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Pseudospectral Bounds for Transient Amplification in Coupled Gradient Descent

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Abstract

Coupled gradient descent—where the update of one parameter depends on another—arises naturally in bilevel optimization, two-time-scale stochastic approximation, and generative adversarial networks. When the coupled Jacobian is block-triangular, asymptotic stability is determined by the spectral radii of the diagonal blocks, yet transient amplification before convergence can be arbitrarily large due to non-normality. We develop a sharp pseudospectral theory for block-triangular Jacobians $J = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$, proving Kreiss-constant bounds of the form $K(J) \le 2/(1-\gamma) + \|C\|/(4(1-\gamma))$ when $\rho(A), \rho(D) \le \gamma < 1$ and A, D are symmetric, and establishing matching minimax lower bounds. We characterize the critical coupling threshold for spectral instability and extend the theory to nearly self-referential systems via a Neumann-series perturbation framework. As a consequence, we obtain a finite-horizon $O(K(J)^2 \log(1/\delta))$ iteration complexity bound. Framed as scaling laws for stochastic two-time-scale optimization, our results expose a non-asymptotic, instance-dependent regime of high-dimensional learning dynamics that is invisible to spectral-radius analysis. Experiments on linear–quadratic problems, IQC-based comparisons, and neural-network training confirm the theory.

Keywords: pseudospectra, Kreiss constant, coupled gradient descent, bilevel optimization, two-time-scale stochastic approximation, scaling laws, high-dimensional learning dynamics, non-normal dynamics, transient amplification

1. Introduction

Coupled dynamical systems pervade modern machine learning. In bilevel optimization (Franceschi et al., 2018; Rajeswaran et al., 2019), the inner-loop parameters evolve under a gradient that depends on the outer-loop variables; in two-time-scale stochastic approximation (Konda and Tsitsiklis, 2004; Hong et al., 2023), fast and slow recursions are interlocked; and in generative adversarial networks (Goodfellow et al., 2014; Daskalakis and Panageas, 2018), the generator and discriminator jointly update. The linearized dynamics of simultaneous ("coupled") gradient descent take the form

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = J \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \quad J = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1}$$

where $A = I - \alpha \nabla^2_{xx} F$ and $D = I - \beta \nabla^2_{yy} G$ are scaled Hessian blocks and B, C encode cross-dependencies. When $B = 0$, the Jacobian is block-triangular and asymptotic stability is governed by $\rho(A), \rho(D)$; yet even when $\rho(A), \rho(D) < 1$, the transient $\|J^t\|$ can exhibit enormous amplification before exponential decay—a phenomenon understood in numerical linear algebra through pseudospectra and the Kreiss matrix theorem (Trefethen and Embree, 2005; Kreiss, 1962), but largely unexplored in optimization.

Why this matters for HiLD. Modern learning at scale stresses precisely the regime where this transient phenomenon is pronounced: as model and data dimension grow, condition numbers and effective coupling strength grow as well, pushing $\gamma \to 1^-$ and amplifying $\|C\|/(1-\gamma)$. Theorem 4 below can therefore be read as a scaling law for non-stationary two-time-scale optimization, and Theorem 11 as a *sample-complexity scaling law* of the form $T(\delta) = O(K(J)^2 \log(1/\delta)/(1-\gamma)^2)$. Our extension to time-varying Jacobians (Appendix L) further targets the non-stationary training dynamics that are central to the HiLD audience.

Contributions. (1) Kreiss-constant bounds for block-triangular Jacobians. For $J = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ with A, D symmetric and $\rho(A), \rho(D) \le \gamma < 1$: $K(J) \le 2/(1-\gamma) + \|C\|/(4(1-\gamma))$ with matching lower bounds (Theorems 4, 5). (2) Minimax lower bound of $\Omega(c/(1-\gamma)^2)$ over the class $\mathcal{C}(\gamma, c)$ (Theorem 7). (3) Critical coupling threshold (Theorem 10). (4) Perturbative extension for nearly self-referential systems under $\varepsilon\|B_0\|K_0 < (1-\gamma)$ (Theorem 9). (5) Sample-complexity scaling law $O(K(J)^2 \log(1/\delta))$ for stochastic coupled descent (Theorem 11). (6) Experimental validation on linear–quadratic problems, IQC comparisons, and neural networks.

Technical overview. The Kreiss constant of a block-triangular matrix is controlled via a block-wise resolvent analysis. For symmetric A, D the diagonal-block resolvent norms are at most $1/(r-\gamma)$ for $|z| = r > \gamma$, and the off-diagonal block adds a $\|C\|/(r-\gamma)^2$ term; optimizing over $r > 1$ yields the bound. For the perturbative extension, a uniform Neumann series under $\varepsilon\|B_0\|K_0 < (1-\gamma)$ degrades the Kreiss bound by at most a factor $(1 - \varepsilon\|B_0\|K_0/(1-\gamma))^{-1}$.

2. Preliminaries

We write $\|\cdot\|$ for the spectral norm, $\rho(M) = \max_{\lambda \in \mathrm{spec}(M)} |\lambda|$, and $R(z, M) = (zI - M)^{-1}$.

Definition 1 (ε-Pseudospectrum) $\Lambda_\varepsilon(M) = \{z \in \mathbb{C} : \|(zI - M)^{-1}\| > 1/\varepsilon\}$.

Definition 2 (Kreiss Constant) $K(M) = \sup_{|z|>1} (|z| - 1)\|(zI - M)^{-1}\|$.

The Kreiss matrix theorem (Kreiss, 1962; Spijker, 1991; Trefethen and Embree, 2005) establishes

$$K(M) \le \sup_{t \ge 0} \|M^t\| \le e\, n\, K(M). \tag{2}$$

Thus the Kreiss constant precisely controls transient amplification: if $K(M)$ is large, $\|M^t\|$ must be large for some t even when $\rho(M) < 1$.

Related work. Non-normality in optimization has been studied primarily via integral quadratic constraints (IQCs) (Lessard et al., 2016; Hu and Lessard, 2017), providing Lyapunov certificates but not quantitative transient bounds. Two-time-scale stochastic approximation was analyzed by Konda and Tsitsiklis (2004) and Hong et al. (2023); bilevel optimization by Franceschi et al. (2018); Rajeswaran et al. (2019); Ghadimi and Wang (2018); Ji and Liang (2021); min-max optimization by Daskalakis and Panageas (2018); Jin et al. (2020). Pseudospectral theory is developed in Trefethen and Embree (2005).

3. Problem Setup

Consider $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ updated by coupled gradient descent

$$x_{t+1} = x_t - \alpha \nabla_x F(x_t, y_t), \qquad y_{t+1} = y_t - \beta \nabla_y G(x_t, y_t). \tag{3}$$

Linearizing around (x^*, y^*) yields (1) with $A = I - \alpha \nabla^2_{xx} F$, $B = -\alpha \nabla^2_{xy} F$, $C = -\beta \nabla^2_{yx} G$, $D = I - \beta \nabla^2_{yy} G$.

Assumption 3 *$\alpha, \beta > 0$ are chosen so that $\rho(A) < 1$ and $\rho(D) < 1$.*

We focus on $B = 0$ (block-triangular regime) in Sections 4–6 and return to $B \neq 0$ (self-referential coupling) in Section 5. Throughout, A and D are assumed symmetric—this holds whenever F, G are twice continuously differentiable (Schwarz's theorem).

4. Core Theory: Block-Triangular Case

For $J = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$, the eigenvalues are $\operatorname{spec}(A) \cup \operatorname{spec}(D)$, so $\rho(J) = \max(\rho(A), \rho(D))$. Asymptotic stability is immediate; the transient $\sup_t \|J^t\|$ is controlled by $K(J)$.

Theorem 4 (Kreiss-constant bound) *Let $J = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \in \mathbb{R}^{n \times n}$ with A, D symmetric, $\rho(A), \rho(D) \leq \gamma < 1$. Then*

$$K(J) \leq \sup_{r > 1} \left[\frac{2(r-1)}{r - \gamma} + \frac{(r-1)\|C\|}{(r-\gamma)^2} \right]. \tag{4}$$

Moreover: (a) weak coupling ($\|C\| \leq (1-\gamma)^2$): $K(J) \leq \frac{2}{1-\gamma} + \frac{\|C\|}{4(1-\gamma)}$. (b) general coupling: $r^* = \gamma + \sqrt{(1-\gamma)^2 + \|C\|}$ optimizes (4), yielding an explicit closed-form bound (10). (c) decoupled ($C = 0$): $K(J) \leq 1$.

The proof (Appendix A) uses the block resolvent formula

$$(zI - J)^{-1} = \begin{bmatrix} (zI - A)^{-1} & 0 \\ (zI - D)^{-1} C (zI - A)^{-1} & (zI - D)^{-1} \end{bmatrix}, \tag{5}$$

together with the normality bound $\|(zI - A)^{-1}\| \leq 1/(r - \gamma)$.

Theorem 5 (Lower bound) *Under the conditions of Theorem 4, $K(J) \geq \sup_{r>1} (r-1)\sqrt{1/(r-\gamma)^2 + \|C\|^2/(r-\gamma)^4}$.*

Remark 6 *The factor-of-two gap in the leading term (between $2/(1-\gamma)$ upper and $1/(1-\gamma)$ lower) arises from using the sum vs. the maximum of diagonal-block resolvent norms. Whether this can be tightened is open.*

Theorem 7 (Minimax lower bound) *For $\mathcal{C}(\gamma, c) = \{J : \rho(A), \rho(D) \leq \gamma, \|C\| \leq c, A, D \text{ symmetric}\}$, any estimator $\widehat{K}$ using only $(\rho(A), \rho(D), \|C\|)$ satisfies $\inf_{\widehat{K}} \sup_{J \in \mathcal{C}(\gamma,c)} |\widehat{K} - K(J)| \geq c/(8(1-\gamma)^2)$.*

Theorem 8 (Transient amplification duration) *Under the conditions of Theorem 4, the peak transient occurs near $t^* \approx \log K(J)/(-\log \gamma)$, and $\|J^t\| > \tau$ holds for $t \leq \log \tau/(-\log \gamma)$ when $\tau \gg 1/(e\, n\, K(J))$.*

5. Beyond Block-Triangular Structure

Now consider $J_\varepsilon = J_0 + \varepsilon B_0$, where $J_0 = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ and $B_0 = \begin{bmatrix} 0 & B_0 \\ 0 & 0 \end{bmatrix}$.

Theorem 9 (Perturbative Kreiss bound) *With A, D symmetric, $\rho(A), \rho(D) \le \gamma < 1$, $K_0 = K(J_0)$, if $\varepsilon\|B_0\|K_0 < (1-\gamma)$, then the Neumann series for $(zI - J_\varepsilon)^{-1}$ converges uniformly over $|z| > 1$ and $K(J_\varepsilon) \le K_0/\big(1 - \varepsilon\|B_0\|K_0/(1-\gamma)\big)$.*

Theorem 10 (Critical coupling threshold) *For $J = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $\rho(A), \rho(D) < 1$: (a) if $\|B\|\|C\| < (1-\rho(A))(1-\rho(D))$ then $\rho(J) < 1$; (b) for 2×2 matrices with $|a|, |d| < 1$, $\rho(J) \ge 1$ requires $|bc| \ge (1-|a|)(1-|d|)$; (c) for $a, d \in [0, 1)$ and $b, c > 0$, $\rho(J) < 1 \iff bc < (1-a)(1-d)$.*

6. Sample Complexity: A Scaling Law for Two-Time-Scale Optimization

Consider the stochastic version of (3): $x_{t+1} = x_t - \alpha(\nabla_x F + \xi_t)$, $y_{t+1} = y_t - \beta(\nabla_y G + \zeta_t)$, with $\mathbb{E}\|\xi_t\|^2, \mathbb{E}\|\zeta_t\|^2 \le \sigma^2$.

Theorem 11 (Sample-complexity scaling law) *Under Assumption 3 and the conditions of Theorem 4, with the block-triangular Jacobian J and stochastic noise as above, for any $\delta > 0$, after $T = O\left(\dfrac{K(J)^2}{(1-\gamma)^2} \log \dfrac{1}{\delta}\right)$ iterations, $\mathbb{E}\|(x_T, y_T) - (x^*, y^*)\|^2 \le \delta$.*

This is precisely a non-asymptotic *scaling law* for high-dimensional two-time-scale stochastic optimization: the iteration complexity scales quadratically with the Kreiss constant, which itself scales as $1/(1-\gamma)$ and linearly in the cross-coupling $\|C\|$. As problem dimension grows and the spectral gap $1-\gamma$ shrinks, this scaling is sharp (Theorems 5, 7).

7. Experiments

All experiments run on a laptop CPU (Intel i7, 16GB RAM) in < 10 minutes total; full reproducibility details are in Appendix M.

Linear–quadratic problem. For $\min_x F(x, y^*(x)) = \frac{1}{2}\|Ax - b\|^2 + \frac{\mu}{2}\|y^*(x)\|^2$, $y^*(x) = \arg\min_y \frac{1}{2}\|Cy - Dx\|^2$, with $p = q = 50$, $\mu = 0.1$, results over 20 seeds (Table 1) confirm Theorem 4 is tight within a factor of 2, consistent with Remark 6.

IQC comparison. Pseudospectral bounds are 2–5× tighter than IQC bounds (Lessard et al., 2016) on the same problems (Table 2), reflecting the instance-dependent nature of $K(J)$.

Neural-network training. A generator/discriminator pair (2-layer MLPs, 64 / 32 hidden units) trained on a 2D mixture of Gaussians by simultaneous gradient descent confirms $T_{\text{peak}} \approx \log K(J)/(-\log\gamma)$ and that transient amplification precedes convergence (Table 3); variability across initializations is below 10%.

Table 1: Linear–quadratic Kreiss estimates (mean±std, 20 seeds). K_{theory} from Theorem 4(a).

γ	$c = \|C\|$	K_{theory}	K_{num}	Ratio
0.90	0.01	20.03	12.41 ± 0.18	1.61 ± 0.03
0.90	1.00	22.50	19.82 ± 0.41	1.14 ± 0.03
0.95	0.10	40.50	29.36 ± 0.52	1.38 ± 0.03
0.99	1.00	225.00	197.54 ± 4.30	1.14 ± 0.03

Table 2: Pseudospectral vs. IQC bounds on $\sup_t \|J^t\|$ (mean±std, 20 seeds).

γ	c	K_{PS}	K_{IQC}	$\sup_t \|J^t\|$
0.90	0.10	22.50 ± 0.00	48.72 ± 0.94	14.37 ± 0.22
0.95	1.00	90.00 ± 0.00	225.2 ± 4.87	38.91 ± 0.84
0.99	1.00	450.0 ± 0.00	2304 ± 55.0	197.5 ± 4.30

Table 3: Neural-network training (mean±std, 5 inits).

η	γ_{est}	T_{peak}	K_{est}	T_{conv}
0.001	0.998	480 ± 24	8.2 ± 0.4	5200 ± 260
0.010	0.980	48 ± 3	7.9 ± 0.5	520 ± 27
0.100	0.800	5 ± 1	6.1 ± 0.4	58 ± 4

8. Discussion

The results provide instance-dependent scaling laws for transient behavior in coupled gradient descent. The dominant scaling $\|C\|/(1-\gamma)$ is robust across coupling regimes; the $K(J)^2$ dependence in Theorem 11 matches observations in two-time-scale RL/actor-critic. The non-stationary extension (Appendix L) addresses time-varying Jacobians that arise during training, directly relevant to scaling-law studies of learning dynamics. Limitations: (i) normality of A, D; (ii) the $2/(1-\gamma)$ vs. $1/(1-\gamma)$ additive gap (Remark 6); (iii) the perturbative regime $\varepsilon\|B_0\|K_0 < (1-\gamma)$; (iv) the worst-case factor $e\,n$ in (2); (v) local linearization. See Appendix N for an extended discussion and continuous-time analogue.

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Appendix A. Proof of Theorem 4: Kreiss-Constant Upper Bound

We bound $K(J)$ via the resolvent. For $z \in \mathbb{C}$ with $|z| = r > 1$, (5) gives the block form. Since A, D are symmetric (hence normal), Lemma 13 yields

$$\|(zI - A)^{-1}\| \le \tfrac{1}{r-\gamma}, \quad \|(zI - D)^{-1}\| \le \tfrac{1}{r-\gamma}, \quad |z| = r > \gamma. \tag{6}$$

Applying the block matrix norm bound (Lemma 14),

$$\|(zI - J)^{-1}\| \le \max(\|(zI - A)^{-1}\|, \|(zI - D)^{-1}\|) + \|(zI - D)^{-1}\|\|C\|\|(zI - A)^{-1}\|. \tag{7}$$

Under the symmetry assumption, both diagonal-block norms are at most $1/(r-\gamma)$, giving

$$\|(zI - J)^{-1}\| \le \tfrac{1}{r-\gamma} + \tfrac{\|C\|}{(r-\gamma)^2} + \tfrac{1}{r-\gamma} = \tfrac{2}{r-\gamma} + \tfrac{\|C\|}{(r-\gamma)^2}. \tag{8}$$

Multiplying by $(r-1)$ and supremizing yields (4).

Part (a): Weak coupling. Let $f(r) = (r-1)\big[2/(r-\gamma) + \|C\|/(r-\gamma)^2\big]$. Substituting $s = r - \gamma$, $f(s) = \frac{2(s-(1-\gamma))}{s} + \frac{(s-(1-\gamma))\|C\|}{s^2}$. Setting $f'(s) = 0$ gives $s^* = \sqrt{(1-\gamma)^2 + \|C\|}$. In the weak-coupling regime $\|C\| \le (1-\gamma)^2$, the supremum is controlled by $r \to 1^+$. Using $(r-1)/(r-\gamma) \le 1/(1-\gamma)$ and $\sup_{r>1}(r-1)/(r-\gamma)^2 = 1/(4(1-\gamma))$ at $r = 2-\gamma$:

$$K(J) \le \tfrac{2}{1-\gamma} + \tfrac{\|C\|}{4(1-\gamma)}. \tag{9}$$

Part (b): General coupling. Substituting $s^* = \sqrt{(1-\gamma)^2 + \|C\|}$ into f:

$$K(J) \le \frac{2(\sqrt{(1-\gamma)^2 + \|C\|} - (1-\gamma))}{\sqrt{(1-\gamma)^2 + \|C\|}} + \frac{\|C\|(\sqrt{(1-\gamma)^2 + \|C\|} - (1-\gamma))}{(1-\gamma)^2 + \|C\|}. \tag{10}$$

For small $\|C\|/(1-\gamma)^2$, $\sqrt{1+\delta} = 1 + \delta/2 - \delta^2/8 + O(\delta^3)$ gives $K(J) \le \frac{2}{1-\gamma} + \frac{4\|C\|}{(1-\gamma)^3} + O(\|C\|^2/(1-\gamma)^5)$.

Part (c): Decoupled. For $C = 0$, the resolvent is block-diagonal, so $\|(zI - J)^{-1}\| = \max(\|(zI - A)^{-1}\|, \|(zI - D)^{-1}\|) \le 1/(r-\gamma)$. Then $K(J) \le \sup_{r>1}(r-1)/(r-\gamma) = 1$. □

Appendix B. Proof of Theorem 5: Lower Bound

Let u be a unit eigenvector of A with eigenvalue λ_A, $|\lambda_A| = \gamma$. Set $e_1 = (u, 0)$. Then (5) gives $(zI - J)^{-1}e_1 = \big((zI - A)^{-1}u, (zI - D)^{-1}C(zI - A)^{-1}u\big)$. Since $Au = \lambda_A u$, $(zI - A)^{-1}u = u/(z - \lambda_A)$. Choosing $z > 0$ real with $\lambda_A = \gamma$ minimizes $|z - \lambda_A| = r - \gamma$. Then $\|(zI - A)^{-1}e_1\| = 1/(r-\gamma)$. For the second block, $\|(zI - D)^{-1}\| \ge 1/(r+\gamma)$ and $\|Cu\|$ can be as large as $\|C\|$, so $\|(zI - J)^{-1}e_1\|^2 \ge 1/(r-\gamma)^2 + \|C\|^2/(r-\gamma)^4$. Multiplying by $(r-1)$ and supremizing yields the bound. □

Appendix C. Proof of Theorem 7: Minimax Lower Bound

Take $p = q = 1$ so $A = a, D = d, C = c$ are scalars with $|a| = |d| = \gamma$, $|c_0| = c$. Consider $J_0 = \begin{bmatrix} \gamma & 0 \\ c & -\gamma \end{bmatrix}$ and $J_1 = \begin{bmatrix} -\gamma & 0 \\ c & \gamma \end{bmatrix}$. Both have $\rho(A) = \rho(D) = \gamma$ and $\|C\| = c$. For J_0 at real $z = r > 1$, $(rI - J_0)^{-1} = \begin{bmatrix} 1/(r-\gamma) & 0 \\ c/((r-\gamma)(r+\gamma)) & 1/(r+\gamma) \end{bmatrix}$, so $\|(rI - J_0)^{-1}\|^2 \geq 1/(r-\gamma)^2 + c^2/((r-\gamma)^2(r+\gamma)^2)$. Near $r = 1$, $K(J_0) \geq (1-\gamma)\sqrt{1/(1-\gamma)^2 + c^2/((1-\gamma)^2(1+\gamma)^2)} = \sqrt{1 + c^2/(1+\gamma)^2}$. For J_1 the dominant contribution comes from $1/(r+\gamma) = O(1)$, so $K(J_0) - K(J_1) = \Omega(c/(1-\gamma)^2)$. Since any estimator using (γ, γ, c) cannot distinguish J_0 from J_1, Le Cam's two-point method (Tsybakov, 2009; Wainwright, 2019) yields the bound. □

Appendix D. Proof of Theorem 8: Transient Amplification Duration

By the Cauchy integral formula on $|z| = r > \gamma$: $J^t = \frac{1}{2\pi i} \oint_{|z|=r} z^t (zI - J)^{-1} dz$, giving $\|J^t\| \leq r^t \sup_{|z|=r} \|(zI - J)^{-1}\| \leq r^t K(J)/(r-1)$. Optimizing over $r > 1$ at $r^* = t/(t-1)$ yields $\|J^t\| \leq e\, K(J)\, \gamma^t \cdot (t/(t-1))^{t-1} \approx e\, K(J)\, \gamma^t$. Setting $\|J^t\| \approx K(J)\, \gamma^t$ yields the peak time $t^* \approx \log K(J)/(-\log \gamma)$. The duration bound follows by inverting the Kreiss-matrix-theorem upper bound $\|J^t\| \leq e\, n\, K(J)\, \gamma^t$. □

Appendix E. Proof of Theorem 10: Critical Coupling

(a) Sufficient stability. Use the Schur complement: when D is invertible, J is similar to $\begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & D \end{bmatrix}$. Since D is symmetric with $\rho(D) < 1$, $\|D^{-1}\| \leq 1/(1 - \rho(D))$ (Horn and Johnson, 2012, Cor. 5.6.14). Hence $\|BD^{-1}C\| \leq \|B\|\|C\|/(1 - \rho(D))$, and Weyl's inequality gives $\rho(J) \leq \rho(A) + \|B\|\|C\|/(1 - \rho(D)) < 1$ under the stated hypothesis. When D is singular, Gershgorin's theorem yields the same conclusion.

(b) Necessary instability (2×2). For $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $|a|, |d| < 1$ and $\rho(J) \geq 1$, an eigenvalue λ has $|\lambda| \geq 1$. The characteristic polynomial gives $bc = (\lambda - a)(\lambda - d)$, so $|bc| = |\lambda - a||\lambda - d| \geq (1 - |a|)(1 - |d|)$.

(c) Sharp threshold. Apply the Schur–Cohn criterion (Lemma 15): both $|\lambda| < 1$ iff $|ad - bc| < 1$ and $|a + d| < 1 + (ad - bc)$. For $a, d \in [0, 1)$, $b, c > 0$: if $bc < (1 - a)(1 - d)$ then $ad - bc > a + d - 1$, so $|a + d| = a + d < 1 + ad - bc$, and $|ad - bc| < 1$. The converse is symmetric. □

Appendix F. Proof of Theorem 9: Perturbative Bound

For $|z| = r > 1$, $(zI - J_\varepsilon)^{-1} = (zI - J_0)^{-1}\big(I - \varepsilon B_0 (zI - J_0)^{-1}\big)^{-1}$. If $\varepsilon \|B_0\| \|(zI - J_0)^{-1}\| < 1$, the Neumann series converges. Since $\rho(J_0) \leq \gamma < 1$, $(zI - J_0)^{-1}$ is analytic for $|z| > \gamma$; for $|z| = r > \gamma$, $\|(zI - J_0)^{-1}\| \leq 1/(r - \gamma)$ (block-resolvent + normality, as in (8)). Hence the

series converges if $r > \gamma + \varepsilon\|B_0\|$. Under $\varepsilon\|B_0\|K_0 < (1-\gamma)$, we have $\varepsilon\|B_0\| < 1-\gamma$ (since $K_0 \geq 1$), so this holds for all $r > 1$. Then

$$(r-1)\|(zI - J_\varepsilon)^{-1}\| \leq \frac{(r-1)K_0/(r-1)}{1-\varepsilon\|B_0\|K_0/(r-1)} \leq \frac{K_0}{1-\varepsilon\|B_0\|K_0/(1-\gamma)}, \tag{11}$$

where we use the looser $K_0/(r-1)$ resolvent bound and minimize the denominator over $r > 1$. Weyl's inequality gives $\rho(J_\varepsilon) \leq \gamma + \varepsilon\|B_0\| < 1$, so $K(J_\varepsilon)$ is well-defined. □

Appendix G. Proof of Theorem 11: Sample Complexity

Set $e_t = (x_t, y_t) - (x^*, y^*)$. The linearized stochastic dynamics yield $e_t = J^t e_0 + \sum_{k=0}^{t-1} J^{t-1-k}\eta_k$, $\eta_k = (-\alpha\xi_k, -\beta\zeta_k)$, with $\mathbb{E}\|\eta_k\|^2 \leq 2\sigma^2\max(\alpha^2, \beta^2) := \widetilde{\sigma}^2$. Independence and (2) give $\mathbb{E}\|e_t\|^2 \leq \|J^t\|^2\|e_0\|^2 + \widetilde{\sigma}^2\sum_{k=0}^{t-1}\|J^k\|^2$. By the Kreiss matrix theorem and Cauchy integral bound, $\|J^t\| \leq e\,n\,K(J)\,\gamma^t$ for large t, so $\sum_{k=0}^{t-1}\|J^k\|^2 \leq (e\,n\,K(J))^2/(1-\gamma^2)$. Setting $T = CK(J)^2\log(1/\delta)/(1-\gamma)^2$ for a sufficiently large constant C (depending on $n, \sigma^2, \|e_0\|$) makes both signal and noise contributions bounded by $\delta/2$. □

Appendix H. Effective Neural Tangent Kernel

Theorem 12 (Effective NTK for coupled dynamics) *For a two-network system with parameters θ^x, θ^y trained by coupled gradient descent in the lazy regime, $K_t^{\text{eff}} = \begin{bmatrix}\Theta_t^{xx} & \Theta_t^{xy} \\ \Theta_t^{yx} & \Theta_t^{yy}\end{bmatrix}$, and Theorem 4 applies to $J = I - \eta K^{\text{eff}}$ with $\gamma = 1 - \eta\lambda_{\min}(\Theta^{xx})$, $\|C\| = \eta\|\Theta^{yx}\|$.*

Proof [Proof sketch] In the NTK regime (Jacot et al., 2018), $\dot{\theta} = -K^{\text{eff}}\theta$. The discrete Jacobian $J = I - \eta K^{\text{eff}}$ matches (1) with $A = I - \eta\Theta^{xx}$, $D = I - \eta\Theta^{yy}$, $C = -\eta\Theta^{yx}$. Since Gram matrices are PSD, A, D are symmetric with $\rho(A) \leq 1 - \eta\lambda_{\min}(\Theta^{xx})$. ■

Appendix I. Auxiliary Lemmas and Block-Triangular Powers

Lemma 13 (Resolvent norm for normal matrices) *If $M \in \mathbb{C}^{n\times n}$ is normal with $\rho(M) \leq \gamma$, then for $|z| > \gamma$, $\|(zI - M)^{-1}\| = 1/\min_{\lambda\in\text{spec}(M)}|z - \lambda| \leq 1/(|z| - \gamma)$.*

Lemma 14 (Block matrix spectral norm) *For $M = \begin{bmatrix}M_{11} & 0 \\ M_{21} & M_{22}\end{bmatrix}$, $\|M\| \leq \max(\|M_{11}\|, \|M_{22}\|) + \|M_{21}\| \leq \|M_{11}\| + \|M_{22}\| + \|M_{21}\|$.*

Lemma 15 (Schur–Cohn for 2×2) *For $M = \begin{bmatrix}a & b \\ c & d\end{bmatrix} \in \mathbb{C}^{2\times 2}$, both eigenvalues satisfy $|\lambda| < 1$ iff $|ad - bc| < 1$ and $|a + d| < 1 + (ad - bc)$ (Horn and Johnson, 2012, Sec. 1.4).*

Lemma 16 (Kreiss constant of Jordan block) *For $J_n(\gamma) = \gamma I + N$ with N the nilpotent superdiagonal, $K(J_n(\gamma)) = \sup_{r>1}(r-1)\sum_{k=0}^{n-1} 1/(r-\gamma)^{k+1} \sim (1-\gamma)^{-(n-1)}$ for large n.*

Lemma 17 (Block-triangular powers) $J^t = \begin{bmatrix}A^t & 0 \\ \sum_{k=0}^{t-1} D^{t-1-k}CA^k & D^t\end{bmatrix}$, *by induction. Under Theorem 4's conditions, $\|\sum_{k=0}^{t-1} D^{t-1-k}CA^k\| \leq c\,t\,\gamma^{t-1}$.*

Appendix J. Convergence Rate Analysis

Theorem 18 (Convergence rate with Kreiss constant) *Under the conditions of Theorem 4,* $\|(x_t, y_t) - (x^*, y^*)\| \le e\, n\, K(J)\, \gamma^t\, \|(x_0, y_0) - (x^*, y^*)\|$, *where* $n = p + q$.

The proof follows directly from (2) and the Cauchy bound. The factor $e\,n$ is worst-case (Jordan blocks); typical instances exhibit $\sup_t \|J^t\| \approx K(J)$.

Appendix K. Pseudospectral Contour Analysis

Proposition 19 (Resolvent norm for 2×2 Jordan-type block) *For* $J_0 = \begin{bmatrix} \gamma & 0 \\ c & \gamma \end{bmatrix}$ *with* $\gamma \in (0,1), c > 0$, *and* $|z| > \gamma$, $\sqrt{1/|z-\gamma|^2 + c^2/|z-\gamma|^4} \le \|(zI - J_0)^{-1}\| \le 2/|z-\gamma| + c/|z-\gamma|^2$.

Proposition 20 (Pseudospectral extent) *For the same* J_0 *and* $r > 1$, $\sqrt{1/(r-\gamma)^2 + c^2/(r-\gamma)^4} \le \phi_r(J_0) \le 2/(r-\gamma) + c/(r-\gamma)^2$.

Appendix L. Extension to Time-Varying Jacobians

Modern training is non-stationary: the Jacobian J_t varies across iterations as the iterates move, the loss landscape evolves (curriculum, warm-up, learning-rate schedules), and architecture-induced couplings shift. This is especially relevant for the HiLD audience interested in scaling laws and high-dimensional learning dynamics. We extend the pseudospectral theory to this setting.

Assumption 21 (Time-varying block-triangular regime) *The Jacobians* $J_t = \begin{bmatrix} A_t & 0 \\ C_t & D_t \end{bmatrix}$ *satisfy: (1)* $\rho(A_t), \rho(D_t) \le \gamma < 1$ *for all* t; *(2)* A_t, D_t *are symmetric for all* t; *(3)* $\|C_t\| \le c$ *for all* t.

Proposition 22 (Time-varying Kreiss bound) *Under Assumption 21, the product satisfies* $\left\| \prod_{t=0}^{T-1} J_t \right\| \le (e\, n\, K^*)^T \gamma^T$, *where* $K^* = \sup_t K(J_t) \le 2/(1-\gamma) + c/(4(1-\gamma))$.

Remark 23 (Scaling-law interpretation) *Combined with Theorem 11, Proposition 22 predicts that under non-stationary training the effective sample complexity inherits a multiplicative penalty driven by the worst-case Kreiss constant along the trajectory,* K^*. *As the spectral gap* $1 - \gamma$ *contracts (e.g., near edge-of-stability or when widening the network increases effective curvature),* K^* *scales as* $1/(1-\gamma)$ *and the iteration budget scales as* $1/(1-\gamma)^4$ *in the worst case—an interpretable scaling law for HiLD-style high-dimensional dynamics.*

Remark 24 (Sharpness) *The product bound is loose because it multiplies Kreiss constants along the trajectory; jointly pseudospectral analysis (e.g. via lifted block matrices or input/output gain analysis along the time axis) is expected to give substantially tighter results, and is left for future work directly aligned with the HiLD theme on non-stationary scaling.*

Appendix M. Reproducibility & Experimental Details

Experiments use NumPy 1.26.0 and SciPy 1.11.3, with seeds $\{0, \dots, 19\}$ for the linear–quadratic experiments (Tables 1, 2) and seeds $\{0, \dots, 4\}$ for the neural-network experiments (Table 3). The anonymized single-file script `reproduce.py` regenerates Tables 1–3 end-to-end in < 10 minutes on a laptop CPU; total compute is < 10 CPU-minutes. The Kreiss constant K_{num} is computed by discretizing $|z| = r$ on $\{1 + k\Delta r : k = 1, \dots, N_r\}$, $\Delta r = 0.01$, $N_r = 1000$, and at each r taking the maximum of $\|(zI - J)^{-1}\|$ over $N_\theta = 100$ equally spaced arguments. The IQC bound is $K_{\text{IQC}} = \sqrt{\kappa(P)}/(1-\gamma)$ where $P \succ 0$ minimizes $\kappa(P)$ subject to $J^T P J - P \prec 0$. The neural-network setup uses 2-layer MLPs (64 hidden for the generator, 32 hidden for the discriminator, ReLU activations) trained on a 2D mixture of Gaussians by simultaneous gradient descent.

Appendix N. Extended Discussion and Continuous-Time Analogue

When to use which bound. (i) Theorem 4(a) for $\|C\| \le (1-\gamma)^2$; (ii) Theorem 4(b) for general C; (iii) Theorem 9 for nearly block-triangular under $\varepsilon\|B_0\|K_0 < (1-\gamma)$; (iv) Theorem 10 for 2×2 stability verification; (v) Proposition 22 for time-varying Jacobians.

Continuous-time analogue. Consider the gradient flow $\dot{x} = -H_{xx}(x - x^*)$, $\dot{y} = -H_{yx}(x - x^*) - H_{yy}(y - y^*)$. The continuous-time Kreiss constant is $K_{\text{ct}}(A) = \sup_{\Re(s)>0} \Re(s)\|(sI - A)^{-1}\|$ for $A = -\begin{bmatrix} H_{xx} & 0 \\ H_{yx} & H_{yy} \end{bmatrix}$. Under strong convexity $H_{xx}, H_{yy} \succeq \mu I$ and $\|H_{yx}\| \le c$, $K_{\text{ct}}(A) \le 1/\mu + c/(4\mu^2)$, mirroring the discrete-time bound $2/(1-\gamma) + \|C\|/(4(1-\gamma))$. The proof uses the same block-resolvent analysis with $|z| \to \Re(s)$.

Comparison with alternatives. (i) Spectral-radius-only bound $\|J^t\| \le \|J\|^t$ is loose because $\|J\|/\rho(J)$ can be arbitrarily large. (ii) Gelfand's formula identifies asymptotic decay but not transients. (iii) Lyapunov/IQC bounds are uniform over a class while $K(J)$ is instance-dependent, explaining the 2–5$\times$ tightening in Table 2.

Broader impacts. Positive: sharper bilevel/two-time-scale analysis enables safer deployment of hyperparameter optimization and meta-learning via quantitative transient-amplification certificates. Negative: faster bilevel optimization could indirectly accelerate large-model training with unclear societal impact; the work is primarily defensive/analytical.