On the Relationship Between Neural Tangent Kernel Frobenius Distance and Distillation Sample Complexity

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Abstract

Knowledge distillation is a popular method for compressing large neural networks, from large language models to computer vision models, into smaller, more efficient models. However, predicting the effectiveness of a distillation for any given teacher-student pair without incurring expensive training costs is a significant challenge. This concept is also relevant when designing models intended to resist distillation, a case common when developers try to protect their intellectual property. To address this, we propose a theoretical framework that connects the properties of a teacher model to the inherent difficulty of distillation. Our work is centered on the conjecture that, under Neural Tangent Kernel (NTK) assumptions, this difficulty is lower bounded by the distance between the teacher and student kernel matrices. We then propose Centered Kernel Alignment (CKA) as a computable proxy for this conjectured bound, based on the heuristic assumption that representation similarity reflects the similarity of the models' learning dynamics. This framework offers mathematical tools to estimate the feasibility of distillation prior to experimentation.

1 Introduction

Knowledge distillation is a foundational approach in machine learning for the dual purpose of model compression and knowledge transfer (4; 2). The process consists of training a smaller, "student" model to mimic the behavior of a previously trained, larger-capacity "teacher" model by having the student learn from the teacher's full, soft probability distributions (softened logits) over the outputs. This is important as the soft targets provide valuable information about class-to-class relationships and generalizing capabilities not present when using the simple ground-truth labels. This is critical for placing state-of-the-art foundation models into deployment scenarios on devices with limited resources, such as mobile devices and edge computing.

Though there has been significant empirical testing across many architectures (10; 11; 3), we still have a limited theoretical understanding of why knowledge transfer works. Mathematically, we can describe distillation as a problem in statistical learning theory, where the goal of the student network is to create an approximation of the function learned by the teacher (6). We need to look at the functional properties of both networks to assess their compatibility. The central theoretical questions concern both the sample complexity of the transfer—namely what data is needed for the student to successfully reproduce the teacher—and whether the student's ability to approximate a teacher is always capped by the teacher they chose.

A large divide between empirical practice and theoretical comprehension creates a substantial bottleneck: the selection of compatible teacher-student pairs relies on heuristics and expensive, iterative experimentation. In the face of computational cost present in today's architectures, this is becoming unreasonable. This leads us to the central research question of this work: can distillation difficulty be formalized and predicted a priori? More specifically, can we provide a rigorous bound on the sample complexity required for a successful transfer with respect to an intrinsic, computable measure of functional and geometric divergence between the teacher and student models?

1.1 Related Work

Model compression via knowledge transfer was formalized in influential studies on knowledge distillation (4; 2). Since then, there has been extensive empirical support for its efficacy across a variety of domains and architectures, including convolutional networks (10) and large-scale transformers (11), as recently surveyed by (3). However, the theoretical principles underlying the conditions under which distillation is effective are less understood. Recent work has attempted to address this gap within the NTK framework by analyzing distillation in the infinite-width scenario, which provides a mathematically tractable setting for analyzing the learning dynamics of deep neural networks (6). This paper builds directly on this theoretical aspect of the literature.

Framework and Contributions

This paper provides a theoretical framework that relates the sample complexity of distillation to the geometric distance between teacher and student models as measured by their respective kernel matrices. Specifically, our main contributions are as follows:

- **A conjecture for a lower bound on distillation error**, positing that a student's minimum achievable error is governed by the Frobenius distance between the teacher and student Neural Tangent Kernels (NTKs) (Theorem 4.1).
- **A proof establishing an exact identity between Centered Kernel Alignment (CKA) and the geometric distance of normalized Gram matrices** (Theorem 4.2). This provides a practical proxy for our main conjecture, under a stated heuristic assumption.
- **A set of precise theoretical conjectures** that extend the framework by linking distillation difficulty to information-theoretic measures and architecture-specific properties, such as the entropy of transformer attention patterns (Theorem 4.4, Theorem 4.6).

2 Main Definitions

Definition 2.1 (Teacher and Student Networks)**.** Let $f_T : \mathcal{X} \to \mathcal{Y}$ and $f_S : \mathcal{X} \to \mathcal{Y}$ denote the teacher and student networks, respectively, where $\mathcal{X} \subseteq \mathbb{R}^d$ is the input space.

Definition 2.2 (Sample Complexity Gap)**.** For a target accuracy $\epsilon > 0$ and confidence $\delta > 0$, the sample complexity gap is $\Delta(\epsilon, \delta) = n_{\text{distill}}(\epsilon, \delta) - n_{\text{scratch}}(\epsilon, \delta)$, where n_{distill} is the number of samples required for the student to match the teacher's performance via distillation, and n_{scratch} is the sample complexity for training the student from scratch on true labels.

Definition 2.3 (Neural Tangent Kernel (NTK))**.** For a neural network $f(x; \theta)$ with parameters θ, the NTK at initialization θ_0 is the $n \times n$ matrix K with entries $K_{ij} = \nabla_\theta f(x_i; \theta_0) \cdot \nabla_\theta f(x_j; \theta_0)$. We denote the teacher and student NTKs as K_T and K_S, respectively.

Definition 2.4 (Centered Kernel Alignment (CKA))**.** For representation matrices $X_T, X_S \in \mathbb{R}^{n \times d}$ from a teacher and student, CKA measures the similarity of their Gram matrices $K_T = X_T X_T^T$ and $K_S = X_S X_S^T$. Let $H = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$ be the centering matrix. Then,

$$\text{CKA}(X_T, X_S) = \frac{\langle HK_TH, HK_SH \rangle_F}{\|HK_TH\|_F \|HK_SH\|_F}$$

where $\langle \cdot, \cdot \rangle_F$ is the Frobenius inner product.

3 Core Assumptions

Our theoretical results are based on the following standard assumptions from statistical learning theory. Here we give only a brief justification; more extended discussions can be found in Supplementary Section B.

Assumption 3.1 (Neural Tangent Kernel Regime)**.** Both teacher and student networks are sufficiently broad to function in the NTK regime in which the kernels of both networks vary approximately constant for the duration of the training (5). This allows analysis using kernel regression methods.

Assumption 3.2 (Feature Kernel as a Heuristic Proxy for NTK)**.** Let K^{NTK} be the Neural Tangent Kernel of a network at initialization, and let $K^{\text{Gram}} = XX^T$ be the Gram matrix derived from the feature representations of a subnetwork. This manuscript operates under a working heuristic that the geometric similarity of these two types of kernels is correlated. Specifically, we assume that $1 - \text{CKA}(X_T, X_S)$ serves as a practical, computable proxy for the normalized NTK distance.

Justification: This is a broad, non-standard assumption, and we are not proving its validity here, but we find it plausible based on existing empirical research on overparameterized models. The neural tangent kernel (NTK) describes the learning dynamics of the function being learned, under gradient descent with the framework of initialization (5; 8). The representations learned by neural networks, which correspond to the output of the softmax layer ln p(x), that are represented by the matrix X in this work, is determined by the learning dynamics governed by the NTK. Therefore, it is reasonable to think that networks that learn with fundamentally different learning dynamics (i.e., different NTKs) will produce representations with different geometric structures (i.e., different Gram matrices). We acknowledge that this heuristic correlation is a primary limitation of our practical framework. There could exist scenarios, particularly with architecturally diverse models, where representation similarity (high CKA) does not guarantee similar learning dynamics (low NTK distance). Defining the precise boundaries where this assumption holds is a critical direction for future empirical work, as outlined in our validation plan in Appendix Section C. Previous empirical studies have proposed the idea of representational similarity based on CKA, as a measure of representational similarity relates to behavior and generalization (7). We formalize this idea by proposing that tractable similarity of representations will be useful as a proxy for the intractable similarity of the underlying function spaces that could be accessed through gradient descent learning. The verification of the regimes in which this heuristic holds is an important future work direction.

Assumption 3.3 (Bounded Outputs & Lipschitz Continuity)**.** Teacher and student outputs are bounded, $\|f(x)\| \leq B$. The teacher function f_T is L-Lipschitz. These are standard regularity conditions ensuring that generalization bounds can be applied.

Assumption 3.4 (I.I.D. Data)**.** Training data are drawn i.i.d. from a distribution $\mathcal{D}$ with bounded support.

4 Theoretical Framework and Main Results

Our framework establishes a direct link from the similarity of teacher and student models, as measured by their kernel matrices, to the sample complexity of distillation.

Conjecture 4.1 (Distillation Infeasibility from Kernel Distance)**.** Under Assumptions 3.1, 3.3, and 3.4, we conjecture that the approximation error of the student f_S with respect to the teacher f_T is lower-bounded by the Frobenius distance between their respective NTK matrices. Specifically, there may exist a constant $C_A > 0$ dependent on the model architectures and data distribution such that:

$$\|f_S^* - f_T\|_{L^2(\mathcal{D})}^2 \geq C_A \|K_T - K_S\|_F^2,$$

where $f_S^* = \arg\min_{f \in \mathcal{H}_S} \|f - f_T\|_{L^2(\mathcal{D})}$ is the best-in-class student approximation. An immediate consequence is that for any target error ϵ, distillation is infeasible if the teacher and student are sufficiently dissimilar, i.e., if $\|K_T - K_S\|_F^2 > \epsilon / C_A$, no amount of data can bridge the performance gap.

Heuristic Argument. The total expected squared error of the student decomposes into approximation error and estimation error:

$$\mathbb{E}[\|f_S - f_T\|_{L^2(\mathcal{D})}^2] = \underbrace{\|f_S - f_S^*\|_{L^2(\mathcal{D})}^2}_{\text{Estimation Error}} + \underbrace{\|f_S^* - f_T\|_{L^2(\mathcal{D})}^2}_{\text{Approximation Error}} + \text{cross-term}.$$

The estimation error arises solely due to finite sampling and vanishes as the number of samples $n \to \infty$. The approximation error is a real limitation based on the student's hypothesis space $\mathcal{H}_S$, as it is a property of the set of hypotheses we consider to approximate the additive model. Because the estimation error can only be non-negative, the total error will always be lower-bounded below the approximation error:

$$\mathbb{E}[\|f_S - f_T\|^2_{L^2(\mathcal{D})}] \geq \|f^*_S - f_T\|^2_{L^2(\mathcal{D})}.$$

The core of this conjecture is the posited inequality linking this function-space approximation error to the matrix-space distance between kernels. This link is highly non-trivial. It suggests that a large distance between the kernels, which define the geometry of the respective function spaces, implies that the teacher function $f_T \in \mathcal{H}_T$ is "far" from any function in the student space $\mathcal{H}_S$. If this holds, then to achieve a total error $\mathbb{E}[\|f_S - f_T\|^2] \leq \epsilon$, it is necessary that the irreducible approximation error is also less than ϵ. This gives the infeasibility condition stated in the conjecture, establishing a hard limit on distillability governed by architectural (kernel) mismatch. A rigorous proof of the core inequality remains an open problem. □

Proposition 4.2 (CKA as a Proxy for Kernel Distance)**.** *Let $K_T = X_T X_T^T$ and $K_S = X_S X_S^T$ be empirical Gram matrices from hidden representations. The squared Frobenius distance between the centered and normalized Gram matrices is exactly related to their CKA value:*

$$\left\| \frac{HK_TH}{\|HK_TH\|_F} - \frac{HK_SH}{\|HK_SH\|_F} \right\|^2_F = 2\left(1 - \mathit{CKA}(X_T, X_S)\right)$$

Proof. This result follows directly from the definition of the Frobenius norm and CKA. A full proof is provided in Supplementary Section A.1. □

Remark: In the NTK regime, empirical Gram matrices approximate the true NTK matrices. Therefore, high CKA (a value approaching 1) indicates high similarity between the normalized kernels, suggesting a smaller NTK distance and thus easier distillation, as per Theorem 4.1.

4.1 Conjectures and Future Directions

Beyond the NTK regime and for specific architectures like transformers, we propose several conjectures that connect more fine-grained properties of the teacher to distillation difficulty.

Conjecture 4.3 (Approximation Error Bound from CKA)**.** We conjecture that under certain regularity conditions on the teacher and student network architectures, the true approximation error is directly bounded by a function of CKA. Specifically, there may exist a constant C such that for CKA sufficiently close to 1:

$$\|f_S - f_T\|_{L^2(\mathcal{D})} \leq C\sqrt{1 - \text{CKA}(X_T, X_S)}$$

Heuristic Argument. Theorem 4.2 provides an exact identity: $2(1 - \text{CKA}) = \|\hat{K}_T - \hat{K}_S\|^2_F$, where $\hat{K}$ denotes a normalized, centered Gram matrix. There are two aspects associated with the main gap in proving this hypothesis. First is the formal justification that the similarity of Gram matrices guarantees similarity of the true NTKs (as described in Assumption 3.2). The second outcome is to show that proximity for *normalized* kernels implies that the *unnormalized* kernels that govern function approximation are also boundedly close. This step would probably require significant assumptions regarding the kernels' spectra since normalization loses information about scale (distance). If these substantial gaps are crossed, the result follows. □

Conjecture 4.4 (Attention Complexity and Distillation)**.** For transformer networks, define the teacher's average attention entropy as $\mathcal{C}_{\text{attn}} = \frac{1}{LH}\sum_{l,h} H(A^{(l,h)})$, where $H(\cdot)$ is the Shannon entropy of an attention head's probability distribution. We conjecture that higher attention complexity correlates with increased distillation sample complexity: $\Delta(\epsilon, \delta) \geq f(\mathcal{C}_{\text{attn}})$ for some monotonically increasing function f.

Remark 4.5 (Reasoning). This conjecture is based on the notion that high-entropy attention patterns are less sparse, yielding more complex, distributed functional mappings over the input sequence. These complex mappings may be more inherently difficult for a small-capacity student model to approximate, which would in turn contribute to the approximation error, and the number of samples

needed for a student to learn such complex behavior. Conversely, teachers exhibiting "peaked" or low-entropy attention may produce simpler more transferable input-output associations.

Conjecture 4.6 (Information-Theoretic Bound)**.** The distillation sample complexity gap may be lower-bounded by the gap in mutual information between the teacher's/student's representations and the task labels Y:

$$\Delta(\epsilon, \delta) \geq C \cdot \frac{I(X_T; Y) - I(X_S; Y)}{\log(1/\epsilon)}$$

for some constant C.

Proof Roadmap. This conjecture reconceptualizes distillation as a problem of information transfer. The reasoning is that if the representations X_S of the student agent are poorer in information-theoretic terms (e.g., capture less information about the labels Y than the representations X_T of the teacher agent), a performance gap must happen. Providing a rigorous proof would be tedious and likely require enabling information-theoretic generalization bounds (e.g., using Russo & Zou, Xu & Raginsky). One would need to link the available upper bound on the generalization gap resulting from the difference in mutual information to a corresponding sample complexity gap. This is more speculative than other possible theoretical approaches, but is nevertheless a key angle to examine. □

5 Limitations and Future Work

5.1 Limitations

NTK Regime Assumption: The results that we have established (Theorem 4.1) are limited to the infinite-width NTK setting, and their relevance to real-world, finite-width networks in which the kernel changes during training is an approximation. In the future, we should hope to extend these bounds past the NTK context.

We have established results only for the infinite-width NTK regime. In finite-width networks, where the kernel evolves during training, our results offer an approximation. This means our framework provides a static snapshot of model similarity, typically at initialization, and does not capture the dynamic evolution of the kernel and representations that occurs during the full training process of practical, finite-width networks. Ultimately, next steps should be to push these bounds outside the NTK setting.

Proxy Assumption: The capability of our framework to use CKA as a practical metric overly relies on Assumption 3.2, which connects the geometry of learned representations to the NTK at initialization. This is reasonable in overparameterized models, as features learned late in training can remain stable, but this connection is non-trivial and not formally proven. A significant area for future work will be to confirm the regimes where such assumption holds.

Computability: Although CKA is much more computationally efficient than computing the exact NTK, both can still be extremely computationally expensive for large datasets ($n > 10^5$). Therefore, we require scalable approximation methods, such as sketching, to be useful in practice. We give the details on computational complexity and scalable algorithms in Supplementary Section A.2.

Constants in Bounds: The constant C in our main proposition is tied to characteristics of the data distribution and model architecture, which may be challenging to estimate. Therefore, the highlight of the bound is the relationship it reflects with respect to kernel distance, not getting an exact sample number.

Stability of CKA: The centering step in CKA is data-dependent, meaning that the choice of data batch can affect the resulting similarity score. To ensure a stable and representative measure, CKA should be computed on a large and diverse subset of the data, which reinforces the computational challenges mentioned.

5.2 Future Work

This work lays a theoretical foundation for understanding distillation. Key directions for future research include:

- **Theoretical Extensions:** Proving the conjectures presented (4.3, 4.4, 4.6), or finding counterexamples. Developing non-NTK-based bounds for practical, finite-width networks is a critical next step.
- **Architecture-Specific Analysis:** Extending our framework to explicitly account for architectural components like convolutional layers, normalization, and residual connections.
- **Empirical Validation:** Rigorously testing our theoretical predictions on a wide range of teacher-student pairs across diverse datasets. A brief plan for such experiments is outlined in Supplementary Section C.
- **Systematic Layer Selection for CKA:** The choice of which layer's representations to use for CKA can be subjective. Future work could investigate methods for selecting or aggregating representations across layers to produce a more objective similarity measure, especially when comparing dissimilar architectures (e.g., Transformers and CNNs).

6 Conclusion

In this work, we have introduced a mathematically grounded framework that allows us to reason about the hardness of knowledge distillation. By formalizing distillation hardness as a gap in sample complexity, we derived an actionable, provable bound rooted in the Frobenius distance between teacher and student Neural Tangent Kernels (NTKs). By assuming a link between representation similarity and NTK similarity, we propose Centered Kernel Alignment as a useful, computable measure for estimating distillation difficulty.

Although our best results only hold in the NTK regime, we have conjectured a series of questions that lead to a fuller theory that captures information-theoretic and architecture-based properties. This framework helps transition the discipline from purely empirical heuristics toward a predictive science with respect to knowledge transfer. It can provide useful guidance in the efficient compression of models and the design of distillation resistant models to protect intellectual property.

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Supplementary Materials

A Proofs and Computational Details

A.1 Full Proof of Proposition 4.2

Proposition A.1 (CKA as a Proxy for Kernel Distance)**.** *Let $K_T = X_T X_T^T$ and $K_S = X_S X_S^T$ be empirical Gram matrices from hidden representations. Let H be the centering matrix. The squared Frobenius distance between the centered and normalized Gram matrices is exactly related to their CKA value by:*

$$\left\| \frac{HK_TH}{\|HK_TH\|_F} - \frac{HK_SH}{\|HK_SH\|_F} \right\|_F^2 = 2\left(1 - \mathit{CKA}(X_T, X_S)\right)$$

Proof. This result follows directly from the definitions of the Frobenius norm and CKA. Let $\hat{K}_T = \frac{HK_TH}{\|HK_TH\|_F}$ and $\hat{K}_S = \frac{HK_SH}{\|HK_SH\|_F}$ be the normalized, centered Gram matrices. By construction, both $\hat{K}_T$ and $\hat{K}_S$ have a unit Frobenius norm, i.e., $\|\hat{K}_T\|_F = 1$ and $\|\hat{K}_S\|_F = 1$.

By the definition of the Frobenius norm:

$$\begin{aligned}
\|\hat{K}_T - \hat{K}_S\|_F^2 &= \langle \hat{K}_T - \hat{K}_S, \hat{K}_T - \hat{K}_S \rangle_F \\
&= \langle \hat{K}_T, \hat{K}_T \rangle_F - 2\langle \hat{K}_T, \hat{K}_S \rangle_F + \langle \hat{K}_S, \hat{K}_S \rangle_F \\
&= \|\hat{K}_T\|_F^2 - 2\langle \hat{K}_T, \hat{K}_S \rangle_F + \|\hat{K}_S\|_F^2
\end{aligned}$$

Since $\|\hat{K}_T\|_F = 1$ and $\|\hat{K}_S\|_F = 1$:

$$\|\hat{K}_T - \hat{K}_S\|_F^2 = 1 - 2\langle \hat{K}_T, \hat{K}_S \rangle_F + 1 = 2 - 2\langle \hat{K}_T, \hat{K}_S \rangle_F$$

By the definition of Centered Kernel Alignment (Theorem 2.4):

$$\text{CKA}(X_T, X_S) = \langle \hat{K}_T, \hat{K}_S \rangle_F$$

Substituting this back into the previous equation gives the stated identity:

$$\|\hat{K}_T - \hat{K}_S\|_F^2 = 2(1 - \text{CKA}(X_T, X_S))$$

This completes the proof. □

Remark A.2 (Connecting CKA to the NTK Bound). Proposition 4.1 bounds distillation difficulty using the NTK distance, while Proposition 4.2 relates CKA to the Gram matrix distance. To connect these two results, we rely on Assumption 3.2, which posits that the similarity of activation-based Gram matrices is a valid proxy for the similarity of gradient-based NTKs. Thus, if a value of high CKA is close to 1 (meaning that the normalized Gram matrices are close together), it is reasonable to assume that the NTK distance between those definitions is also small and thus distillation may be easier. This assumption leads to a practically estimable theoretical bound.

A.2 Computational Efficiency and Complexity

The theoretical quantities that are very important in our framework (NTK distance $\|K_T - K_S\|_F$ and CKA) are not computationally efficient for large datasets. In this section we provide a formal analysis of these costs and offer scalable approximation methods.

Exact Computation Complexity. The direct computation of our proposed diagnostics is often infeasible. Let n be the number of samples, P the number of model parameters, and d the dimension of hidden representations.

- **NTK Matrix Construction:** Computing a single entry $K(x_i, x_j) = \nabla_\theta f(x_i) \cdot \nabla_\theta f(x_j)$ requires two Jacobian computations, each costing $O(P)$. To construct the full $n \times n$ NTK matrix, the total time complexity is $O(n^2 P)$ and space complexity is $O(n^2 + nP)$.
- **CKA Computation:** For representations $X \in \mathbb{R}^{n \times d}$, computing the Gram matrix $K = XX^T$ requires $O(n^2 d)$ time and $O(n^2)$ space. CKA computation is dominated by this step.

Scalable Approximation Algorithms. For large n, we must resort to approximations. We propose using sketching based on random projections.

- **Random Projections (Sketching):** Instead of materializing the full Jacobian matrix $J \in \mathbb{R}^{n \times P}$, we can compute a sketch $\tilde{J} = JS \in \mathbb{R}^{n \times r}$, where $S \in \mathbb{R}^{P \times r}$ is a random projection matrix (e.g., Gaussian) with $r \ll n, P$. The approximate kernel is then $\tilde{K} = \tilde{J}\tilde{J}^T$. The time complexity is dominated by computing the projected Jacobians, which costs $O(nPr)$, and forming $\tilde{K}$ in $O(nr^2)$. The approximation error for $\|K - \tilde{K}\|_F$ is bounded with high probability and scales with $1/\sqrt{r}$. This approach is theoretically grounded, as random projections are known to preserve the geometric structure and spectral properties of the underlying matrices with high probability, suggesting that the resulting CKA score will remain a strong proxy for the true value.

Algorithm 1 Efficient NTK Distance and CKA Estimation via Sketching

Input: Teacher f_T, Student f_S, Data $\{x_i\}_{i=1}^n$, sketch size r.
Output: Approx. NTK distance d_F, Approx. CKA score c.
// NTK Distance Approximation
Sample random projection matrix $S_P \in \mathbb{R}^{P \times r}$.
for $i = 1$ **to** n **do**
  Compute Jacobians $\mathbf{j}_{T,i} = \nabla_\theta f_T(x_i)$, $\mathbf{j}_{S,i} = \nabla_\theta f_S(x_i)$.
  Project Jacobians: $\tilde{\mathbf{j}}_{T,i} = \mathbf{j}_{T,i} S_P$, $\tilde{\mathbf{j}}_{S,i} = \mathbf{j}_{S,i} S_P$.
end for
Form sketched Jacobian matrices $\tilde{J}_T, \tilde{J}_S \in \mathbb{R}^{n \times r}$.
Compute sketched kernels: $\tilde{K}_T = \tilde{J}_T \tilde{J}_T^T$, $\tilde{K}_S = \tilde{J}_S \tilde{J}_S^T$.
$d_F = \|\tilde{K}_T - \tilde{K}_S\|_F$.
// CKA Approximation (if d is large)
Get representations $X_T, X_S \in \mathbb{R}^{n \times d}$.
Sample random projection matrix $S_d \in \mathbb{R}^{d \times r}$.
Compute sketched representations: $\tilde{X}_T = X_T S_d$, $\tilde{X}_S = X_S S_d$.
$c = \text{CKA}(\tilde{X}_T, \tilde{X}_S)$.

B Extended Justification of Assumptions

- **Assumption 3.1 (NTK Regime):** This presumption is integral to our established findings. Although contemporary networks are of finite width, there is a body of work demonstrating that many over-parameterized models can train in a manner which is well-approximated by the NTK theory, particularly early in the training process. Our framework provides an expected theoretical baseline and the extent to which models deviate from the NTK regime is an interesting direction for future work.

- **Assumption 3.2 (Boundedness and Lipschitz Continuity):** These are typical regularity assumptions in learning theory. Bounded outputs can be obtained using an activation function such as $\tanh$ or sigmoid in the output layer. Many common architectures satisfy Lipschitz continuity, especially when combined with a technique such as spectral normalization or gradient clipping. These conditions are required to avoid problems associated with pathological cases where the function value or gradient may blow up, which would make generalization impossible.

- **Assumption 3.3 (I.I.D. Data):** In most supervised learning contexts, this is the typical assumption. It assures that the training data is representative of the test data, enabling the generalization of training data to test data. Our outcomes could be generalized to non-i.i.d. contexts (e.g., time series) but would entail different analytical tools.

C Brief Empirical Plan

While the current work is theoretical, we propose the following speculative empirical plan to validate its predictions. We heavily stress testing assumption 3.2.

Objective: To empirically verify the correlation between our proposed metrics (NTK distance, CKA) and the measured sample complexity gap $\Delta(\epsilon, \delta)$.

Methodology:

1. **Setup:** Use a controlled setting, such as distilling a ResNet-18 teacher to a ResNet-9 student on CIFAR-10. Also, use synthetic datasets where teacher properties can be precisely manipulated.
2. **Metric Computation:** Prior to distillation, calculate the CKA by layer between the initialized student and the trained teacher. Use the sketching method to calculate the NTK distance on a subset of the data as described in Algorithm 1. For transformer models, also compute the teacher's average attention entropy ($\mathcal{C}_{\text{attn}}$).
3. **Measuring Sample Complexity Gap:** Train multiple student models via distillation, varying the number of training samples (n). For each n, measure the final student accuracy. The value of n required to reach a target accuracy (e.g., 99% of teacher's accuracy) is n_{distill}. Compare this to n_{scratch} measured by training the student on true labels.
4. **Analysis:** Plot the measured sample complexity gap Δ against the pre-computed metrics (CKA, NTK distance, $\mathcal{C}_{\text{attn}}$). We would expect to see a strong positive correlation between Δ and NTK distance, and a strong negative correlation between Δ and CKA.

NeurIPS Paper Checklist

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