXiv:2505.18346*, 2025.

Preetum Nakkiran, Prayaag Venkat, Sham M. Kakade, and Tengyu Ma. Optimal regularization can mitigate double descent. In *International Conference on Learning Representations*, 2021.

Junsoo Oh, Jerry Song, and Chulhee Yun. From linear to nonlinear: Provable weak-to-strong generalization through feature learning. In *High-dimensional Learning Dynamics 2025*, 2025.

Long Ouyang, Jeffrey Wu, Xu Jiang, Diogo Almeida, Carroll Wainwright, Pamela Mishkin, Chong Zhang, Sandhini Agarwal, Katarina Slama, Alex Ray, et al. Training language models to follow instructions with human feedback. *Advances in neural information processing systems*, 35: 27730–27744, 2022.

Divyansh Pareek, Simon S Du, and Sewoong Oh. Understanding the gains from repeated self-distillation. *Advances in Neural Information Processing Systems*, 37:7759–7796, 2024.

Mary Phuong and Christoph Lampert. Towards understanding knowledge distillation. In *International conference on machine learning*, pp. 5142–5151. PMLR, 2019.

Alec Radford, Jong Wook Kim, Chris Hallacy, Aditya Ramesh, Gabriel Goh, Sandhini Agarwal, Girish Sastry, Amanda Askell, Pamela Mishkin, Jack Clark, et al. Learning transferable visual models from natural language supervision. In *International conference on machine learning*, pp. 8748–8763. PmLR, 2021.

Changho Shin, John Cooper, and Frederic Sala. Weak-to-strong generalization through the data-centric lens. In *The Thirteenth International Conference on Learning Representations*, 2025.

Jong-Chyi Su and Subhransu Maji. Adapting models to signal degradation using distillation. *arXiv preprint arXiv:1604.00433*, 2016.

Jialiang Tang, Shuo Chen, Gang Niu, Hongyuan Zhu, Joey Tianyi Zhou, Chen Gong, and Masashi Sugiyama. Direct distillation between different domains. In *Computer Vision – ECCV 2024*, pp. 154–172, Cham, 2025. Springer Nature Switzerland.

Colin Wei, Kendrick Shen, Yining Chen, and Tengyu Ma. Theoretical analysis of self-training with deep networks on unlabeled data. In *International Conference on Learning Representations*, 2021.

David Xing Wu and Anant Sahai. Provable weak-to-strong generalization via benign overfitting. In *The Thirteenth International Conference on Learning Representations*, 2025.

Qing Xu, Min Wu, Xiaoli Li, Kezhi Mao, and Zhenghua Chen. Reinforced cross-domain knowledge distillation on time series data. In *The Thirty-eighth Annual Conference on Neural Information Processing Systems*, 2024.

Chuanpeng Yang, Yao Zhu, Wang Lu, Yidong Wang, Qian Chen, Chenlong Gao, Bingjie Yan, and Yiqiang Chen. Survey on knowledge distillation for large language models: methods, evaluation, and application. *ACM Transactions on Intelligent Systems and Technology*, 2024.

Fan Yang, Hongyang R. Zhang, Sen Wu, Christopher Re, and Weijie J. Su. Precise high-dimensional asymptotics for quantifying heterogeneous transfers. *Journal of Machine Learning Research*, 26 (113):1–88, 2025.

A THE USE OF LARGE LANGUAGE MODELS (LLMS)

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B PROOFS

B.1 BASIC TOOLS

Preliminary definitions and auxiliary lemmas used in the proofs of the main results are provided in this section.

Lemma 2. *(Lemma B.26 in Bai & Silverstein (2010)) Let $\mathbf{C}$ be an $M \times M$ deterministic matrix and $\mathbf{z} \in \mathbb{R}^M$ be a random vector of independent entries. Assume that $\mathbb{E}x_i = 0, \mathbb{E}|x_i|^2 = 1$, and $\mathbb{E}|x_i|^\ell \le C_\ell$. Then for any $\ell \ge 1$,*

$$\mathbb{E}|\mathbf{z}^\mathsf{T}\mathbf{C}\mathbf{z} - \mathrm{Tr}\mathbf{C}|^\ell \le c_\ell\big((C_4\mathrm{Tr}\mathbf{C}\mathbf{C}^*)^{\ell/2} + C_{2\ell}\mathrm{Tr}(\mathbf{C}\mathbf{C}^*)^{\ell/2}\big),$$

where c_ℓ is a constant depending on ℓ only.

Before stating the subsequent results, it is convenient to introduce the notion of stochastic domination.

Definition 1. *Let $\chi = \chi^{(p)},\ \zeta = \zeta^{(p)}$ be two families of p-dependent random variables. We say that χ is stochastically dominated by ζ if for all small $c > 0$ and large constant $\ell > 0$,*

$$\mathbb{P}\big(|\chi^{(p)}| > p^c|\zeta^{(p)}|\big) \le p^{-\ell}$$

for all large p. If χ is stochastically dominated by ζ, we use the notation $\chi \prec \zeta$ or $\chi = O_\prec(\zeta)$. We say an event $\mathcal{E}_p$ holds with high probability if

$$\mathbb{P}(\mathcal{E}_p^C) \le p^{-\ell} \text{ for any fixed } \ell > 0.$$

Lemma 3. *(Lemma 22 in Yang et al. 2025) Let $\mathbf{Z}$ satisfies Assumption 1(a). We further suppose that $\tau \le M/N < (1+\tau)^{-1}$. Denote the eigenvalues of $\mathbf{Z}\mathbf{Z}^\mathsf{T}$ by $\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\mathsf{T}) \le \cdots \le \lambda_{\max}(\mathbf{Z}\mathbf{Z}^\mathsf{T})$. Then we have*

$$(\sqrt{M} - \sqrt{N})^2 - O_\prec(\sqrt{N}) \le \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\mathsf{T}) \le \lambda_{\max}(\mathbf{Z}\mathbf{Z}^\mathsf{T}) \le (\sqrt{M} + \sqrt{N})^2 + O_\prec(\sqrt{N})$$

with high probability.

Lemma 4. *(Corollary 25 in Yang et al. (2025)) Suppose $\boldsymbol{\varepsilon}_1, ..., \boldsymbol{\varepsilon}_t$ are independent random vectors satisfying Assumption 1(c). Then, we have that for any deterministic vector $\mathbf{v} \in \mathbb{R}^N$,*

$$|\mathbf{v}^\mathsf{T}\boldsymbol{\varepsilon}_i| \prec \sigma\|\mathbf{v}\|,\ i = 1, ..., t,$$

and for any deterministic matrix $\mathbf{B} \in \mathbb{R}^{N\times N}$,

$$|\boldsymbol{\varepsilon}_i^\mathsf{T}\mathbf{B}\boldsymbol{\varepsilon}_j - \delta(i=j)\sigma^2\mathrm{Tr}\mathbf{B}| \prec \sigma^2\|\mathbf{B}\|_\mathsf{F},\ \text{for } i, j \in [t].$$

Moreover, for any deterministic vector $\mathbf{v}$, we have

$$|\mathbf{v}^\mathsf{T}\boldsymbol{\varepsilon}_i| \prec \sigma\|\mathbf{v}\|,\ i \in [t].$$

Definition 2. *Let $\mathbf{A}_p, \mathbf{B}_p \in \mathbb{R}^{p\times p}$ be sequences of random or deterministic symmetric real matrices. We say $\mathbf{A}_p, \mathbf{B}_p$ are equivalent, if*

$$\frac{1}{p}\mathrm{Tr}\mathbf{D}_p(\mathbf{A}_p - \mathbf{B}_p) = o_{a.s.}(1) \text{ and } \mathbf{u}^\mathsf{T}(\mathbf{A}_p - \mathbf{B}_p)\mathbf{v} = o_{a.s.}(1)$$

for any sequence of deterministic matrices $\mathbf{D}_p$ and all deterministic vectors $\mathbf{u}, \mathbf{v}$ such that

$$\limsup_p \|\mathbf{D}_p\| < \infty, \limsup_p \max\{\|\mathbf{u}\|, \|\mathbf{v}\|\} < \infty.$$

Lemma 5. *(Theorem 3.7 in Knowles & Yin (2017)) We denote by ϱ the probability measure associated with m determined in Lemma 1. Let $\mathbf{X} = \mathbf{\Sigma}^{1/2}\mathbf{Z} \in \mathbb{R}^{M\times N}$. Suppose Assumption 1(a)(c) and Assumption 2 hold. Then for any fixed $z \in \mathbb{C}$ such that*

$$\operatorname{dist}\big(\Re z, \operatorname{supp}(\varrho)\big) > c \tag{14}$$

for some small constant $c > 0$, we have

$$\mathbf{u}^{\mathsf{T}}\big(\mathbf{Q}(z) - \mathbf{\Pi}(z)\big)\mathbf{v} \prec N^{-1/2}\sqrt{\frac{\Im m(z)}{\Im z}}\|\mathbf{u}\|\|\mathbf{v}\| \lesssim N^{-1/2}\|\mathbf{u}\|\|\mathbf{v}\|. \tag{15}$$

uniformly for all deterministic vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^M$, where

$$\mathbf{Q}(z) = \left(\frac{1}{N}\mathbf{X}\mathbf{X}^{\mathsf{T}} - z\mathbf{I}_M\right)^{-1},\ \mathbf{\Pi}(z) = -(z\mathbf{I}_M + zm\mathbf{\Sigma})^{-1}.$$

Remark 3. *When $\Im z = 0$ and equation 14 holds, since there exists an open set containing z on which both $\mathbf{Q}(z)$ and $\mathbf{\Pi}(z)$ are analytic, equation 15 still holds. This property will be frequently used in the subsequent proofs.*

Lemma 5 shows that $\mathbf{\Pi}(z)$ is a deterministic equivalent of $\mathbf{Q}(z)$. For technical reasons, we further require the following result.

Lemma 6. *Suppose the conditions in Lemma 5 hold. $\mathbf{A}$ denotes a deterministic $M \times M$ matrix with bounded spectral norm. For any fixed complex numbers $\tilde{z}_1, \tilde{z}_2 \in \mathbb{C}\backslash\mathbb{R}^+$, we have for all deterministic vectors $\mathbf{u}, \mathbf{v}$,*

$$\mathbf{u}^{\mathsf{T}}\big(\mathbf{Q}(\tilde{z}_1)\mathbf{A}\mathbf{Q}(\tilde{z}_2) - \mathbf{\Pi}(\tilde{z}_1)\mathcal{S}(\mathbf{A})\mathbf{\Pi}(\tilde{z}_2)\big)\mathbf{v} = o_{a.s.}(\|\mathbf{u}\|\|\mathbf{v}\|), \tag{16}$$

where

$$\mathcal{S}(\mathbf{A}) = \mathbf{A} + \frac{\frac{1}{N}\operatorname{Tr}\mathbf{\Sigma}\mathbf{\Pi}(\tilde{z}_1)\mathbf{A}\mathbf{\Pi}(\tilde{z}_2)}{\big(1 + \frac{1}{N}\operatorname{Tr}\mathbf{\Sigma}\mathbf{\Pi}(\tilde{z}_2)\big)\big(1 + \frac{1}{N}\operatorname{Tr}\mathbf{\Sigma}\mathbf{\Pi}(\tilde{z}_1)\big) - \frac{1}{N}\operatorname{Tr}\mathbf{\Sigma}\mathbf{\Pi}(\tilde{z}_1)\mathbf{\Sigma}\mathbf{\Pi}(\tilde{z}_2)}\mathbf{\Sigma}. \tag{17}$$

Moreover, for any deterministic matrix $\mathbf{C} \in \mathbb{R}^{M\times M}$ satisfying $\|\mathbf{C}\| \le C$ for some constant C, we have

$$\frac{1}{M}\operatorname{Tr}\mathbf{C}[\mathbf{Q}(\tilde{z}_1)\mathbf{A}\mathbf{Q}(\tilde{z}_2) - \mathbf{\Pi}_1(\tilde{z}_1)\mathcal{S}(\mathbf{A})\mathbf{\Pi}_2(\tilde{z}_2)] = o_{a.s.}(1). \tag{18}$$

The proof of this lemma is deferred to Appendix B.7.

Remark 4. *Lemma 6 provides the deterministic equivalent of $\mathbf{Q}(\tilde{z}_1)\mathbf{A}\mathbf{Q}(\tilde{z}_2)$. Lin & Pan (2024) established the local laws for the $\mathbf{Q}(\tilde{z}_1)\mathbf{A}\mathbf{Q}(\tilde{z}_2)$. However, their results require $\Re\tilde{z}_1, \Re\tilde{z}_2$ to be sufficiently close to $\operatorname{supp}(\varrho)$ and $\Im\tilde{z}_1, \Im\tilde{z}_2$ to be bounded below by N^{-1+c}, where c is any fixed constant. Lemma 6 extends the result to other regions.*

B.2 Proof of Theorem 1

To simplify notation, we set $z_1 = -\lambda_{\mathsf{t}}, z_2 = -\lambda_{\mathsf{s}}$. Recalling equation 2 and equation 3, we get

$$\begin{aligned}
\boldsymbol{\beta}_{\mathsf{s}} - \boldsymbol{\beta}_2 &= \frac{1}{N_2}\mathbf{Q}_2\big[\xi\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}}\boldsymbol{\beta}_{\mathsf{t}} + (1-\xi)\mathbf{X}_2(\mathbf{X}_2^{\mathsf{T}}\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2)\big] - \boldsymbol{\beta}_2 \\
&= \frac{1}{N_2}\mathbf{Q}_2\left[\frac{1}{N_1}\xi\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}}\mathbf{Q}_1\mathbf{X}_1(\mathbf{X}_1^{\mathsf{T}}\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1) + (1-\xi)\mathbf{X}_2(\mathbf{X}_2^{\mathsf{T}}\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2)\right] - \boldsymbol{\beta}_2 \\
&= \xi\big[(\mathbf{I}_M + z_2\mathbf{Q}_2)(\mathbf{I}_M + z_1\mathbf{Q}_1)\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2\big] + \underbrace{\xi\frac{1}{N_1}(\mathbf{I}_M + z_2\mathbf{Q}_2)\mathbf{Q}_1\mathbf{X}_1\boldsymbol{\varepsilon}_1}_{\boldsymbol{a}_5} \\
&\quad + (1-\xi)\big[(\mathbf{I}_M + z_2\mathbf{Q}_2)\boldsymbol{\beta}_2 - \boldsymbol{\beta}_2\big] + \underbrace{(1-\xi)\frac{1}{N_2}\mathbf{Q}_2\mathbf{X}_2\boldsymbol{\varepsilon}_2}_{\boldsymbol{a}_6} \\
&= \xi\boldsymbol{\gamma} + \underbrace{\xi z_2\mathbf{Q}_2\boldsymbol{\gamma}}_{\boldsymbol{a}_1} + \underbrace{\xi z_1\mathbf{Q}_1\boldsymbol{\beta}_1}_{\boldsymbol{a}_2} + \underbrace{z_2\mathbf{Q}_2\boldsymbol{\beta}_2}_{\boldsymbol{a}_3} + \underbrace{\xi z_1 z_2\mathbf{Q}_2\mathbf{Q}_1\boldsymbol{\beta}_1}_{\boldsymbol{a}_4} + \boldsymbol{a}_5 + \boldsymbol{a}_6.
\end{aligned} \tag{19}$$

By this, we decompose $\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}})$ as follows:

$$\begin{aligned}\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}}) &= (\boldsymbol{\beta}_{\mathsf{s}}-\boldsymbol{\beta}_2)^{\mathsf{T}}\boldsymbol{\Sigma}_2(\boldsymbol{\beta}_{\mathsf{s}}-\boldsymbol{\beta}_2)\\ &= \xi^2\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2\boldsymbol{\gamma}+2\sum_{i=1}^{6}b_i+\sum_{i=1}^{6}h_{ii}+\sum_{1\le i\ne j\le 6}h_{ij},\end{aligned}$$

where

$$b_i=\xi\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2\boldsymbol{a}_i,\ h_{ii}=\left\|\boldsymbol{\Sigma}_2^{1/2}\boldsymbol{a}_i\right\|^2,\ h_{ij}=\boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{\Sigma}_2\boldsymbol{a}_j.$$

Next, we compute the limit of each term above.

Let $n\in\mathbb{N}^+$. According to the Definition 1 and the Borel–Cantelli lemma, we have

$$\chi(n)=o_{a.s.}(1)\text{ if }\chi(n)\prec n^{-c}$$

for any constant $c>0$. By this, the limits of b_1,b_2,b_3,b_4 can be readily obtained using Lemma 5:

$$\begin{aligned}b_1&=\xi^2z_2\boldsymbol{\gamma}^{\mathsf{T}}\mathbf{Q}_2\boldsymbol{\Sigma}_2\boldsymbol{\gamma}=\xi^2z_2\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Pi}_2\boldsymbol{\Sigma}_2\boldsymbol{\gamma}+o_{a.s.}(1),\\ b_2&=\xi^2z_1\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2\mathbf{Q}_1\boldsymbol{\beta}_1=\xi^2z_1\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2\boldsymbol{\Pi}_1\boldsymbol{\beta}_1+o_{a.s.}(1),\\ b_3&=\xi z_2\boldsymbol{\beta}_2^{\mathsf{T}}\mathbf{Q}_2\boldsymbol{\Sigma}_2\boldsymbol{\gamma}=\xi z_2\boldsymbol{\beta}_2^{\mathsf{T}}\boldsymbol{\Pi}_2\boldsymbol{\Sigma}_2\boldsymbol{\gamma}+o_{a.s.}(1),\\ b_4&=\xi^2z_1z_2\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2\mathbf{Q}_2\mathbf{Q}_1\boldsymbol{\beta}_1=\xi^2z_1z_2\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1\boldsymbol{\beta}_1+o_{a.s.}(1),\end{aligned}$$

where the last identity is due to

$$\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2[\mathbf{Q}_2\mathbf{Q}_1-\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1]\boldsymbol{\beta}_1=\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2\big[(\mathbf{Q}_2-\boldsymbol{\Pi}_2)\mathbf{Q}_1+\boldsymbol{\Pi}_2(\mathbf{Q}_1-\boldsymbol{\Pi}_1)\big]\boldsymbol{\beta}_1=o_{a.s.}(1).$$

We now consider the terms contributing to $\mathbf{Var}$. By Lemma 4 and the identity

$$\frac{1}{N_i}\mathbf{Q}_i\mathbf{X}_i\mathbf{X}_i^{\mathsf{T}}=\mathbf{I}_M+z_1\mathbf{Q}_i,\ i=1,2. \tag{20}$$

we find that

$$\begin{aligned}&\Big|h_{55}-\frac{\xi^2\sigma^2}{N_1^2}\mathrm{Tr}(\mathbf{I}_M+z_2\mathbf{Q}_2)\boldsymbol{\Sigma}_2(\mathbf{I}_M+z_2\mathbf{Q}_2)\mathbf{Q}_1\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}}\mathbf{Q}_1\Big|\\ &\prec\frac{1}{M^2}\|(\mathbf{I}_M+z_2\mathbf{Q}_2)\boldsymbol{\Sigma}_2(\mathbf{I}_M+z_2\mathbf{Q}_2)(\mathbf{Q}_1+z_1\mathbf{Q}_1^2)\|_{\mathsf{F}}\lesssim\frac{1}{\sqrt{M}}.\end{aligned} \tag{21}$$

For any deterministic matrix $\mathbf{C}\in\mathbb{R}^{M\times M}$ satisfying $\|\mathbf{C}\|$ is bounded, and having the spectral decomposition

$$\mathbf{C}=\sum_{i=1}^{M}\lambda_i\mathbf{u}_i\mathbf{v}_i^{\mathsf{T}},$$

we have by Lemma 5 that

$$\begin{aligned}\frac{1}{M}\mathrm{Tr}\mathbf{C}\mathbf{Q}_1\mathbf{Q}_2&=\frac{1}{M}\sum_{i=1}^{M}\lambda_i\mathrm{Tr}\mathbf{u}_i\mathbf{v}_i^{\mathsf{T}}\mathbf{Q}_1\mathbf{Q}_2\\ &=\frac{1}{M}\sum_{i=1}^{M}\lambda_i\mathbf{v}_i^{\mathsf{T}}\mathbf{Q}_1\mathbf{Q}_2\mathbf{u}_i\\ &=\frac{1}{M}\sum_{i=1}^{M}\lambda_i\mathbf{v}_i^{\mathsf{T}}\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2\mathbf{u}_i+o_{a.s.}(1)\\ &=\frac{1}{M}\mathrm{Tr}\mathbf{C}\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2+o_{a.s.}(1).\end{aligned} \tag{22}$$

Similarly, by recalling the notation $\mathbf{\Pi}_i\mathcal{S}_i(\mathbf{I}_M)\mathbf{\Pi}_i = \mathbf{\Pi}_i'$ for $i = 1, 2$, one may check by Lemma 6 that

$$\begin{aligned}
\frac{1}{M}\mathrm{Tr}\mathbf{C}\mathbf{Q}_2\mathbf{Q}_1^2 &= \frac{1}{M}\mathrm{Tr}\mathbf{C}\mathbf{\Pi}_2\mathbf{\Pi}_1' + o_{a.s.}(1),\\
\frac{1}{M}\mathrm{Tr}\mathbf{C}\mathbf{Q}_1\mathbf{Q}_2^2 &= \frac{1}{M}\mathrm{Tr}\mathbf{C}\mathbf{\Pi}_1\mathbf{\Pi}_2',\\
\frac{1}{M}\mathrm{Tr}\mathbf{Q}_2\mathbf{C}\mathbf{Q}_2\mathbf{Q}_1^2 &= \frac{1}{M}\mathrm{Tr}\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{C})\mathbf{\Pi}_2\mathbf{\Pi}_1' + o_{a.s.}(1),\\
\frac{1}{M}\mathrm{Tr}\mathbf{Q}_1\mathbf{C}\mathbf{Q}_1\mathbf{Q}_2^2 &= \frac{1}{M}\mathrm{Tr}\mathbf{\Pi}_1\mathcal{S}_1(\mathbf{C})\mathbf{\Pi}_1\mathbf{\Pi}_2' + o_{a.s.}(1).
\end{aligned} \tag{23}$$

Then by Lemma 5, Lemma 6, equation 20 and equation 21, for $\xi \neq 0$, we have

$$\begin{aligned}
\frac{h_{55}}{\xi^2\sigma^2} &= \frac{1}{N_1}\mathrm{Tr}(\mathbf{I}_M + z_2\mathbf{Q}_2)\mathbf{\Sigma}_2(\mathbf{I}_M + z_2\mathbf{Q}_2)(\mathbf{Q}_1 + z_1\mathbf{Q}_1^2)\\
&= \frac{1}{N_1}\mathrm{Tr}\left[\mathbf{\Sigma}_2 + z_2\mathbf{\Sigma}_2\mathbf{Q}_2 + z_2\mathbf{Q}_2\mathbf{\Sigma}_2 + z_2^2\mathbf{Q}_2\mathbf{\Sigma}_2\mathbf{Q}_2\right]\left[\mathbf{\Pi}_1 + z_1\mathbf{\Pi}_1'\right] + o_{a.s.}(1)\\
&= \frac{1}{N_1}\mathrm{Tr}\left[\mathbf{\Sigma}_2 + 2z_2\mathbf{\Sigma}_2\mathbf{\Pi}_2 + z_2^2\mathbf{\Pi}_2\mathcal{S}_2(\mathbf{\Sigma}_2)\mathbf{\Pi}_2\right][\mathbf{\Pi}_1 + z_1\mathbf{\Pi}_1']\\
&+ o_{a.s.}(1).
\end{aligned}$$

Likewise, we have by Lemma 5 that

$$\begin{aligned}
h_{66} &= (1-\xi)^2\sigma^2\frac{1}{N_2}\mathrm{Tr}\mathbf{\Sigma}_2(\mathbf{I}_M + z_2\mathbf{Q}_2)\mathbf{Q}_2\\
&= (1-\xi)^2\sigma^2\frac{1}{N_2}\mathrm{Tr}\mathbf{\Sigma}_2(\mathbf{\Pi}_2 + z_2\mathbf{\Pi}_2') + o_{a.s.}(1).
\end{aligned}$$

Let $d = \min\left\{\mathrm{dist}\left(z_1, \mathbb{R}^+\right), \mathrm{dist}\left(z_2, \mathbb{R}^+\right)\right\}$. According to Lemma 4 and the fact following from equation 20 that

$$\frac{1}{\sqrt{N_i}}\|\mathbf{Q}_i\mathbf{X}_i\| = \sqrt{\|\mathbf{Q}_i + z_i\mathbf{Q}_i^2\|} \leq \sqrt{d^{-1} + d^{-2}|z_i|} \lesssim 1,$$

one has for $j = 1, 2, 3, 4$,

$$\begin{aligned}
|h_{5j}| = |h_{j5}| &\prec \frac{\sigma}{M}\|\mathbf{X}_1^\mathsf{T}\mathbf{Q}_1(\mathbf{I}_M + z_1\mathbf{Q}_2)\mathbf{\Sigma}_2\boldsymbol{a}_j\| \lesssim \frac{1}{\sqrt{M}},\\
|h_{6j}| = |h_{j6}| &\prec \frac{\sigma}{M}\|\mathbf{X}_2^\mathsf{T}\mathbf{Q}_2\mathbf{\Sigma}_2\boldsymbol{a}_j\| \lesssim \frac{1}{\sqrt{M}},
\end{aligned}$$

and

$$|b_5 + b_6| \prec \frac{1}{\sqrt{M}}.$$

Using Lemma 4 again, it can be shown that

$$\begin{aligned}
|h_{65}| = |h_{56}| &\prec \frac{\sigma^2}{M^2}\|\mathbf{X}_2^\mathsf{T}\mathbf{Q}_2\mathbf{\Sigma}_2(\mathbf{I}_M + z_2\mathbf{Q}_2)\mathbf{Q}_1\mathbf{X}_1\|_\mathsf{F}\\
&\lesssim \frac{\sigma^2}{M}\sqrt{\frac{N_2\|\mathbf{X}_2^\mathsf{T}\mathbf{Q}_2\mathbf{\Sigma}_2(\mathbf{I}_M + z_2\mathbf{Q}_2)\mathbf{Q}_1\mathbf{X}_1\|^2}{M^2}} \lesssim \frac{1}{\sqrt{M}}.
\end{aligned}$$

Therefore, we get

$$b_5 + b_6 + h_{65} + h_{56} + \sum_{j=1}^{4}(h_{5j} + h_{j_5} + h_{6j} + h_{j6}) = o_{a.s.}(1).$$

We now turn to the terms $h_{ii}, i = 1, 2, 3, 4$. By Lemma 6, we have

$$\begin{aligned}
h_{11} &= \xi^2 z_2^2 \boldsymbol{\gamma}^\mathsf{T} \mathbf{Q}_2 \boldsymbol{\Sigma}_2 \mathbf{Q}_2 \boldsymbol{\gamma} \\
&= \xi^2 z_2^2 \boldsymbol{\gamma}^\mathsf{T} \boldsymbol{\Pi}_2 \mathcal{S}_2(\boldsymbol{\Sigma}_2) \boldsymbol{\Pi}_2 \boldsymbol{\gamma} + o_{a.s.}(1), \\
h_{22} &= \xi^2 z_1^2 \boldsymbol{\beta}_1^\mathsf{T} \mathbf{Q}_1 \boldsymbol{\Sigma}_2 \mathbf{Q}_1 \boldsymbol{\beta}_1 \\
&= \xi^2 z_1^2 \boldsymbol{\beta}_1^\mathsf{T} \boldsymbol{\Pi}_1 \mathcal{S}_1(\boldsymbol{\Sigma}_2) \boldsymbol{\Pi}_1 \boldsymbol{\beta}_1 + o_{a.s.}(1), \\
h_{33} &= z_2^2 \boldsymbol{\beta}_2^\mathsf{T} \mathbf{Q}_2 \boldsymbol{\Sigma}_2 \mathbf{Q}_2 \boldsymbol{\beta}_2 \\
&= z_2^2 \boldsymbol{\beta}_2^\mathsf{T} \boldsymbol{\Pi}_2 \mathcal{S}_2(\boldsymbol{\Sigma}_2) \boldsymbol{\Pi}_2 \boldsymbol{\beta}_2 + o_{a.s.}(1), \\
h_{44} &= \xi^2 z_1^2 z_2^2 \boldsymbol{\beta}_1^\mathsf{T} \mathbf{Q}_1 \mathbf{Q}_2 \boldsymbol{\Sigma}_2 \mathbf{Q}_2 \mathbf{Q}_1 \boldsymbol{\beta}_1 \\
&= \xi^2 z_1^2 z_2^2 \mathbb{E} \boldsymbol{\beta}_1^\mathsf{T} \mathbf{Q}_1 \boldsymbol{\Pi}_2 \mathcal{S}_2(\boldsymbol{\Sigma}_2) \boldsymbol{\Pi}_2 \mathbf{Q}_1 \boldsymbol{\beta}_1 + o_{a.s.}(1) \\
&= \xi^2 z_1^2 z_2^2 \boldsymbol{\beta}_1^\mathsf{T} \boldsymbol{\Pi}_1 \mathcal{S}_1(\boldsymbol{\Pi}_2 \mathcal{S}_2(\boldsymbol{\Sigma}_2) \boldsymbol{\Pi}_2) \boldsymbol{\Pi}_1 \boldsymbol{\beta}_1 + o_{a.s.}(1).
\end{aligned}$$

Similarly, one can obtain the limits of the remaining terms in $h_{ij}, 1 \le i, j \le 6$:

$$\begin{aligned}
h_{12} = h_{21} &= \xi^2 z_1 z_2 \boldsymbol{\gamma}^\mathsf{T} \mathbf{Q}_2 \boldsymbol{\Sigma}_2 \mathbf{Q}_1 \boldsymbol{\beta}_1 \\
&= \xi^2 z_1 z_2 \boldsymbol{\gamma}^\mathsf{T} \boldsymbol{\Pi}_2 \boldsymbol{\Sigma}_2 \boldsymbol{\Pi}_1 \boldsymbol{\beta}_1 + o_{a.s.}(1), \\
h_{13} = h_{31} &= \xi z_2^2 \boldsymbol{\beta}_2^\mathsf{T} \mathbf{Q}_2 \boldsymbol{\Sigma}_2 \mathbf{Q}_2 \boldsymbol{\gamma} \\
&= \xi z_2^2 \boldsymbol{\beta}_2^\mathsf{T} \boldsymbol{\Pi}_2 \mathcal{S}_2(\boldsymbol{\Sigma}_2) \boldsymbol{\Pi}_2 \boldsymbol{\gamma} + o_{a.s.}(1), \\
h_{14} = h_{41} &= \xi^2 z_1 z_2^2 \boldsymbol{\gamma}^\mathsf{T} \mathbf{Q}_2 \boldsymbol{\Sigma}_2 \mathbf{Q}_2 \mathbf{Q}_1 \boldsymbol{\beta}_1 \\
&= \xi^2 z_1 z_2^2 \boldsymbol{\gamma}^\mathsf{T} \boldsymbol{\Pi}_2 \mathcal{S}_2(\boldsymbol{\Sigma}_2) \boldsymbol{\Pi}_2 \boldsymbol{\Pi}_1 \boldsymbol{\beta}_1 + o_{a.s.}(1), \\
h_{23} = h_{32} &= \xi z_1 z_2 \boldsymbol{\beta}_1^\mathsf{T} \mathbf{Q}_1 \boldsymbol{\Sigma}_2 \mathbf{Q}_2 \boldsymbol{\beta}_2 \\
&= \xi z_1 z_2 \boldsymbol{\beta}_1^\mathsf{T} \boldsymbol{\Pi}_1 \boldsymbol{\Sigma}_2 \boldsymbol{\Pi}_2 \boldsymbol{\beta}_2 + o_{a.s.}(1), \\
h_{24} = h_{42} &= \xi^2 z_1^2 z_2 \boldsymbol{\beta}_1^\mathsf{T} \mathbf{Q}_1 \boldsymbol{\Sigma}_2 \mathbf{Q}_2 \mathbf{Q}_1 \boldsymbol{\beta}_1 \\
&= \xi^2 z_1^2 z_2 \boldsymbol{\beta}_1^\mathsf{T} \mathbf{Q}_1 \boldsymbol{\Sigma}_2 \boldsymbol{\Pi}_2 \mathbf{Q}_1 \boldsymbol{\beta}_1 + o_{a.s.}(1) \\
&= \xi^2 z_1^2 z_2 \boldsymbol{\beta}_1^\mathsf{T} \boldsymbol{\Pi}_1 \mathcal{S}_1(\boldsymbol{\Sigma}_2 \boldsymbol{\Pi}_2) \boldsymbol{\Pi}_1 \boldsymbol{\beta}_1 + o_{a.s.}(1), \\
h_{34} = h_{43} &= \xi z_1 z_2^2 \boldsymbol{\beta}_2^\mathsf{T} \mathbf{Q}_2 \boldsymbol{\Sigma}_2 \mathbf{Q}_2 \mathbf{Q}_1 \boldsymbol{\beta}_1 \\
&= \xi z_1 z_2^2 \boldsymbol{\beta}_2^\mathsf{T} \boldsymbol{\Pi}_2 \mathcal{S}_2(\boldsymbol{\Sigma}_2) \boldsymbol{\Pi}_2 \boldsymbol{\Pi}_1 \boldsymbol{\beta}_1 + o_{a.s.}(1).
\end{aligned}$$

Combining the above estimates, we conclude the proof of Theorem 1.

B.3 Proof of Theorem 2

We use the same notation as in Appendix B.2. Note that $\boldsymbol{\gamma} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2 = 0$. Denoting

$$\mathbf{H} = \xi z_1 \mathbf{Q}_1 + z_2 \mathbf{Q}_2 + \xi z_1 z_2 \mathbf{Q}_2 \mathbf{Q}_1, \tag{24}$$

by equation 19, we have

$$\boldsymbol{\beta}_\mathsf{s} - \boldsymbol{\beta} = \mathbf{H}\boldsymbol{\beta} + \boldsymbol{a}_5 + \boldsymbol{a}_6.$$

Hence, the excess risk becomes

$$\begin{aligned}
\mathbf{ER}(\boldsymbol{\beta}_\mathsf{s}) &= \|\boldsymbol{\Sigma}_2^{1/2}(\boldsymbol{\beta}_\mathsf{s} - \boldsymbol{\beta})\|^2 \\
&= \boldsymbol{\beta}^\mathsf{T} \mathbf{H}^\mathsf{T} \boldsymbol{\Sigma}_2 \mathbf{H} \boldsymbol{\beta} + 2 \sum_{i=5,6} \boldsymbol{\beta}^\mathsf{T} \mathbf{H}^\mathsf{T} \boldsymbol{\Sigma}_2 \boldsymbol{a}_i + \sum_{i=5,6} h_{ii}.
\end{aligned}$$

Using Lemma 4, by Assumption 3 we have

$$\boldsymbol{\beta}^\mathsf{T} \mathbf{H}^\mathsf{T} \boldsymbol{\Sigma}_2 \mathbf{H} \boldsymbol{\beta} - \frac{\tilde{\sigma}^2}{M} \mathrm{Tr} \mathbf{H}^\mathsf{T} \boldsymbol{\Sigma}_2 \mathbf{H} \prec \frac{\tilde{\sigma}^2}{M} \|\mathbf{H}^\mathsf{T} \boldsymbol{\Sigma}_2 \mathbf{H}\|_\mathsf{F} \lesssim \frac{1}{\sqrt{M}}.$$

By equation 24, we have

$$\begin{aligned}\frac{1}{M}\mathrm{Tr}\boldsymbol{\Sigma}_2\mathbf{H}\mathbf{H}^\mathsf{T} &= \frac{1}{M}\Big[\xi^2 z_1^2 \mathrm{Tr}\boldsymbol{\Sigma}_2\mathbf{Q}_1^2 + \xi z_1 z_2 \mathrm{Tr}\boldsymbol{\Sigma}_2[\mathbf{Q}_1\mathbf{Q}_2 + \mathbf{Q}_2\mathbf{Q}_1] \\ &\quad + z_2^2\mathrm{Tr}\boldsymbol{\Sigma}_2\mathbf{Q}_2^2 + \xi^2 z_1^2 z_2 \mathrm{Tr}\boldsymbol{\Sigma}_2[\mathbf{Q}_2\mathbf{Q}_1^2 + \mathbf{Q}_1^2\mathbf{Q}_2] \\ &\quad + 2\xi z_1 z_2^2 \mathrm{Tr}\mathbf{Q}_2\boldsymbol{\Sigma}_2\mathbf{Q}_2\mathbf{Q}_1 + \xi^2 z_1^2 z_2^2 \mathrm{Tr}\mathbf{Q}_2\boldsymbol{\Sigma}_2\mathbf{Q}_2\mathbf{Q}_1^2\Big] \\ &= \sum_{i=1}^{6} t_i,\end{aligned}$$

where

$$t_1 = \frac{1}{M}\xi^2 z_1^2 \mathrm{Tr}\boldsymbol{\Sigma}_2\mathbf{Q}_1^2, t_2 = \frac{2}{M}\xi z_1 z_2 \mathrm{Tr}\mathbf{Q}_1\mathbf{Q}_2, t_3 = \frac{z_2^2}{M}\mathrm{Tr}\boldsymbol{\Sigma}_2\mathbf{Q}_2^2,$$

$$t_4 = 2\frac{\xi^2}{M} z_1^2 z_2 \mathrm{Tr}\boldsymbol{\Sigma}_2\mathbf{Q}_2\mathbf{Q}_1^2, t_5 = \frac{2\xi z_1 z_2^2}{M}\mathrm{Tr}\mathbf{Q}_2\boldsymbol{\Sigma}_2\mathbf{Q}_2\mathbf{Q}_1, t_6 = \frac{\xi^2 z_1^2 z_2^2}{M}\mathrm{Tr}\mathbf{Q}_2\boldsymbol{\Sigma}_2\mathbf{Q}_2\mathbf{Q}_1^2.$$

We next consider the terms $t_i, i = 1, ..., 6$. In the subsequent proof, we shall make use of Lemma 5, Lemma 6 and the property that $\boldsymbol{\Sigma}_2\boldsymbol{\Pi}_2 = \boldsymbol{\Pi}_2\boldsymbol{\Sigma}_2$.

By equation 22, we have

$$t_2 = \frac{2\xi z_1 z_2}{M}\mathrm{Tr}\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2\boldsymbol{\Sigma}_2 + o_{a.s.}(1).$$

$$t_1 = \xi^2 z_1^2 \frac{1}{M}\mathrm{Tr}\boldsymbol{\Sigma}_2\boldsymbol{\Pi}_1' + o_{a.s.}(1).$$

The limits of t_3, t_4, t_5, t_6 can be derived by equation 23:

$$t_3 = \frac{z_2^2}{M}\mathrm{Tr}\boldsymbol{\Sigma}_2\boldsymbol{\Pi}_2' + o_{a.s.}(1),$$

$$t_4 = 2\frac{\xi^2 z_1^2 z_2}{M}\mathrm{Tr}\boldsymbol{\Sigma}_2\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1' + o_{a.s.}(1),$$

$$t_5 = \frac{2\xi z_1 z_2^2}{M}\mathrm{Tr}\boldsymbol{\Pi}_2\mathcal{S}_2(\boldsymbol{\Sigma}_2)\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1 + o_{a.s.}(1),$$

and

$$t_6 = \frac{\xi^2 z_1^2 z_2^2}{M}\mathrm{Tr}\boldsymbol{\Pi}_2\mathcal{S}_2(\boldsymbol{\Sigma}_2)\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1' + o_{a.s.}(1).$$

Using Lemma 4, we find

$$|\boldsymbol{\beta}^\mathsf{T}\mathbf{H}^\mathsf{T}\boldsymbol{\Sigma}_2\boldsymbol{a}_5| \prec \frac{1}{M^{3/2}}\|\mathbf{H}^\mathsf{T}\boldsymbol{\Sigma}_2(\mathbf{I}_M + z_2\mathbf{Q}_2)\mathbf{Q}_1\mathbf{X}_1\|_\mathsf{F} \lesssim \frac{1}{\sqrt{M}},$$

$$|\boldsymbol{\beta}^\mathsf{T}\mathbf{H}^\mathsf{T}\boldsymbol{\Sigma}_2\boldsymbol{a}_6| \prec \frac{1}{M^{3/2}}\|\mathbf{Q}_2\mathbf{X}_2\|_\mathsf{F} \lesssim \frac{1}{\sqrt{M}}.$$

Therefore, the terms $\boldsymbol{\beta}^\mathsf{T}\mathbf{H}^\mathsf{T}\boldsymbol{\Sigma}_2\boldsymbol{a}_i, i = 5, 6$ are ignorable. The proof is now complete.

B.4 Proof of Theorem 3

Letting $\lambda_\mathsf{t} = \lambda_\mathsf{s} = 0$, by equation 3, we obtain

$$\begin{aligned}\boldsymbol{\beta}_\mathsf{s} &= \xi\boldsymbol{\beta}_1^{\mathsf{OLS}} + (1-\xi)\boldsymbol{\beta}_2^{\mathsf{OLS}} \\ &= \xi(\mathbf{X}_1\mathbf{X}_1^\mathsf{T})^{-1}\mathbf{X}_1(\mathbf{X}_1^\mathsf{T}\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1) + (1-\xi)(\mathbf{X}_2\mathbf{X}_2^\mathsf{T})^{-1}\mathbf{X}_2(\mathbf{X}_2^\mathsf{T}\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2) \\ &= \xi\boldsymbol{\beta}_1 + (1-\xi)\boldsymbol{\beta}_2 + \xi(\mathbf{X}_1\mathbf{X}_1^\mathsf{T})^{-1}\mathbf{X}_1\boldsymbol{\varepsilon}_1 + (1-\xi)(\mathbf{X}_2\mathbf{X}_2^\mathsf{T})^{-1}\mathbf{X}_2\boldsymbol{\varepsilon}_2.\end{aligned}$$

Plugging this into $\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}})$, one may obtain that

$$\begin{aligned}\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}}) &= \|\boldsymbol{\Sigma}_2^{1/2}(\boldsymbol{\beta}_2 - \boldsymbol{\beta}_{\mathsf{s}})\|^2 \\ &= \left\|\boldsymbol{\Sigma}_2^{1/2}[\xi\boldsymbol{\gamma} + \xi(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\mathbf{X}_1\boldsymbol{\varepsilon}_1 + (1-\xi)(\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}})^{-1}\mathbf{X}_2\boldsymbol{\varepsilon}_2]\right\|^2 \\ &= \widehat{\mathbf{Bias}} + h_1 + h_2 + 2h_3 + 2h_4 + 2h_5,\end{aligned}$$

where

$$\begin{aligned}h_1 &= \xi^2\boldsymbol{\varepsilon}_1^{\mathsf{T}}\mathbf{X}_1^{\mathsf{T}}(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\mathbf{X}_1\boldsymbol{\varepsilon}_1, \\ h_2 &= (1-\xi)^2\boldsymbol{\varepsilon}_2^{\mathsf{T}}\mathbf{X}_2^{\mathsf{T}}(\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2(\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}})^{-1}\mathbf{X}_2\boldsymbol{\varepsilon}_2, \\ h_3 &= \xi^2\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\mathbf{X}_1\boldsymbol{\varepsilon}_1, \\ h_4 &= \xi(1-\xi)\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2(\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}})^{-1}\mathbf{X}_2\boldsymbol{\varepsilon}_2, \\ h_5 &= \xi(1-\xi)\boldsymbol{\varepsilon}_1^{\mathsf{T}}\mathbf{X}_1^{\mathsf{T}}(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2(\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}})^{-1}\mathbf{X}_2\boldsymbol{\varepsilon}_2.\end{aligned}$$

By Lemmas 3-4, we have with high probability,

$$\begin{aligned}\left|h_2 - (1-\xi)^2\sigma^2\mathrm{Tr}(\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2\right| &= \left|h_2 - (1-\xi)^2\sigma^2\mathrm{Tr}(\mathbf{Z}_2\mathbf{Z}_2)^{\mathsf{T}}\right| \\ &\prec (1-\xi)^2\sigma^2\|(\mathbf{Z}_2\mathbf{Z}_2^{\mathsf{T}})^{-1}\|_{\mathsf{F}} \\ &= (1-\xi)^2\sigma^2\sqrt{\sum_{i=1}^{M}\lambda_i^{-2}\big(\mathbf{Z}_2\mathbf{Z}_2^{\mathsf{T}}\big)} \\ &\lesssim (1-\xi)^2\sigma^2\frac{1}{\sqrt{M}}\mathrm{Tr}(\mathbf{Z}_2\mathbf{Z}_2^{\mathsf{T}})^{-1} \\ &\lesssim \frac{1}{\sqrt{M}}.\end{aligned} \tag{25}$$

Lemma 5 implies that with high probability,

$$\mathrm{Tr}(\mathbf{Z}_2\mathbf{Z}_2^{\mathsf{T}})^{-1} = \frac{M}{N_2 - M} + o_{a.s.}(1).$$

Combining this with equation 25, we obtain with high probability,

$$h_2 = (1-\xi)^2\sigma^2\frac{M}{N_2 - M}\big(1 + o_{a.s.}(1)\big).$$

Similarly, one may derive with high probability,

$$\begin{aligned}|h_5| &\prec \sigma^2\|\mathbf{X}_1^{\mathsf{T}}(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2(\mathbf{X}_2\mathbf{X}_2^{\mathsf{T}})^{-1}\mathbf{X}_2\|_{\mathsf{F}} \\ &= \sigma^2\sqrt{\mathrm{Tr}(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}} \lesssim \frac{1}{\sqrt{M}}\widehat{\mathbf{Var}}.\end{aligned}$$

Using Lemmas 3-4, the following estimate holds with high probability,

$$\begin{aligned}\left|h_1 - \xi^2\sigma^2\mathrm{Tr}(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2\right| &\prec \|\boldsymbol{\Sigma}_2(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\|_{\mathsf{F}} \\ &\lesssim \sqrt{M\|\boldsymbol{\Sigma}_2\|^2\|\boldsymbol{\Sigma}_1\|^{-2}\|(\mathbf{Z}_1\mathbf{Z}_1^{\mathsf{T}})^{-1}\|^2} \lesssim \frac{1}{\sqrt{M}}.\end{aligned}$$

Then by Lemma 5, one has with high probability,

$$\begin{aligned}\mathrm{Tr}(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2 &= \frac{1}{N_1}\mathrm{Tr}\left(\frac{1}{N_1}\mathbf{Z}_1\mathbf{Z}_1^{\mathsf{T}}\right)^{-1}\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_1^{-1/2} \\ &= \frac{1}{N_1}\frac{N_1}{N_1 - M}\mathrm{Tr}\boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_1^{-1} + o_{a.s.}(1).\end{aligned}$$

Therefore, for $\xi \neq 0$, we get with high probability,

$$\frac{h_1}{\xi^2\sigma^2}h_1 = \mathrm{Tr}(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\boldsymbol{\Sigma}_2 + o_{a.s.}(1) = \frac{1}{N_1 - M}\mathrm{Tr}\boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_1^{-1} + o_{a.s.}(1).$$

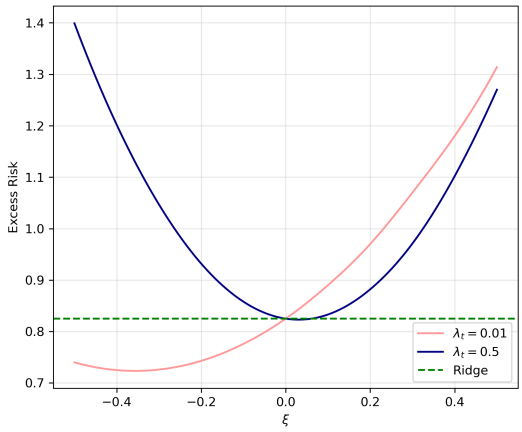


Figure 3: Theoretical excess risk for different λ_{t}. Settings: $(M, N_1, N_2) = (200, 200, 600)$, $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \mathbf{I}_M$, $\lambda_{\mathrm{s}} = 0.5$, SNR=4, $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 \sim \mathcal{N}(0, \frac{4}{M})$, $\sigma^2 = 1$.

We note that

$$\|\boldsymbol{\gamma}\|^2 \gtrsim \widehat{\mathbf{Bias}} = \xi^2 \|\boldsymbol{\Sigma}_2^{1/2}\boldsymbol{\gamma}\|^2 \gtrsim \lambda_{\min}(\boldsymbol{\Sigma}_2)\|\boldsymbol{\gamma}\|^2 \gtrsim \|\boldsymbol{\gamma}\|^2.$$

Since $\sigma_M(\boldsymbol{\Sigma}_1) \lesssim 1, \sigma_1(\boldsymbol{\Sigma}_2) \lesssim 1$, it is easy to see $\widehat{\mathbf{Var}} \sim 1$. Using Lemmas 3-4, we get with high probability

$$\begin{aligned}
|h_3| &\prec \xi^2\sigma\|\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Sigma}_2(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\mathbf{X}_1\| \\
&\leq \xi^2\sigma\sqrt{\widehat{\mathbf{Bias}}}\|\boldsymbol{\Sigma}_2^{1/2}\|\|(\mathbf{X}_1\mathbf{X}_1^{\mathsf{T}})^{-1}\mathbf{X}_1\| \\
&\lesssim \frac{\sqrt{\widehat{\mathbf{Bias}}}}{M^{1/4}}\frac{1}{M^{1/4}} \leq \frac{\widehat{\mathbf{Bias}}}{\sqrt{M}} + \frac{1}{\sqrt{M}} \\
&\lesssim \frac{1}{\sqrt{M}}(\widehat{\mathbf{Bias}} + \widehat{\mathbf{Var}}).
\end{aligned}$$

Similarly, we can estimate with high probability

$$|h_4| \prec \frac{1}{\sqrt{M}}(\widehat{\mathbf{Bias}} + \widehat{\mathbf{Var}}).$$

Combining the above estimates on $h_i, i = 1, 2, 3, 4, 5$, the proof of Theorem 3 is completed.

B.5 Proof of Corollary 2

We take the derivative of $\mathbf{ER}(\boldsymbol{\beta}_{\mathrm{s}})$ with respect to ξ, and evaluate it at $\xi = 0$:

$$\frac{\partial}{\partial\xi}\mathbf{ER}(\boldsymbol{\beta}_{\mathrm{s}})\Big|_{\xi=0} = \sigma^2\left(\lambda_{\mathrm{t}}\lambda_{\mathrm{s}}\underline{m}_1\mathrm{SNR} - \frac{M}{N_2}\right)\frac{\mathrm{d}}{\mathrm{d}z}\big(z\underline{m}_2(z)\big)\Big|_{z=-\lambda_{\mathrm{s}}} + o_{a.s.}(1).$$

Since

$$z\underline{m}_2(z) = \int \frac{z}{x-z}\mathrm{d}\varrho_{\mathrm{MP},2}(x) = -1 + \int \frac{x}{x-z}\mathrm{d}\varrho_{\mathrm{MP},2}(x),$$

we hence get that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z\underline{m}_2(z)) > 0\big|_{z=-\lambda_{\mathrm{s}}}. \tag{26}$$

Therefore, $\frac{\partial}{\partial\xi}\mathbf{ER}(\boldsymbol{\beta}_{\mathrm{s}})|_{\xi=0}$ and $\lambda_{\mathrm{t}}\lambda_{\mathrm{s}}\mathrm{SNR} - \frac{M}{N_2}$ share the same sign almost surely. That is, the limiting optimal value of ξ is negative when equation 9 holds, which establishes Corollary 2.

We provide an example in Figure 3 to illustrate this corollary. Specifically, when $\lambda_{\mathrm{t}} = 0.5$, the limiting optimal value of ξ is positive, whereas when $\lambda_{\mathrm{t}} = 0.01$, it becomes negative.

B.6 Details for Proposition 1 and Remark 2

Recall that $\mathbf{ER}_0$ denotes the excess risk of the ridge regression model trained solely on the target domain data.

Assumption A.1. *When $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ are deterministic, we assume that :*

$$\Big|\boldsymbol{\beta}_1^\mathsf{T}\lambda_\mathsf{t}\lambda_\mathsf{s}\big[\boldsymbol{\Pi}_1\boldsymbol{\Sigma}_2\boldsymbol{\Pi}_2 - \lambda_\mathsf{s}\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2\mathcal{S}_2(\boldsymbol{\Sigma}_2)\boldsymbol{\Pi}_2\big]\boldsymbol{\beta}_2 - \frac{\sigma^2}{N_2}\mathrm{Tr}[\boldsymbol{\Sigma}_2(\boldsymbol{\Pi}_2 - \lambda_\mathsf{s}\boldsymbol{\Pi}_2')]$$
$$+\boldsymbol{\beta}_2^\mathsf{T}[\lambda_\mathsf{s}^2\boldsymbol{\Pi}_2\mathcal{S}_2(\boldsymbol{\Sigma}_2)\boldsymbol{\Pi}_2 - \lambda_\mathsf{s}\boldsymbol{\Pi}_2\boldsymbol{\Sigma}_2]\boldsymbol{\gamma}\Big| > c,$$

where c is a positive constant.

Assumption A.2. *If $\boldsymbol{\beta} = \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ is random, we assume that*

$$\Big|\frac{\tilde{\sigma}^2}{M}\Big[\lambda_\mathsf{t}\lambda_\mathsf{s}\mathrm{Tr}\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2\boldsymbol{\Sigma}_2 - \lambda_\mathsf{t}\lambda_\mathsf{s}^2\mathrm{Tr}\boldsymbol{\Pi}_2\mathcal{S}_2(\boldsymbol{\Sigma}_2)\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1\Big] - \frac{\sigma^2}{N_2}\mathrm{Tr}[\boldsymbol{\Sigma}_2(\boldsymbol{\Pi}_2 - \lambda_\mathsf{s}\boldsymbol{\Pi}_2')]\Big| > c,$$

where c is a positive constant.

Proof of Proposition 1: (i) Suppose the conditions in Theorem 1 hold. Note that

$$\frac{\partial}{\partial\xi}\mathbf{ER}(\boldsymbol{\beta}_\mathsf{s})\big|_{\xi=0} = 2\boldsymbol{\beta}_1^\mathsf{T}\big[\lambda_\mathsf{t}\lambda_\mathsf{s}\boldsymbol{\Pi}_1\boldsymbol{\Sigma}_2\boldsymbol{\Pi}_2 - \lambda_\mathsf{t}\lambda_\mathsf{s}^2\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2\mathcal{S}_2(\boldsymbol{\Sigma}_2)\boldsymbol{\Pi}_2\big]\boldsymbol{\beta}_2$$
$$+ 2\boldsymbol{\beta}_2^\mathsf{T}[\lambda_\mathsf{s}^2\boldsymbol{\Pi}_2\mathcal{S}_2(\boldsymbol{\Sigma}_2)\boldsymbol{\Pi}_2 - \lambda_\mathsf{s}\boldsymbol{\Pi}_2\boldsymbol{\Sigma}_2]\boldsymbol{\gamma} - \frac{2\sigma^2}{N_2}\mathrm{Tr}[\boldsymbol{\Sigma}_2(\boldsymbol{\Pi}_2 - \lambda_\mathsf{s}\boldsymbol{\Pi}_2')] + o_{a.s.}(1).$$

Under the conditions in Theorem 1 and Assumption A.1, the asymptotic excess risk is a quadratic function whose minimizer is bounded away from 0. Therefore, $\min_\xi \mathbf{ER}(\boldsymbol{\beta}_\mathsf{s})$ is strictly less than $\mathbf{ER}_0$ almost surely.

(ii) Similarly, suppose Assumption A.2, under the conditions of Theorem 2, the inequality equation 13 holds by noticing that

$$\frac{\partial}{\partial\xi}\mathbf{ER}(\boldsymbol{\beta}_\mathsf{s})\big|_{\xi=0} = \frac{2\tilde{\sigma}^2}{M}\Big[\lambda_\mathsf{t}\lambda_\mathsf{s}\mathrm{Tr}\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2\boldsymbol{\Sigma}_2 - \lambda_\mathsf{t}\lambda_\mathsf{s}^2\mathrm{Tr}\boldsymbol{\Pi}_2\mathcal{S}_2(\boldsymbol{\Sigma}_2)\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1\Big] - \frac{2\sigma^2}{N_2}\mathrm{Tr}[\boldsymbol{\Sigma}_2(\boldsymbol{\Pi}_2 - \lambda_\mathsf{s}\boldsymbol{\Pi}_2')]$$
$$+ o_{a.s.}(1).$$

Further discussion on Remark 2: To clarify the dependence of Assumption A.1 on the geometry of $\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ and the noise strength σ^2, we consider a simple example in which $\boldsymbol{\Sigma}_2 = \mathbf{I}_M$. Then we have

$$\begin{aligned}\frac{\partial}{\partial\xi}\mathbf{ER}(\boldsymbol{\beta}_\mathsf{s})\big|_{\xi=0} &= \lambda_\mathsf{s}\lambda_\mathsf{t}(\underline{m}_2 - \lambda_\mathsf{s}\underline{m}_2')\boldsymbol{\beta}_1^\mathsf{T}\boldsymbol{\Pi}_1\boldsymbol{\beta}_2 - \frac{\sigma^2 M}{N_2}(\underline{m}_2 - \lambda_\mathsf{s}\underline{m}_2') - \lambda_\mathsf{s}(\underline{m}_2 - \lambda_\mathsf{s}\underline{m}_2')\boldsymbol{\beta}_2^\mathsf{T}\boldsymbol{\gamma}\\ &\quad + o_{a.s.}(1)\\ &= \underbrace{\big(\lambda_\mathsf{s}\lambda_\mathsf{t}\boldsymbol{\beta}_1^\mathsf{T}\boldsymbol{\Pi}_1\boldsymbol{\beta}_2 - \frac{\sigma^2 M}{N_2} - \lambda_\mathsf{s}\boldsymbol{\beta}_2^\mathsf{T}\boldsymbol{\gamma}\big)}_{e}\frac{\mathrm{d}}{\mathrm{d}z}(z\underline{m}_2(z))\big|_{z=-\lambda_\mathsf{s}} + o_{a.s.}(1),\end{aligned} \tag{27}$$

where $\underline{m}_2(z)$ is defined in equation 8. Recalling equation 26, we have $|\frac{\partial}{\partial\xi}\mathbf{ER}(\boldsymbol{\beta}_\mathsf{s})|_{\xi=0}| > c$ if $|e| > C$ for some constant C. Below, we discuss two cases, when $\boldsymbol{\Sigma}_1 = \mathbf{I}_M$ and when $\boldsymbol{\Sigma}_1 \neq \mathbf{I}_M$:

- $\boldsymbol{\Sigma}_1 = \mathbf{I}_M$. The term e becomes

$$e = \lambda_\mathsf{s}\lambda_\mathsf{t}\underline{m}_1\boldsymbol{\beta}_1^\mathsf{T}\boldsymbol{\beta}_2 - \frac{\sigma^2 M}{N_2} - \lambda_\mathsf{s}\boldsymbol{\beta}_2^\mathsf{T}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2).$$

  Recall that the limiting ridge risk is minimized at $\lambda_\mathsf{s}^* = \frac{\sigma^2 M}{N_2\|\boldsymbol{\beta}_2\|^2}$, with asymptotic excess risk $\sigma^2\frac{M}{N_2}\underline{m}_2(-\lambda_\mathsf{s}^*)$ (Hastie et al., 2022). Taking $\lambda_\mathsf{s} = \lambda_\mathsf{s}^*$, we have

$$e = \lambda_\mathsf{s}^*(\lambda_\mathsf{t}\underline{m}_1 - 1)\boldsymbol{\beta}_1^\mathsf{T}\boldsymbol{\beta}_2.$$

Note that

$$\lambda_{\mathsf{t}}\underline{m}_1 - 1 = -\int \frac{x}{x+\lambda_{\mathsf{t}}}\mathrm{d}\varrho_{\mathsf{MP},1}(x) < 0.$$

Therefore, in a small neighborhood containing 0, $\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}})$ is monotonic in ξ, indicating that the teacher's supervision is helpful – even outperforming the optimal ridge regression – provided that $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are not asymptotically orthogonal.

- $\boldsymbol{\Sigma}_1 \neq \mathbf{I}_M$. By taking $\lambda_{\mathsf{s}} = \lambda_{\mathsf{s}}^*$, e becomes

$$e = \lambda_{\mathsf{s}}^*(\lambda_{\mathsf{t}}\boldsymbol{\beta}_1^{\mathsf{T}}\boldsymbol{\Pi}_1\boldsymbol{\beta}_2 - \boldsymbol{\beta}_2^{\mathsf{T}}\boldsymbol{\beta}_1) = -\lambda_{\mathsf{s}}^*\sum_{i=1}^{M}\frac{m_1\sigma_i}{1+m_1\sigma_i}\boldsymbol{\beta}_1^{\mathsf{T}}\mathbf{u}_i\mathbf{u}_i^{\mathsf{T}}\boldsymbol{\beta}_2, \tag{28}$$

where m_1 is determined by Lemma 1 and $\boldsymbol{\Sigma}_1 = \sum_{i=1}^{M}\sigma_i\mathbf{u}_i\mathbf{u}_i^{\mathsf{T}}$ represents the spectral decomposition of $\boldsymbol{\Sigma}_1$. By equation 28, the alignment of $\boldsymbol{\beta}_i$ $(i=1,2)$ with the eigenvectors of $\boldsymbol{\Sigma}_1$ determines whether Assumption A.1 is satisfied. Therefore, given $\lambda_{\mathsf{s}} = \lambda_{\mathsf{s}}^*$, under the "help" of covariate shift, even if $\boldsymbol{\beta}_1^{\mathsf{T}}\boldsymbol{\beta}_2 = 0$, it may still be possible to find a ξ such that $\mathbf{ER}(\boldsymbol{\beta}_{\mathsf{s}}) < \mathbf{ER}_0$, a.s. By comparing with the case where $\boldsymbol{\Sigma}_1 = \mathbf{I}_M$, we find that the presence of covariate shift can, in some cases, be beneficial.

B.7 Proof of Lemma 6

The following result, which is an immediate consequence of Lemma 1, will be used in the proof below:

$$-zm = \left(1 + \frac{1}{N}\mathrm{Tr}\boldsymbol{\Sigma}\boldsymbol{\Pi}(z)\right)^{-1}. \tag{29}$$

We abuse notation by writing z_1 and z_2 for $\tilde{z}_1$ and $\tilde{z}_2$, respectively, whenever there is no risk of ambiguity. Without loss of generality, we assume $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ and z_1, z_2 lie on the negative real axis, as the other cases follow by analogous arguments.

Using standard techniques of martingale decomposition (see, e.g., Bai & Silverstein (2010)), we can prove the almost sure convergence of the random part:

$$\mathbf{u}^{\mathsf{T}}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{v} = \mathbf{u}^{\mathsf{T}}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{v} + o_{a.s.}(1). \tag{30}$$

Therefore, it suffices to consider the term $\mathbf{u}^{\mathsf{T}}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{v}$. Let $\sigma_1 \geq \cdots \geq \sigma_M$ denote the eigenvalues of $\boldsymbol{\Sigma}$. For the sequence of deterministic matrices, we denote $\mathbf{A}_M = o(1)$ if $\|\mathbf{A}_M\| \to 0$. Since

$$\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2) = \mathbf{Q}(z_1)\mathbf{A}\boldsymbol{\Pi}(z_2) + \mathbf{Q}(z_1)\mathbf{A}\big(\mathbf{Q}(z_2) - \boldsymbol{\Pi}(z_2)\big), \tag{31}$$

we obtain by Lemma 5 that

$$\begin{aligned}\mathbf{u}^{\mathsf{T}}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{v} &= \mathbf{u}^{\mathsf{T}}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\boldsymbol{\Pi}(z_2)\mathbf{v} + \mathbf{u}^{\mathsf{T}}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}(\mathbf{Q}(z_2) - \boldsymbol{\Pi}(z_2))\mathbf{v} \\ &= \mathbf{u}^{\mathsf{T}}\boldsymbol{\Pi}(z_1)\mathbf{A}\boldsymbol{\Pi}(z_2)\mathbf{v} + \mathbf{u}^{\mathsf{T}}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}(\mathbf{Q}(z_2) - \boldsymbol{\Pi}(z_2))\mathbf{v} + o(1),\end{aligned} \tag{32}$$

where the second identity follows from Lemma 5, the Dominated Convergence Theorem and the fact that

$$\|\boldsymbol{\Pi}(z_1)\| = \max_i \left|z_1 + z_1 m(z_1)\sigma_i\right|^{-1} \leq |z_1|^{-1},\ \|\mathbf{A}\boldsymbol{\Pi}(z_2)\mathbf{v}\| \leq \|\mathbf{A}\|\|\boldsymbol{\Pi}(z_2)\| \leq |z_2|^{-1}\|\mathbf{A}\|.$$

Therefore, our task reduces to finding the deterministic equivalent of

$$\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\big(\mathbf{Q}(z_2) - \boldsymbol{\Pi}(z_2)\big).$$

Denote

$$\mathbf{X}_{-k} = \mathbf{X} - \mathbf{x}_k\mathbf{e}_k^{\mathsf{T}},\ \mathbf{Q}_{-k}(z) = \left(\frac{\mathbf{X}_{-k}\mathbf{X}_{-k}^{\mathsf{T}}}{N} - z\mathbf{I}_M\right)^{-1}.$$

By Sherman-Morrison formula, one may easily check that

$$\begin{aligned}\mathbf{Q}(z) &= \mathbf{Q}_{-k}(z) - \frac{\frac{1}{N}\mathbf{Q}_{-k}(z)\mathbf{x}_k\mathbf{x}_k^{\mathsf{T}}\mathbf{Q}_{-k}(z)}{1 + \frac{1}{N}\mathbf{x}_k^{\mathsf{T}}\mathbf{Q}_{-k}(z)\mathbf{x}_k}, \\ \mathbf{Q}(z)\mathbf{x}_k &= \frac{\mathbf{Q}_{-k}(z)\mathbf{x}_k}{1 + \frac{1}{N}\mathbf{x}_k^{\mathsf{T}}\mathbf{Q}_{-k}(z)\mathbf{x}_k}.\end{aligned} \tag{33}$$

We show here the following result for future use:

$$\frac{1}{N}\mathbb{E}\mathrm{Tr}\mathbf{C}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2) = \frac{1}{N}\mathbb{E}\mathrm{Tr}\mathbf{C}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2) + o(1), \tag{34}$$

where $\mathbf{C} \in \mathbb{R}^{M\times M}$ is a deterministic matrix with $\|\mathbf{C}\| \le C$ for some constant C. We decompose

$$\begin{aligned}&\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2) - \mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\\ =&[\mathbf{Q}(z_1) - \mathbf{Q}_{-1}(z_1)]\mathbf{A}\mathbf{Q}(z_2) + \mathbf{Q}_{-1}(z_1)\mathbf{A}[\mathbf{Q}_{-1}(z_2) - \mathbf{Q}_{-1}(z_2)].\end{aligned}$$

Applying the identity

$$\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{A}^{-1},$$

we have for $i = 1, 2$, and $\widetilde{\mathbf{C}} \in \mathbb{R}^{M\times M}$ with finite spectral norm (where $\widetilde{\mathbf{C}}$ may be a deterministic matrix, or a random matrix that is either dependent on or independent of $\mathbf{X}$),

$$\frac{1}{N}|\mathrm{Tr}[\mathbf{Q}(z_i) - \mathbf{Q}_{-1}(z_i)]\widetilde{\mathbf{C}}| = \frac{1}{N^2}|\mathbf{x}_1^\mathsf{T}\mathbf{Q}(z_i)\widetilde{\mathbf{C}}\mathbf{Q}_{-1}(z_1)\mathbf{x}_1| \le \frac{C}{N^2}\|\mathbf{x}_1\|^2 = o_{a.s.}(1).$$

We denote $d = \min\{\mathrm{dist}(z_1, \mathbb{R}^+), \mathrm{dist}(z_2, \mathbb{R}^+)\}$. One may easily check that $d \sim 1$. Then by

$$\frac{1}{N}|\mathrm{Tr}[\mathbf{Q}(z_i) - \mathbf{Q}_{-1}(z_i)]\widetilde{\mathbf{C}}| \le \frac{M}{N}(\|\mathbf{Q}(z_i)\widetilde{\mathbf{C}}\| + \|\mathbf{Q}_{-1}(z_i)\widetilde{\mathbf{C}}\|) \le \frac{2M}{dN}, \text{ for } i = 1, 2,$$

and the Dominated Convergence Theorem, we obtain equation 34. By similar arguments, we get for any deterministic unit vectors $\mathbf{u}, \mathbf{v}$,

$$\begin{aligned}\mathbf{u}^\mathsf{T}\mathbb{E}\widetilde{\mathbf{C}}\mathbf{Q}(z_i)\mathbf{C}\mathbf{v} &= \mathbf{u}^\mathsf{T}\mathbb{E}\widetilde{\mathbf{C}}\mathbf{Q}_{-k}(z_i)\mathbf{C}\mathbf{v} + o(1)\\ &= \mathbf{u}^\mathsf{T}\widetilde{\mathbf{C}}\mathbf{\Pi}(z_i)\mathbf{C}\mathbf{v} + o(1),\ i = 1, 2,\\ \mathbf{u}^\mathsf{T}\mathbb{E}\widetilde{\mathbf{C}}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{C}\mathbf{v} &= \mathbf{u}^\mathsf{T}\mathbb{E}\widetilde{\mathbf{C}}\mathbf{Q}_{-k}(z_1)\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{C}\mathbf{v} + o(1),\end{aligned} \tag{35}$$

where $\widetilde{\mathbf{C}}$ and $\mathbf{C}$ are deterministic $M \times M$ matrices with finite spectral norms.

We denote

$$b_k = \frac{1}{N}\mathbf{x}_k^\mathsf{T}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k,\ \tilde{b} = \frac{1}{N}\mathbb{E}\mathbf{x}_k^\mathsf{T}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k,$$

$$\mathsf{b}_k = \frac{1}{N}\mathbf{x}_k^\mathsf{T}\mathbf{Q}_{-k}(z_1)\mathbf{x}_k, \tilde{\mathsf{b}} = \frac{1}{N}\mathbb{E}\mathbf{x}_k^\mathsf{T}\mathbf{Q}_{-k}(z_1)\mathbf{x}_k.$$

It follows directly from the proof of equation 34 that

$$\begin{aligned}\tilde{b} &= \frac{1}{N}\mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_2) = \frac{1}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}(z_2) + o(1),\\ \tilde{\mathsf{b}} &= \frac{1}{N}\mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_2) = \frac{1}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}(z_2) + o(1).\end{aligned} \tag{36}$$

Recalling equation 29, we rewrite $\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\big(\mathbf{Q}(z_2) - \mathbf{\Pi}(z_2)\big)$ as

$$\begin{aligned}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\big(\mathbf{Q}(z_2) - \mathbf{\Pi}(z_2)\big) &= \mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\big(\mathbf{I}_M - \mathbf{Q}^{-1}(z_2)\mathbf{\Pi}(z_2)\big)\\ &= \mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\big(\mathbf{\Pi}^{-1}(z_2) - \mathbf{Q}^{-1}(z_2)\big)\mathbf{\Pi}(z_2)\\ &= \mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\bigg(-\frac{1}{N}\mathbf{X}\mathbf{X}^\mathsf{T} - z_2 m\mathbf{\Sigma}\bigg)\mathbf{\Pi}(z_2)\\ &= \mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\frac{\mathbf{\Sigma}\mathbf{\Pi}(z_2)}{1 + \frac{1}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}(z_2)} - \frac{1}{N}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{X}\mathbf{X}^\mathsf{T}\mathbf{\Pi}(z_2).\end{aligned} \tag{37}$$

An application of equation 33 yields that

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T} &= \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\frac{\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T}}{1+b_k}\\ &= \frac{1}{N(1+\tilde{b})}\sum_{k=1}^{N}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T}\Big[1+\frac{\tilde{b}-b_k}{(1+b_k)}\Big]\\ &= \frac{1}{N(1+\tilde{b})}\Big[\sum_{k=1}^{N}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T}+\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{X}\mathbf{B}\mathbf{X}^\mathsf{T}\Big]\\ &= \frac{1}{1+\tilde{b}}\big(\mathbb{E}\mathbf{F}_1+\mathbb{E}\mathbf{F}_2\big),\end{aligned} \tag{38}$$

where $\mathbf{B}=\mathrm{diag}\big(\tilde{b}-b_1,...,\tilde{b}-b_N\big)$, and

$$\mathbf{F}_1=\frac{1}{N}\sum_{k=1}^{N}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T},\ \mathbf{F}_2=\frac{1}{N}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{X}\mathbf{B}\mathbf{X}^\mathsf{T}.$$

We now bound the spectral norm of $\mathbf{F}_2$. Define the event

$$\mathcal{E}=\Big\{\frac{1}{N}\|\mathbf{Z}\mathbf{Z}^\mathsf{T}\|\le 2\big(1+\sqrt{\frac{M}{N}}\big)^2\Big\}.$$

We then have

$$\begin{aligned}\|\mathbb{E}\mathbf{F}_2\|\le\mathbb{E}\|\mathbf{F}_2\|&\le\frac{1}{N}\frac{\|\mathbf{A}\|}{d^2}\mathbb{E}\|\mathbf{X}\mathbf{B}\mathbf{X}^\mathsf{T}\|\\ &\le\frac{\|\mathbf{A}\|\|\mathbf{\Sigma}\|}{d^2}\big[4(1+\sqrt{\phi})^2\mathbb{E}\|\mathbf{B}\|\delta(\mathcal{E})+\frac{1}{N}\mathbb{E}\|\mathbf{B}\|\|\mathbf{Z}\mathbf{Z}^\mathsf{T}\|\delta(\mathcal{E}^C)\big]\\ &\le C\mathbb{E}\max_k|\tilde{b}-b_k|+\frac{1}{N}\sqrt{\mathbb{E}\max_k|\tilde{b}-b_k|^2\mathbb{E}\|\mathbf{Z}\mathbf{Z}^\mathsf{T}\|^2\delta(\mathcal{E}^C)}.\end{aligned}$$

By using the inequality that (see e.g. Bai & Silverstein (2010))

$$\mathbb{P}(\mathcal{E}^C)\le N^{-\ell}\text{ for any }\ell>0,$$

we have

$$\begin{aligned}\mathbb{E}\|\mathbf{Z}\mathbf{Z}^\mathsf{T}\|^2\delta(\mathcal{E}^C)\le\mathbb{E}\|\mathbf{Z}\mathbf{Z}^\mathsf{T}\|_\mathsf{F}^2\delta(\mathcal{E}^C)&\le\sqrt{\mathbb{E}\|\mathbf{Z}\mathbf{Z}^\mathsf{T}\|_\mathsf{F}^4\mathbb{P}(\mathcal{E}^C)}\\ &\le N^{100}o(N^{-101})=o(N^{-1}).\end{aligned} \tag{39}$$

It can be shown by Lemma 2 that for $\ell\ge 1$,

$$\begin{aligned}\mathbb{P}(|\tilde{b}-b_k|>t)&\le\frac{\mathbb{E}|\mathbf{z}_k^\mathsf{T}\mathbf{\Sigma}^{1/2}\mathbf{Q}_{-k}\mathbf{\Sigma}^{1/2}\mathbf{z}_k-\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-k}(z)|^\ell}{(Nt)^\ell}\\ &=t^{-\ell}\frac{\mathbb{E}[\mathbb{E}_{-k}|\mathbf{z}_k^\mathsf{T}\mathbf{\Sigma}^{1/2}\mathbf{Q}_{-k}\mathbf{\Sigma}^{1/2}\mathbf{z}_k-\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-k}(z)|^\ell]}{N^\ell}\\ &\le t^{-\ell}C\frac{\mathbb{E}[(\mathrm{Tr}\mathbf{Q}_{-k}^2)^{\ell/2}+\mathrm{Tr}(\mathbf{Q}_{-k})^\ell]}{N^\ell}\\ &\le Ct^{-\ell}N^{-\ell/2},\end{aligned}$$

where we use the fact that

$$\mathrm{Tr}\big(\mathbf{Q}_{-k}(z_2)\big)^\ell\le M\|\mathbf{Q}_{-k}(z_2)\|^\ell\le\frac{M}{d^\ell}.$$

By taking a large enough ℓ, we have

$$\begin{aligned}\mathbb{E}\max_k|\tilde{b}-b_k|&=\Big(\int_{t\le N^{-1/4}}+\int_{t>N^{-1/4}}\Big)\mathbb{P}(\max_k|\tilde{b}-b_k|>t)\mathrm{d}t\\ &\le N^{-1/4}+\int_{t>N^{-1/4}}\sum_{k=1}^{N}\mathbb{P}(|\tilde{b}-b_k|>t)\mathrm{d}t\\ &\le 2N^{-1/4}.\end{aligned} \tag{40}$$

Similarly, one may obtain

$$\mathbb{E}\max_k |\tilde{b} - b_k|^2 = o(1). \tag{41}$$

This, along with equation 39 and equation 40, implies that

$$\|\mathbb{E}\mathbf{F}_2\| = o(1).$$

By using equation 33, we rewrite

$$\begin{aligned}
\mathbb{E}\mathbf{F}_1 &= \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T} \\
&= \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\big[\mathbf{Q}_{-k}(z_1) - \frac{1}{N}\frac{\mathbf{Q}_{-k}(z_1)\mathbf{x}_k\mathbf{x}_k^\mathsf{T}\mathbf{Q}_{-k}(z_1)}{1+\mathsf{b}_k}\big]\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T} \\
&= \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\mathbf{Q}_{-k}(z_1)\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T} - \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\frac{\frac{1}{N}\mathbf{Q}_{-k}(z_1)\mathbf{x}_k\mathbf{x}_k^\mathsf{T}\mathbf{Q}_{-k}(z_1)}{1+\mathsf{b}_k}\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T} \\
&= \mathbb{E}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{\Sigma} - \frac{1}{(1+\tilde{\mathsf{b}})N}\sum_{k=1}^{N}\mathbb{E}\frac{1}{N}\mathbf{Q}_{-k}(z_1)\mathbf{x}_k\mathbf{x}_k^\mathsf{T}\mathbf{Q}_{-k}(z_1)\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T} \\
&\quad - \frac{1}{(1+\tilde{\mathsf{b}})N^2}\sum_{k=1}^{N}\mathbb{E}\mathbf{Q}_{-k}(z_1)\mathbf{x}_k\mathbf{x}_k^\mathsf{T}\mathbf{Q}_{-k}(z_1)\mathbf{A}\mathbf{Q}_{-k}(z_2)\mathbf{x}_k\mathbf{x}_k^\mathsf{T}\frac{(\mathsf{b}_k - \tilde{\mathsf{b}})}{1+\mathsf{b}_k} \\
&= \mathbb{E}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{\Sigma} - \frac{1}{1+\tilde{\mathsf{b}}}\big(\mathbb{E}\mathsf{F}_1 + \mathbb{E}\mathsf{F}_2\big),
\end{aligned} \tag{42}$$

where

$$\mathsf{F}_1 = \frac{1}{N}\mathbf{Q}_{-1}(z_1)\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_2)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{x}_1\mathbf{x}_1^\mathsf{T},$$

$$\mathsf{F}_2 = \frac{1}{N}\mathbf{Q}_{-1}(z_1)\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_2)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\frac{(\mathsf{b}_1 - \tilde{\mathsf{b}})}{1+\mathsf{b}_1}.$$

We first consider $\mathbb{E}\mathsf{F}_2$. Let $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ denote a pair of unit vectors satisfying

$$\tilde{\mathbf{u}}, \tilde{\mathbf{v}} = \arg\max_{\|\tilde{\mathbf{u}}\|=\|\tilde{\mathbf{v}}\|=1} |\tilde{\mathbf{u}}^\mathsf{T}\mathbb{E}\mathsf{F}_2\tilde{\mathbf{v}}|,$$

and let $\mathbf{y} = \mathbf{Q}_{-1}(z_1)\tilde{\mathbf{u}} = (y_1, ..., y_M)^\mathsf{T}$. Using the Burkholder's inequality (Burkholder, 1973), we have

$$\begin{aligned}
\mathbb{E}|\mathbf{y}^\mathsf{T}\mathbf{x}_1|^4 = \mathbb{E}\left|\sum_{i=1}^{M} y_i x_{i1}\right|^4 &\le c\mathbb{E}\left|\sum_{i=1}^{M} y_i^2\right|^2 + c\mathbb{E}\sum_{i=1}^{M}|y_i x_{i1}|^4 \\
&\le C\mathbb{E}\|\mathbf{y}\|^4 + C\mathbb{E}\sum_{i=1}^{M} y_i^4 \lesssim 1,
\end{aligned}$$

where we use the inequality

$$\sum_{i=1}^{M} y_i^4 \le \left(\sum_{i=1}^{M} y_i^2\right)^2 \le \|\mathbf{y}\|^4.$$

Likewise, we have $\mathbb{E}|\mathbf{x}_1^\mathsf{T}\tilde{\mathbf{v}}|^4 \lesssim 1$. It follows from Lemma 2 that

$$\mathbb{E}|\mathsf{b}_1 - \tilde{\mathsf{b}}|^\ell \le \frac{c}{N^\ell}[(\mathrm{Tr}\mathbf{Q}_{-1}^2(z_1))^{\ell/2} + \mathrm{Tr}\mathbf{Q}_{-1}^\ell(z_1)] \le \frac{C}{N^{\ell/2}},$$

and

$$\mathbb{E}|\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_1)\mathbf{x}_1|^\ell \le C\mathbb{E}|\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_1)\mathbf{x}_1 - \mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_1)|^\ell + C|\mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_1)|^\ell \lesssim N^\ell.$$

Since $\mathsf{b}_k > 1$, we can bound the spectral norm of $\mathbb{E}\mathsf{F}_2$ as

$$\begin{aligned}\|\mathbb{E}\mathsf{F}_2\| &= |\tilde{\mathbf{u}}^\mathsf{T}\mathbb{E}\mathsf{F}_2\tilde{\mathbf{v}}| \le \mathbb{E}|\tilde{\mathbf{u}}^\mathsf{T}\mathsf{F}_2\tilde{\mathbf{v}}| \\ &\le \frac{1}{N}\mathbb{E}|\tilde{\mathbf{u}}^\mathsf{T}\mathbf{Q}_{-1}(z_1)\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\tilde{\mathbf{v}}||\mathsf{b}_1-\tilde{\mathsf{b}}| \\ &\le \frac{1}{N}\sqrt{\mathbb{E}|\mathbf{y}^\mathsf{T}\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\tilde{\mathbf{v}}|^2\mathbb{E}|\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{x}_1(\mathsf{b}_1-\tilde{\mathsf{b}})|^2} \\ &\le \frac{1}{N}\sqrt{\sqrt{\mathbb{E}|\mathbf{y}^\mathsf{T}\mathbf{x}_1|^4\mathbb{E}|\mathbf{x}_1^\mathsf{T}\tilde{\mathbf{v}}|^4}\sqrt{\mathbb{E}|\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{x}_1|^4\mathbb{E}|\mathsf{b}_1-\tilde{\mathsf{b}}|^4}} \\ &\le C\frac{1}{N}o(N) = o(1).\end{aligned}$$

Therefore, it suffices to find the deterministic equivalent of $\mathbb{E}\mathsf{F}_1$. We recall the definition above equation 31 that $\mathbf{A}_M = o(1)$ if $\|\mathbf{A}_M\| = o(1)$. Let $\mathbb{E}_{-1}(\cdot) = \mathbb{E}[\cdot|\mathbf{x}_2, ..., \mathbf{x}_N]$. We have

$$\begin{aligned}\mathbb{E}\mathsf{F}_1 &= \frac{1}{N}\mathbb{E}\mathbf{Q}_{-1}(z_1)\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{x}_1\mathbf{x}_1^\mathsf{T} \\ &= \frac{1}{N}\mathbb{E}\mathbf{Q}_{-1}(z_1)\big[\mathbb{E}_{-1}\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{x}_1\mathbf{x}_1^\mathsf{T}\big] \\ &= \frac{1}{N}\mathbb{E}\mathbf{Q}_{-1}(z_1)\mathbf{\Sigma}^{1/2}\mathbb{E}_{-1}[\mathbf{z}_1\mathbf{z}_1^\mathsf{T}\mathbf{\Sigma}^{1/2}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{\Sigma}^{1/2}\mathbf{z}_1\mathbf{z}_1^\mathsf{T}]\mathbf{\Sigma}^{1/2} \\ &= \frac{1}{N}\mathbb{E}\mathbf{Q}_{-1}(z_1)\big[\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\big]\mathbf{\Sigma} \\ &+ \frac{1}{N}\mathbb{E}\mathbf{Q}_{-1}(z_1)\mathbf{\Sigma}\big[\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)+\mathbf{Q}_{-1}(z_2)\mathbf{A}\mathbf{Q}_{-1}(z_1)\big]\mathbf{\Sigma} \\ &+ \frac{1}{N}(\mathbb{E}z_{11}^4-3)\mathbb{E}\mathbf{Q}_{-1}(z_1)\mathbf{\Sigma}^{1/2}\mathrm{diag}(\mathbf{\Sigma}^{1/2}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{\Sigma}^{1/2})\mathbf{\Sigma}^{1/2} \\ &= \frac{1}{N}\mathbb{E}\big[\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\big]\mathbf{Q}_{-1}(z_1)\mathbf{\Sigma}+o(1) \\ &= \frac{1}{N}\left[\mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\right]\mathbf{\Pi}(z_1)\mathbf{\Sigma}+o(1),\end{aligned} \tag{43}$$

where the last identity is due to equation 34, equation 35 and

$$\begin{aligned}&\frac{1}{N}\mathbb{E}\big[\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\big]\mathbf{Q}_{-1}(z_1)\mathbf{\Sigma} \\ =&\frac{1}{N}\mathbb{E}\big[\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)-\mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\big]\mathbf{Q}_{-1}(z_1)\mathbf{\Sigma} \\ &+\frac{1}{N}[\mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)]\mathbf{Q}_{-1}(z_1)\mathbf{\Sigma} \\ =&\frac{1}{N}\mathbb{E}[\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)]\mathbf{\Pi}(z_1)\mathbf{\Sigma}+o(1).\end{aligned}$$

By equation 36, equation 38, equation 42 and equation 43 and the fact that $\|\mathbf{\Pi}(z_2)\|$ is bounded, we have

$$\begin{aligned}&\frac{1}{N}\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{X}\mathbf{X}^\mathsf{T}\mathbf{\Pi}(z_2) \\ =&\frac{1}{(1+\tilde{b})}\mathbb{E}\mathbf{F}_1\mathbf{\Pi}(z_2)+o(1) \\ =&\frac{1}{1+\tilde{b}}\left[\mathbb{E}\mathbf{Q}_{-1}(z_1)\mathbf{A}\mathbf{Q}_{-1}(z_2)\mathbf{\Sigma}\mathbf{\Pi}(z_2)-\frac{1}{1+\tilde{\mathsf{b}}}\mathbb{E}\mathsf{F}_1\mathbf{\Pi}(z_2)\right]+o(1) \\ =&\frac{\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)\mathbf{\Sigma}\mathbf{\Pi}(z_2)}{1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}(z_2)}-\frac{\frac{1}{N}[\mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)]\mathbf{\Pi}(z_1)\mathbf{\Sigma}\mathbf{\Pi}(z_2)}{(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}(z_2))(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}(z_1))}+o(1).\end{aligned}$$

This, along with equation 31, equation 37, leads to

$$\begin{aligned}&\mathbb{E}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2) \\ =&\mathbf{\Pi}(z_1)\mathbf{A}\mathbf{\Pi}(z_2)+\frac{\frac{1}{N}[\mathbb{E}\mathrm{Tr}\mathbf{\Sigma}\mathbf{Q}(z_1)\mathbf{A}\mathbf{Q}(z_2)]\mathbf{\Pi}(z_1)\mathbf{\Sigma}\mathbf{\Pi}(z_2)}{(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}(z_2))(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma}\mathbf{\Pi}(z_1))}+o(1).\end{aligned} \tag{44}$$

Multiplying both sides of the above equation on the left by $\mathbf{\Sigma}$, and taking the trace, we obtain

$$\begin{aligned}&\frac{1}{N}\mathbb{E}\mathrm{Tr}\mathbf{\Sigma Q}(z_1)\mathbf{AQ}(z_2)\\=&\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1)\mathbf{A\Pi}(z_2)+\frac{\frac{1}{N}[\mathbb{E}\mathrm{Tr}\mathbf{\Sigma Q}(z_1)\mathbf{AQ}(z_2)]\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1)\mathbf{\Sigma\Pi}(z_2)}{(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_2))(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1))}+o(1).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{N}\mathbb{E}\mathrm{Tr}\mathbf{\Sigma Q}(z_1)\mathbf{AQ}(z_2)\\=&\left(1-\frac{\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1)\mathbf{\Sigma\Pi}(z_2)}{(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_2))(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1))}\right)^{-1}\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1)\mathbf{A\Pi}(z_2)+o(1).\end{aligned}\tag{45}$$

Plugging equation 45 into equation 44, we get

$$\begin{aligned}&\mathbb{E}\mathbf{Q}(z_1)\mathbf{AQ}(z_2)=\mathbf{\Pi}(z_1)\mathbf{A\Pi}(z_2)\\&+\frac{\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1)\mathbf{A\Pi}(z_2)}{\big(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_2)\big)\big(1+\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1)\big)-\frac{1}{N}\mathrm{Tr}\mathbf{\Sigma\Pi}(z_1)\mathbf{\Sigma\Pi}(z_2)}\mathbf{\Pi}(z_1)\mathbf{\Sigma\Pi}(z_2)+o(1).\end{aligned}\tag{46}$$

The result equation 16 follows by combining the equation 46 with equation 30. Now we prove equation 18. Using a proof analogous to that of equation 30, we can obtain that

$$\frac{1}{M}\mathrm{Tr}\mathbf{C}\big[\mathbf{Q}(z_1)\mathbf{AQ}_2(z_2)-\mathbb{E}\mathbf{Q}(z_1)\mathbf{AQ}(z_2)\big]=o_{a.s.}(1).\tag{47}$$

We denote the spectral decomposition of $\mathbf{C}$ by

$$\mathbf{C}=\sum_{i=1}^{M}\lambda_i\mathbf{u}_i\mathbf{v}_i^{\mathsf{T}}.$$

By equation 46, we have

$$\begin{aligned}\frac{1}{M}\mathrm{Tr}\mathbf{C}\mathbb{E}\mathbf{Q}(z_1)\mathbf{AQ}(z_2)&=\frac{1}{M}\mathrm{Tr}\sum_{i=1}^{M}\lambda_i\mathbf{u}_i\mathbf{v}_i^{\mathsf{T}}\mathbb{E}\mathbf{Q}(z_1)\mathbf{AQ}(z_2)\\&=\frac{1}{M}\sum_{i=1}^{M}\lambda_i\mathbf{v}_i^{\mathsf{T}}\mathbb{E}\mathbf{Q}(z_1)\mathbf{AQ}(z_2)\mathbf{u}_i\\&=\frac{1}{M}\sum_{i=1}^{M}\lambda_i\mathbf{v}_i^{\mathsf{T}}\mathbf{\Pi}(z_1)\mathcal{S}(\mathbf{A})\mathbf{\Pi}(z_2)\mathbf{u}_i+o(1)\\&=\frac{1}{M}\mathrm{Tr}\mathbf{C\Pi}(z_1)\mathcal{S}(\mathbf{A})\mathbf{\Pi}(z_2)+o(1).\end{aligned}$$

This, along with equation 47, establishes equation 18. The proof is completed.

C Additional Experimental Details

C.1 Demonstration of Theorem 3

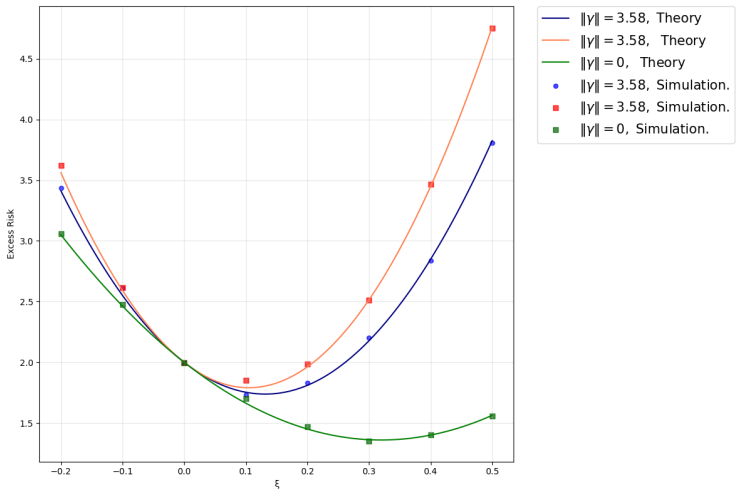


Figure 4: Theoretical predictions (solid curves) versus simulation results (scatter points, averaged over 100 independent trials) for ridgeless regression. We set $(M, N_1, N_2) = (400, 600, 600)$, $\boldsymbol{\beta}_2 = \frac{4}{\sqrt{M}}(1, ..., 1)^\mathsf{T}$, $\sigma^2 = 1$ and $\boldsymbol{\Sigma}_2 = \mathbf{I}_M$. We label the case $\|\boldsymbol{\gamma}\| = 3.58$ as $\boldsymbol{\gamma} = \frac{-8}{\sqrt{M}}(1, ..., 1, 0, ..., 0)^\mathsf{T}$ with the first $M/5$ entries equal to 1. The orange and green curves correspond to the setting where $\boldsymbol{\Sigma}_1 = \mathrm{diag}(4, ..., 4, \frac{1}{4}, ..., \frac{1}{4})$, with the first half of the diagonal entries equal to 4 and the second half equal to $\frac{1}{4}$. The dark blue curve corresponds to the setting where $\boldsymbol{\Sigma}_1 = 4\mathbf{I}_M$.

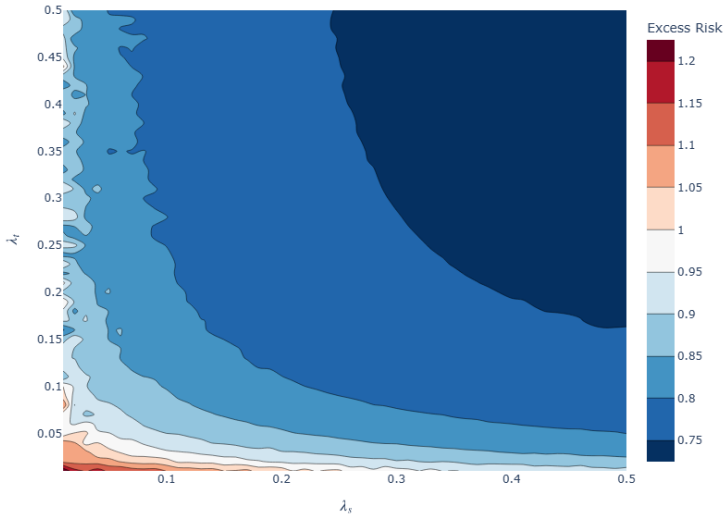
Figure 4 presents empirical results that support 3. The gap between the orange and green curves quantifies the impact of model shift on the excess risk. Furthermore, the gap between the dark blue and orange curves reflects the role of the term $\mathrm{Tr}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}_2$ as characterized in Theorem 3.

C.2 Impact of regularization parameters

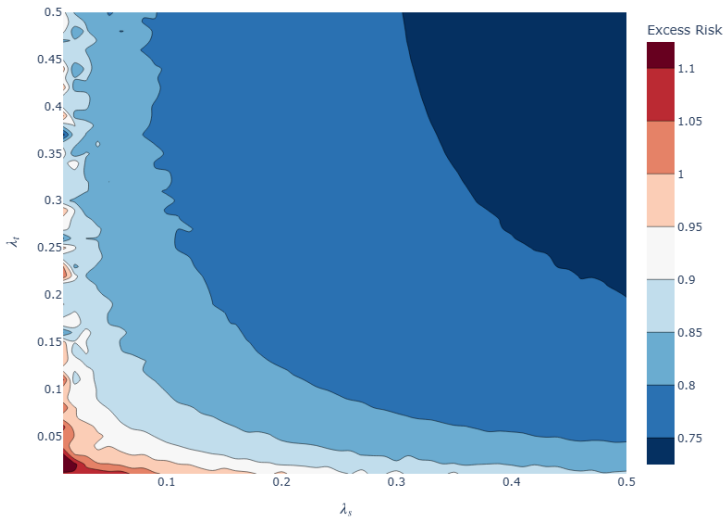
To examine the impact of the regularization parameters $\lambda_\mathsf{t}, \lambda_\mathsf{s}$, we plot the empirical excess risk of the student model for $(\lambda_\mathsf{t}, \lambda_\mathsf{s}) \in [0.01, 0.5]^2$ in Figures 5-7 (averaged over 5 trials), corresponding to $\xi = 0.5, -0.5$ and 1.5, respectively. We set $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 \sim \mathcal{N}(0, \frac{1}{M}\mathbf{I}_M)$, $(M, N_1, N_2) = (400, 300, 200)$, $\sigma^2 = 1$. We set $\boldsymbol{\Sigma}_2 = \mathbf{I}_M$ in the absence of covariate shift. Under covariate shift, we set $\boldsymbol{\Sigma}_1 = \mathrm{diag}(d_1, ..., d_M)$, where

$$d_i = 0.64\delta(i \leq M/2) + 0.25\delta(M/2 < i \leq M).$$

From these figures, we observe that when $\xi > 1$, the influence of λ_t becomes large. In contrast, in the case $\xi = -0.5$, λ_s almost dominates the variation of the excess risk, reflecting a weaker impact of the teacher's guidance (anti-learning against the teacher's supervision).

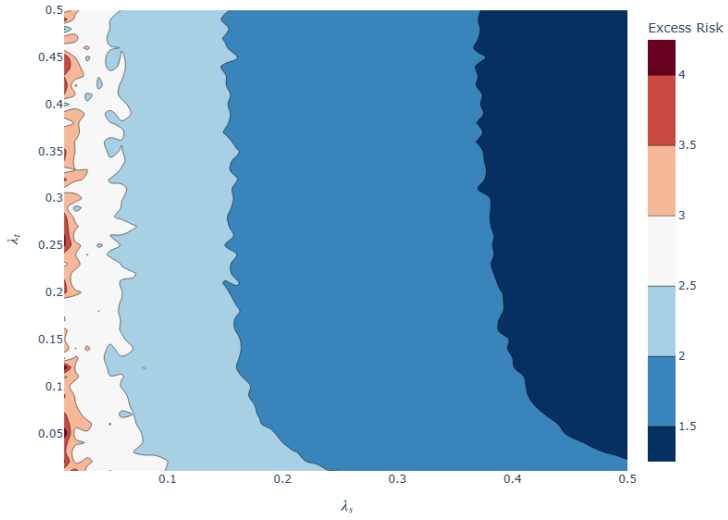


(a) With covariate shift

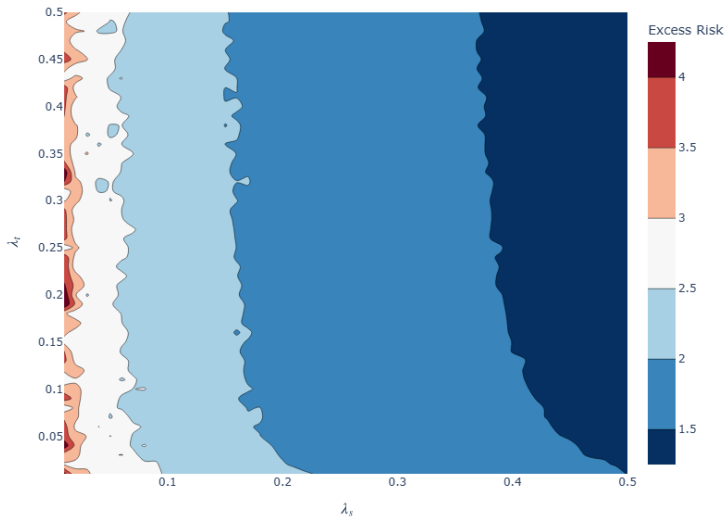


(b) Without covariate shift

Figure 5: Excess risk when $\xi = 0.5$.

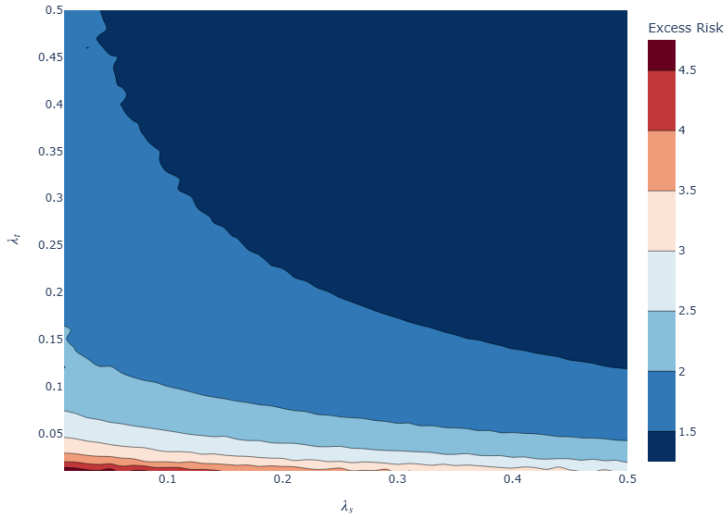


(a) With covariate shift

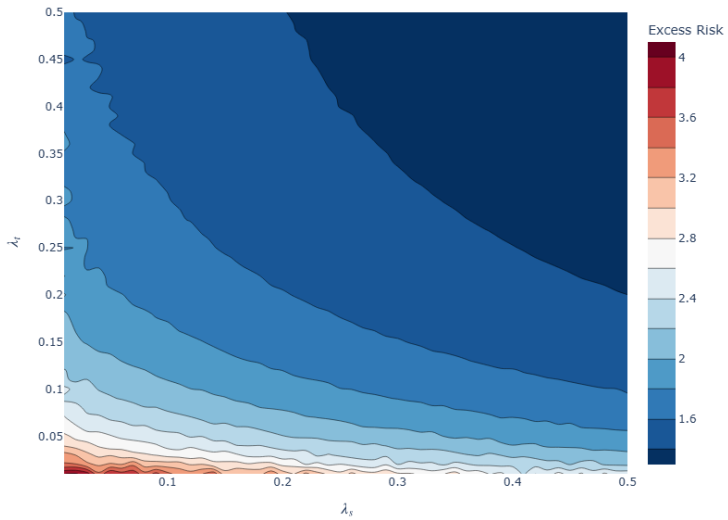


(b) Without covariate shift

Figure 6: Excess risk when $\xi = -0.5$.



(a) With covariate shift



(b) Without covariate shift

Figure 7: Excess risk when $\xi = 1.5$.