Hyperbolic Fine-Tuning for Large Language Models

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Abstract

Large language models (LLMs) have demonstrated remarkable performance on various tasks. However, it remains an open question whether the default Euclidean space is the most suitable choice for embedding tokens in LLMs. In this study, we investigate the non-Euclidean characteristics of LLMs. Our findings reveal that token frequency follows a power-law distribution, with high-frequency tokens clustering near the origin and low-frequency tokens positioned farther away. Additionally, token embeddings exhibit a high degree of hyperbolicity, indicating a latent tree-like structure in the embedding space. Motivated by these observations, we propose to efficiently fine-tune LLMs in hyperbolic space to better exploit the underlying complex structures. However, we find that this hyperbolic fine-tuning cannot be achieved through the naive application of exponential and logarithmic maps when the embedding and weight matrices both reside in Euclidean space. To address this technical issue, we introduce hyperbolic low-rank efficient fine-tuning, HypLoRA, which performs low-rank adaptation directly on the hyperbolic manifold, preventing the cancellation effect produced by consecutive exponential and logarithmic maps and thereby preserving hyperbolic modeling capabilities. Extensive experiments across various base models and two different reasoning benchmarks, specifically arithmetic and commonsense reasoning tasks, demonstrate that HypLoRA substantially improves LLM performance.

1 Introduction

Large language models (LLMs) such as GPT-4 [1], LLaMA [2], Gemma [3], and Qwen [4] have demonstrated remarkable capabilities in understanding and generating human-like text [5, 6, 7]. Despite their impressive capabilities, these models often rely on Euclidean geometry for token representation, which may inadequately capture the inherently complex and hierarchical nature of real-world data structures [8, 9, 10, 11]. Consider how words naturally organize into nested categories with varying levels of abstraction: abstract concepts like "fruit" occupy higher positions in the semantic hierarchy, while specific instances such as "apple" or "banana" populate the lower levels. Representing such structures effectively is crucial for understanding the semantics of language in LLMs.

Recent advancements suggest that non-Euclidean geometries, particularly hyperbolic spaces [11, 12, 13, 14, 15, 16, 17, 18, 19, 20], offer promising alternatives for modeling hierarchical data. Hyperbolic space, distinguished by its negative curvature, is especially well-suited for representing tree-like

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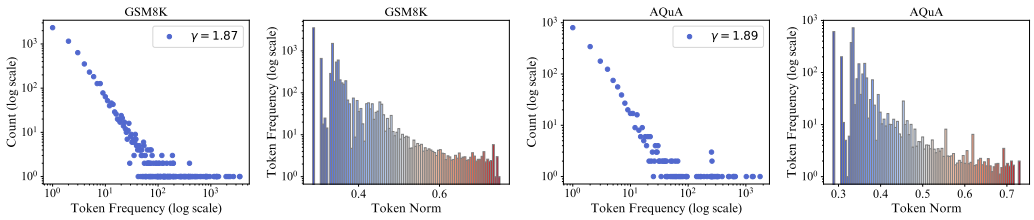


Figure 1: Token frequency distribution and token frequency vs. norm analysis for GSM8K and AQuA datasets in LLaMA3-8B. The left panels show the token frequency distributions (power-law distribution), while the right panels illustrate the relationship between token frequency and the corresponding norms. This visualization reveals the underlying geometric structure of the token embeddings. For additional data analysis and visualizations, please refer to Appendix A.

hierarchical data due to its exponential volume growth and geometric prior. This geometric property makes hyperbolic space especially useful for tasks involving complex, hierarchically structured information.

Proposed Analysis Framework. In this work, we first delve into how LLMs interact with token embeddings and explore the extent to which these embeddings exhibit non-Euclidean characteristics. We approach this from both a global and local perspective. At the `global` level, we analyze the overall distribution of tokens by frequency and investigate how these frequencies are arranged across the embedding space. At the `local` level, we measure the hyperbolicity of the metric space spanned by each input prompt, where the hyperbolicity of the Euclidean embedding serves as a proxy for evaluating the distance or dissimilarity between the underlying embedding structure and a tree-like hierarchy [21, 22, 14].

Our analysis in Section 4 reveals several key insights. `First`, token frequency follows a power-law distribution, which generally implies an underlying hierarchical structure similar to a branching tree [23, 24, 25, 26]. High-frequency tokens (e.g., abstract concepts, function words) tend to be located near the origin of the embedding space, while low-frequency tokens (e.g., specific terms) are farther away, as demonstrated in Table 1. `Second`, our investigation of hyperbolicity (δ values) in Table 2 demonstrates that LLM token embeddings exhibit significant tree-like properties.

Based on our findings above, a natural consideration is to develop hyperbolic LLMs that explicitly incorporate a hyperbolic inductive bias[2]. However, training LLMs from scratch can be resource-intensive [27, 28, 29]. As a more resource-efficient alternative, we propose to build the first low-rank adaptation fine-tuning method in hyperbolic space. This approach is particularly advantageous given that existing LLMs are all Euclidean, and not all downstream tasks require hyperbolic geometry in their fine-tuning. By employing hyperbolic adapters for specific tasks on Euclidean LLMs, we can leverage the benefits of both geometries while maintaining computational efficiency.

Challenges. Adapting LLMs in non-Euclidean embedding spaces with classic techniques, *i.e.* applying exponential and logarithmic maps within tangent space [30, 31, 32, 33, 34] for weight adaptation is problematic in this case. This approach fails to fully capture the hyperbolic geometry, as the exponential and logarithmic maps are mutually inverse and can be canceled with consecutive operations[3]. Consequently, the inherent properties of the hyperbolic space are not effectively preserved, limiting the potential benefits of incorporating non-Euclidean geometries into the adaptation process.

Proposed Method. To address this limitation, we introduce HypLoRA to perform low-rank adaptation directly on the hyperbolic manifold without transformation to the tangent space, thus preserving hyperbolic modeling capabilities and counteracting the cancellation effect. HypLoRA integrates hyperbolic geometry into existing LLMs, implicitly introducing high-order interaction and accounting

[2]The connection between power-law distribution and hyperbolic geometry is elaborated in Section 4.3.

[3]The cancellation effect occurs because standard hyperbolic neural network [31, 30] approaches apply transformations in the tangent space at the origin, requiring the sequence: Euclidean embedding $\rightarrow$ exponential map to hyperbolic space $\rightarrow$ logarithmic map to tangent space $\rightarrow$ linear transformation $\rightarrow$ exponential map back to hyperbolic space $\rightarrow$ projection to Euclidean space. When these operations are chained together, the maps are mutually inverse and effectively cancel out, reducing the entire sequence to approximately the original Euclidean transformation BAx without preserving the beneficial hyperbolic geometry.

for the token hierarchies, enabling them to benefit from hyperbolic characteristics while minimizing additional computational costs.

To summarize, our main contributions are twofold: (1) We conduct a comprehensive investigation into the hyperbolic characteristics of token embeddings in LLMs, revealing their inherent tree-like structure and strong hyperbolic properties. (2) We propose HypLoRA, a parameter-efficient fine-tuning method that integrates hyperbolic geometry into LLMs while preserving hyperbolic modeling capabilities. We conduct extensive experiments on various models and two different tasks, specifically arithmetic reasoning and commonsense reasoning, demonstrating clear advantages over competitive baselines.

2 Related Work

Hyperbolic Representation Learning and Foundation Models. Hyperbolic geometry has been successfully applied to various neural network architectures and models [17, 19, 16], including shallow hyperbolic neural networks [13, 31, 35, 36, 37, 38], hyperbolic CNNs [39, 40, 41], hyperbolic GNNs [30, 42, 43], and hyperbolic attention networks or Transformers [44, 35, 36, 45]. These models leverage the inductive biases of hyperbolic geometry to achieve remarkable performance on various tasks and applications [30, 20, 46, 14, 15, 47, 48, 49, 50]. Recent efforts have focused on adapting LLMs and CLIP [51] to hyperbolic spaces. Key advancements include developing more expressive hyperbolic image-text representations [52], enabling compositional entailment learning for deeper vision-language understanding [53], designing safety-aware hyperbolic frameworks for content moderation [54], and creating core modules to facilitate the construction of novel hyperbolic foundation models [55]. However, training LLMs from scratch remains computationally expensive [56, 57]. The computational complexity increases further when considering Riemannian optimization [56, 57, 58] and additional hyperbolic operations, like Möbius addition.

Geometric Analysis of Language Model Embeddings. Prior work has made important observations about the geometry of embeddings that helped motivate our research. Reif et al. [59] demonstrated that BERT embeddings contain distinct syntactic and semantic subspaces and showed evidence of tree-like parse structures, while Gao et al. [60] revealed that token embeddings tend to cluster in a narrow cone during training, leading to representation degeneration. Building on these geometric insights, Rudman et al. [61] introduced IsoScore to formally quantify how uniformly embeddings utilize the ambient vector space. Additionally, Puccetti et al. [62] analyzed outlier dimensions in Transformers and showed their correlation with token frequencies. While these works provide crucial foundations for understanding embedding geometry, our work differs in that we specifically quantify and leverage the natural hyperbolicity of token embeddings.

Parameter-Efficient Fine-Tuning (PEFT) and LoRA. Fine-tuning LLMs [63, 64, 2] for downstream tasks poses significant challenges due to their massive number of parameters. To address this issue, PEFT methods have been proposed, which aim to train a small subset of parameters while achieving better performance compared to full fine-tuning. PEFT methods can be broadly categorized into prompt-based methods [65, 66, 67], adapter-based methods [68, 69], and reparameterization-based methods [29, 70, 71]. Among these, the reparameterization-based LoRA [29] has gained significant attention due to its simplicity, effectiveness, and compatibility with existing model architectures. Variants of LoRA, such as LoRA+[72], DoRA [73], and AdaLoRA [74], have been proposed to improve its performance and efficiency. Recent research has also investigated ensembles of multiple LoRAs [75, 76] and quantization techniques [77, 78, 79]. The proposed method is a foundational algorithm that is orthogonal to existing approaches and can potentially be combined with various LoRA variants to exploit their complementary strengths and achieve superior performance.

3 Preliminary

This section introduces the concepts used in our study, including the Lorentz model of hyperbolic geometry and the LoRA adapter.

Hyperbolic Geometry. Unlike the flat Euclidean geometry, hyperbolic geometry is characterized by a constant negative curvature. We utilize the Lorentz model, also known as the hyperboloid model, for our study due to its ability to effectively capture hierarchical structures and maintain numerical stability [12, 35, 80]. The Lorentz model in n dimensions with curvature $-1/K(K > 0)$ is defined

as:

$$\mathcal{L}_K^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{x} \rangle_{\mathcal{L}} = -K, x_0 > 0\}, \tag{1}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is the Lorentzian inner product, given by: $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{L}} = -x_0 y_0 + \sum_{i=1}^{n} \mathbf{x}_i \mathbf{y}_i$.

Tangent Space. In the Lorentz model $\mathcal{L}_K^n$, the tangent space at a point $\mathbf{x}$ is denoted as $\mathcal{T}_{\mathbf{x}}\mathcal{L}_K^n$. It is defined as the set of all vectors $\mathbf{u}$ that are orthogonal to $\mathbf{x}$ under the Lorentzian inner product:

$$\mathcal{T}_{\mathbf{x}}\mathcal{L}_K^n := \{\mathbf{u} \in \mathbb{R}^{n+1} : \langle \mathbf{u}, \mathbf{x} \rangle_{\mathcal{L}} = 0\}. \tag{2}$$

To facilitate projection between the hyperboloid and its tangent spaces, one can utilize two critical mappings: the exponential and logarithmic maps. The *exponential map* at $\mathbf{x}$, denoted $\exp_{\mathbf{x}}^K$, projects a vector from the tangent space $\mathcal{T}_{\mathbf{x}}\mathcal{L}_K^n$ back onto the hyperboloid. Conversely, the *logarithmic map*, denoted $\log_{\mathbf{x}}^K$, maps a point on the hyperboloid to the tangent space at $\mathbf{x}$. The relevant formulas are given in Appendix D.1.

LoRA Adapter. The LoRA adapter provides an efficient approach for modifying large LLMs with minimal computational overhead. Instead of retraining the entire model, LoRA focuses on adjusting specific components within the model's architecture to transform an input $\mathbf{x}$ into an output $\mathbf{z}$. In practice, LoRA targets the weight matrices found in each Transformer layer of an LLM. Typically, the weight W of the Transformer, which resides in the dimensions $\mathbb{R}^{d \times k}$, is adapted through a low-rank approximation. This is achieved by introducing an additional term, ΔW, to the original weight matrix:

$$\mathbf{z} = W_{\text{LoRA}}(\mathbf{x}) = W\mathbf{x} + \Delta W\mathbf{x} = W\mathbf{x} + BA\mathbf{x}. \tag{3}$$

Here, $B \in \mathbb{R}^{d \times r}$ and $A \in \mathbb{R}^{r \times k}$ represent two smaller, learnable matrices where r—the rank of these matrices—is significantly less than either d or k. This design choice ensures that $r \ll \min(d, k)$, thereby reducing the complexity of the model adaptation. During the fine-tuning process, only the matrices A and B are adjusted, while the pre-existing weights W are kept frozen. This method significantly decreases the number of parameters that need to be trained, from dk to $(d + k)r$, enhancing the efficiency of the fine-tuning process. As a result, LoRA enables the targeted adaptation of LLMs, allowing them to transform an input $\mathbf{x}$ into an output $\mathbf{z}$ while maintaining high performance and adapting to new tasks or datasets with a fraction of the computational resources typically required.

4 Investigation

In this section, we present an in-depth investigation of token embeddings in LLMs from both global and local perspectives. Our goal is to uncover the geometric structures underlying pretrained token representations, specifically examining the global distribution of token frequencies and their spatial arrangement, as well as the local hyperbolicity of token embeddings across various datasets.

4.1 Global Token Statistics

We begin by investigating the global distribution of token frequencies in the context of arithmetic reasoning datasets, focusing on datasets such as GSM8K [81], AQuA [82], MAWPS [83], and SVAMP [84]. We also provide a broader analysis across different types of datasets and LLMs in Appendix A. Figure 1 (left) presents the distribution of token frequencies, with a power-law exponent of approximately $\gamma \approx 1.9$, as estimated by the `powerlaw` package [85]. In such distributions, the exponent γ controls how quickly token frequencies decline: smaller values of γ (closer to 1) indicate a more gradual decay where frequent tokens dominate, while larger values signify a sharper decline, with most tokens being rare.

This power-law behavior aligns with the tree-like hierarchical nature of language [11, 14, 86, 49, 20]. High-frequency tokens often correspond to more abstract or general concepts, while low-frequency tokens represent specific or rare terms. This distribution naturally suggests a hierarchical organization of the token space, where general concepts serve as the "roots" and specific terms "branch out" as we move through the hierarchy.

Empirical Observation. To better understand the relationship between token frequency and their spatial arrangement within the embedding space, we calculate the average token frequency as a function of their distance from the origin. The results are shown in Figure 1 (right), indicating that more frequent tokens tend to have smaller norms and vice versa. Table 1 provides representative

Table 1: Mean, Minimum, and Maximum Frequency and Norm Values for Different Models and Groups. Group 1: *to, in, have, that, and, is, for*, Group 2: *how, much, many, time, cost*, Group 3: *animal, fruit, number, color, size*, Group 4: *dog, cow, apple, banana, 380, 480, purple, red, medium, small, large*.

Model	Group	Frequency (Mean [Min∼Max])	Norm (Mean [Min∼Max])
Gemma-7B	Group 1	4934.4 [1838 ∼ 8539]	3.160 [3.060 ∼ 3.299]
	Group 2	2709.4 [474 ∼ 6681]	3.561 [3.488 ∼ 3.627]
	Group 3	292.0 [34 ∼ 1191]	3.765 [3.623 ∼ 3.887]
	Group 4	114.3 [25 ∼ 284]	3.998 [3.660 ∼ 4.520]
LLaMA-7B	Group 1	4993.9 [1838 ∼ 8547]	0.951 [0.793 ∼ 1.060]
	Group 2	2712.6 [474 ∼ 6683]	1.222 [1.118 ∼ 1.299]
	Group 3	299.8 [34 ∼ 1200]	1.325 [1.274 ∼ 1.428]
	Group 4	139.1 [26 ∼ 286]	1.364 [1.326 ∼ 1.417]
LLaMA3-8B	Group 1	4937.4 [1838 ∼ 8547]	0.353 [0.330 ∼ 0.396]
	Group 2	2710.0 [474 ∼ 6683]	0.456 [0.394 ∼ 0.499]
	Group 3	292.6 [34 ∼ 1191]	0.499 [0.452 ∼ 0.549]
	Group 4	97.1 [13 ∼ 284]	0.569 [0.499 ∼ 0.675]
LLaMA-13B	Group 1	4993.9 [1838 ∼ 8547]	1.027 [0.833 ∼ 1.255]
	Group 2	2712.6 [474 ∼ 6683]	1.429 [1.346 ∼ 1.489]
	Group 3	299.8 [34 ∼ 1200]	1.494 [1.453 ∼ 1.532]
	Group 4	139.1 [26 ∼ 286]	1.501 [1.470 ∼ 1.526]

tokens with different norm ranges within the embedding space of different base models. The results presented in Table 1 demonstrate several critical findings. *First*, we observe a statistically significant separation between functional/abstract words (Group 1) and specific terms (Group 4) across all models, with Group 1 consistently exhibiting the smallest embedding norms and highest frequencies, while Group 4 shows the largest norms and lowest frequencies. *Second*, the relative ordering of groups remains consistent across all examined models, with Group 1 $<$ Group 2 $<$ Group 3 $<$ Group 4 in terms of embedding norms, despite absolute magnitude variations. Most notably, even across different architectural families (LLaMA vs. Gemma), the hierarchical organization principle remains preserved, though with different absolute scales, where Gemma-7B exhibits systematically larger embedding norms (mean Group 1 norm: 3.160) compared to LLaMA models (mean Group 1 norm: $0.353 \sim 1.027$), yet maintains the same relative hierarchical structure.

Conclusion. These findings suggest that the spatial organization of token embeddings reflects the inherent hierarchical relationships in language, supporting the hypothesis that token embedding in LLMs exhibits a tree-like structure, with spatial positioning aligned with token frequency and specificity. It is worth noting, however, that a power-law distribution of token frequency alone does not guarantee the emergence of a hierarchical token embedding, as it also depends on the training objectives. Our analysis demonstrates that the hierarchy is strongly correlated with token frequencies, which can be understood through the lens of LLMs tokenization and co-occurrence pattern learning during training. While the exact mechanisms underlying this relationship require further investigation in future work, the spatial distribution of token embeddings remains crucial as it provides the primary motivation for our methodological approach.

4.2 δ-Hyperbolicity of Local Token Embeddings

To rigorously quantify the hierarchical nature of token embeddings, we examine the δ-hyperbolicity of the space spanned by the token embedding. δ-Hyperbolicity, introduced by Gromov [87], is a measure that captures the degree to which a metric space deviates from an exact tree structure. Lower values of δ imply a space more similar to a perfect tree, while higher values indicate deviation from a tree-like structure. A brief explanation of δ-hyperbolicity can be found on Wikipedia[4].

We compute δ-hyperbolicity using the four-point condition, which compares the Gromov products between any four points a, b, c, and w in the metric space. Specifically, the hyperbolicity is defined as:

$$[a, c]_w \geq \min([a, b]_w, [b, c]_w) - \delta, \tag{4}$$

[4] https://en.wikipedia.org/wiki/Hyperbolic_metric_space

Table 2: Comparison of δ-Hyperbolicity across various metric spaces and datasets. The left table provides reference values for baseline metric spaces, allowing for a clearer interpretation of hyperbolicity in the analyzed datasets in the right table.

Metric Space	**Hyperbolicity(δ)**
Sphere Space	0.99 ± 0.01
Random Graph	0.62 ± 0.34
PubMed Graph	0.40 ± 0.45
Scale-free Graph	0.00
Tree Graph	0.00

Hyperbolicity(δ)	MAWPS	SVAMP	GSM8K	AQuA
LLaMA-7B	0.08 ± 0.02	0.09 ± 0.01	0.10 ± 0.01	0.10 ± 0.01
LLaMA-13B	0.08 ± 0.01	0.09 ± 0.01	0.09 ± 0.01	0.10 ± 0.01
Gemma-7B	0.11 ± 0.01	0.11 ± 0.01	0.11 ± 0.01	0.12 ± 0.01
LLaMA3-8B	0.06 ± 0.01	0.07 ± 0.01	0.07 ± 0.01	0.08 ± 0.01
Average	0.08 ± 0.01	0.09 ± 0.01	0.09 ± 0.01	0.10 ± 0.01

where the Gromov product $[a, b]_w$ is:

$$[a, b]_w = \frac{1}{2}(d(a, w) + d(b, w) - d(a, b)). \tag{5}$$

Quantitative Analysis. To measure the hyperbolicity of token embeddings, we apply this algorithm to various open-source LLMs. Following the methodologies proposed by Khrulkov et al. [14] and Cetin et al. [15], we estimate δ-hyperbolicity using the efficient algorithm introduced by Fournier et al. [88]. To ensure scale invariance, we normalize δ by the diameter of the embedding space, $\mathrm{diam}(X)$, yielding a relative measure: $\delta_{rel} = \frac{2\delta}{\mathrm{diam}(X)}$. This relative measure ranges from 0 to 1, with values closer to 0 indicating a highly hyperbolic (tree-like) structure, and values near 1 indicating a non-hyperbolic, flat structure. Following previous works [14], we employ Euclidean distance as a measure of the shortest distance. To further validate the correctness of this approach, we generate a series of random graphs with predefined hyperbolicity, embed them using a graph neural network (GNN), and then compute the hyperbolicity in Euclidean space. Details of this process are provided in Appendix B. Our experiments reveal a positive correlation between the hyperbolicity of the embeddings and the original graphs. Consequently, we utilize this method as a proxy for estimating the hyperbolicity of token embeddings. In our analysis, we calculate hyperbolicity at the prompt level, treating each token within a prompt as a point in the metric space spanned by the embeddings. By averaging the hyperbolicity across all prompts, we assess the overall hyperbolic structure of token embeddings in each dataset.

Conclusion. Our results, as shown in Table 2, reveal that token embeddings exhibit significant hyperbolicity, suggesting that the embedding space has a strong tree-like structure. This observation further corroborates our findings from the global token statistics, where the arrangement of tokens in the embedding space mirrors hierarchical relationships seen in language data. We also provide the hyperbolicity analysis of the final hidden layer in Appendix A.3.

4.3 Connection between Power-law Distribution and Hyperbolic Geometry

The observation of a power-law distribution in token frequencies, as discussed in Section 4, is not merely a statistical curiosity. It has deep connections to the underlying geometry of the data, particularly to hyperbolic spaces, which are well-suited for representing hierarchical structures [11, 23, 89, 90]. For instance, Nickel and Kiela [11] highlighted that the existence of power-law degree distributions can often be traced back to hierarchical structures. Similarly, Ravasz and Barabási [86] established that the scaling law $P(k) \sim k^{-\gamma}$ can signify the co-existence of a hierarchy of nodes with varying degrees of clustering. Krioukov et al. [23] further strengthened this connection by showing that the exponent of the power-law degree distribution is a function of the hyperbolic space curvature. Building on this geometric understanding, Papadopoulos et al. [90] demonstrated that complex (scale-free) network topologies naturally emerge when networks grow within an underlying hyperbolic metric space, and importantly, that the resulting hyperbolic embedding of these dynamic scale-free networks facilitates highly efficient greedy forwarding.

To formalize this connection with hyperbolic geometry, we can consider embedding tokens in a hyperbolic space. A common model for hyperbolic space is the Poincaré disk model ($\mathbb{H}^2$) with curvature $K = -1$. In such a space, both the circumference $C(r)$ and area $A(r)$ of a circle of radius r exhibit exponential growth:

$$C(r) = 2\pi \sinh(r) \sim e^r \quad \text{as } r \to \infty, \tag{6}$$

$$A(r) = 2\pi(\cosh(r) - 1) \sim e^r \quad \text{as } r \to \infty. \tag{7}$$

Table 3: Accuracy comparison of various LLMs on arithmetic reasoning tasks. The percentage following each dataset indicates the proportion of prompts relative to the total number of inference prompts. M.AVG represents the micro-average accuracy. For a full comparison table, please see Appendix F.

Base Model	PEFT Method	# Params (%)	MAWPS(8.5%)	SVAMP(35.6%)	GSM8K(46.9%)	AQuA(9.0%)	M.AVG
GPT-3.5	None	None	87.4	69.9	56.4	38.9	62.3
LLaMA-7B	LoRA	0.83	**81.9**	48.2	38.3	18.5	43.7
	HypLoRA	0.83	79.2	**49.3**	**39.2**	**20.7**	**44.6**
LLaMA-13B	LoRA	0.67	**83.5**	54.7	48.5	18.5	51.0
	HypLoRA	0.67	83.4	**55.8**	**49.7**	**21.5**	**52.2**
Gemma-7B	LoRA	0.79	89.5	74.2	61.4	34.3	65.9
	HypLoRA	0.79	**89.5**	**75.6**	**68.8**	**46.5**	**71.0**
LLaMA3-8B	LoRA	0.70	**91.2**	**81.2**	69.8	42.1	73.2
	HypLoRA	0.70	89.1	80.6	**72.6**	**46.9**	**74.5**
Gemma3-4B	LoRA	1.04	89.0	**80.3**	67.7	43.5	71.8
	HypLoRA	1.04	**90.8**	78.9	**67.8**	**48.4**	**72.0**

If we consider token embeddings in a hyperbolic space with polar coordinates (r, θ), where $r \in \mathbb{R}^+$ is the radial coordinate (correlating with token frequency) and $\theta \in [0, 2\pi]$ is the angular coordinate (encoding semantic similarity), the radial distribution of tokens follows $p(r) \sim e^{-\zeta r}$, where $\zeta > 0$ relates to the hyperbolic curvature K. The frequency function $k(r)$ for tokens at radius r is then given by $k(r) \sim e^{-r}$. Through coordinate transformation, we can derive the power-law frequency distribution:

$$P(k) \sim P(r) \left| \frac{dr}{dk} \right| \sim k^{-\gamma}.$$

The relationship between the hyperbolic curvature and the power-law exponent γ can be given by $\gamma = 2 + \frac{1}{\zeta}$ (as concluded by Krioukov et al. [23] in the context of complex networks, where ζ relates to the effective temperature or network structure). This relationship underscores the theoretical connection between the power-law behavior observed in token frequencies and the inherent hyperbolic geometry of the embedding space.

Hyperbolic space offers distinct advantages for modeling language hierarchies, especially when addressing the structural and spatial constraints of token co-occurrence: **(1) Separation of Low-Frequency Tokens.** Tokens with low frequencies, which typically represent more specific or granular concepts, require clear separation from each other to maintain semantic clarity. **(2) Proximity to High-Frequency Hypernyms.** Simultaneously, these low-frequency tokens should remain close to their corresponding high-frequency hypernyms or function words. Hyperbolic space is uniquely suited for capturing these dual constraints due to its exponential volume growth, which inherently supports hierarchical structure and allows for ample separation of specific entities while keeping them close to their parent categories. This contrasts with Euclidean space, where such arrangements can lead to crowding or distortion of distances.

Overall Conclusion. Through these analyses, we demonstrate that token embeddings in LLMs exhibit hierarchical organization and significant hyperbolicity. This understanding not only sheds light on the geometric nature of token embeddings but also motivates the development of methods that can better capture and preserve these underlying geometric properties.

5 Hyperbolic Fine-tuning for LLMs

The core technique in the LoRA adapter involves linear transformations. The conventional approach to implementing linear transformations in the Lorentz model of hyperbolic geometry is through operations in the tangent space, while maintaining the learnable weights in Euclidean space [31, 30]. However, this approach presents a significant challenge for our application. Since the hidden states of LLMs exist in Euclidean space, we would need to project these states to hyperbolic space and subsequently map them back to the tangent space. This process results in consecutive logarithmic and exponential mappings $(\log_{\mathbf{o}}^K(\exp_{\mathbf{o}}^K(\mathbf{x})))$, which effectively cancel each other out, reducing the method to the original LoRA approach and nullifying any benefits from hyperbolic geometry.

Direct Lorentz Low-rank Transformation (LLR). To overcome this limitation, we propose a Direct Lorentz Low-rank Transformation (LLR) that operates directly on the hyperbolic manifold without

Table 4: Comparison on Commonsense Reasoning Tasks. These datasets contain relatively similar amounts of data, so we use AVG to represent the average accuracy.

Base Model	PEFT Method	# Params (%)	BoolQ	PIQA	SIQA	HellaSwag	WinoGrande	ARC-e	ARC-c	OBQA	AVG
GPT-3.5	None	None	73.1	85.4	68.5	78.5	66.1	89.8	79.9	74.8	77.0
LLaMA3-8B	LoRA	0.70	70.8	85.2	79.9	91.7	84.3	84.2	71.2	79.0	80.8
	HypLoRA (Ours)	0.70	**74.1**	**87.6**	**80.6**	**94.5**	**84.7**	**90.4**	**81.2**	**85.2**	**84.8**
Gemma3-4B	LoRA	1.04	68.1	83.2	77.2	88.9	**80.5**	84.5	69.9	83.6	79.5
	HypLoRA (Ours)	1.04	**70.0**	**84.3**	**79.2**	**91.5**	80.3	**89.1**	**75.9**	**86.4**	**82.5**
Qwen2.5-7B	LoRA	0.71	**73.4**	**89.5**	79.5	93.6	84.1	92.8	82.0	87.0	85.2
	HypLoRA (Ours)	0.71	72.8	89.3	**79.8**	**94.8**	**84.4**	**95.5**	**87.5**	**90.8**	**87.0**

relying on tangent space mappings. This approach allows us to perform low-rank adaptation while preserving the advantages of hyperbolic geometry:

$$\begin{aligned}\mathbf{z}^E &= W_{\text{LoRA}}(\mathbf{x}^E) = W\mathbf{x}^E + \Delta W\mathbf{x}^E \\ &= W\mathbf{x}^E + \Pi^K_{H\to E}(\mathbf{LLR}(BA, \Pi^K_{E\to H}(\mathbf{x}^E))),\end{aligned} \tag{8}$$

where $\mathbf{LLR}$ represents our Direct Lorentz Low-Rank Transformation that operates directly on the hyperbolic representation $\mathbf{x}^H = \Pi^K_{E\to H}(\mathbf{x}^E)$:

$$\mathbf{LLR}(BA, \mathbf{x}^H) = (\sqrt{\|BA\mathbf{x}^H_s\|_2^2 + K}, BA\mathbf{x}^H_s), \tag{9}$$

where $\mathbf{x}^H_s$ is the space-like component of $\mathbf{x}^H$, i.e., $\mathbf{x}^H_s = \mathbf{x}^H_{[1:n]}$ without the first time-like dimension $\mathbf{x}^H_{[0:1]}$. The operators $\Pi^K_{E\to H}$ and $\Pi^K_{H\to E}$ represent projections from Euclidean to hyperbolic space and vice versa, respectively. While exponential maps and logarithmic maps are commonly used for these projections, we simplify the process by utilizing stereographic projection and its inverse [91] as a more computationally efficient alternative for LLM adaptation, where the detailed formula is given in Appendix D.1. It can be verified that $\mathbf{LLR}(BA, \mathbf{x}^H) \in \mathcal{L}^n$, ensuring that our transformation remains within the Lorentz model of hyperbolic space. This transformation primarily affects the space-like dimensions, functioning similarly to a Lorentz rotation. The linear transformation is inspired by hyperbolic neural networks [35, 45, 92]. For efficient integration with LLMs, the transformation removes normalization and non-linear activation terms in [35], varying curvatures in [45], and orthogonal constraints in [92]. Our main contribution lies in applying hyperbolic low-rank adaptation for LLMs, while the specific linear transformation itself is flexible, and other transformations on the manifold could also be compatible with our approach.

By adapting in the hyperbolic domain, HypLoRA captures more complex hierarchical relationships than traditional Euclidean-based methods, as detailed in Proposition 1. Additionally, the low-rank nature of the adaptation matrices A and B promotes parameter efficiency, making HypLoRA well-suited for LLMs.

Time Complexity. HypLoRA has similar theoretical time complexity as the Euclidean LoRA, which is $\mathcal{O}(r \cdot (d + k))$, where d and k represent the input and output dimensions, respectively. However, in practical implementation, HypLoRA introduces additional computations due to the logarithmic and exponential maps. These additional operations, nevertheless, can be completed within $\mathcal{O}(N)$ where N is the number of input tokens.

Proposition 1. *Let $\mathbf{x} \in \mathbb{R}^d$ denote the input token embeddings. The HypLoRA adaptation, applied to $\mathbf{x}$, involves a sequence of projection into hyperbolic space, a Direct Lorentz Low-rank Transformation (LLR), and projection back to Euclidean space. Due to the non-linear nature of these hyperbolic operations, the effective transformation applied by HypLoRA introduces higher-order terms with respect to $\mathbf{x}$. As detailed in Appendix E, these terms exhibit explicit dependency on the L2 norm, $\|\mathbf{x}\|_2$, of the input embeddings. This norm-dependent, higher-order modification enables HypLoRA to capture hierarchical relationships in the embedding space, thereby achieving natural alignment with the underlying hyperbolic geometry of the token representations.*

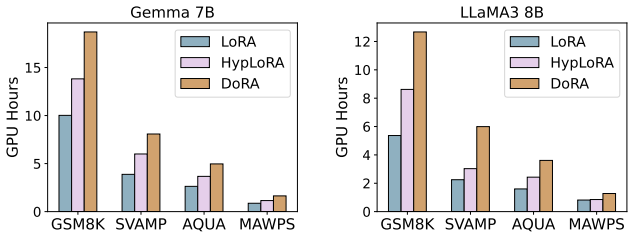
5.1 Experimental Settings

Datasets. Following the experiment setup outlined in [93], we utilize two high-quality datasets, Math10K and Commonsense170K, tailored for mathematical and commonsense reasoning, respectively. Math10K consists of training data from GSM8K [81], MAWPS, MAWPS-single [83], and

Table 5: Results for varying curvature K on the Gemma3-4B model

Dataset	K=0.5	K=1.0
MAWPS	90.7	**91.9**
SVAMP	**78.9**	78.0
GSM8K	67.8	**68.8**
AQuA	**48.4**	41.7
M.AVG	**71.9**	71.6

Figure 2: GPU (A100) usage during inference



1,000 samples from AQuA [82], augmented with ChatGPT-generated step-by-step rationales to reinforce reasoning capabilities. The test set includes GSM8K, AQuA, MAWPS, and SVAMP [84], ensuring no overlap with the training data. Commonsense170K is constructed by reformatting samples from BoolQ, PIQA, SIQA, HellaSwag, WinoGrande, ARC-e, ARC-c, and OBQA using standardized templates that outline the task, content, and answer, resulting in 170K training samples. The test datasets are drawn from the same sources, with strict separation from training samples.
Model Comparison. We evaluate the performance of LLaMA-7B, LLaMA-13B, LLaMA3-8B, Gemma-7B, Qwen2.5-7B, and Gemma3-4B base models. Except for Gemma-7B, we use the *instruct* variants of all models to ensure consistency with instruction-tuned behavior. For fine-tuning methods, we compare with LoRA [29].

Implementation Details. To ensure consistency and comparability, our experimental setup closely followed the training configurations outlined in Hu et al. [93]. Across all fine-tuning tasks, we employed the AdamW optimizer with a learning rate of 3×10^{-4} and trained for a total of three epochs. LoRA modules (and consequently, HypLoRA adapters) were integrated into both the Multi-Head Attention (MHA) and MLP layers of the foundation models. A key hyperparameter for HypLoRA is the curvature K (defining the hyperbolic curvature as $-1/K$), which was determined by searching the set $\{0.5, 1.0\}$. For evaluation, final scores were micro-averaged for arithmetic reasoning and averaged for commonsense reasoning across the datasets, thereby giving equal weight to each individual prompt, regardless of the varying number of questions per dataset (e.g., 1,319 in GSM8K versus 238 in MAWPS). All experiments were executed on a single NVIDIA A100 GPU with 80GB of memory.

5.2 Experimental Results

Table 3 summarizes our key experimental outcomes on arithmetic reasoning tasks, while Table 4 presents results for commonsense reasoning benchmarks. Our primary comparison contrasts LoRA and HypLoRA to demonstrate the effectiveness of the proposed approach, with additional baselines provided in Appendix F. Our findings consistently demonstrate that HypLoRA achieves superior performance compared to the standard LoRA across a diverse range of LLMs and reasoning tasks. This empirical evidence supports the theoretical advantages outlined in Proposition 1, suggesting that adapting LLMs within the hyperbolic domain allows for a more effective capture of the complex, often hierarchical, relationships.

Arithmetic Reasoning Performance. On arithmetic reasoning tasks, as indicated by results in Table 3, HypLoRA shows notable efficacy, especially on datasets recognized for their complexity, such as GSM8K and AQuA. These datasets demand robust multi-step reasoning and a nuanced understanding of numerical and textual relationships. The enhanced performance of HypLoRA in these areas aligns with its design; by operating in hyperbolic space, it can better model the hierarchical structure of problems and distinguish subtle yet critical differences in input embeddings. This is further corroborated by the theoretical analysis (Appendix E), which posits that HypLoRA introduces higher-order, norm-dependent terms. These terms allow the model to develop a more refined sensitivity to token importance and inter-token relationships.

Commonsense Reasoning Performance. The robust performance of HypLoRA extends to commonsense reasoning, as detailed in Table 4. For the Gemma3-4B model, HypLoRA achieved an average accuracy of 82.5% across all datasets, surpassing LoRA's 79.5%. Similarly, on the Qwen2.5-7B model, HypLoRA obtained an average of 87.0% compared to LoRA's 85.2%. These improvements are distributed across various commonsense benchmarks, including notable gains on datasets like ARC-c and OBQA for Gemma3-4B, and ARC-c, ARC-e, and OBQA for Qwen2.5-7B. Commonsense

reasoning often relies on understanding implicit relationships and contextual nuances, which may not always be explicitly hierarchical but still benefit from the richer representational capacity offered by hyperbolic geometry. The ability of HypLoRA to better discern these subtleties, likely due to the mechanisms described in Proposition 1, contributes to these observed performance gains, showcasing the broad applicability of hyperbolic fine-tuning.

The Impact of Curvature on Performance. Curvature in hyperbolic space is a key hyperparameter in HypLoRA, directly affecting its capacity to model underlying structures and geometries. To evaluate its impact, we experiment with a learnable curvature initialised with different curvature values on the Gemma3-4B model, as shown in Table 5, where the curvature is defined as $-1/K$. Our results demonstrate that curvature does influence model performance. For Gemma-7B and Gemma-3-4B, a curvature value of 0.5 consistently yields the best overall performance across both arithmetic and commonsense reasoning benchmarks. Similarly, for LLaMa3-8B, 0.5 proves optimal. In commonsense reasoning benchmarks, a curvature of 1.0 performs best for LLaMa3-8B and Qwen2.5-7B.

Efficiency. In Section 5, we analyze the time complexity of our approach, which remains consistent with that of LoRA. However, during actual inference, HypLoRA incurs additional computational overhead due to operations such as the exponential and logarithmic mappings, or the inverse stereographic and stereographic projections when using the stereographic method. These operations introduce some additional runtime, particularly for larger models. The GPU hours for inference on four datasets are presented in Figure 2. Despite this overhead, our method demonstrates improved efficiency when compared to the previous competitive model, DoRA. Notably, HypLoRA still outperforms DoRA in terms of both runtime and overall efficiency. Besides, all models can be fine-tuned in approximately one hour for optimal training efficiency.

6 Conclusion

In this work, we investigated the non-Euclidean geometric properties inherent in LLM token embeddings, confirming their strong hyperbolic characteristics, which suggest underlying hierarchical structures. Building on these insights, we introduced HypLoRA, a novel hyperbolic low-rank adaptation technique. HypLoRA performs fine-tuning directly on the hyperbolic manifold. Extensive experiments show that HypLoRA significantly improves LLM performance on arithmetic reasoning and commonsense tasks. By leveraging the hyperbolic structure of the data, HypLoRA enhances the model's ability to capture and utilize intricate relationships, leading to better reasoning capabilities. While the theoretical time complexity aligns with LoRA, the practical application of hyperbolic projections introduces a slight computational overhead. This is a manageable aspect that can be readily addressed through optimized numerical libraries or by exploring alternative, more computationally efficient projection techniques without sacrificing the geometric benefits.

Broader Impact. Enhancing reasoning-oriented LLMs can help education, scientific assistance, and safer decision-support systems, but the same improvements may also accelerate misuse (e.g., automating complex disinformation or amplifying biased advice) and increase energy consumption because of added hyperbolic projections. We therefore advocate releasing checkpoints and code with usage guidelines (as in our public repo), tracking compute budgets when scaling HypLoRA further, and pairing capability gains with evaluations focused on safety, robustness, and fairness.

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