Exponential Spectral Pursuit: An Effective Initialization Method for Sparse Phase Retrieval

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Abstract

Sparse phase retrieval aims to reconstruct an n-dimensional k-sparse signal from its phaseless measurements. For most of the existing reconstruction algorithms, their sampling complexity is known to be dominated by the initialization stage. In this paper, in order to improve the sampling complexity for initialization, we propose a novel method termed exponential spectral pursuit (ESP). Theoretically, our method offers a tighter bound of sampling complexity compared to the state-of-the-art ones, such as the truncated power method. Moreover, it empirically outperforms the existing initialization methods for sparse phase retrieval.

1. Introduction

Phase retrieval aims to reconstruct a signal $\mathbf{x}$ from the measurements $\mathbf{y} = [y_1, \cdots, y_m]^\top$, where

$$y_i = |\mathbf{a}_i^* \mathbf{x}|, \ \ i = 1, 2, \cdots, m, \tag{1}$$

and $\mathbf{a}_i \in \mathbb{C}^n$ is the measurement vector. This problem arises in many fields of science and engineering (Bunk et al., 2007; Millane, 1990; Shechtman et al., 2015; Zhang & Liang, 2016). For example, in astronomical imaging (Dainty & Fienup, 1987; Guo et al., 2021), phase retrieval was employed to reconstruct high-resolution images from telescope observations. While in electron microscopy (Miao et al., 2008; Varnavides et al., 2023; Yu et al., 2010), it was used to enhance the contrast and resolution of images. Generally, if the number of samples equals the dimension of the target signal, there will exist multiple non-trivial solutions (Bates, 1982; Bruck & Sodin, 1979; Hayes, 1982; Hofstetter, 1964) to (1), making it highly ill-posed. Approaches to overcome this under-determined issue mainly include using more measurements (Bendory & Eldar, 2017; Candès et al., 2015a; Fu et al., 2021; Goldstein & Studer, 2018), or imposing prior knowledge of the target signal, such as sparsity and so forth (Cai et al., 2016; Ohlsson et al., 2012; Tong et al., 2021; Xu et al., 2024; Zhang et al., 2022).

With the great success of random matrix theory in the field of compressive sensing (Candes & Tao, 2005; Donoho, 2006), it has been proposed to introduce random Gaussian measurements in phase retrieval. For example, Candès et al. (2015b) considered complex random Gaussian measurements and solved the complex system via a gradient descent-like approach. Specifically, it starts from a careful initialization via spectral method, while refining this solution iteratively by means of an greedy principle. Theoretical analysis indicates that at least

$$m = \Omega(n \log n)^{1} \tag{2}$$

samples are required in order to recover $\mathbf{x}$ exactly.

In many practical scenarios, the target signal is naturally sparse (such as images and radio signals), offering potential to obtain a lower sampling complexity by exploiting sparsity (Moravec et al., 2007). Indeed, Eldar & Mendelson (2014); Truong & Scarlett (2020) showed that, in order to recover an n-dimensional k-sparse signal (i.e., with at most k non-zero entries) from its magnitude-only Gaussian measurements, the information-theoretic bound of sampling complexity is

$$m = \Omega(k \log n), \tag{3}$$

which is significantly lower than (2). However, practical algorithms for sparse phase retrieval, such as AltMin (Netrapalli et al., 2015), SPARTA (Wang et al., 2018), and SWF (Yuan et al., 2019), still require

$$m = \Omega(k^2 \log n) \tag{4}$$

samples to perform the recovery task.

The difference between (3) and (4) is known as the statistical-to-computational gap, which results from the initialization stage. In particular, algorithms like SPARTA and

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Proceedings of the 41st International Conference on Machine Learning, Vienna, Austria. PMLR 235, 2024.

[1]Throughout this paper, we use $\Omega(\cdot)$ for asymptotic lower bound and $\Theta(\cdot)$ for both asymptotic lower bound and upper bound, i.e., $m = \Omega(f(s))$ means $|m/f(s)| \le C$ for some universal constant $C > 0$ when s is sufficient large, while $m = \Theta(f(s))$ means $C' \le |m/f(s)| \le C''$ for some universal constant $C', C'' > 0$ in the case when s is sufficient large.

SWF require (4) to obtain a good estimate within the δ-neighborhood of the target signal $\mathbf{x}$. Whereas in the iterative stage, they only need

$$m = \Omega(k \log n) \tag{5}$$

samples to reconstruct $\mathbf{x}$ exactly. The difference of sampling complexity between the initialization and iterative stages also happens to SAM (Cai et al., 2022a), HTP (Cai et al., 2022b), CoPRAM (Jagatap & Hegde, 2019), and so forth. To date, the sampling complexity of practical phase retrieval algorithms has been dominated by the initialization stage.

Recently, Wu and Rebeschini (Wu & Rebeschini, 2021) presented a new method called Hadamard WF (HWF) for initialization. They suggested that the sampling complexity of initialization can be reduced, depending on the maximum and minimum magnitudes of non-zeros in $\mathbf{x}$, denoted by $|x_{\max}|$ and $|x_{\min}|$, respectively. In particular, when $|x_{\min}|$ is on the order of $\frac{\|\mathbf{x}\|}{\sqrt{k}}$, the sampling complexity in the initialization stage is

$$m = \Omega\left(\max\left\{\frac{\|\mathbf{x}\|^2}{|x_{\max}|^2} k \log n, \frac{\|\mathbf{x}\|}{|x_{\max}|}\sqrt{k}\log^3 n\right\}\right). \tag{6}$$

Thus, if $|x_{\max}|$ is on the order of $\|x\|$, the cost for initialization can be reduced to $m = \Omega(k \log n)$ (ignoring the $\log^3 n$ term), which matches the information-theoretical bound (3).

More recently, Cai et al. (2022c) relaxed the condition on $|x_{\min}|$ and proposed the truncated power (TP) method for initialization. Specifically, TP uses the truncated spectrum

$$\mathbf{Y} = \sum_{j=1}^{m} {y_j}^2 \mathbf{1}(\eta_1\|\mathbf{x}\| \le y_j \le \eta_2\|\mathbf{x}\|)\mathbf{a}_j\mathbf{a}_j^*, \tag{7}$$

where η_1 and η_2 are pre-specified hyper-parameters. They suggested that given the sampling complexity in (6), TP produces a good initialization $\mathbf{z}$ falling into the δ-neighborhood of $\mathbf{x}$, without relying on any assumption on $|x_{\min}|$.

In this paper, with an aim of enhancing the result in (6), we develop a new algorithm termed Exponential Spectral Pursuit (ESP). By employing an exponential-based spectrum (see (9)), ESP provides a δ-neighborhood initialization if

$$m = \Omega\left(\frac{\|\mathbf{x}\|^2}{|x_{\max}|^2} k \log n\right). \tag{8}$$

The advantage of our ESP method is threefold. First of all, it eliminates the $\log^3 n$ term in (6), offering a tighter bound of sampling complexity. Second, it needs not to tune hyper-parameters (e.g., η_1, η_2 in (7)), making the algorithm easier to implement than TP. Third, while the aforementioned results are based solely on the real case, our result applies to the complex case as well.

Algorithm 1 Exponential Spectral Pursuit

Input: sparsity k, samples $\mathbf{y}$, and sampling matrix $\mathbf{A}$.
Step 1: Search an index $i_{\max}$ corresponding to the largest diagonal element of $\mathbf{L}$.
Step 2: Select an index set S corresponding to the most significant k entries in the $i_{\max}$-th column of $\mathbf{L}$.
Step 3: Use the principle eigenvector of $\mathbf{L}_S$ as the estimate of $\mathbf{z} \in \mathbb{C}^n$ and re-scale it to $\|\mathbf{z}\| = \lambda$.
Output: $\mathbf{z}$.

The remainder of the paper is organized as follows. In Section 2, we introduce our algorithm and provide detailed interpretation for it. Section 3 contains theoretical analyses for each step of Algorithm 1 and a discussion on the results. Numerical simulations and analysis are conducted in Section 4. Finally, we conclude our paper in Section 5.

2. Algorithm

Before we proceed to the details of our ESP method, we first explain some notations used throughout this paper. Denote $[n]$ as the set $\{1, 2, \cdots, n\}$. For a set $S \subseteq [n]$, let $\mathbb{C}^S$ be the subspace of $\mathbb{C}^n$ spanned by vectors supported on S, i.e., $\{\mathbf{x} \in \mathbb{C}^n | \text{supp}(\mathbf{x}) \subseteq S\}$. Define $\mathbf{a}_S$ as a vector keeping all elements of $\mathbf{a}$ indexed by S while setting others to zero. For any matrix $\mathbf{A}$, define $\mathbf{A}_S$ as the matrix which keeps all rows and columns of $\mathbf{A}$ indexed by S and sets others to zero. For any vector $\mathbf{x} \in \mathbb{C}^n$, denote $\mathbf{x}^*$, $\|\mathbf{x}\|$, and $\|\mathbf{x}\|_0$ as its conjugate transpose, ℓ_2- and ℓ_0-norm, respectively. As an extension of Gaussian random vector, we define the n-dimensional complex Gaussian random vector $\mathbf{u} \in \mathcal{CN}(n)$ as $\mathbf{u} = \mathcal{R}(\mathbf{u}) + \mathrm{i}\mathcal{I}(\mathbf{u})$, where i is the imaginary unity. $\mathcal{R}(\mathbf{u})$ and $\mathcal{I}(\mathbf{u}) \in \mathcal{N}(\mathbf{0}, \frac{1}{2}\mathbf{I})$ are independent n-dimensional Gaussian random vectors.

Define a spectrum $\mathbf{L} \in \mathbb{C}^{n\times n}$ as follows:

$$\mathbf{L} := \frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{2} - \exp\left(-\frac{y_j^2}{\lambda^2}\right)\right)\mathbf{a}_j\mathbf{a}_j^*, \tag{9}$$

where $\lambda^2 := \frac{1}{m}\sum_{j=1}^{m} y_j^2$. Then, we introduce the proposed ESP method, which is composed of three steps: 1) diagonal search, 2) support recovery, and 3) signal estimation, as specified in Algorithm 1. In each step, $\mathbf{L}$ plays an important role. Since $\mathbf{L}$ involves an interaction between random vectors a_j's and variables y_j^2/λ^2's, it is natural to consider the expected version for analysis. However, precisely deriving the expectation of $\mathbf{L}$ is challenging. An alternative is to consider a proxy spectrum of $\mathbf{L}$:

$$\tilde{\mathbf{L}} := \frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{2} - \exp\left(-\frac{y_j^2}{\|\mathbf{x}\|^2}\right)\right)\mathbf{a}_j\mathbf{a}_j^*, \tag{10}$$

which has a closed-form expectation (Gao & Xu, 2017)

$$\mathbb{E}[\tilde{\mathbf{L}}] = \frac{\mathbf{x}\mathbf{x}^*}{4\|\mathbf{x}\|^2}. \tag{11}$$

Noting that λ^2 is close to $\|\mathbf{x}\|^2$ (Appendix Lemma B.1), we can roughly view $\frac{\mathbf{x}\mathbf{x}^*}{4\|\mathbf{x}\|^2}$ as the expectation of $\mathbf{L}$. Based on this approximation, ESP can be interpreted as follows:

1) **Step 1** searches an index $i_{\max}$ corresponding to the largest diagonal element of $\mathbf{L}$. Denote f_i as the i-th diagonal entry of $\mathbf{L}$. Then, its expectation can be approximated by that of the i-th diagonal entry of $\tilde{\mathbf{L}}$. By (11), we have

$$\mathbb{E}[f_i] \approx \mathbb{E}[\tilde{f}_i] \overset{(11)}{=} \frac{|x_i|^2}{4\|\mathbf{x}\|^2}, \tag{12}$$

which means that the most significant diagonal element $f_{i_{\max}}$ of $\mathbf{L}$ is roughly $\frac{|x_{\max}|^2}{4\|\mathbf{x}\|^2}$. In fact, this suffices to establish a relationship between $|x_{i_{\max}}|$ and $|x_{\max}|$. As will be shown in Proposition 3.3, given $\Omega(\frac{\|\mathbf{x}\|^2}{|x_{\max}|^2} k \log n)$ samples, $|x_{i_{\max}}| > \frac{1}{2}|x_{\max}|$ holds with high probability.

2) **Step 2** aims to recover $\mathrm{supp}(\mathbf{x})$ based on the $i_{\max}$-th column of $\mathbf{L}$ determined in **Step 1**. Ideally, it requires the $i_{\max}$-th column of $\mathbf{L}$ to have elements supported on $\mathrm{supp}(\mathbf{x})$ being more significant than those supported on $[n]\setminus \mathrm{supp}(\mathbf{x})$. To analyze this condition, we consider the expectation of the $i_{\max}$-th column of $\mathbf{L}$, denoted by $\mathbb{E}(\mathbf{L}\mathbf{e}_{i_{\max}})$, where $\mathbf{e}_{i_{\max}} \in \mathbb{R}^n$ denotes a vector with the $i_{\max}$-th entry being one and zero otherwise. As before, we use $\mathbb{E}(\tilde{\mathbf{L}}\mathbf{e}_{i_{\max}})$ to approximate $\mathbb{E}(\mathbf{L}\mathbf{e}_{i_{\max}})$:

$$\mathbb{E}(\mathbf{L}\mathbf{e}_{i_{\max}})_i \approx \mathbb{E}(\tilde{\mathbf{L}}\mathbf{e}_{i_{\max}})_i \overset{(11)}{=} \begin{cases} \frac{x^*_{i_{\max}} x_i}{4\|\mathbf{x}\|^2} & i \in \mathrm{supp}(\mathbf{x}) \\ 0, & i \notin \mathrm{supp}(\mathbf{x}) \end{cases}, \tag{13}$$

which indicates a non-trivial gap $\frac{x^*_{i_{\max}} x_i}{4\|\mathbf{x}\|^2}$ for distinguishing whether an index $i \in \mathrm{supp}(\mathbf{x})$ or not. However, if some nonzero elements x_i's are extremely small (e.g., $|x_i| = \Theta(\frac{\|\mathbf{x}\|}{k^5})$), this gap may be too small to identify those i's. To address this issue, we slightly modify the goal of **Step 2**. Instead of recovering all support indices in $\mathrm{supp}(\mathbf{x})$, we only choose those in

$$S_\gamma := \left\{ j \in \mathrm{supp}(\mathbf{x}) \,\middle|\, |x_j| \geq \frac{\gamma\|\mathbf{x}\|}{2\sqrt{k}} \right\}, \tag{14}$$

where $\gamma \in (0, 1)$ is a constant to be specified. In fact, identifying S_γ already suffices to produce a good estimate of $\mathbf{x}$ in **Step 3** (see Theorem 3.6), yet with an improved sampling complexity compared to (6).

3) **Step 3** is designed to produce a good estimate of $\mathbf{x}$. It is trivial that $\mathbf{x}$ is the principle eigenvector of $\mathbf{x}\mathbf{x}^*$. Thus, based on the recovered support set S in **Step 2**, we perform eigenvalue decomposition on $\mathbf{L}_S$ and let $\mathbf{z}$ be the principle eigenvector with ℓ_2-norm re-scaled to λ. As long as S contains sufficient number of support indices $\mathbf{x}$, $\mathbf{z}$ falls into the δ-neighborhood of $\mathbf{x}$, implying a good estimation of $\mathbf{x}$ (see Theorem 3.6).

We mention that the spectrum $\mathbf{L}$ has been studied thoroughly for the generic phase retrieval problem (Gao & Xu, 2017). In this paper, we extend it to sparse phase retrieval and propose the ESP method for the initialization based on $\mathbf{L}$. Both in theory and practice, ESP is demonstrated to outperform the state-of-the-art ones in the sparse scenario.

3. Main results

3.1. Preliminaries

In this section, we provide theoretical results and the corresponding sketch proofs. Before we start, we first define the distance between two complex vectors.

Definition 3.1. Let $\mathbf{u} \in \mathbb{C}^n$ and $\mathbf{v} \in \mathbb{C}^n$ be two complex vectors. Then, the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathrm{dist}(\mathbf{u}, \mathbf{v}) = \min_{\phi \in [0, 2\pi)} \|\mathbf{u} - e^{i\phi}\mathbf{v}\|.$$

Based on this definition, a good initialization $\mathbf{z}$ falling into the δ-neighborhood of $\mathbf{x}$ satisfies

$$\mathrm{dist}(\mathbf{x}, \mathbf{z}) < \delta\|\mathbf{x}\|. \tag{15}$$

In the analyses of phase retrieval algorithms, (15) is often the goal of initialization and also the prerequisite for the iterative refining stage; See, e.g., (Wang et al., 2018).

In the following, we introduce the definition of the k-sparse spectral norm of $\mathbf{L}$, which is useful for analyzing the eigenvalue decomposition on $\mathbf{L}_S$ in **Step 3**.

Definition 3.2. Let $\mathbf{L} \in \mathbb{C}^{n\times n}$. Define the k-sparse spectral norm of $\mathbf{L}$ as

$$\tau(\mathbf{L}, k) = \max\{|\lambda_{\max}(\mathbf{L}, k)|, |\lambda_{\min}(\mathbf{L}, k)|\}, \tag{16}$$

where $\lambda_{\max}(\mathbf{L}, k) = \max_{\mathbf{u}\in\mathbb{C}^n, \|\mathbf{u}\|=1, \|\mathbf{u}\|_0 \leq k} \mathbf{u}^*\mathbf{A}\mathbf{u}$ and $\lambda_{\min}(\mathbf{L}, k) = \min_{\mathbf{u}\in\mathbb{C}^n, \|\mathbf{u}\|=1, \|\mathbf{u}\|_0 \leq k} \mathbf{u}^*\mathbf{L}\mathbf{u}$, i.e., the largest and smallest k-sparse eigenvector of $\mathbf{L}$, respectively.

Next, we provide theoretical results for ESP. The fraction

$$s := \frac{|x_{\max}|^2}{\|\mathbf{x}\|^2} \tag{17}$$

characterizing the distribution of nonzero elements in $\mathbf{x}$ plays an important role in our results.

3.2. Step 1: Diagonal search

In **Step 1** of Algorithm 1, we search a single index $i_{\max}$ corresponding to the largest diagonal element of $\mathbf{L}$. The following proposition quantitatively characterizes the relationship between $|x_{i_{\max}}|$ and $|x_{\max}|$.

Proposition 3.3. *Consider* **Step 1** *of the ESP algorithm. If $m \geq \frac{C}{s^2}\log n$, then with probability at least $1-e^{-cm}$,*

$$|x_{i_{\max}}| > \frac{|x_{\max}|}{2}, \tag{18}$$

where c, C are constants.

Throughout the paper, we follow the convention that letters c and C, and their indexed versions (e.g., c_1) indicate positive, universal constants that may vary at each appearance.

Sketch of Proof: Let $\mathbf{f} \in \mathbb{R}^n$ and $\tilde{\mathbf{f}} \in \mathbb{R}^n$ be the vectors consisting of the diagonal elements of $\mathbf{L}$ and $\tilde{\mathbf{L}}$, respectively. Then, observe that

$$\begin{aligned}\frac{|x_{i_{\max}}|^2}{4\|\mathbf{x}\|^2} &= \frac{|x_i|^2}{4\|\mathbf{x}\|^2} + \left(\tilde{f}_i - \frac{|x_i|^2}{4\|\mathbf{x}\|^2}\right) - \left(\tilde{f}_{i_{\max}} - \frac{|x_{i_{\max}}|^2}{4\|\mathbf{x}\|^2}\right) \\ &\quad + \left(\tilde{f}_{i_{\max}} - \tilde{f}_i\right) \\ &\geq \frac{|x_i|^2}{4\|\mathbf{x}\|^2} - 2\max_{l\in[n]}\left|\tilde{f}_l - \frac{|x_l|^2}{4\|\mathbf{x}\|^2}\right| + \left(\tilde{f}_{i_{\max}} - \tilde{f}_i\right).\end{aligned} \tag{19}$$

If the second and the third term on the right-hand side of (19) can be well bounded, we can establish the desired relationship between $|x_{i_{\max}}|$ and $|x_{\max}|$.

For the second term, conducting an concentration analysis for $\tilde{\mathbf{f}}$ yields that for any $\eta \in (0,1)$,

$$\max_{i\in[n]}\left|\tilde{f}_i - \frac{|x_i|^2}{4\|\mathbf{x}\|^2}\right| < \frac{\eta}{3}\frac{|x_{\max}|^2}{\|\mathbf{x}\|^2} \tag{20}$$

holds with high probability.

For the third term in the bracket, however, it is difficult to bound it directly, because $i_{\max}$ is defined by $\mathbf{f}$ but used here as an index for $\tilde{f}$. Nevertheless, we can bound it by relating $\mathbf{f}$ and $\tilde{\mathbf{f}}$ and exploiting the definition of $i_{\max}$. Specifically,

$$\begin{aligned}\tilde{f}_{i_{\max}} - \tilde{f}_i &= (\tilde{f}_{i_{\max}} - \tilde{f}_i) - (f_{i_{\max}} - f_i) + (f_{i_{\max}} - f_i) \\ &\geq (\tilde{f}_{i_{\max}} - \tilde{f}_i) - (f_{i_{\max}} - f_i),\end{aligned} \tag{21}$$

which can be bounded via concentration analysis. That is, for any $\eta \in (0,1)$, it holds with high probability that

$$\max_{1\leq p\neq q\leq n}|(\tilde{f}_p - \tilde{f}_q) - (f_p - f_q)| \leq \frac{\eta}{3}\frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}, \tag{22}$$

which implies that $(\tilde{f}_{i_{\max}} - \tilde{f}_i) - (f_{i_{\max}} - f_i) \geq -\frac{\eta}{3}\frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$. This, together with (19), (20) and (21), leads to that

$$\frac{|x_{i_{\max}}|^2}{4\|\mathbf{x}\|^2} \geq \frac{|x_i|^2}{4\|\mathbf{x}\|^2} - \eta\frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}. \tag{23}$$

Finally, by taking $i = \arg\max_j |x_j|$ and $\eta = \frac{3}{16}$, we arrive at the desired result. $\square$

3.3. Step 2: Support recovery

In **Step 2**, we select a set S of k indices corresponding to the largest k elements (in modulus) in the $i_{\max}$-th column of $\mathbf{L}$. The following proposition shows that $S \supseteq S_\gamma$, where S_γ is defined in (14). That is, **Step 2** recovers all support indices of $\mathbf{x}$ except those of small nonzeros in (i.e., $|x_j| < \frac{\gamma\|\mathbf{x}\|}{2\sqrt{k}}$), implying that the recovered support is nearly exact.

Proposition 3.4. *Suppose that $x_{i_{\max}}$ satisfies (18). Then, if $m \geq \frac{C_\gamma}{s}k\log n$, the recovered index set in* **Step 2** *of the ESP algorithm obeys $S \supseteq S_\gamma$ with probability exceeding $1-\exp(-c_\gamma m)$, where C_γ and c_γ are numerical constant depending on γ.*

Sketch of Proof: Define

$$\begin{cases}\mathbf{q} := \frac{1}{m}\sum_{j=1}^m\left(\frac{1}{2} - \exp\left(-\frac{y_j^2}{\lambda^2}\right)\right)\mathbf{a}_j\mathbf{a}_j^*\mathbf{e}_{i_{\max}}, \\ \tilde{\mathbf{q}} := \frac{1}{m}\sum_{j=1}^m\left(\frac{1}{2} - \exp\left(-\frac{y_j^2}{\|\mathbf{x}\|^2}\right)\right)\mathbf{a}_j\mathbf{a}_j^*\mathbf{e}_{i_{\max}}.\end{cases} \tag{24}$$

It is trivial to see that $\mathbf{q}$ and $\tilde{\mathbf{q}}$ are the $i_{\max}$-th column of $\mathbf{L}$ and $\tilde{\mathbf{L}}$, respectively. To show $S \supseteq S_\gamma$, it suffices to prove the following inequality:

$$|q_i| > |q_j| \quad \forall i \in S_\gamma, \forall j \notin \text{supp}(\mathbf{x}). \tag{25}$$

To that end, we conduct an concentration analysis for $\tilde{q}_i$, which yields that

$$\max_{l\in[n]}|\tilde{q}_l - \mathbb{E}[\tilde{q}_l]| \leq \frac{\epsilon}{2}\times\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|} \tag{26}$$

holds with high probability.

Furthermore, we can bound $\max_{l\in[n]}|q_l - \tilde{q}_l|$ as

$$\max_{l\in[n]}|q_l - \tilde{q}_l| \leq \frac{\epsilon}{2}\times\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|}. \tag{27}$$

From (26) and (27),

$$\max_{l\in[n]}\big||q_l| - |\mathbb{E}[\tilde{q}_l]|\big| \leq \max_{l\in[n]}|q_l - \mathbb{E}[\tilde{q}_l]| \leq \frac{\epsilon\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|} \tag{28}$$

holds with high probability.

Recalling (11) and also employing the result of Proposition 3.3 (i.e., $|x_{i_{\max}}| > \frac{|x_{\max}|}{2}$ in (18)), we have

$$|\mathbb{E}[\tilde{q}_i]| = \begin{cases}\frac{|x_{i_{\max}}x_i|}{4\|\mathbf{x}\|^2} \overset{(18)}{>} \frac{|x_{\max}||x_i|}{8\|\mathbf{x}\|^2}, & i \in \text{supp}(\mathbf{x}), \\ 0, & i \notin \text{supp}(\mathbf{x}).\end{cases} \tag{29}$$

Plugging $\epsilon = \frac{1}{24}$ in (28) and also using the definition of S_γ, we further have

$$\begin{cases} |q_i| > \frac{\gamma |x_{\max}|}{32\sqrt{k}\|\mathbf{x}\|}, & \forall i \in S_\gamma, \\ |q_j| \le \frac{\gamma |x_{\max}|}{32\sqrt{k}\|\mathbf{x}\|}, & \forall j \notin \operatorname{supp}(\mathbf{x}). \end{cases} \tag{30}$$

Since S corresponds to the most significant k entries of $\mathbf{q}$ and $|S_\gamma| \le k$, it is trivial that $S \supseteq S_\gamma$. □

3.4. Step 3: Signal estimation

In **Step 3**, we employ spectral initialization with spectrum $\mathbf{L}_S$ to estimate $\mathbf{x}$. The following proposition offers a condition for the principle eigenvector $\mathbf{z}$ of $\mathbf{L}_S$ to fall into the δ-neighbourhood of $\mathbf{x}$.

Proposition 3.5. *For any $\delta > 0$, let $\gamma = \frac{\delta}{4}$. Suppose the recovered support set S in* **Step 2** *of the ESP algorithm satisfies $|S| = k$ and $S \supseteq S_\gamma$. Then,* **Step 3** *produces a signal estimate $\mathbf{z}$ satisfying dist$(\mathbf{z}, \mathbf{x}) < \delta\|\mathbf{x}\|$ with probability at least $1 - \exp(-c_\delta m)$, provided that $m \ge C_\delta k \log n$.*

Sketch of Proof: Noting that the principle eigenvector of $\mathbb{E}(\tilde{\mathbf{L}}) = \frac{\mathbf{x}\mathbf{x}^*}{4\|\mathbf{x}\|^2}$ is a multiple of $\mathbf{x}$, it is natural to estimate the difference between $\mathbf{L}$ and $\mathbb{E}(\tilde{\mathbf{L}})$. To that end, we conduct spectral analysis for $\mathbf{L} - \mathbb{E}(\tilde{\mathbf{L}})$ and show that for any constant t, it holds with high probability that

$$\tau(\mathbf{L} - \mathbb{E}(\tilde{\mathbf{L}}), k) \le \tau(\mathbf{L} - \tilde{\mathbf{L}}, k) + \tau(\tilde{\mathbf{L}} - \mathbb{E}(\tilde{\mathbf{L}}), k) \le 6t. \tag{31}$$

Recall that $\mathbf{z}$ is the output of **Step 3** and let

$$\begin{cases} \mathbf{z}^0 := \frac{\mathbf{z}}{\lambda}, \\ \mathbf{x}^0 := \frac{\mathbf{x}}{\|\mathbf{x}\|}. \end{cases} \tag{32}$$

Then, noting that $\operatorname{supp}(\mathbf{z}^0) \subseteq S$, we can decompose the distance between $\mathbf{z}^0$ and $\mathbf{x}^0$ as

$$\operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0)^2 = \|\mathbf{x}^0_{S^c}\|^2 + \operatorname{dist}(\mathbf{z}^0, (\mathbf{x}^0)_S)^2. \tag{33}$$

For the first term on the right-hand side of (33), using the definition of S_γ and recalling $S_\gamma \subseteq S$, it can be easily derived that

$$\|\mathbf{x}^0_{S^c}\|^2 < \frac{\gamma^2}{4}. \tag{34}$$

As for the second term, performing the spectral analysis leads to

$$\begin{aligned} &\operatorname{dist}(\mathbf{z}^0, (\mathbf{x}^0)_S)^2 \\ &\le \|\mathbf{x}^0_S\|^2 + 1 - \frac{6\|\mathbf{x}^0_S\|}{\sqrt{9 + 1024\tau^2(\mathbf{L} - \mathbb{E}(\tilde{\mathbf{L}}), k)}}. \end{aligned} \tag{35}$$

Note that

$$1 - \frac{\gamma^2}{4} \le 1 - \|\mathbf{x}^0_{S^c}\|^2 = \|\mathbf{x}^0_S\|^2 \le 1. \tag{36}$$

Further, plugging (31) and (36) into (35) yields that

$$\operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0_S)^2 \le 8192t^2 + \frac{\gamma^2}{4}. \tag{37}$$

By relating (34) and (37), we obtain

$$\begin{aligned} \operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0)^2 &= \operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0_S)^2 + \|\mathbf{x}^0_{S^c}\|^2 \\ &\le 8192t^2 + \frac{\gamma^2}{2}. \end{aligned} \tag{38}$$

Finally, we need to bound the error between λ and $\|\mathbf{x}\|$, i.e., the length of $\mathbf{z}$ and $\mathbf{x}$. It shall be shown that $\big|\lambda - \|\mathbf{x}\|\big| \le \frac{\delta}{2}\|\mathbf{x}\|$ and $\lambda \le 2\|\mathbf{x}\|$ hold with high probability. Therefore, taking $t = \frac{\delta}{512}, \gamma = \frac{\delta}{4}$ in (38), we derive that

$$\begin{aligned} \operatorname{dist}(\mathbf{z}, \mathbf{x}) &\le \operatorname{dist}(\lambda\mathbf{z}^0, \lambda\mathbf{x}^0) + \operatorname{dist}(\lambda\mathbf{x}^0, \mathbf{x}) \\ &\le \lambda \operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0) + \big|\lambda - \|\mathbf{x}\|\big| \\ &\le \frac{\delta}{2}\|\mathbf{x}\| + \frac{\delta}{2}\|\mathbf{x}\| = \delta\|\mathbf{x}\| \end{aligned} \tag{39}$$

holds with high probability, which completes the proof. □

3.5. δ-neighbourhood initialization via ESP

By relating Propositions 3.3-3.5, the following theorem is immediate.

Theorem 3.6. *Considering the phase retrieval problem. For any $\delta > 0$, the output of ESP satisfies dist$(\mathbf{z}, \mathbf{x}) < \delta\|\mathbf{x}\|$ with probability at least $1 - \exp(-c_\delta m)$, provided that $m = \Omega(\frac{\|\mathbf{x}\|^2}{|x_{\max}|^2} k \log n)$.*

The relationship between Propositions 3.3-3.5 and Theorem 3.6 is illustrated in Figure 1. Specifically, via concentration analysis, Proposition 3.3 demonstrates that **Step 1** yields the significant index $i_{\max}$ such that $|x_{i_{\max}}| > \frac{1}{2}|x_{\max}|$, which serves as the prerequisite of **Step 2**. Then, Proposition 3.4 also utilizes concentration analysis to show that **Step 2** produces an estimated support set S containing all indices in S_γ. Finally, through spectral analysis, Proposition 3.5 proves that given the estimated support set S, **Step 3** generates a signal estimate falling into the δ-neighbourhood of $\mathbf{x}$. Combining three propositions leads to Theorem 3.6.

3.6. Discussion

We compare our sampling complexity with existing results. As mentioned, the so far best sampling complexity for δ-neighbourhood initialization of phase retrieval, obtained by Cai et al. (2022c), is given by

$$m = \Omega\left(\max\left\{\frac{k}{s}\log n, \sqrt{\frac{k}{s}}\log^3 n\right\}\right). \tag{40}$$

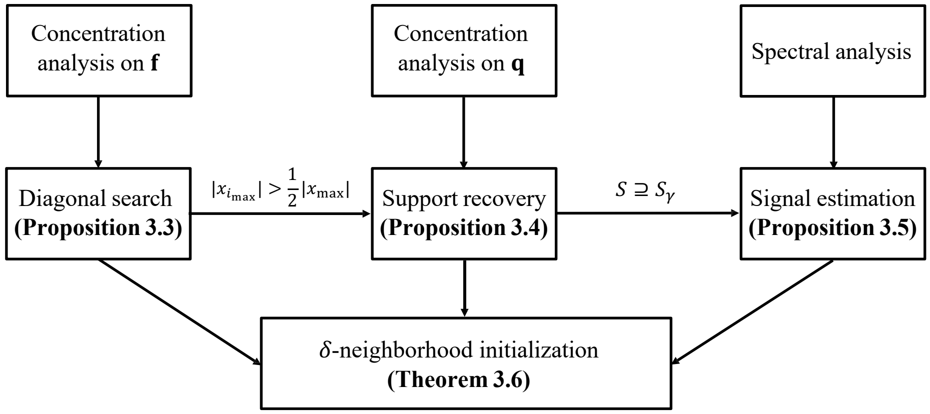


Figure 1: Illustrative diagram of the proof structure for efficient initialization.

In comparison, the sampling complexity of ESP (i.e., (8)) is precisely the former in the $\max$-function of (40). When $k < s\log^4 n$ (i.e., when $\mathbf{x}$ is highly sparse), the latter term in the $\max$-function dominates the former, indicating superiority of the proposed result.

We explain that the $\log^3 n$ term in the sampling complexity of TP results from the spectrum they utilize. Specifically, TP seeks the maximal element of $\frac{1}{m}\sum_{j=1}^{m} y_j^2|a_{ji}|^2$, $i \in [n]$, a typical spectrum in phase retrieval algorithms (see, e.g., SPARTA and CoPRAM). However, this spectrum is heavily tailed as it is the forth power of Gaussian. Thus, TP employs a truncation argument to their analysis in order to derive results similar to Proposition 3.3, which therefore leads to the $\log^3 n$ term. The same phenomenon occurs to Wu & Rebeschini (2021) as well. Whereas in this paper, we introduce a novel exponential spectrum, which has a sub-exponential tail. This enables us to directly derive the concentration inequalities, thus eliminating the $\log^3 n$ term. Despite that, obtaining the final result is not so straightforward. In fact, the challenge lies in deriving the expectation of the proposed spectrum. To address this, we introduce a proxy spectrum that approximates the one used in the practical algorithm, which requires a sophisticated analysis.

Finally, we would like to discuss the initialization relative error δ, which will play an important role in the subsequent stage (i.e., refining stage). In fact, it has been shown that if $\delta < 1/10$, then SPARTA can effectively recover the target signal given sufficient samples ($m = \Omega(k\log n)$). Similar phenomena also happen to other algorithms, such as SWF ($\delta < 1/20$) and HTP ($\delta < 1/12$). Moreover, we hereby point out the dependence on δ in the complexity. It can be discovered from the analysis that ESP needs

$$m = \Omega\left(\frac{\|\mathbf{x}\|^2}{|x_{\max}|^2}\delta^{-2}k\log n\right) \tag{41}$$

samples to produce a desired estimate. This result can be further improved to

$$m = \Omega\left(\max\left\{\frac{\|\mathbf{x}\|^2}{|x_{\max}|^2}, \delta^{-2}\right\}k\log n\right) \tag{42}$$

if we apply the same truncated power method in Cai et al. (2022c) to **Step 3** of Algorithm 1.

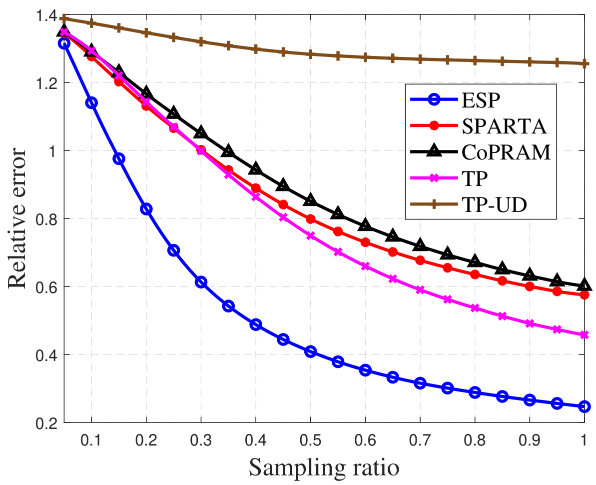
4. Numerical Simulation

In this section, we conduct numerical experiments to evaluate the performance of ESP.

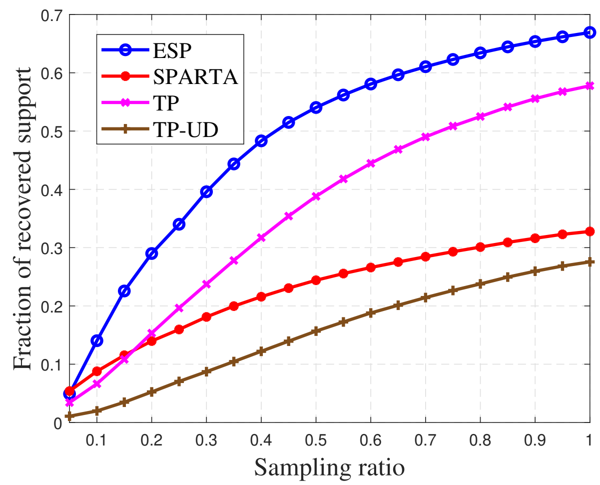
4.1. Experiment settings

In our experiments, the sampling vectors $\{\mathbf{a}_i\}_{i=1}^m$ are n-dimensional standard complex Gaussian random vectors. The input k-sparse signal $\mathbf{x} \in \mathbb{C}^n$ has $\mathrm{supp}(x)$ generated at random and nonzero elements i) drawn from standard complex Gaussian or ii) being 1's, which are called sparse Gaussian and sparse 0-1 signal, respectively. We consider recovering 0-1 signals because it represents a challenging case for phase retrieval.

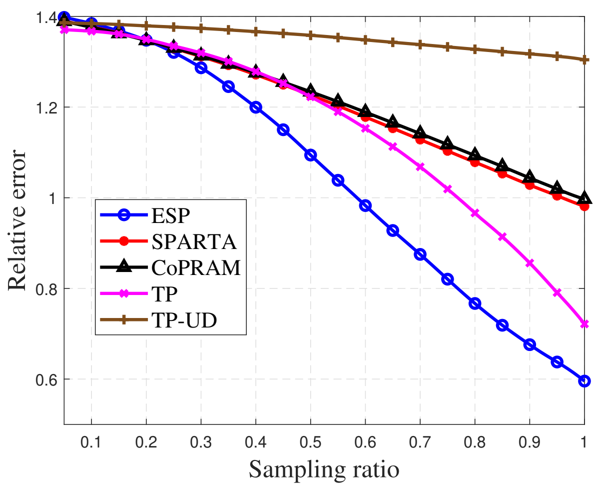
Our first experiment aims at exploring the performance of ESP and compare it with existing methods, including CoPRAM (Jagatap & Hegde, 2019), SPARTA (Wang et al., 2018) and TP (Cai et al., 2022c). For SPARTA and CoPRAM, we only consider the initialization method while ignoring the refining stage. As for the second experiment, we investigate the effect of parameter s $(= \frac{|x_{\max}|^2}{\|x\|^2})$, which plays an important role in the sampling complexity of both ESP and TP (see (8) and (6)). In particular, we consider input signals with $s = \frac{1}{k}$ and $s = \frac{1}{\sqrt{k}}$, respectively. In the third experiment, we further evaluate the performance of ESP and TP under different combinations of sampling number and sparsity. The result is displayed through the phase transition plot.



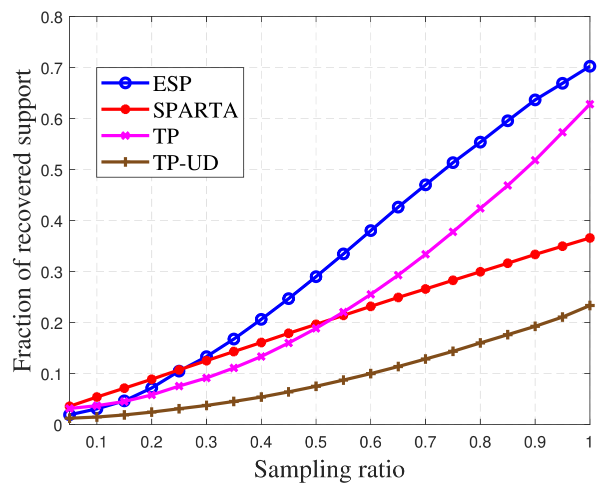
(a) Recovery of sparse Gaussian signal ($k = 10$)



(b) Recovery of sparse Gaussian signal ($k = 10$)



(c) Recovery of sparse 0-1 signal ($k = 10$)



(d) Recovery of sparse 0-1 signal ($k = 10$)

Figure 2: Performance comparison of relative error and fraction of recovered support as a function of sampling ratio.

To compare the performance of different methods, we introduce two metrics: i) relative error and ii) fraction of recovered support. In (Cai et al., 2022c; Wang et al., 2018), the relative error was employed to measure the normalized distance between the signal estimate and ground truth:

$$\text{Relative error} := \frac{\text{dist}(\mathbf{z}, \mathbf{x})}{\|\mathbf{x}\|}.$$

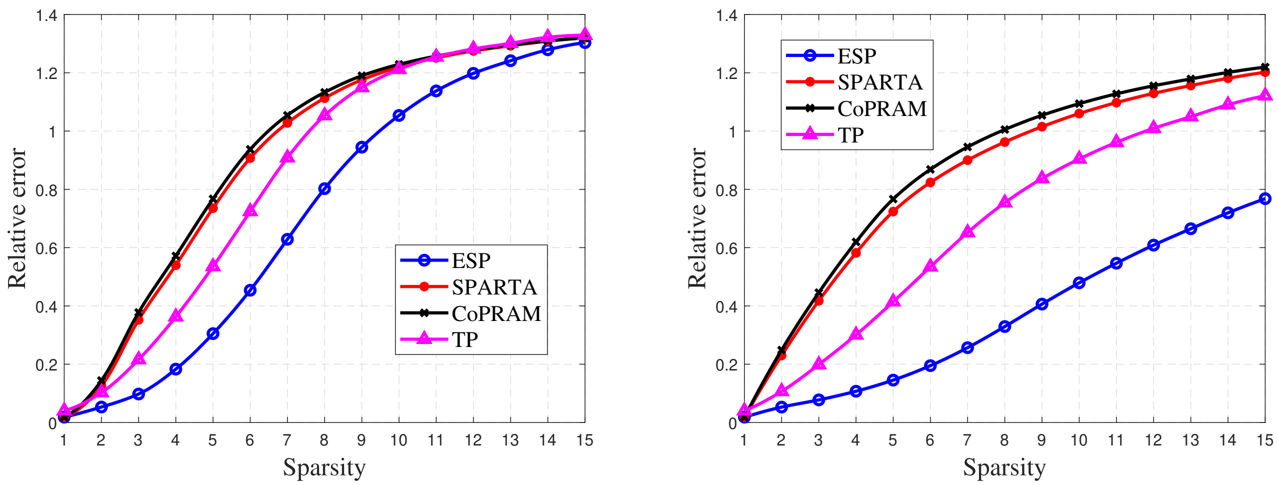
The fraction of recovered support evaluates the percentage of support indices being selected:

$$\text{Fraction of recovered support} := \frac{|\text{supp}(\mathbf{z}) \cap \text{supp}(\mathbf{x})|}{|\text{supp}(\mathbf{x})|}.$$

4.2. Recovery performance v.s. sampling ratio.

We fix the signal dimension $n = 1,000$ and the sparsity $k = 10$, while varying the sampling ratio (i.e., $\frac{m}{n}$) from 0.05 to 1 with step size 0.05. For each sampling ratio, we conduct $1,000$ independent trials and record the (averaged) relative error and fraction of recovered support, respectively. Recall that the original TP involves two hyper-parameters η_1, η_2. We optimized them and set $\eta_1 = 0.2, \eta_2 = 5$. To show the importance of optimization, we use TP-UD to represent TP with un-designed hyper-parameters $\eta_1 = 0.9, \eta_2 = 1.1$ and test its performance.

Figures 2a and 2b show the performance comparison of different methods for recovering sparse Gaussian signals. It can be observed that for all methods under test, the performance improves as the sampling ratio increases. Overall, ESP exhibits the best performance in terms of both the recovery error and fraction of recovered support. CoPRAM and SPARTA have comparable performance, which is inferior to that of TP with optimized hyper-parameters, yet outper-



(a) Case of $s = \frac{1}{k}$ (m =1000)

(b) Case of $s = \frac{1}{\sqrt{k}}$ (m =1000)

Figure 3: Performance comparison of relative error as a function of sparsity. We omit CoPRAM since it uses the same initialization method as SPARTA.

forms that of TP-UD, significantly. In particular, when the sampling ratio is 30%, ESP already recovers 40% support indices. In comparison, the fractions of recovered support for TP, SPARTA and TP-UD are only 24%, 18% and 9%, respectively. The superiority of ESP is due to its diagonal search over the exponential spectral, which, in essence, enlarges the gap between support and non-support elements of the spectrum. Hence, more support indices can be selected.

Figures 2c and 2d depict the performance comparison in recovering sparse 0-1 signals. In general, one observes that all testing methods exhibit a similar behavior as in Figures 2a and 2b, except that the advantage of ESP over the comparative methods is slightly narrowed. In particular, ESP performs better in the high sampling-ratio region, but has comparable performance to TP when the sampling rate is low. This is because the nonzero elements of sparse 0-1 signals have the same magnitude. Consequently, the gap between the support and non-support elements of the spectrum for ESP is essentially in the same order of magnitude as that for TP. In this case, the superiority of ESP in distinguishing support elements becomes less obvious.

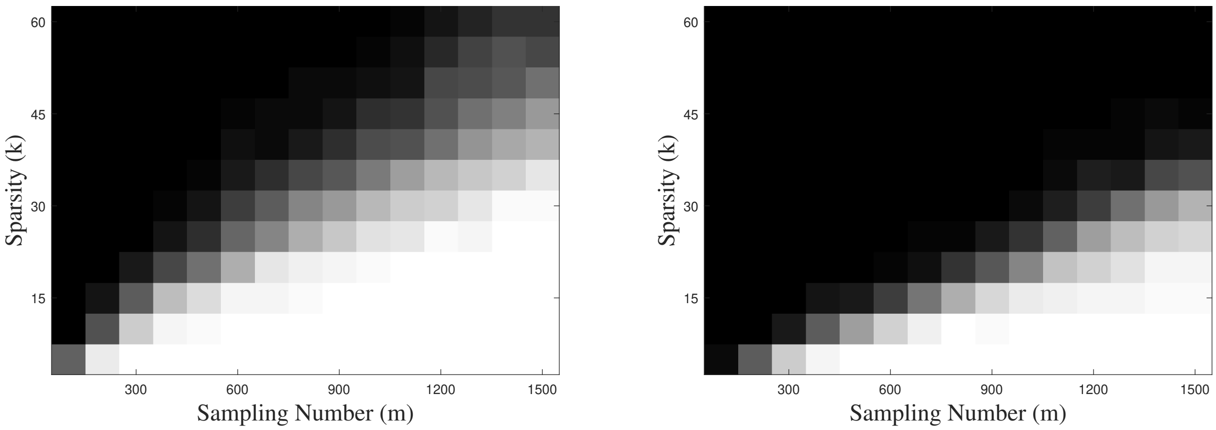
4.3. Effect of s

To explore how the fraction $s := \frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$ influences the performance of signal recovery, we consider two different cases: $s = \frac{1}{k}$ and $s = \frac{1}{\sqrt{k}}$. For the first case, we consider sparse 0-1 signals. As for the second case, we construct input signals in the following way: we first set $\|\mathbf{x}\|^2 = k + \sqrt{k}$ and determine $|x_{\max}|$ according to s, that is, $|x_{\max}| = (\sqrt{k}+1)^{1/2}$. Then, the magnitudes of the remaining $k-1$ nonzero elements in $\mathbf{x}$ are set to 1's. Moreover, we fix the signal dimension $n = 2,000$ and sampling number $m = 1,000$, and vary the sparsity from 1 to 15.

The results of "Relative error v.s. Sparsity" are averaged over 1000 independent trials and displayed in Figures 4a and 4b. One can observe that for all testing algorithms, the relative error goes large as the sparsity level increases. This is consistent with Theorem 3.6 that the lower bound of probability (i.e., $1 - \exp(-c_\delta m) = 1 - \exp(-c_\delta \frac{k}{s} \log n)$) for ESP to achieve a δ-neighbourhood initialization decreases as k goes high. Moreover, while ESP has a similar performance with TP when $s = \frac{1}{k}$, it performs much better when $s = \frac{1}{\sqrt{k}}$. This suggests superiority of our method when s is relatively large, which also matches our discussion in Section 3.6. Specifically, the term of $\sqrt{\frac{k}{s}} \log^3 n$ in (40) dominates $\frac{k}{s} \log n$ when s is large. In this case, ESP requires fewer samples than TP to recover the signal, which in turn suggests better recovery performance of ESP, given the same amount of samples.

4.4. Phase transition plot

With the aim of exploring the performance of ESP and TP under different combinations of sampling number and sparsity, we produce the phase transition plot for these two algorithms. Specifically, we fix the signal dimension as $n = 1000$ and obtain k-sparse target signals from standard complex Gaussian distribution. Meanwhile, we vary the sampling number from 100 to $1,500$ with step size 100 and sparsity from 5 to 60 with step size 5, respectively. In order to measure the performance of ESP and TP, we conduct 200 independent trials and employ the successful recovery rate as the metric, which is defined as the fraction of successful



(a) ESP

(b) TP

Figure 4: The phase transition plots for ESP (left) and TP (right) with signal dimension $n = 2000$. We employ different grey levels to represent different successful recovery rates for each block. Black means that the successful recovery rate is 0, white 1, and gray between 0 and 1.

trials. In particular, a trial is considered to be successful if the relative error does not exceed 0.75.

The phase transition plot is displayed in Figure 4a and 4b, from which one can gain two observations. First of all, ESP holds a noticeably higher successful rate than TP when the sparsity is relatively small compared to the sampling number (e.g., $m = 1,000, k = 30$), which suggests superiority of our method when k is relatively small. In fact, this aligns with our theoretical result, which shows that eliminating the $log^3 n$ factor indeed leads to a reduced sampling complexity empirically. Secondly, when the sparsity is extremely small relative to the sampling number (e.g, $m = 1,000, k = 10$), ESP and TP share comparable successful rates. This perhaps arises from that we use the fixed threshold to judge whether one trial is successful. In this case, the fixed threshold can be too large to distinguish the difference between the performance of ESP and TP.

5. Conclusion

In this paper, we have proposed a novel initialization method called ESP for sparse phase retrieval. Through theoretical analysis, we have shown that ESP produces a δ-neighbourhood initialization of $\mathbf{x}$ when $m = \Omega(\frac{\|\mathbf{x}\|^2}{|x_{\max}|^2} k \log n)$, which improves upon some existing results that depend additionally on a $\log^3 n$ term. Moreover, through empirical simulations, we have demonstrated that ESP has better recovery performance of sparse signals compared to existing methods, while maintaining the ease of implementation. Therefore, ESP is an attractive alternative to TP for the initialization task in phase retrieval.

In Liu et al. (2021), an interesting algorithm was proposed to achieve the information-theoretic sampling complexity (3). However, this algorithm involves exhaustive search over all possible $k \times k$ sub-matrices of an $n \times n$ matrix to find one with the largest eigenvalue, which is NP hard. To date, no practical algorithm has been reported to attain the information-theoretic bound. Whether it is possible to find one such algorithm and thus bridge the statistical-to-computational gap for sparse phase retrieval remains an interesting open question.

Acknowledgements

This work was supported in part by the National Natural Science Foundation of China under Grant 61971146.

Impact Statement

This paper presents work involving with the sampling complexity in the field of the sparse phase retrieval problem. There are many potential societal consequences of our work, none which we feel must be specifically highlighted here.

References

Bates, R. H. T. Fourier phase problems are uniquely solvable in more than one dimensional. i: Underlying theory. *Optik*, 61(3):247–262, 1982.

Bendory, T. and Eldar, Y. C. Phase retrieval from stft measurements via non-convex optimization. In *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.*, pp. 4770–4774, 2017. doi: 10.1109/ICASSP.2017.7953062.

Bruck, Y. and Sodin, L. On the ambiguity of the image reconstruction problem. *Opt. Commun.*, 30(3):304–308, 1979. ISSN 0030-4018. doi: https://doi.org/10.1016/0030-4018(79)90358-4.

Bunk, O., Diaz, A., Pfeiffer, F., David, C., Schmitt, B., Satapathy, D. K., and van der Veen, J. F. Diffractive imaging for periodic samples: retrieving one-dimensional concentration profiles across microfluidic channels. *Acta Crystallogr. A*, 63 Pt 4:306–14, 2007.

Cai, J.-F., Jiao, Y., Lu, X., and You, J. Sample-efficient sparse phase retrieval via stochastic alternating minimization. *IEEE Trans. Signal Process.*, 70:4951–4966, 2022a. doi: 10.1109/TSP.2022.3214091.

Cai, J.-F., Li, J., Lu, X., and You, J. Sparse signal recovery from phaseless measurements via hard thresholding pursuit. *Appl. Comput. Harmon. Anal.*, 56:367–390, 2022b. ISSN 1063-5203. doi: https://doi.org/10.1016/j.acha.2021.10.002.

Cai, J.-F., Li, J., and You, J. Provable sample-efficient sparse phase retrieval initialized by truncated power method. *Inverse Probl.*, 39, 2022c.

Cai, T. T., Li, X., and Ma, Z. Optimal rates of convergence for noisy sparse phase retrieval via thresholded Wirtinger flow. *Ann. Stat.*, 44(5):2221 – 2251, 2016. doi: 10.1214/16-AOS1443.

Candes, E. and Tao, T. Decoding by linear programming. *IEEE Trans. Inf. Theory*, 51(12):4203–4215, 2005. doi: 10.1109/TIT.2005.858979.

Candès, E. J., Eldar, Y. C., Strohmer, T., and Voroninski, V. Phase retrieval via matrix completion. *SIAM Rev.*, 57(2):225–251, 2015a. doi: 10.1137/151005099.

Candès, E. J., Li, X., and Soltanolkotabi, M. Phase retrieval via wirtinger flow: Theory and algorithms. *IEEE Trans. Inf. Theory*, 61(4):1985–2007, 2015b. doi: 10.1109/TIT.2015.2399924.

Dainty, J. and Fienup, J. Phase retrieval and image reconstruction for astronomy. *Image Recovery: Theory Appl.*, 13, Jan. 1987.

Donoho, D. Compressed sensing. *IEEE Trans. Inf. Theory*, 52(4):1289–1306, 2006. doi: 10.1109/TIT.2006.871582.

Eldar, Y. C. and Mendelson, S. Phase retrieval: Stability and recovery guarantees. *Appl. Comput. Harmon. Anal.*, 36(3):473–494, 2014.

Fu, N., Zheng, P., Li, X., and Qiao, L. Weighted phase retrieval of fourier measurement with outliers: Measurement structure and reconstruction algorithm. *IEEE Trans. Instrum. Meas.*, 70:1–14, 2021. doi: 10.1109/TIM.2020.3032914.

Gao, B. and Xu, Z. Phaseless recovery using the gauss–newton method. *IEEE Trans. Signal Process.*, 65 (22):5885–5896, 2017. doi: 10.1109/TSP.2017.2742981.

Goldstein, T. and Studer, C. Phasemax: Convex phase retrieval via basis pursuit. *IEEE Trans. Inf. Theory*, 64 (4):2675–2689, 2018.

Guo, Y., Wu, Y., Li, Y., Rao, X., and Rao, C. Deep phase retrieval for astronomical Shack–Hartmann wavefront sensors. *Mon. Not. R. Astron. Soc.*, 510(3):4347–4354, 12 2021. ISSN 0035-8711. doi: 10.1093/mnras/stab3690.

Hayes, M. The reconstruction of a multidimensional sequence from the phase or magnitude of its fourier transform. *IEEE Trans. Acoust. Speech Signal Processing*, 30 (2):140–154, 1982. doi: 10.1109/tassp.1982.1163863.

Hofstetter, E. Construction of time-limited functions with specified autocorrelation functions. *IEEE Trans. Inf. Theory*, 10(2):119–126, 1964. doi: 10.1109/TIT.1964.1053648.

Jagatap, G. and Hegde, C. Sample-efficient algorithms for recovering structured signals from magnitude-only measurements. *IEEE Trans. Inf. Theory*, 65(7):4434–4456, 2019. doi: 10.1109/TIT.2019.2902924.

Langley, P. Crafting papers on machine learning. In Langley, P. (ed.), *Proceedings of the 17th International Conference on Machine Learning (ICML 2000)*, pp. 1207–1216, Stanford, CA, 2000. Morgan Kaufmann.

Liu, Z., Ghosh, S., and Scarlett, J. Towards sample-optimal compressive phase retrieval with sparse and generative priors. In *Adv. Neural Inf. Process. Syst.*, 2021.

Miao, J., Ishikawa, T., Shen, Q., and Earnest, T. Extending x-ray crystallography to allow the imaging of noncrystalline materials, cells, and single protein complexes. *Annual review of physical chemistry*, 59:387–410, 02 2008. doi: 10.1146/annurev.physchem.59.032607.093642.

Millane, R. P. Phase retrieval in crystallography and optics. *Journal of The Optical Society of America A-optics Image Science and Vision*, 7:394–411, 1990.

Moravec, M. L., Romberg, J. K., and Baraniuk, R. Compressive phase retrieval. In *SPIE Optical Engineering + Applications*, 2007.

Netrapalli, P., Jain, P., and Sanghavi, S. Phase retrieval using alternating minimization. *IEEE Trans. Signal Process.*, 63 (18):4814–4826, 2015. doi: 10.1109/TSP.2015.2448516.

Ohlsson, H., Yang, A. Y., Dong, R., and Sastry, S. S. Cprl – an extension of compressive sensing to the phase retrieval problem. In *Adv. Neural Inf. Process. Syst.*, 2012.

Shechtman, Y., Eldar, Y. C., Cohen, O., Chapman, H. N., Miao, J., and Segev, M. Phase retrieval with application to optical imaging: A contemporary overview. *IEEE Signal Process. Mag.*, 32(3):87–109, 2015. doi: 10.1109/MSP.2014.2352673.

Tong, Z., Liu, Z., Hu, C., Wang, J., and Han, S. Preconditioned deconvolution method for high-resolution ghost imaging. *Photonics Res.*, 9(11):1069, 6 2021.

Truong, L. V. and Scarlett, J. Support recovery in the phase retrieval model: Information-theoretic fundamental limit. *IEEE Trans. Inf. Theory*, 66(12):7887–7910, 2020. doi: 10.1109/TIT.2020.3031218.

Varnavides, G., Ribet, S. M., Zeltmann, S. E., Yu, Y., Savitzky, B. H., Dravid, V. P., Scott, M. C., and Ophus, C. Iterative phase retrieval algorithms for scanning transmission electron microscopy, 2023.

Vershynin, R. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2018. doi: 10.1017/9781108231596.

Wang, G., Zhang, L., Giannakis, G. B., Akçakaya, M., and Chen, J. Sparse phase retrieval via truncated amplitude flow. *IEEE Trans. Signal Process.*, 66(2):479–491, 2018. doi: 10.1109/TSP.2017.2771733.

Wu, F. and Rebeschini, P. Hadamard wirtinger flow for sparse phase retrieval. In *Int. Conf. Artif. Intell. Statist.*, pp. 982–990, 2021.

Xu, M., Dong, D., and Wang, J. Subspace phase retrieval. *IEEE Trans. Inf. Theory*, 70(6):4538–4570, 2024. doi: 10.1109/TIT.2024.3386821.

Yu, R., Kennedy, S., Paganin, D., and Jesson, D. Phase retrieval low energy electron microscopy. *Micron*, 41(3): 232–238, 2010. ISSN 0968-4328. doi: https://doi.org/10.1016/j.micron.2009.10.010.

Yuan, Z., Wang, H., and Wang, Q. Phase retrieval via sparse wirtinger flow. *J. Comput. Appl. Math.*, 355:162–173, 2019.

Zhang, H. and Liang, Y. Reshaped wirtinger flow for solving quadratic system of equations. *Adv. Neural Inf. Process. Syst.*, 29:2622–2630, 2016.

Zhang, Y., Mao, X., Wang, J., and Liu, W. Demo: A flexible deartifacting module for compressed sensing mri. *IEEE J. Sel. Top. Signal Process.*, 16(4):725–736, 2022. doi: 10.1109/JSTSP.2022.3158057.

A. Proofs of Propositions

In this section, we present the proof of each proposition. The lemmas used in the proof is deferred to Section B. In the following, we define

$$
\begin{aligned}
f_i &:= \frac{1}{m}\sum_{j=1}^{m}\big(\frac{1}{2}-\exp(-\frac{y_j^2}{\lambda^2})\big)|a_{ji}|^2,\\
\tilde{f}_i &:= \frac{1}{m}\sum_{j=1}^{m}\big(\frac{1}{2}-\exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\big)|a_{ji}|^2,\\
\mathbf{q} &:= \frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{2}-\exp\left(-\frac{y_j^2}{\lambda^2}\right)\right)\mathbf{a}_j\mathbf{a}_j^*\mathbf{e}_{i_{\max}},\\
\tilde{\mathbf{q}} &:= \frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{2}-\exp\left(-\frac{y_j^2}{\|\mathbf{x}\|^2}\right)\right)\mathbf{a}_j\mathbf{a}_j^*\mathbf{e}_{i_{\max}},
\end{aligned}
$$

for notation simplicity.

A.1. Proof of Proposition 3.3

Proof. Taking $\eta=\frac{3}{16}$ in Lemma B.3 and Lemma B.4 yields that

$$
\begin{aligned}
\frac{|x_{i_{\max}}|^2}{4\|\mathbf{x}\|^2} &= \frac{|x_i|^2}{4\|\mathbf{x}\|^2}+(\tilde{f}_i-\frac{|x_i|^2}{4\|\mathbf{x}\|^2})-(\tilde{f}_{i_{\max}}-\frac{|x_{i_{\max}}|^2}{4\|\mathbf{x}\|^2})+(\tilde{f}_{i_{\max}}-\tilde{f}_i)-(f_{i_{\max}}-f_i)+(f_{i_{\max}}-f_i)\\
&\overset{(a)}{\geq} \frac{|x_i|^2}{4\|\mathbf{x}\|^2}-2\max_{l\in[n]}\left|\tilde{f}_l-\frac{|x_l|^2}{4\|\mathbf{x}\|^2}\right|-\max_{1\le p\neq q\le n}|(\tilde{f}_p-\tilde{f}_q)-(f_p-f_q)|\\
&\overset{(b)}{\geq} \frac{|x_i|^2}{4\|\mathbf{x}\|^2}-\frac{3}{4}\frac{|x_{\max}|^2}{4\|\mathbf{x}\|^2},
\end{aligned}
$$

where (a) comes from the definition of $i_{\max}$ and (b) uses Lemma B.3 and Lemma B.4 with $\eta=\frac{3}{16}$. Let $i=\arg\max_j|x_j|$ and we can easily conclude that $|x_{i_{\max}}|>\frac{1}{2}|x_{\max}|$. □

A.2. Proof of Proposition 3.4

Proof. Take $\epsilon=\frac{1}{16}$ in Lemma B.5 and employ the equation (13), we can conclude that for any $l\in S_\gamma$,

$$
\begin{aligned}
|q_l| &= |\mathbb{E}[\tilde{q}_l]+q_l-\mathbb{E}[\tilde{q}_l]|\\
&\overset{(a)}{\geq} |\mathbb{E}[\tilde{q}_l]|-\max_{i\in[n]}|q_i-\mathbb{E}[\tilde{q}_i]|\\
&\overset{(b)}{\geq} \frac{|x_{i_{\max}}^*||x_l|}{4\|\mathbf{x}\|^2}-\frac{1}{16}\times\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|}\\
&\overset{(c)}{>} \frac{\gamma|x_{\max}|}{16\sqrt{k}\|\mathbf{x}\|}-\frac{1}{16}\times\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|}\\
&\geq \frac{\gamma|x_{\max}|}{32\sqrt{k}\|\mathbf{x}\|},
\end{aligned}
\tag{43}
$$

where (a) comes from the triangle inequality, (b) employs Lemma B.5 with $\epsilon=\frac{1}{16}$ and (c) is based on the assumption that $|x_{i_{\max}}|>\frac{1}{2}|x_{\max}|$ together with the definition of S_γ.

Similarly, for $i\notin \mathrm{supp}(\mathbf{x})$, we can derive that

$$
\begin{aligned}
|q_i| &= |\mathbb{E}[\tilde{q}_i]+q_i-\mathbb{E}[\tilde{q}_i]|\\
&\overset{(a)}{\leq} |\mathbb{E}[\tilde{q}_i]|+\max_{l\in[n]}|q_l-\mathbb{E}[\tilde{q}_l]|
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(b)}{\leq} 0 + \frac{1}{16} \times \frac{\gamma |x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|} \\
&= \frac{\gamma |x_{\max}|}{32\sqrt{k}\|\mathbf{x}\|},
\end{aligned} \tag{44}
$$

where (a) is from the triangle inequality and (b) is due to Lemma B.5 with $\epsilon = \frac{1}{16}$.

Obviously, (43) and (44) indicate that $|q_l| > |q_i|$ for any $l \in S_\gamma$ and $i \notin \operatorname{supp}(\mathbf{x})$. Therefore, ESP can definitely select all indices in S_γ with probability at least $1 - \exp(-c_\gamma m)$ in the second step provided that $m \geq \frac{C_\gamma k}{s} \log n$ for sufficiently large constant C_γ depending on γ. □

A.3. Proof of Proposition 3.5

Proof. Define $\mathbf{x}^0 = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\mathbf{z}^0 = \frac{\mathbf{z}}{\lambda}$. From the assumption ($S_\gamma \subseteq S$) and the definition of S_γ, we have $\|\mathbf{x}^0_{S^c}\|^2 \leq 1 - \|\mathbf{x}^0_{S_\gamma}\|^2 \leq k \times \frac{\gamma^2}{4k} = \frac{\gamma^2}{4}$. Recall that $\mathbf{z}^0$ is the unit eigenvector corresponding to the largest eigenvalue of $\mathbf{L}_S$ and $\tau(\mathbf{\Delta}, k) \leq \frac{3}{64}$ holds with probability at least $1 - \exp(-cm)$ when $m \geq Ck \log n$ from Lemma B.7. Employing Lemma B.9 with $\Lambda = S$, we have

$$
\begin{aligned}
\operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0_S)^2 \leq& \|\mathbf{x}^0_S\|^2 + 1 - \frac{6\|\mathbf{x}^0_S\|}{\sqrt{9 + 1024\tau^2(\mathbf{\Delta}, k)}} \\
&\overset{(a)}{\leq} \max\left\{2 - \frac{2}{\sqrt{1 + \frac{1024}{9}\tau^2(\mathbf{\Delta}, k)}}, 2 - \frac{\gamma^2}{4} - \frac{2\sqrt{1 - \frac{\gamma^2}{4}}}{\sqrt{1 + \frac{1024}{9}\tau^2(\mathbf{\Delta}, k)}}\right\} \\
&\overset{(b)}{\leq} \max\left\{2 - \frac{2}{1 + \frac{1024}{9}\tau^2(\mathbf{\Delta}, k)}, 2 - \frac{\gamma^2}{4} - \frac{2(1 - \frac{\gamma^2}{4})}{1 + \frac{1024}{9}\tau^2(\mathbf{\Delta}, k)}\right\} \\
&= \max\left\{2 - \frac{18}{9 + 1024\tau^2(\mathbf{\Delta}, k)}, 2 - \frac{\gamma^2}{4} - \frac{18(1 - \frac{\gamma^2}{4})}{9 + 1024\tau^2(\mathbf{\Delta}, k)}\right\} \\
&\leq \frac{2048\tau^2(\mathbf{\Delta}, k)}{9 + 1024\tau^2(\mathbf{\Delta}, k)} + \frac{\gamma^2}{4} \\
&\leq \frac{2048}{9}\tau^2(\mathbf{\Delta}, k) + \frac{\gamma^2}{4} \\
&\overset{(c)}{\leq} \frac{2048}{9} \times 36t^2 + \frac{\gamma^2}{4} \\
&= 8192t^2 + \frac{\gamma^2}{4},
\end{aligned} \tag{45}
$$

where (a) comes from $1 - \frac{\gamma^2}{4} \leq \|\mathbf{x}^0_S\|^2 \leq 1$, (b) is because $x^2 - x \geq 0, x \geq 1$, (c) uses Lemma B.7 and holds with probability $1 - \exp(-c_{1,t} m)$ provided that $m \geq C_{1,t} k \log n$.

We then decompose the distance between $\mathbf{z}^0$ and $\mathbf{x}^0$ as

$$
\begin{aligned}
\operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0)^2 &= \operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0_S)^2 + \|\mathbf{x}^0_{S^c}\|^2 \\
&\overset{(a)}{\leq} 8192t^2 + \frac{\gamma^2}{2}.
\end{aligned}
$$

where (a) is based on (45) and the fact $\|\mathbf{x}^0_{S^c}\|^2 \leq \frac{\gamma^2}{4}$. Therefore, let $t = \frac{\delta}{512}$ and $\gamma = \frac{\delta}{4}$, when $m \geq C_\delta k \log n$ it holds with probability at least $1 - \exp(-c_\delta m)$ that

$$
\operatorname{dist}(\mathbf{z}^0, \mathbf{x}^0) \leq \frac{\delta}{4}. \tag{46}
$$

Based on Lemma B.8 and the definition of $\mathbf{z}^0$, if $m \geq C_\delta k \log n$, with probability exceeding $1 - \exp(-c_\delta m)$,

$$\begin{aligned}
\mathrm{dist}(\mathbf{z}, \mathbf{x}) &\leq \mathrm{dist}(\lambda \mathbf{z}^0, \lambda \mathbf{x}^0) + \mathrm{dist}(\lambda \mathbf{x}^0, \mathbf{x}) \\
&\leq \lambda \mathrm{dist}(\mathbf{z}^0, \mathbf{x}^0) + \big|\lambda - \|\mathbf{x}\|\big| \\
&\overset{(a)}{\leq} \frac{\delta}{4}\lambda + \frac{\delta}{2}\|\mathbf{x}\| \\
&\overset{(b)}{\leq} \frac{\delta}{2}\|\mathbf{x}\| + \frac{\delta}{2}\|\mathbf{x}\| \\
&= \delta \|\mathbf{x}\|,
\end{aligned}$$

where (a) comes from (46) and Lemma B.1 with $\beta = \frac{\delta}{2}$ and (b) is based on Lemma B.1 with $\beta = 1$. Therefore, we complete the proof. □

B. Auxiliary lemmas

We first present two lemmas that are useful to characterize the difference between the spectrum $\mathbf{L}$ and $\tilde{\mathbf{L}}$.

B.1. Difference between L and $\tilde{\mathbf{L}}$

From (9) and (10), the difference between $\mathbf{L}$ and $\tilde{\mathbf{L}}$ is due to the different denominators (i.e., λ^2 and $\|\mathbf{x}\|^2$) in their respective exponential terms. The next lemma shows that this difference can be arbitrarily small, given sufficient samples.

Lemma B.1 (Lemma 7.8 of Candès et al. (2015b))**.** *For any constant $\beta > 0$ and any set $\mathcal{S}$ satisfying $|\mathcal{S}| \leq k$, suppose $\mathbf{a}_j$'s are n-dimensional complex Gaussian random vectors. Then, if $m \geq C_\beta k \log k$,*

$$(1-\beta)\|\mathbf{x}\|^2 \leq \frac{1}{m}\sum_{j=1}^{m} |\mathbf{a}_j^* \mathbf{x}|^2 \leq (1+\beta)\|\mathbf{x}\|^2$$

holds for all $\mathbf{x} \in \mathbb{C}^{\mathcal{S}}$ with probability at least $1 - e^{-c_\beta m}$, where c_β and C_β are constant depending on β.

We can extend Lemma B.1 by relating β with $\frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$.

Lemma B.2. *For any constant $\beta > 0$ and any set $\mathcal{S}$ satisfying $|\mathcal{S}| \leq k$, suppose $\mathbf{a}_j$'s are n-dimensional complex Gaussian random vectors. Then, if $m \geq \frac{k\|\mathbf{x}\|^2}{C'_\beta |x_{\max}|^2}\log n$,*

$$\left(1 - \beta\sqrt{\frac{s}{k}}\right)\|\mathbf{x}\|^2 \leq \frac{1}{m}\sum_{j=1}^{m} |\mathbf{a}_j^* \mathbf{x}|^2 \leq \left(1 + \beta\sqrt{\frac{s}{k}}\right)\|\mathbf{x}\|^2$$

holds for all $\mathbf{x} \in \mathbb{C}^{\mathcal{S}}$ with probability at least $1 - e^{-c'_\beta m}$, where c'_β and C'_β are constant related to β and $\frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$.

Proof. For $j = 1, 2, 3, \cdots, n$, since $\mathbf{a}_j \in \mathcal{CN}(n)$, we can derive that $|\mathbf{a}_j^* \mathbf{x}|^2$'s are an sub-exponential random variables with ψ_1 norm $c\|\mathbf{x}\|^2$ and expectations $\mathbb{E}(|\mathbf{a}_j^* \mathbf{x}|^2) = \|\mathbf{x}\|^2$. Hence, from Bernstein's inequality, we have

$$P\left(\left|\frac{1}{m}\sum_{j=1}^{m} |\mathbf{a}_j^* \mathbf{x}|^2 - \|\mathbf{x}\|^2\right| > \beta\sqrt{\frac{s}{k}}\|\mathbf{x}\|^2\right) \leq 2\exp\left(-C'_\beta m \min\{\frac{\beta^2 s\|\mathbf{x}\|^4}{k\|\mathbf{x}\|^4}, \frac{\beta\sqrt{s}\|\mathbf{x}\|^2}{\sqrt{k}\|\mathbf{x}\|^2}\}\right).$$

Recall that $s = \frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$. Therefore, when $m \geq \frac{k}{C'_\beta s}\log n$), with probability exceeding $1 - e^{-c'_\beta m}$, it holds that

$$(1 - \beta\sqrt{\frac{s}{k}})\|\mathbf{x}\|^2 \leq \frac{1}{m}\sum_{j=1}^{m} |\mathbf{a}_j^* \mathbf{x}|^2 \leq (1 + \beta\sqrt{\frac{s}{k}})\|\mathbf{x}\|^2.$$

□

Next, we shall establish several concentration and deviation inequalities on these random variables (vectors).

B.2. Concentration and deviation analysis

Lemma B.3. *Denote* $s = \frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$. *For any constant* $0 < \eta < 1$ *the inequality*

$$\max_{i\in[n]} \left| \tilde{f}_i - \frac{|x_i|^2}{4\|\mathbf{x}\|^2} \right| < \frac{\eta}{3} s$$

holds with probability $1 - \exp(-c_\eta m)$ *provided that* $m \geq \frac{C_\eta}{s^2} \log n$, *where* c_η *and* C_η *are constant depending on* η.

Proof of Lemma B.3. From the definition of $\tilde{f}_i$, we have

$$\left| \tilde{f}_i - \frac{|x_i|^2}{4\|\mathbf{x}\|^2} \right| \leq \left| \frac{1}{2m} \sum_{j=1}^{m} |a_{ji}|^2 - \frac{1}{2} \right| + \left| \frac{1}{m} \sum_{j=1}^{m} \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2}) |a_{ji}|^2 - \frac{1}{2} + \frac{|x_i|^2}{4\|\mathbf{x}\|^2} \right|. \tag{47}$$

It is revealed in (11) that $\mathbb{E}(\tilde{f}_i) = \frac{|x_i|^2}{4\|\mathbf{x}\|^2}$. Note that $|a_{ji}|^2$ and $\exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})|a_{ji}|^2$ are both sub-exponential random variables. Employing the Bernstein's inequality yields

$$P\left(\left| \frac{1}{2m} \sum_{j=1}^{m} |a_{ji}|^2 - \frac{1}{2} \right| > \frac{\eta}{6} s \right) \leq \exp\left(-c_1 \min\left\{ \frac{\eta^2 m s^2}{36 K_1^2}, \frac{\eta m s}{6 K_1} \right\} \right), \tag{48}$$

$$P\left(\left| \frac{1}{m} \sum_{j=1}^{m} \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2}) |a_{ji}|^2 - \frac{1}{2} + \frac{|x_i|^2}{4\|\mathbf{x}^2\|} \right| > \frac{\eta}{6} s \right) \leq \exp\left(-c_2 \min\left\{ \frac{\eta^2 m s^2}{36 K_2^2}, \frac{\eta m s}{6 K_2} \right\} \right), \tag{49}$$

where c_1, c_2 are constants, $K_1 = \max_{j\in[m]} \left\| |a_{ji}|^2 \right\|_{\psi_1}$ and $K_2 = \max_{j\in[m]} \left\| \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2}) |a_{ji}|^2 \right\|_{\psi_1}$ are sub-exponential norms respectively. Denote the aforementioned probabilities in (48) and (49) as Pro_1 and Pro_2. Recall the definition of $\tilde{f}_i$, we derive from (47) that

$$P\left(|\tilde{f}_i - \mathbb{E}(\tilde{f}_i)| > \frac{\eta}{3} s \right) \leq Pro_1 + Pro_2.$$

Taking the union bound for $i \in [n]$ and employ the notation $s = \frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$, we conclude that when $m \geq \frac{C_\eta}{s^2} \log n$, the inequality

$$\max_{i\in[n]} \{ |\tilde{f}_i - \mathbb{E}(\tilde{f}_i)| \} \leq \frac{\eta}{3} s$$

holds with probability over $1 - \exp(-c_\eta m)$, which is definitely the result of this lemma. $\square$

Lemma B.4. *Denote* $s = \frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$. *For any* $0 < \eta < 1$,

$$\max_{1\leq p\neq q\leq n} |(\tilde{f}_p - \tilde{f}_q) - (f_p - f_q)| \leq \frac{\eta}{3} s$$

holds with probability exceeding $1 - \exp(-d_\eta m)$ *as long as* $m \geq \frac{D_\eta}{s^2} \log n$, *where* d_η *and* D_η *are constant related to* η.

Proof of Lemma B.4. From the definition of $\mathbf{f}_0$, for any fixed $p \neq q$,

$$
\begin{aligned}
|(\tilde{f}_p - \tilde{f}_q) - (f_p - f_q)| &= \left| \frac{1}{m} \sum_{j=1}^{m} \big(\exp(-\frac{y_j^2}{\lambda^2}) - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2}) \big) \big(|a_{jp}|^2 - |a_{jq}|^2 \big) \right| \\
&\overset{(a)}{=} \max_{j\in[m]} \exp(-\frac{y_j^2}{\xi}) \frac{1}{\xi^2} |\lambda^2 - \|\mathbf{x}\|^2| \left| \frac{1}{m} \sum_{j=1}^{m} \big(|a_{jp}|^2 - |a_{jq}|^2 \big) \right| \\
&\overset{(b)}{\leq} \frac{1}{\xi} |\lambda^2 - \|\mathbf{x}\|^2| \left| \frac{1}{m} \sum_{j=1}^{m} \big(|a_{jp}|^2 - |a_{jq}|^2 \big) \right| \\
&\overset{(c)}{\leq} \frac{2}{3\|\mathbf{x}\|^2} \frac{\|\mathbf{x}\|^2}{2} \left| \frac{1}{m} \sum_{j=1}^{m} \big(|a_{jp}|^2 - |a_{jq}|^2 \big) \right| \\
&= \frac{1}{3} \left| \frac{1}{m} \sum_{j=1}^{m} \big(|a_{jp}|^2 - |a_{jq}|^2 \big) \right|,
\end{aligned}
\tag{50}
$$

where (a) comes from Lagrange's mean value formula with $F(p) = \exp(-\frac{y_j^2}{p})$, (b) is due to the numeric inequality $xe^{-x} \leq 1$, for $x \geq 0$ and (c) employs Lemma B.1 with $\delta = \frac{1}{2}$. Note that for any $j \in [m]$,$|a_{jp}|^2 - |a_{jq}|^2$ is a sub-exponential variable with constant sub-exponential norm. From Bernstein's inequality,

$$
P\left(\left| \frac{1}{m} \sum_{j=1}^{m} \big(|a_{jp}|^2 - |a_{jq}|^2 \big) \right| > \frac{\eta}{3} s \right) \leq \exp\left(c_3 \min\left\{ \frac{\eta^2 m s^2}{9K_3^2}, \frac{\eta m s}{3K_3} \right\} \right), \tag{51}
$$

where $K_3 = \max_{j\in[m]} \left\| |a_{jp}|^2 - |a_{jq}|^2 \right\|_{\psi_1}$ is the sub-exponential norm and c_3 is a constant. Take the union bound of (51), it can be shown that when $m \geq \frac{D_\eta}{s^2} \log n$,

$$
\max_{1\leq p\neq q\leq n} |(\tilde{f}_p - \tilde{f}_q) - (f_p - f_q)| \leq \frac{\eta}{3} s
$$

holds with probability exceeding $1 - \exp(-d_\eta m)$. $\square$

Lemma B.5. *Denote $s = \frac{|x_{\max}|^2}{\|\mathbf{x}\|^2}$. Then for any $\epsilon > 0$,*

$$
\max_{l\in[n]} |q_l - \mathbb{E}[\tilde{q}_l]| \leq \epsilon \frac{\gamma |x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|} \tag{52}
$$

holds with probability exceeding $1 - \exp(-d_\epsilon m)$ provided that $m \geq D_\epsilon \frac{k}{\gamma^2 s} \log n$, where d_ϵ and D_ϵ are constant depending on ϵ.

Proof of Lemma B.5. The proof of this Lemma is divided into two cases: $l = i_{\max}$ and $l \neq i_{\max}$.

† **Case 1:** $l \neq i_{\max}$

To begin with, we introduce $\tilde{q}_l$ as

$$
\tilde{q}_l = \frac{1}{m} \sum_{j=1}^{m} \big(\frac{1}{2} - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2}) \big) a_{jl} a_{ji_{\max}}^*.
$$

Note that both $a_{jl} a_{ji_{\max}}^*$ and $\big(\frac{1}{2} - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2}) \big) a_{jl} a_{ji_{\max}}^*$ are sub-exponential random variables. Employing Bernstein's inequality, we can derive

$$P\left(\left|\frac{1}{m}\sum_{j=1}^{m} a_{jl}a_{ji_{\max}}^{*}\right| > \frac{\epsilon}{2}\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|}\right) \le \exp\left(-c_4\min\left\{\frac{\epsilon^2\gamma^2 m|x_{\max}|^2}{16K_4^2 k\|\mathbf{x}\|^2}, \frac{\epsilon\gamma m|x_{\max}|}{4K_4\sqrt{k}\|\mathbf{x}\|}\right\}\right), \tag{53}$$

$$P\left(|\tilde{q}_l - \mathbb{E}[\tilde{q}_l]| > \frac{\epsilon}{2}\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|}\right) \le \exp\left(-c_5\min\left\{\frac{\epsilon^2\gamma^2 m|x_{\max}|^2}{16K_5^2 k\|\mathbf{x}\|^2}, \frac{\epsilon\gamma m|x_{\max}|}{4K_5\sqrt{k}\|\mathbf{x}\|}\right\}\right), \tag{54}$$

where $K_4 = \max_{j\in[m]}\left\|a_{jl}a_{ji_{\max}}^{*}\right\|_{\psi_1}$, $K_5 = \max_{j\in[m]}\left\|\exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})a_{jl}a_{ji_{\max}}^{*}\right\|_{\psi_1}$ are sub-exponential norms and c_4, c_5 are two constant. Take the union bound of (54) for $l \in [n], l \neq i_{\max}$ and consider all possible cases for $i_{\max}$, we have

$$\max_{j\in[n]}\max_{l\in[n], l\neq j}|\tilde{q}_l - \mathbb{E}[\tilde{q}_l]| \le \frac{\epsilon}{2}\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|} \tag{55}$$

holds with probability at least $1-\exp(-d_{1,\epsilon}m)$ provided that $m \ge D_{1,\epsilon}\frac{k}{\gamma^2 s}\log n$.
Next, through numerical analysis of $q_l - \tilde{q}_l$, we can derive that

$$\begin{aligned}
|q_l - \tilde{q}_l| &= \left|\frac{1}{m}\sum_{j=1}^{m}\big(\exp(-\frac{y_j^2}{\lambda^2}) - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\big)a_{jl}a_{ji_{\max}}^{*}\right| \\
&\le \max_{j\in[m]}\left|\exp(-\frac{y_j^2}{\lambda^2}) - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\right|\left|\frac{1}{m}\sum_{j=1}^{m}a_{jl}a_{ji_{\max}}^{*}\right| \\
&\overset{(a)}{\le}\left|\frac{1}{m}\sum_{j=1}^{m}a_{jl}a_{ji_{\max}}^{*}\right| \overset{(b)}{\le}\frac{\epsilon}{2}\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|}
\end{aligned} \tag{56}$$

where (a) is based on the fact that $\exp(-\frac{y_j^2}{\lambda^2}) \in [0,1]$ and $\exp(-\frac{y_j^2}{\|\mathbf{x}\|^2}) \in [0,1]$, (b) is based on (53) and holds for all indices $j \in [n], j \neq i_{\max}$ with probability exceeding $1-\exp(-d_{2,\epsilon}m)$ by taking the union bound, provided that $m \ge D_{2,\epsilon}\frac{k}{\gamma^2 s}\log n$. Combining (55) and (56), we complete the proof of Lemma B.5 in this case.

† **Case 2:** $l = i_{\max}$

Similar to (55), we can derive that

$$|\tilde{q}_{i_{\max}} - \mathbb{E}(\tilde{q}_{i_{\max}})| \le \frac{\epsilon}{2}\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|} \tag{57}$$

holds with probability exceeding $1-\exp(d_{3,\epsilon}m)$ when $m \ge D_{3,\epsilon}\frac{k}{\gamma^2 s}$. Moreover, since $\mathbb{E}(\frac{1}{m}\sum_{j=1}^{m}|a_{ji_{\max}}|^2) = 1$, (53) is transformed as

$$P\left(\left|\frac{1}{m}\sum_{j=1}^{m}|a_{ji_{\max}}|^2 - 1\right| > \frac{\epsilon}{6}\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|}\right) \le \exp\left(-c_4^{'}\min\left\{\frac{\epsilon^2\gamma^2 m|x_{\max}|^2}{144(K_4^{'})^2 k\|\mathbf{x}\|^2}, \frac{\epsilon\gamma m|x_{\max}|}{12K_4^{'}\sqrt{k}\|\mathbf{x}\|}\right\}\right), \tag{58}$$

where $K_4^{'} = \left\||a_{ji_{\max}}|^2\right\|_{\psi_1}$ is the sub-exponential norm. Take the union bound of (58), we can derive that

$$\left|\frac{1}{m}\sum_{j=1}^{m}|a_{ji_{\max}}|^2 - 1\right| \le \frac{\epsilon}{6}\frac{\gamma|x_{\max}|}{2\sqrt{k}\|\mathbf{x}\|} \tag{59}$$

holds with probability exceeding $1-\exp(-c_\epsilon m)$ if $m \ge C_\epsilon\frac{k}{\gamma^2 s}\log n$. Finally, we conduct the numerical analysis for

$|q_{i_{\max}} - \tilde{q}_{i_{\max}}|$ below.

$$
\begin{aligned}
|q_{i_{\max}} - \tilde{q}_{i_{\max}}| &= \left| \frac{1}{m} \sum_{j=1}^{m} \big(\exp(-\frac{y_j^2}{\lambda^2}) - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2}) \big) |a_{ji_{\max}}|^2 \right| \\
&\overset{(a)}{=} \max_{j\in[m]} \exp(-\frac{y_j^2}{\xi}) \frac{1}{\xi^2} |\lambda^2 - \|\mathbf{x}\|^2| \left| \frac{1}{m} \sum_{j=1}^{m} |a_{ji_{\max}}|^2 \right| \\
&\overset{(b)}{\leq} \frac{1}{\xi} |\lambda^2 - \|\mathbf{x}\|^2| \left| \frac{1}{m} \sum_{j=1}^{m} |a_{ji_{\max}}|^2 \right| \\
&\overset{(c)}{\leq} \frac{1}{(1 - \frac{\epsilon\gamma}{12}\frac{\sqrt{s}}{\sqrt{k}})\|\mathbf{x}\|^2} \frac{\epsilon\gamma}{12} \frac{\sqrt{s}}{\sqrt{k}} \|\mathbf{x}\|^2 \big(1 + \frac{\epsilon}{6} \frac{\gamma\sqrt{s}}{2\sqrt{k}}\big) \\
&\overset{(d)}{\leq} 3 \times \frac{\epsilon\gamma}{12} \frac{\sqrt{s}}{\sqrt{k}} = \frac{\epsilon\gamma\sqrt{s}}{4\sqrt{k}},
\end{aligned} \tag{60}
$$

where (a) is from Lagrange's mean value formula with $F(p) = \exp(-\frac{y_j^2}{p})$, (b) is due to the numeric inequality $xe^{-x} \leq 1$, for $x \geq 0$, (c) employs Lemma B.2 with $\beta = \frac{\epsilon\gamma}{12}$ and (59), (d) is based on the fact that $\frac{\epsilon\gamma}{12}\frac{\sqrt{s}}{\sqrt{k}} < \frac{1}{2}$. It is worth noting that (60) holds with probability exceeding $1 - \exp(-d_{4,\epsilon} m)$ provided $m \geq D_{4,\epsilon} \frac{k}{\gamma^2 s} \log n$. Therefore, combining (56) and (60), we complete the proof. □

Lemma B.6. *The following sparse optimization problem*

$$
\max \mathbf{u}^* \left[\frac{\mathbf{x}\mathbf{x}^*}{4\|\mathbf{x}\|^2} \right] \mathbf{u} \quad \textit{s.t. } \|\mathbf{u}\| = \frac{1}{m} \sum_{j=1}^{m} y_j^2, \ \|\mathbf{u}\|_0 \leq k
$$

attains its optimum provided that $\mathbf{u}$ *is a multiple of* $\mathbf{x}$.

Proof of Lemma B.6. Suppose that $\text{supp}(\mathbf{u}) = T_0$ and $\text{supp}(\mathbf{x}) = T$. The optimization problem can be rewrite as

$$
\max_{|T_0|\leq k} \max_{\text{supp}(\mathbf{u}) = T_0} \mathbf{u}^* \left[\frac{\mathbf{x}_{T_0}\mathbf{x}_{T_0}^*}{4\|\mathbf{x}\|^2} \right] \mathbf{u} \quad \text{s.t. } \|\mathbf{u}\| = \frac{1}{m} \sum_{j=1}^{m} y_j^2.
$$

Obviously, the result of the inner maximization can be directly acquired as the maximal eigenvalue of $\frac{\mathbf{x}_{T_0}\mathbf{x}_{T_0}^*}{4\|\mathbf{x}\|^2}$, which is $\|\mathbf{u}\|^2 \frac{\|\mathbf{x}_{T_0}\|^2}{4\|\mathbf{x}\|^2}$. Hence, it suffices to optimize

$$
\max_{|T_0|\leq k} \left(\frac{1}{m} \sum_{j=1}^{m} y_j^2 \right)^2 \frac{\|\mathbf{x}_{T_0}\|^2}{4\|\mathbf{x}\|^2}
$$

The optimal value is attained when $T_0 = T$. Therefore, the solution of this problem is a multiple of the unit eigenvector corresponding to the largest eigenvalue of $\frac{\mathbf{x}\mathbf{x}^*}{4\|\mathbf{x}\|^2}$. Since eigenvectors for the largest eigenvalue of $\frac{\mathbf{x}\mathbf{x}^*}{4\|\mathbf{x}\|^2}$ are multiples of $\mathbf{x}$. We complete the proof by combining the above two claims. □

Lemma B.7. *Define* $\mathbf{\Delta} = \mathbf{L} - \mathbb{E}(\tilde{\mathbf{L}})$. *For any* $0 < t < 1$, *when* $m \geq C_t k \log n$,

$$
\tau(\mathbf{\Delta}, k) \leq 6t
$$

holds with probability at least $1 - \exp(-c_t m)$.

Proof of Lemma B.7. Divide $\mathbf{\Delta}$ as $\mathbf{\Delta} = \mathbf{L} - \tilde{\mathbf{L}} + \tilde{\mathbf{L}} - \mathbb{E}(\tilde{\mathbf{L}})$ and suppose $\mathbf{\Delta}_1 = \mathbf{L} - \tilde{\mathbf{L}}, \mathbf{\Delta}_2 = \tilde{\mathbf{L}} - \mathbb{E}(\tilde{\mathbf{L}})$. Define the set $\mathbb{C}^{n,k} = \{\mathbf{x} \in \mathbb{C}^n \big| \|\mathbf{x}\|_0 \le k, \|\mathbf{x}\| = 1\}$, we then separately analyze $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$.

Recall that $\mathbf{\Delta}_1 = \frac{1}{m}\sum_{j=1}^m \{\exp(-\frac{y_j^2}{\lambda^2}) - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\}\mathbf{a}_j\mathbf{a}_j^*$, we first construct the ϵ-net N_ϵ $(0 < \epsilon < \frac{1}{2})$ for $\mathbb{C}^{n,k}$ (i.e., for any $\mathbf{u} \in \mathbb{C}^{n,k}$, there exists an $\mathbf{u}' \in N_\epsilon$ such that $\|\mathbf{u} - \mathbf{u}'\| \le \epsilon$). From the covering theory, $|N_\epsilon| \le \binom{n}{k}(\frac{3}{\epsilon})^k \le (\frac{3en}{\epsilon k})^k$ (Vershynin, 2018, Corollary 4.2.13). Moreover, using the definition of $\tau(\mathbf{\Delta}_1, k)$, we assume that there exists an eigenvector $\mathbf{u}_0 \in \mathbb{C}^{n,k}$ such that

$$\tau(\mathbf{\Delta}_1, k) = \mathbf{u}_0^*\mathbf{\Delta}_1\mathbf{u}_0.$$

From the definition of N_ϵ, we can find an $\mathbf{u}_0' \in N_\epsilon$ such that $\|\mathbf{u}_0 - \mathbf{u}_0'\| \le \epsilon$. Hence, we have

$$\begin{aligned}
\tau(\mathbf{\Delta}_1, k) &= \mathbf{u}_0^*\mathbf{\Delta}_1\mathbf{u}_0 \\
&= (\mathbf{u}_0 - \mathbf{u}_0')^*\mathbf{\Delta}_1\mathbf{u}_0' + \mathbf{u}_0^*\mathbf{\Delta}_1(\mathbf{u}_0 - \mathbf{u}_0') + (\mathbf{u}_0')^*\mathbf{\Delta}_1\mathbf{u}_0' \\
&\overset{(a)}{\le} 2\epsilon\tau(\mathbf{\Delta}_1, k) + (\mathbf{u}_0')^*\mathbf{\Delta}_1\mathbf{u}_0',
\end{aligned}$$

where (a) is due to the property $\|\mathbf{u}_0 - \mathbf{u}_0'\| \le \epsilon$ and the definition of $\tau(\mathbf{\Delta}_1, k)$. This relationship can be equivalently expressed as

$$\tau(\mathbf{\Delta}_1, k) \le \frac{1}{1 - 2\epsilon} \max_{\mathbf{u} \in N_\epsilon} \mathbf{u}^*\mathbf{\Delta}_1\mathbf{u}. \tag{61}$$

For any fixed $\mathbf{u} \in N_\epsilon$, we can derive that

$$P\left(\frac{1}{m}\sum_{j=1}^m |\mathbf{a}_j^*\mathbf{u}|^2 > \|\mathbf{u}\|^2 + \frac{t}{2}\right) \overset{(a)}{\le} \exp\left(-c_6 \min\left\{\frac{t^2 m}{4K_6^2}, \frac{tm}{2K_6}\right\}\right), \tag{62}$$

where (a) comes from the fact that $\|\mathbf{u}\| = 1$ and $K_6 = \max_{j \in [m]} \||\mathbf{a}_j^*\mathbf{u}|^2\|_{\psi_1}$ is the sub-exponential norm. Taking the union bound for $\mathbf{u} \in N_\epsilon$ yield

$$P\left(\max_{\mathbf{u} \in N_\epsilon} \frac{1}{m}\sum_{j=1}^m |\mathbf{a}_j^*\mathbf{u}|^2 > \|\mathbf{u}\|^2 + \frac{t}{2}\right) \le (\frac{3en}{\epsilon k})^k \exp\left(-c_6 \min\left\{\frac{t^2 m}{4K_6^2}, \frac{tm}{2K_6}\right\}\right), \tag{63}$$

which indicates that when $m = C_t k \log(n/k)$, with probability exceeding $1 - \exp(-c_t m)$, it holds that

$$\max_{\mathbf{u} \in N_\epsilon} \frac{1}{m}\sum_{j=1}^m |\mathbf{a}_j^*\mathbf{u}|^2 \le 1 + \frac{t}{2}. \tag{64}$$

Finally, we can estimate $\tau(\mathbf{\Delta}_1, k)$ as

$$\begin{aligned}
(1 - 2\epsilon)\tau(\mathbf{\Delta}_1, k) &\overset{(61)}{\le} \max_{\mathbf{u} \in N_\epsilon} \mathbf{u}^*\mathbf{\Delta}_1\mathbf{u} \\
&= \max_{\mathbf{u} \in N_\epsilon} \mathbf{u}^*\left(\frac{1}{m}\sum_{j=1}^m \left\{\exp(-\frac{y_j^2}{\lambda^2}) - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\right\}\mathbf{a}_j\mathbf{a}_j^*\right)\mathbf{u} \\
&= \max_{\mathbf{u} \in N_\epsilon} \frac{1}{m}\sum_{j=1}^m \left\{\exp(-\frac{y_j^2}{\lambda^2}) - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\right\}|\mathbf{a}_j^*\mathbf{u}|^2 \\
&\le \max_{\mathbf{u} \in N_\epsilon} \max_{j \in [m]} \left|\exp(-\frac{y_j^2}{\lambda^2}) - \exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\right| (\frac{1}{m}\sum_{j=1}^m |\mathbf{a}_j^*\mathbf{u}|^2) \\
&\overset{(a)}{=} \max_{j \in [m]} \exp(-\frac{y_j^2}{\xi})\frac{y_j^2}{\xi^2}|\lambda^2 - \|\mathbf{x}\|^2| \max_{\mathbf{u} \in N_\epsilon} \frac{1}{m}\sum_{j=1}^m |\mathbf{a}_j^*\mathbf{u}|^2 \\
&\overset{(b)}{\le} \frac{1}{\xi}|\lambda^2 - \|\mathbf{x}\|^2|(1 + \frac{t}{2}) \overset{(c)}{\le} \frac{\frac{t}{2}\|\mathbf{x}\|^2}{(1 - \frac{t}{2})\|\mathbf{x}\|^2}(1 + \frac{t}{2}) \le 2t.
\end{aligned}$$

where (a) is from Lagrange's mean value formula with $F(p)=\exp(-\frac{y_j^2}{p})$, (b) comes from the inequality $xe^{-x}\leq 1, x\geq 0$ and (64), (c) employs Lemma B.1 with $\xi\in[(1-\frac{t}{2})\|\mathbf{x}\|^2,(1+\frac{t}{2})\|\mathbf{x}\|^2]$. Let $\epsilon=\frac{1}{4}$, we can conclude that

$$\tau(\mathbf{\Delta}_1,k)\leq 4t \tag{65}$$

holds with probability exceeding $1-\exp(c_{1,t}m)$ provided that $m\geq C_{1,t}k\log n$. Next, we analyze $\tau(\mathbf{\Delta}_2,k)$. Similar to (61), we can get

$$\tau(\mathbf{\Delta}_2,k)\leq\frac{1}{1-2\epsilon}\max_{\mathbf{u}\in N_\epsilon}\mathbf{u}^*\mathbf{\Delta}_2\mathbf{u}. \tag{66}$$

Recall that $\mathbf{\Delta}_2=\frac{1}{m}\sum_{j=1}^m\left(\frac{1}{2}-\exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\right)\mathbf{a}_j\mathbf{a}_j^*-\frac{\mathbf{x}\mathbf{x}^*}{4\|\mathbf{x}\|^2}$. Then, it holds that

$$\begin{aligned}(1-2\epsilon)\tau(\mathbf{\Delta}_2,k)&\overset{(66)}{\leq}\max_{\mathbf{u}\in N_\epsilon}\mathbf{u}^*\mathbf{\Delta}_2\mathbf{u}\\&=\max_{\mathbf{u}\in N_\epsilon}\left[\frac{1}{m}\sum_{j=1}^m\big(\frac{1}{2}-\exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\big)|\mathbf{a}_j^*\mathbf{u}|^2-\frac{|\mathbf{x}^*\mathbf{u}|^2}{4\|\mathbf{x}\|^2}\right].\end{aligned} \tag{67}$$

Since for any $j\in[m]$,$\big(\frac{1}{2}-\exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\big)|\mathbf{a}_j^*\mathbf{u}|^2$ is sub-exponential random variable with constant norm, employing Bernstein's inequality and taking the union bound for $\mathbf{u}\in N_\epsilon$ yield that

$$P\left(\max_{\mathbf{u}\in N_\epsilon}\left[\frac{1}{m}\sum_{j=1}^m\big(\frac{1}{2}-\exp(-\frac{y_j^2}{\|\mathbf{x}\|^2})\big)|\mathbf{a}_j^*\mathbf{u}|^2-\frac{|\mathbf{x}^*\mathbf{u}|^2}{4\|\mathbf{x}\|^2}\right]>t\right)\leq(\frac{3en}{\epsilon k})^k\exp\left(-c_7\min\left\{\frac{t^2m}{K_7^2},\frac{tm}{K_7}\right\}\right). \tag{68}$$

Combining (67) and (68) and let $\epsilon=\frac{1}{4}$, we conclude that when $m\geq C_{2,t}k\log n$,

$$\tau(\mathbf{\Delta}_2,k)\leq 2t \tag{69}$$

holds with probability exceeding $1-\exp(c_{2,t}m)$. Hence, we get

$$\tau(\mathbf{\Delta},k)\leq\tau(\mathbf{\Delta}_1,k)+\tau(\mathbf{\Delta}_2,k)\leq 6t. \tag{70}$$

□

B.3. Eigen decomposition

Lemma B.8. *Suppose* $\mathbf{w}_j\in\mathbb{C}^n, j=1,2,3$*, then we have*

$$\textit{dist}(\mathbf{w}_1,\mathbf{w}_2)\leq\textit{dist}(\mathbf{w}_1,\mathbf{w}_3)+\textit{dist}(\mathbf{w}_2,\mathbf{w}_3).$$

Proof of Lemma B.8. From the definition of $\text{dist}(\mathbf{w}_1,\mathbf{w}_2)$, we assume $\phi_{1,3}\in[0,2\pi)$ to be

$$\text{dist}(\mathbf{w}_1,\mathbf{w}_3)=\min_{\phi\in[0,2\pi)}\|\mathbf{w}_1-e^{i\phi}\mathbf{w}_3\|=\|\mathbf{w}_1-e^{i\phi_{1,3}}\mathbf{w}_3\|.$$

Then, from the triangle inequality of norm,

$$\begin{aligned}\text{dist}(\mathbf{w}_1,\mathbf{w}_2)&=\min_{\phi\in[0,2\pi)}\|\mathbf{w}_1-e^{i\phi}\mathbf{w}_2\|\\&\leq\min_{\phi\in[0,2\pi)}\left[\|\mathbf{w}_1-e^{i\phi_{1,3}}\mathbf{w}_3\|+\|e^{i\phi_{1,3}}\mathbf{w}_3-e^{i\phi}\mathbf{w}_2\|\right]\\&=\text{dist}(\mathbf{w}_1,\mathbf{w}_3)+\min_{\phi\in[0,2\pi)}\|\mathbf{w}_3-e^{i(\phi-\phi_{1,3})}\mathbf{w}_2\|\\&=\text{dist}(\mathbf{w}_1,\mathbf{w}_3)+\text{dist}(\mathbf{w}_2,\mathbf{w}_3).\end{aligned}$$

The proof is thus complete. □

Lemma B.9. *Define $\mathbf{\Delta} = \mathbf{L} - \mathbb{E}(\tilde{\mathbf{L}})$ and $\mathbf{x}^0 = \frac{\mathbf{x}}{\|\mathbf{x}\|}$. Suppose supp$(\mathbf{x}) = T$ and $\Lambda \subset [n]$ satisfying $|\Lambda| = k, \Lambda \cap T \neq \emptyset$, and $\|\mathbf{x}_\Lambda^0\| \geq \frac{\sqrt{3}}{2}$. $\mathbf{z}^0$ is the unit eigenvector corresponding to the largest eigenvalue of $\mathbf{L}_\Lambda$. If $\tau(\mathbf{\Delta}, k) \leq \frac{3}{64}$, then the distance between $\mathbf{x}_\Lambda^0$ and $\mathbf{z}_\Lambda^0$ can be bounded as*

$$dist(\mathbf{z}_\Lambda^0, \mathbf{x}_\Lambda^0)^2 \leq \|\mathbf{x}_\Lambda^0\|^2 + 1 - \frac{6\|\mathbf{x}_\Lambda^0\|}{\sqrt{9 + 1024\tau^2(\mathbf{\Delta}, k)}} \tag{71}$$

Proof of Lemma B.9. We first estimate the eigenvalue of $\mathbf{L}_\Lambda$. For the largest eigenvalue $\lambda_{\max} = \lambda_1(\mathbf{L}_\Lambda)$, we have

$$\begin{aligned}\lambda_{\max} &= \lambda_1(\mathbf{L}_\Lambda) \\ &\overset{(a)}{\geq} \lambda_1(\mathbb{E}[(\tilde{\mathbf{L}})_\Lambda]) + \lambda_n(\mathbf{\Delta}_\Lambda) \\ &\overset{(b)}{\geq} \frac{\|\mathbf{x}_\Lambda\|^2}{4\|\mathbf{x}\|^2} - \tau(\mathbf{\Delta}, k) \\ &\overset{(c)}{\geq} \frac{3}{16} - \tau(\mathbf{\Delta}, k),\end{aligned} \tag{72}$$

where (a) comes from $\mathbf{L}_\Lambda = \mathbb{E}[(\tilde{\mathbf{L}})_\Lambda] + \mathbf{\Delta}_\Lambda$ and Weyl's inequality, (b) is due to the definition of $\tau(\mathbf{\Delta}, k)$ and (c) is because $\|\mathbf{x}_\Lambda^0\| \geq \frac{\sqrt{3}}{2}$.

For other eigenvalues, we can similarly derive that for $j \in [n]$,

$$\lambda_j(\mathbf{L}_\Lambda) \leq \lambda_j(\mathbb{E}[(\tilde{\mathbf{L}})_\Lambda]) + \lambda_1(\mathbf{\Delta}_\Lambda) \leq \tau(\mathbf{\Delta}, k). \tag{73}$$

Since $\|\mathbf{z}_\Lambda^0\| = 1$ and $\|\mathbf{x}_\Lambda^0\| \leq 1$, we can employ space decomposition technique and get

$$\mathbf{z}_\Lambda^0 = r_1 \frac{\mathbf{x}_\Lambda^0}{\|\mathbf{x}_\Lambda^0\|} + r_2 \mathbf{u}, \tag{74}$$

where $\mathbf{u}^* \mathbf{x}_\Lambda^0 = 0$, $\|\mathbf{u}\| = 1$, $|r_1|^2 + |r_2|^2 = 1$ and supp$(\mathbf{u}) \subset \Lambda$. Multiple both sides with $\lambda_{\max}$, the relationship becomes

$$\lambda_{\max} \mathbf{z}_\Lambda^0 = r_1 \frac{\lambda_{\max} \mathbf{x}_\Lambda^0}{\|\mathbf{x}_\Lambda^0\|} + r_2 \lambda_{\max} \mathbf{u}. \tag{75}$$

Recall that $\lambda_{\max} \mathbf{z}_\Lambda^0 = \mathbf{L}_\Lambda \mathbf{z}_\Lambda^0$ since $\mathbf{z}_\Lambda^0$ is the eigenvector of $\mathbf{L}_\Lambda$ with $\lambda_{\max}$ as the eigenvalue. Therefore,

$$\lambda_{\max} \mathbf{z}_\Lambda^0 = \mathbf{L}_\Lambda \mathbf{z}_\Lambda^0 = r_1 \frac{\mathbf{L}_\Lambda \mathbf{x}_\Lambda^0}{\|\mathbf{x}_\Lambda^0\|} + r_2 \mathbf{L}_\Lambda \mathbf{u}. \tag{76}$$

Combining (75) and (76), we have

$$r_1 \frac{\lambda_{\max} \mathbf{x}_\Lambda^0}{\|\mathbf{x}_\Lambda^0\|} + r_2 \lambda_{\max} \mathbf{u} = r_1 \frac{\mathbf{L}_\Lambda \mathbf{x}_\Lambda^0}{\|\mathbf{x}_\Lambda^0\|} + r_2 \mathbf{L}_\Lambda \mathbf{u}.$$

By taking the inner product with $\mathbf{u}$, it becomes

$$r_2 \lambda_{\max} = r_1 \frac{\mathbf{u}^* \mathbf{L}_\Lambda \mathbf{x}_\Lambda^0}{\|\mathbf{x}_\Lambda^0\|} + r_2 \mathbf{u}^* \mathbf{L}_\Lambda \mathbf{u}.$$

This is equivalent to

$$\begin{aligned}|r_2| &= |r_1| \frac{|\mathbf{u}^*(\mathbb{E}[(\tilde{\mathbf{L}})_\Lambda] + \mathbf{\Delta}_\Lambda)\mathbf{x}_\Lambda^0| / \|\mathbf{x}_\Lambda^0\|}{|\lambda_{\max} - \mathbf{u}^* \mathbf{L}_\Lambda \mathbf{u}|} \\ &\overset{(a)}{=} |r_1| \frac{|\mathbf{u}^* \mathbf{\Delta}_\Lambda \mathbf{x}_\Lambda^0| / \|\mathbf{x}_\Lambda^0\|}{|\lambda_{\max} - \mathbf{u}^* \mathbf{L}_\Lambda \mathbf{u}|},\end{aligned} \tag{77}$$

where (a) is because $\mathbf{x}_\Lambda^0$ is the eigenvector of $\mathbb{E}[(\tilde{\mathbf{L}})_\Lambda]$ and $\mathbf{u}^*\mathbf{x}_\Lambda^0 = 0$. Since $\mathbf{u}$ is perpendicular to $\mathbf{x}_\Lambda^0$, we estimate $\mathbf{u}^*\mathbf{L}_\Lambda\mathbf{u}$ as

$$\begin{aligned}\mathbf{u}^*\mathbf{L}_\Lambda\mathbf{u} &= \mathbf{u}^*\mathbb{E}[(\tilde{\mathbf{L}})_\Lambda]\mathbf{u} + \mathbf{u}^*\boldsymbol{\Delta}_\Lambda\mathbf{u} \\ &\leq \lambda_2(\mathbb{E}[(\tilde{\mathbf{L}})_\Lambda]) + \tau(\boldsymbol{\Delta}, k) \\ &= \tau(\boldsymbol{\Delta}, k) \end{aligned} \tag{78}$$

Combining (72), (73) and (78) and define $\beta = \frac{|\mathbf{u}^*\boldsymbol{\Delta}_\Lambda\mathbf{x}_\Lambda^0|/\|\mathbf{x}_\Lambda^0\|}{|\lambda_{\max} - \mathbf{u}^*\mathbf{L}_\Lambda\mathbf{u}|}$ we have

$$\beta \leq \frac{\tau(\boldsymbol{\Delta}, k)}{\frac{3}{16} - 2\tau(\boldsymbol{\Delta}, k)} \leq \frac{32\tau(\boldsymbol{\Delta}, k)}{3}, \tag{79}$$

where the second inequality comes from the assumption $\tau(\boldsymbol{\Delta}, k) \leq \frac{3}{64}$. From (77) and using $|r_1|^2 + |r_2|^2 = 1$, we have $|r_1|^2 = 1/(1+\beta^2)$. Therefore,

$$\begin{aligned}\mathrm{dist}(\mathbf{z}_\Lambda^0, \mathbf{x}_\Lambda^0)^2 &= \min_\phi \|\mathbf{z}_\Lambda^0 - e^{i\phi}\mathbf{x}_\Lambda^0\|^2 &(80)\\ &= \min_\phi \|\mathbf{x}_\Lambda^0\|^2 + 1 - 2\mathcal{R}\left[e^{i\phi}(\mathbf{x}_\Lambda^0)^*\mathbf{z}_\Lambda^0\right] \\ &= \|\mathbf{x}_\Lambda^0\|^2 + 1 - 2\left|(\mathbf{x}_\Lambda^0)^*\mathbf{z}_\Lambda^0\right| \\ &\overset{(74)}{=} \|\mathbf{x}_\Lambda^0\|^2 + 1 - 2|r_1|\|\mathbf{x}_\Lambda^0\| \\ &\overset{(79)}{\leq} \|\mathbf{x}_\Lambda^0\|^2 + 1 - 2\frac{\|\mathbf{x}_\Lambda^0\|}{\sqrt{1 + \frac{1024}{9}\tau^2(\boldsymbol{\Delta}, k)}} \\ &= \|\mathbf{x}_\Lambda^0\|^2 + 1 - \frac{6\|\mathbf{x}_\Lambda^0\|}{\sqrt{9 + 1024\tau^2(\boldsymbol{\Delta}, k)}}, &(81)\end{aligned}$$

Hence, we complete the proof. □