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Supercongruences and complex multiplication

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ABSTRACT

We study congruences involving truncated hypergeometric series of the form

$$ {}_3F_2\left(\begin{matrix} 1/2, \ 1/2, \ 1/2 \\ 1, \ 1 \end{matrix}; \lambda\right)_{(mp^s-1)/2} := \sum_{k=0}^{(mp^s-1)/2} ((1/2)_k/k!)^3 \lambda^k $$

where p is a prime and m, s are positive integers. These truncated hypergeometric series are related to the arithmetic of a family of K3 surfaces. For special values of λ, with $s = 1$, our congruences are stronger than those predicted by the theory of formal groups, because of the presence of elliptic curves with complex multiplications. They generalize a conjecture made by Stienstra and Beukers for the $\lambda = 1$ case and confirm some other supercongruence conjectures at special values of λ.

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1. Introduction

The hypergeometric series ${}_rF_{r-1}$ is defined as

$$ {}_rF_{r-1}\left(\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_{r-1} \end{matrix}; \lambda\right) := \sum_{k=0}^{\infty} \left(\frac{(a_1)_k(a_2)_k \cdots (a_r)_k}{k!(b_1)_k(b_2)_k \cdots (b_{r-1})_k}\right) \lambda^k $$

where $(a)_k := a(a+1)\cdots(a+k-1)$ and where none of the b_i is 0 or a negative integer [5]. The truncated hypergeometric series ${}_rF_{r-1}\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{matrix}; \lambda\right)_n$ is the degree n polynomial in λ obtained by truncating the hypergeometric series to the sum over k from 0 to n.

In this paper, we study the arithmetic of ${}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; \lambda\right)_n$; these values are related to a family of K3 surfaces

$$ S_\lambda : W^2 = X_1X_2X_3(X_1 - X_2)(X_2 - X_3)(X_3 - \lambda X_1) $$

with generic Picard number 19, that has been studied in [4,17]. The variation of the complex structure of this family is depicted by its Picard–Fuchs differential equation, which is an order-3 ordinary differential equation. Up to multiplication by a scalar, its unique holomorphic solution near 0 is ${}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; \lambda\right)$. Moreover, the Picard–Fuchs equation of the family S_λ is projectively equivalent to the symmetric square of the Picard–Fuchs equation of

$$ E_\lambda : y^2 = (x-1)\left(x^2 - \frac{1}{1-\lambda}\right); $$

see [17]. In terms of arithmetic, if we let $A_p(\lambda) = \#(S_\lambda/\mathbb{F}_p) - p^2 - 1$ and $a_p(\lambda) = p + 1 - \#(E_\lambda/\mathbb{F}_p)$, then $A_p(\lambda) = \left(\frac{1-\lambda}{p}\right)(a_p(\lambda)^2 - p)$ [4].[1]

Deuring's argument [10, p. 255] shows that for any $\lambda \in \mathbb{F}_p$,

$$ A_p(\lambda) \equiv {}_3F_2(\lambda)_{p-1} \equiv {}_3F_2(\lambda)_{\frac{p-1}{2}} \pmod{p}. $$

More generally Dwork showed in [11] that for any $\lambda \in \mathbb{Z}_p$, there is a p-adic number $\gamma(\lambda)$ such that

$$ {}_3F_2(\lambda)_{mp^s-1} \equiv \gamma(\lambda) \cdot {}_3F_2(\lambda^p)_{mp^{s-1}-1} \pmod{p^s} \tag{1} $$

for all integers $m, s \geq 1$.

It can be shown that these congruences come from a formal group structure attached to S_λ, as constructed by Stienstra [24]. In particular, when ${}_3F_2(\lambda)_{p-1} \not\equiv 0 \pmod{p}$ (i.e. p *ordinary* for S_λ), one can use the so-called Shioda–Inose structure of the K3

[1] The surfaces $X_{\tilde\lambda}$ with affine model $X_{\tilde\lambda} : s^2 = xy(x+1)(y+1)(x+\tilde\lambda y)$ studied in [4] are isomorphic to S_λ via $\lambda = -\tilde\lambda$, $X_1 = 1$, $X_2 = -1/y$, $X_3 = x/y$, and $W = s/y^3$. Note that our λ is the negative of the $\tilde\lambda$ in [4].

surfaces S_λ to show that $\gamma(\lambda) = \left(\frac{1-\lambda}{p}\right) \cdot \alpha_p(\lambda)^2$, with $\alpha_p(\lambda)$ being the unit root of $X^2 - a_p(\lambda)X + p = 0$.

At special values of λ, stronger congruences have been observed for the truncations ${}_3F_2(\lambda)_n$. Such congruences, that are stronger than what can be predicted from the formal group structure, are known as *supercongruences*. For example, Stienstra and Beukers conjectured the following supercongruences involving truncated hypergeometric series in [25], corresponding to our $\lambda = 1$ case: for odd primes p,

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} ; 1 \right)_{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 \equiv b_p \pmod{p^2} $$

where b_p is the pth coefficient of the weight-3 cusp form $\eta(4z)^6$, where $\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = e^{2\pi i z}$, is the eta function. This conjecture was proved by Van Hamme in [29], with subsequent proofs by Ishikawa [14] and Ahlgren [1]. More recently, using a different technique, it is shown in [19] that for any prime $p \equiv 1 \pmod 4$,

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} ; 1 \right)_{\frac{p-1}{2}} = -\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3} $$

where $\Gamma_p(\cdot)$ denotes the p-adic Gamma function; there is a similar expression for primes which are congruent to 3 modulo 4.

Similarly, Z.-W. Sun conjectured (see remark 1.4 in [26]) a congruence for the $\lambda = 64$ case:

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} ; 64 \right)_{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 (64)^k \equiv a_p \pmod{p^2} $$

where $a_p = 0$ if $p \equiv 3, 5, 6 \pmod 7$ and $a_p = 4x^2 - 2p$ where $p = x^2 + 7y^2$, $x, y \in \mathbb{Z}$, if $p \equiv 1, 2, 4 \pmod 7$. In fact, this a_p is just the pth coefficient of $\eta(z)^3\eta(7z)^3$.

We show that such supercongruences occur for ${}_3F_2(\lambda)_n$ whenever the elliptic curve E_λ has complex multiplications (CM):

Theorem 1. *Let λ be an algebraic number such that E_λ has complex multiplications. Let p be a prime and let E_λ have a model defined over $\mathbb{Z}_p$ with good reduction modulo $p\mathbb{Z}_p$. Then*

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} ; \lambda \right)_{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 \lambda^k \equiv \left(\frac{1-\lambda}{p} \right) \alpha_p(\lambda)^2 \pmod{p^2} $$

where $\alpha_p(\lambda) \in \mathbb{Z}_p$ is the unit root of $X^2 - [p+1-\#(E_\lambda/\mathbb{F}_p)]X + p = 0$ if E_λ is ordinary at p; and $\alpha_p(\lambda) = 0$ if E_λ is supersingular at p.

This result confirms the conjecture of Sun mentioned above. The rational values of λ such that E_λ has CM are $\lambda = -1, 4, -8, 64, \frac{1}{4}, \frac{-1}{8}$, and $\frac{1}{64}$ [4]; but note that this theorem applies also to algebraic CM values of λ and primes p such that λ can be embedded in $\mathbb{Z}_p$. For example, E_λ is CM when $\lambda = \frac{7}{8} + \frac{5\sqrt{2}}{8}$, and Theorem 1 applies to both embeddings of λ in $\mathbb{Z}_p$ for $p \equiv \pm 1 \pmod 8$.

At each CM value λ with $|\lambda| < 1$ in each embedding, there is a Ramanujan-type formula of the form $\sum_{k=0}^{\infty}(ak+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3 \lambda^k = \frac{b}{\pi}$ where a, b are algebraic numbers depending on λ. Corresponding supercongruences for $\sum_{k=0}^{\frac{p-1}{2}}(ak+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3 \lambda^k$ have been obtained in [8].

We derive the following corollary to Theorem 1 in section 4:

Corollary 2. *Let H_k be the harmonic sum $\sum_{j=1}^{k}\frac{1}{j}$. If E_λ is a CM elliptic curve, then for almost all primes p such that λ embeds in $\mathbb{Z}_p$,*

$$\sum_{i=0}^{\frac{p-1}{2}}\binom{2i}{i}^3\left(\frac{\lambda}{64}\right)^i\left(6(H_{2i}-H_i)+\left(\frac{\left(\frac{\lambda}{64}\right)^{p-1}-1}{p}\right)\right) \equiv 0 \pmod{p}.$$

Below is one simple, special case of these congruences for $\lambda = 64$.

Corollary 3. *For all primes $p > 3$, we have*

$$\sum_{i=1}^{\frac{p-1}{2}}\binom{2i}{i}^3\sum_{j=1}^{i}\frac{1}{i+j} \equiv 0 \pmod{p}. \tag{2}$$

In general, such congruences are difficult to prove. For similar work, see [1,3] and Remark 1 of [18].

Here are some other well-known examples of supercongruences. Beukers conjectured that for all odd primes p

$${}_4F_3\left(\begin{matrix}\frac{1-p}{2}, \frac{1-p}{2}, \frac{1+p}{2}, \frac{1+p}{2} \\ 1, 1, 1\end{matrix}; 1\right) \equiv c_p \pmod{p^2}$$

where the left hand side[2] is the $\frac{p-1}{2}$th Apéry number $\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}^2\binom{(p-1)/2+k}{k}^2$ and c_p is the pth coefficient of the weight-4 modular form $\eta(2z)^4\eta(4z)^4$.

In [22], Rodriguez-Villegas made many supercongruence conjectures, including that for all odd primes p

[2] Note that this hypergeometric series terminates after the $\frac{p-1}{2}$th term, because of the negative integer argument $\frac{1-p}{2}$, while Rodriguez-Villegas's conjecture that follows is a genuine truncation.

$$ {}_4F_3\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix}; 1\right)_{(p-1)/2} \equiv c_p \pmod{p^3}. $$

Ahlgren and Ono proved the modulo p^2 conjecture of Beukers using Gaussian hypergeometric functions (see [3] and [21, Chapter 11]). Kilbourn applied these methods to prove the modulo p^3 conjecture of Rodriguez-Villegas [15], and McCarthy proved another of Rodriguez-Villegas's modulo p^3 conjectures using a p-adic analogue of Gaussian hypergeometric functions [20].

We end our introduction with another motivation for supercongruences. It is known that the coefficients of weight-k noncongruence modular forms satisfy the so-called Atkin and Swinnerton-Dyer congruences [6,23]. These congruences are supercongruences if $k > 2$ [23] and have played an important role in understanding the characterizations of genuine noncongruence modular forms [16].

The paper is organized as follows. We present some background in Section 2. Section 3 discusses supercongruences and uses a theorem of Coster and van Hamme to show that the function ${}_3F_2(\lambda)_n$ exhibits supercongruences whenever E_λ has CM. In section 4, we relate these supercongruences to some interesting combinatorial congruences.

2. Preliminaries

2.1. Legendre polynomials

Let $P_n(x)$ denote the nth Legendre polynomial, which can be defined by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ [5,9,28]. Equivalently, the degree n Legendre polynomial can be defined as

$$ P_n(x) := {}_2F_1\left(\begin{matrix} -n, \ n+1 \\ 1 \end{matrix}; \frac{1-x}{2}\right) = \sum_{k=0}^{n} \binom{n}{k}\binom{-n-1}{k}\left(\frac{1-x}{2}\right)^k. \tag{3} $$

The first few Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. These polynomials form an important class of orthogonal polynomials and have several nice properties; one relevant to our application is that they have generating function $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$. Because of this, special values of $P_n(x)$ show up in certain expansions of differential forms on elliptic curves.

2.2. The Atkin and Swinnerton-Dyer congruences

For elliptic curves of the form $\mathcal{E} : y^2 = x(x^2 + Ax + B)$ defined over $\mathbb{Z}_p$ with $t = x/y$ as a local parameter at the point at infinity (where t has a simple zero), Coster and van Hamme showed that the coefficients of the t-expansion of the invariant differential form $-\frac{dx}{2y}$ of $\mathcal{E}$ come from special values of Legendre polynomials (see formula (1) of [9]). Explicitly,

$$-\frac{dx}{2y} = \sum_{k=0}^{\infty} a_k t^k \frac{dt}{t} = \sum_{k=0}^{\infty} P_k\left(\tfrac{A}{\sqrt{A^2-4B}}\right)(\sqrt{A^2-4B}\,)^k t^{2k+1}\frac{dt}{t}, \tag{4}$$

where $a_{2k+1} = P_k\left(\tfrac{A}{\sqrt{A^2-4B}}\right)(\sqrt{A^2-4B}\,)^k$ and $a_{2k} = 0$.

The Atkin and Swinnerton-Dyer congruences (ASD) for elliptic curves (Theorem 4 of [6]) imply that if $\mathcal{E}$ has good reduction modulo p, then for all positive integers m and for $s \geq 0$,

$$a_{mp^{s+1}} - A_p a_{mp^s} + p a_{mp^{s-1}} \equiv 0 \pmod{p^{s+1}} \tag{5}$$

where $A_p = p + 1 - \#(\mathcal{E}/\mathbb{F}_p)$. We define a_k to be 0 if k is not integral, as may happen for the final term if $s = 0$.

Essentially, the ASD congruences say that for fixed p and m, terms of the sequence $\{a_{mp^s}\}$ satisfy a three-term congruence with increasing p-adic precision as s increases. The ASD congruences generalize the Hecke recursion: Fourier coefficients b_n of weight $k = 2$, normalized Hecke newforms with trivial nebentypus satisfy the three-term recursion, for all positive integers m, for $s \geq 0$, and for all p,

$$b_{mp^{s+1}} - b_p b_{mp^s} + p b_{mp^{s-1}} = 0. \tag{6}$$

In the ASD congruences for an elliptic curve $\mathcal{E}$, we distinguish two cases. If the middle coefficient A_p is divisible by p, we say that $\mathcal{E}$ is *supersingular* at p or simply that p is supersingular. Otherwise, we say $\mathcal{E}$ is *ordinary* at p or that p is ordinary. Dwork's congruences, in which consecutive ratios of certain terms in a sequence converge to a p-adic limit, are related to ASD congruences at ordinary primes. If p is ordinary and is unramified in $K_p := \mathbb{Q}_p(\sqrt{A^2-4B}\,)$, let β_p be the p-adic unit root of $T^2 - [p + 1 - \#(\mathcal{E}/\mathbb{F}_p)]T + p$. Then the ASD congruences imply that $a_{mp^s} \equiv \beta_p \cdot a_{mp^{s-1}} \pmod{p^s}$. Using the relation between a_{2k+1} and the Legendre polynomial, we have for any good odd ordinary prime p for $\mathcal{E}$ unramified at K_p

$$P_{\frac{mp^s-1}{2}}\left(\frac{A}{\sqrt{A^2-4B}}\right) \equiv {\chi_p}^{mp^{s-1}} \cdot \beta_p \cdot P_{\frac{mp^{s-1}-1}{2}}\left(\frac{A}{\sqrt{A^2-4B}}\right) \pmod{p^s}, \tag{7}$$

where $\chi_p \in K_p$ is the (not necessarily primitive) order-4 root of unity satisfying $\chi_p \equiv \left(\frac{1}{\sqrt{A^2-4B}}\right)^{\frac{p-1}{2}} \pmod{p}$.

2.3. Clausen formula

It follows from the well-known Clausen formula for hypergeometric series and a Pfaff transformation that

$${}_2F_1\left(\begin{matrix}-a, \ a+1 \\ 1\end{matrix}; \frac{1 \pm \sqrt{1-x}}{2}\right)^2 = {}_3F_2\left(\begin{matrix}\frac{1}{2}, -a, \ a+1 \\ 1, \ 1\end{matrix}; x\right). \tag{8}$$

See equation (3.3) of [8] for a derivation of this formula.

Thus,

$$P_n(\sqrt{1-\lambda})^2 = {}_3F_2\left(\begin{matrix}\frac{1}{2}, & -n, & n+1\\ & 1, & 1\end{matrix}; \lambda\right). \tag{9}$$

The following congruence of degree $(p-1)/2$ polynomials holds coefficient-wise.

Lemma 4. *Let p be any odd prime. Then*

$${}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1-p}{2}, & \frac{1+p}{2}\\ & 1, & 1\end{matrix}; x\right) \equiv {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ & 1, & 1\end{matrix}; x\right)_{\frac{p-1}{2}} \pmod{p^2\mathbb{Z}_p[x]}.$$

Proof. We use Zudilin's observation about rising factorials (see Lemma 1 in [7], [18]),

$$\begin{aligned}\left(\frac{1}{2}+\epsilon\right)_k &= \left(\frac{1}{2}+\epsilon\right)\left(\frac{1}{2}+\epsilon+1\right)\cdots\left(\frac{1}{2}+\epsilon+k-1\right)\\ &= \left(\frac{1}{2}\right)_k\left(1+2\epsilon\sum_{j=1}^{k}\frac{1}{2j-1}+O(\epsilon^2)\right),\end{aligned}$$

to expand $(\frac{1\pm p}{2})_k$ in terms of $(\frac{1}{2})_k$. When we take the product $(\frac{1-p}{2})_k(\frac{1+p}{2})_k$, the coefficients of p^1 cancel; and so the product is congruent to $(\frac{1}{2})_k{}^2$ modulo p^2, which establishes the coefficient-wise congruence. □

3. Supercongruences

To prove our main theorem, we use the following theorem of Coster and van Hamme.

Theorem 5 *(Coster and van Hamme, [9]). Let p be an odd prime. Let d be a square-free positive integer such that $(\frac{-d}{p}) = 1$. Let K be an algebraic number field such that $\sqrt{-d} \in K$ and $K \subset \mathbb{Q}_p$. Consider the elliptic curve*

$$\mathcal{E} : Y^2 = X(X^2 + AX + B)$$

with $A, B \in K$, where A and $\Delta = A^2 - 4B$ are p-adic units. Let ω and ω' be a basis of periods of $\mathcal{E}$ and suppose that $\tau = \omega'/\omega \in \mathbb{Q}(\sqrt{-d})$ (which implies that the curve has complex multiplication), τ has positive imaginary part, and $A = 3\wp(\frac{1}{2}\omega)$, $\sqrt{\Delta} = \wp(\frac{1}{2}\omega' + \frac{1}{2}\omega) - \wp(\frac{1}{2}\omega')$, where $\wp$ is the Weierstrass $\wp$-function. Let $\pi, \bar{\pi} \in \mathbb{Q}(\sqrt{-d})$ such that $\pi\bar{\pi} = p$, with $\bar{\pi}$ a p-adic unit, $\pi = u_1 + v_1\tau$, and $\pi\tau = u_2 + v_2\tau$ with u_1, v_1, u_2, v_2 integers and v_1 even. Then we have

$$P_{\frac{mp^r-1}{2}}\left(\frac{A}{\sqrt{\Delta}}\right) \equiv \varepsilon^{mp^{r-1}} \cdot \bar{\pi} \cdot P_{\frac{mp^{r-1}-1}{2}}\left(\frac{A}{\sqrt{\Delta}}\right) \pmod{\pi^{2r}}, \tag{10}$$

where m and r are positive integers, with m odd, and $\varepsilon = (\sqrt{-1})^{-u_2v_2+v_2+p-2}$, where $P_n(x)$ is the nth Legendre polynomial.

The main point of the theorem is the existence of supercongruences arising from an elliptic curve $\mathcal{E}$ with complex multiplication. While Coster and van Hamme interpreted the congruence as inclusion in an ideal of the ring of integers of K, we interpret all of our congruences p-adically. Since the number field K embeds into $\mathbb{Q}_p$ and π is just p times a p-adic unit under this embedding, we may simply replace (mod π^{2r}) with (mod p^{2r}) when we view the congruence p-adically. These congruences are twice as strong as formal group theory predicts.

Note that ε and $\bar{\pi}$ in the theorem above must correspond to $\pm\chi_p$ and $\pm\beta_p$ (with the same sign) in our notation. The conditions that τ has positive imaginary part, that $A = 3\wp(\frac{1}{2}\omega)$, and that $\sqrt{\Delta} = \wp(\frac{1}{2}\omega' + \frac{1}{2}\omega) - \wp(\frac{1}{2}\omega')$, can always be satisfied by a suitable choice of the basis of periods; and we can additionally ensure that $\varepsilon = \chi_p$ and $\bar{\pi} = \beta_p$.

Proposition 6. *For CM values λ of the family $E_\lambda : y^2 = (x-1)(x^2 - \frac{1}{1-\lambda})$, such that $\lambda \in \mathbb{Z}_p$ and p is ordinary, for all positive integers m and s with m odd,*

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1-mp^s}{2}, & \frac{1+mp^s}{2} \\ & 1,\ 1 & \end{matrix}; \lambda\right) \equiv \left(\frac{1-\lambda}{p}\right)\cdot \alpha_p(\lambda)^2 \cdot {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1-mp^{s-1}}{2}, & \frac{1+mp^{s-1}}{2} \\ & 1,\ 1 & \end{matrix}; \lambda\right) \pmod{p^{2s}} $$

where $\alpha_p(\lambda)$ is the unit root of $X^2 - [p+1-\#(E_\lambda/\mathbb{F}_p)]X + p = 0$.

Proof. Letting $X = x-1$, $Y = y$, the elliptic curve E_λ can be rewritten as $Y^2 = X(X^2 + 2X + \frac{\lambda}{\lambda-1})$. Then we have $A = 2, B = \frac{\lambda}{\lambda-1}$, $\Delta = A^2 - 4B = \frac{4}{1-\lambda}$, and $\frac{A}{\sqrt{A^2-4B}} = \sqrt{1-\lambda}$.

By our assumptions that E_λ has complex multiplication, that $\lambda \in \mathbb{Z}_p$, and that p is ordinary, we satisfy the conditions of Theorem 5: $K = \mathbb{Q}(\lambda)$ embeds into $\mathbb{Q}_p$ and $\lambda \not\equiv 1 \pmod p$, so Δ is a p-adic unit. Combining (9) and (10), we have

$$ \begin{aligned} {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1-mp^s}{2}, & \frac{1+mp^s}{2} \\ & 1,\ 1 & \end{matrix}; \lambda\right) &= P_{\frac{mp^s-1}{2}}(\sqrt{1-\lambda})^2 \\ &\equiv (\varepsilon^{mp^{s-1}}\bar{\pi})^2 \cdot P_{\frac{mp^{s-1}-1}{2}}(\sqrt{1-\lambda})^2 \pmod{p^{2s}} \\ &= ({\chi_p}^{mp^{s-1}}\beta_p)^2 \cdot {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1-mp^{s-1}}{2}, & \frac{1+mp^{s-1}}{2} \\ & 1,\ 1 & \end{matrix}; \lambda\right) \\ &= \left(\frac{1-\lambda}{p}\right)\cdot \alpha_p(\lambda)^2 \cdot {}_3F_2\left(\begin{matrix}\frac{1}{2}, & \frac{1-mp^{s-1}}{2}, & \frac{1+mp^{s-1}}{2} \\ & 1,\ 1 & \end{matrix}; \lambda\right). \end{aligned} $$

For the final equality, note that we have chosen A and B so that $\beta_p = \alpha_p(\lambda)$ and so that ${\chi_p}^2$ is the Legendre symbol $\left(\frac{1-\lambda}{p}\right)$, which does not change when we raise it to the odd power mp^{s-1}. $\square$

Proof of Theorem 1. Note that for all primes p that are ordinary for E_λ, Theorem 1 follows from Proposition 6 and Lemma 4. For primes p that are supersingular for E_λ, we can conclude from the ASD congruence that $P_{\frac{p-1}{2}}(\sqrt{1-\lambda}) \equiv 0 \pmod p$. Hence ${}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ 1, 1\end{matrix}; \lambda\right)_{\frac{p-1}{2}} \equiv {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1-p}{2}, \frac{1+p}{2}\\ 1, 1\end{matrix}; \lambda\right) = P_{\frac{p-1}{2}}(\sqrt{1-\lambda})^2 \equiv 0 \pmod{p^2}$, which concludes the proof. □

We note that this establishes, modulo p^2, all cases of Conjecture 5.2 of [26] by Z.-W. Sun. These conjectures can be written as

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \left(\frac{\lambda}{64}\right)^k \equiv \begin{cases} \left(\frac{c}{p}\right)(4a^2-2p) \pmod{p^2} & \text{if } \left(\frac{p}{D}\right)=1 \text{ where } a^2+Db^2=p \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{D}\right)=-1 \end{cases},$$

with appropriate choices of $D \in \mathbb{Z}_+$ and character $\left(\frac{c}{p}\right)$. Note that $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \left(\frac{\lambda}{64}\right)^k = {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ 1, 1\end{matrix}; \lambda\right)_{\frac{p-1}{2}}$ via the identity $\frac{(1/2)_k{}^3}{k!^3} = \binom{2k}{k}^3 \frac{1}{64^k}$. These conjectures address the λ-values $\lambda = -8, 1, -\frac{1}{8}, 4, \frac{1}{4}, 64, \frac{1}{64}, -1$, which are all of the CM values for E_λ over $\mathbb{Q}$, as verified in [4], with the exception of the degenerate case $\lambda = 1$, for which E_λ is not an elliptic curve. The supercongruence for $\lambda = 1$ was proved by Van Hamme in [30] and by Ono in [21].

If E_λ has CM over $K = \mathbb{Q}(\sqrt{-D})$, then the attached 2-dimensional representation ρ decomposes into 2 Grossencharacters when ρ is restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. At splitting primes p, which are precisely the ordinary primes of E_λ, the trace of the Frobenius is $\alpha_p(\lambda)+\beta_p(\lambda)$, where both $\alpha_p(\lambda)$ and $\beta_p(\lambda)$ are in the ring of integers of the quadratic field K and have absolute value $\sqrt{p}$. In the case that K has class number 1 (all Sun λ values correspond to class number 1 cases), then ideals $(\alpha_p(\lambda))$ and $(\beta_p(\lambda))$ are the two distinct prime ideals above p. That is, $\alpha_p(\lambda) = a + b\sqrt{-D}$ and $\beta_p(\lambda) = a - b\sqrt{-D} = \frac{p}{\alpha_p(\lambda)}$, where a and b are integers or half integers depending on $D \equiv 1$ or $3 \pmod 4$, such that $a^2 + b^2D = p$. In the ordinary case, our congruences involve $\alpha_p(\lambda)^2$, which is just $a^2 - Db^2 + 2ab\sqrt{-D}$. Using $\beta_p(\lambda)^2 = a^2 - Db^2 - 2ab\sqrt{-D} \equiv 0 \pmod{p^2}$ and $a^2 + b^2D = p$, we have $\alpha_p(\lambda)^2 \equiv 4a^2 - 2p \pmod{p^2}$, which, along with the character $\left(\frac{1-\lambda}{p}\right)$, is the target of Z.-W. Sun's congruences in the case that $\left(\frac{p}{D}\right) = 1$. In the case that $\left(\frac{p}{D}\right) = -1$, p is a supersingular prime of E_λ and so ${}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ 1, 1\end{matrix}; \lambda\right)_{\frac{p-1}{2}} \equiv 0 \pmod{p^2}$, establishing the other half of Z.-W. Sun's congruences.

Alternatively, we note that Ono has explicitly identified the values $\alpha_p(\lambda)$, for all CM curves E_λ with $\lambda \in \mathbb{Z}$, in Theorem 6 of [21]. These values $\alpha_p(\lambda)$ determine the formal group structure and the ASD congruences (i.e., that $a_p(\lambda) \equiv \left(\frac{1-\lambda}{p}\right)\alpha_p(\lambda)^2 \pmod p$); combining this with Coster and Van Hamme's supercongruences gives another proof of Sun's conjectures, that $a_p(\lambda) \equiv \left(\frac{1-\lambda}{p}\right)\alpha_p(\lambda)^2 \pmod{p^2}$.

Theorem 1, and the following Conjecture 7, apply not only to the cases considered by Ono and Z.-W. Sun, which correspond to CM values of λ over $\mathbb{Q}$, but also to infinitely many other algebraic CM values of λ, for those primes p such that λ embeds in $\mathbb{Z}_p$ with

$\lambda \not\equiv 0, 1 \pmod p$. This is satisfied by almost all primes p such that there is a prime ideal $\mathfrak{p}$ above p in $\mathbb{Q}(\lambda)$ with inertia degree 1.

Based on numeric evidence, we have

Conjecture 7. *For CM values λ of the family $E_\lambda : y^2 = (x-1)(x^2 - \frac{1}{1-\lambda})$, such that $\lambda \in \mathbb{Z}_p$ and p is ordinary, for all positive integers m and s with m odd,*

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} ; \lambda \right)_{\frac{mp^s-1}{2}} \equiv \left(\left(\frac{1-\lambda}{p} \right) \alpha_p(\lambda)^2 \right) {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} ; \lambda \right)_{\frac{mp^{s-1}-1}{2}} \pmod{p^{3s}} $$

where $\alpha_p(\lambda)$ is the unit root of $X^2 - [p + 1 - \#(E_\lambda/\mathbb{F}_p)]X + p = 0$.

4. Corollaries

An idea of Gessel for dealing with the supercongruences of the Apéry numbers

$$ c_n = {}_4F_3\left(\begin{matrix} -n, -n, 1+n, 1+n \\ 1, 1, 1 \end{matrix} ; 1 \right) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 $$

is as follows [12]. He identified the auxiliary sequence $d_n = 2\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (H_{n+k} - H_{n-k})$, where H_k is the harmonic sum $\sum_{j=1}^{k} \frac{1}{j}$, and showed that $c_{k+pn} \equiv (c_k + pnd_k)c_n \pmod{p^2}$ where $0 \le k < p$. Using the idea of Ishikawa [13], we take $k = n = \frac{p-1}{2}$. It follows that when $c_{(p-1)/2} \not\equiv 0 \pmod p$, we have the supercongruence $c_{(p^2-1)/2} \equiv c_{(p-1)/2}^2 \pmod{p^2}$ precisely when $d_{(p-1)/2} \equiv 0 \pmod p$, which follows from the p-adic properties of harmonic sums. In [3], Ahlgren and Ono also need an entity similar to $d_{(p-1)/2}$ to be zero modulo p, which they established using a binomial coefficient identity proved by the WZ method [2].

In the above examples, supercongruences of a sequence c_n were shown to be equivalent to congruences of an auxiliary sequence d_n; and the congruences for d_n were proved using whatever method applied in each case. Similarly, the supercongruence in Theorem 1 for the sequence $a_n = \sum_{i=0}^{n} \binom{2i}{i}^3 (\frac{\lambda}{64})^i$ is equivalent to the auxiliary congruence in Corollary 2 for the sequence $d_n = \sum_{i=0}^{n} \binom{2i}{i}^3 (\frac{\lambda}{64})^i (6(H_{2i} - H_i) + \frac{(\lambda/64)^{p-1}-1}{p})$. However, we proved our supercongruence using the theorem of Coster and Van Hamme, and thus obtain our auxiliary congruence. We know of no direct proof of Corollary 2; we expect a proof for each fixed individual λ might require some combinatorial identity and additional intelligent guesses of WZ pairs to prove the identity, see [1,3].

Lemma 8. *For the sequence $a_n = \sum_{i=0}^{n} \binom{2i}{i}^3 (\frac{\lambda}{64})^i$, we introduce the auxiliary sequence $d_n = \sum_{i=0}^{n} \binom{2i}{i}^3 (\frac{\lambda}{64})^i (6(H_{2i} - H_i) + \frac{(\lambda/64)^{p-1}-1}{p})$. Then for any prime p, any k with $\frac{p-1}{2} \le k < p$, and any n,*

$$ a_{k+pn} \equiv a_k a_n + pd_k \sum_{i=0}^{n} i \binom{2i}{i}^3 \left(\frac{\lambda}{64} \right)^i \pmod{p^2}. $$

Proof. Notice we can write $a_{k+pn} - a_k a_n$ as the telescoping sum $\sum_{i=1}^{n} T_{k,i}$, where

$$\begin{aligned} T_{k,n} &= (a_{k+pn} - a_k a_n) - (a_{k+p(n-1)} - a_k a_{n-1}) \\ &= (a_{k+pn} - a_{k+p(n-1)}) - a_k(a_n - a_{n-1}) \\ &= \sum_{i=-p+k+1}^{k} \binom{2i+2pn}{i+pn}^3 \left(\frac{\lambda}{64}\right)^{i+pn} - \left(\sum_{i=0}^{k} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i\right) \binom{2n}{n}^3 \left(\frac{\lambda}{64}\right)^n \end{aligned}$$

Using the condition that $\frac{p-1}{2} \le k < p$, we notice that $\binom{2i+2pn}{i+pn} \equiv 0 \pmod{p}$ if $-p+k+1 < i < 0$. Simplifying modulo p^2, these terms disappear and we can factor.

$$T_{k,n} \equiv \sum_{i=0}^{k} \left(\binom{2i+2pn}{i+pn}^3 \left(\frac{\lambda}{64}\right)^{pn} - \binom{2n}{n}^3 \left(\frac{\lambda}{64}\right)^n \binom{2i}{i}^3 \right) \left(\frac{\lambda}{64}\right)^i \pmod{p^2}$$

The factor $\binom{2i+2pn}{i+pn}^3$ may be rewritten as $\frac{-\Gamma_p(1+2i+2pn)^3}{\Gamma_p(1+i+pn)^6}\binom{2n}{n}^3$, where Γ_p is the p-adic Gamma function (see [21, Chapter 11]). Let $T_{k,n} \equiv \left(\frac{\lambda}{64}\right)^n \binom{2n}{n}^3 U_{k,n} \pmod{p^2}$, where

$$U_{k,n} = \sum_{i=0}^{k} \left(\left(\frac{-\Gamma_p(1+2i+2pn)^3}{\Gamma_p(1+i+pn)^6} \right) \left(\frac{\lambda}{64}\right)^{(p-1)n} - \binom{2i}{i}^3 \right) \left(\frac{\lambda}{64}\right)^i.$$

To simplify the p-adic Gamma function modulo p^2, we expand Γ_p in terms of factorials and harmonic sums $H_n = \sum_{i=1}^{n} \frac{1}{i}$. (By convention, $H_0 = 0$.) We also use the congruence, for $p > 3$, that $H_{p-1} \equiv 0 \pmod{p}$. (Wolstenholme has shown this congruence holds modulo p^2, though we only need modulo p [31].)

$$\begin{aligned} \Gamma_p(1+i+pn)^r &\equiv (-1)^{(1+i+pn)r} i!^r (1+pnrH_i) \prod_{j=0}^{n-1} (p-1)!^r (1+pjrH_{p-1}) \pmod{p^2} \\ &\equiv (-1)^{(1+i+pn)r} i!^r (1+pnrH_i)(-1)^{nr} \pmod{p^2} \\ &\equiv (-1)^{(1+i)r} i!^r (1+pnrH_i) \pmod{p^2} \end{aligned}$$

Plugging this into $U_{k,n}$, we have

$$\begin{aligned} U_{k,n} &\equiv \sum_{i=0}^{k} \left(\left(\frac{(2i)!^3(1+6pnH_{2i})}{(i)!^6(1+6pnH_i)} \right) \left(\frac{\lambda}{64}\right)^{(p-1)n} - \binom{2i}{i}^3 \right) \left(\frac{\lambda}{64}\right)^i \pmod{p^2} \\ &\equiv \sum_{i=0}^{k} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \left((1+6pn(H_{2i}-H_i)) \left(\frac{\lambda}{64}\right)^{(p-1)n} - 1 \right) \pmod{p^2} \end{aligned}$$

Using $\left(\frac{\lambda}{64}\right)^{(p-1)n} = \left(1 + p\left(\frac{\left(\frac{\lambda}{64}\right)^{p-1} - 1}{p}\right)\right)^n \equiv 1 + pn\left(\frac{\left(\frac{\lambda}{64}\right)^{p-1} - 1}{p}\right) \pmod{p^2}$,

$$U_{k,n} \equiv \sum_{i=0}^{k} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \left((1+6pn(H_{2i}-H_i))\left(1+pn\left(\frac{\left(\frac{\lambda}{64}\right)^{p-1}-1}{p}\right)\right)-1\right)$$
$$\equiv pn \sum_{i=0}^{k} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \left(6(H_{2i}-H_i)+\left(\frac{\left(\frac{\lambda}{64}\right)^{p-1}-1}{p}\right)\right) \pmod{p^2}$$

So $T_{k,n} \equiv pn\binom{2n}{n}^3(\frac{\lambda}{64})^n d_k \pmod{p^2}$. Combining this congruence with the telescoping sum $a_{k+pn} - a_k a_n = \sum_{i=1}^{n} T_{k,i}$ completes the proof of the lemma. □

Using this lemma, we show the equivalence of Theorem 1 and Corollary 2.

Proof of Corollary 2. We consider $T_{k,n}$ with $k = \frac{p-1}{2}$ and $n = 1$. By definition, $T_{\frac{p-1}{2},1} = a_{\frac{3p-1}{2}} - a_{\frac{p-1}{2}} a_{\frac{3-1}{2}}$; we can rewrite this, modulo p^2, as $P_{\frac{3p-1}{2}}(\sqrt{1-\lambda})^2 - P_{\frac{p-1}{2}}(\sqrt{1-\lambda})^2 P_{\frac{3-1}{2}}(\sqrt{1-\lambda})^2$. Since the sequence $P_{\frac{n-1}{2}}(\sqrt{1-\lambda})$ satisfies ASD congruences, we know that $T_{\frac{p-1}{2},1} \equiv 0 \pmod{p}$. However, Theorem 1 is precisely the information we need to conclude that $T_{\frac{p-1}{2},1} \equiv 0 \pmod{p^2}$ whenever λ is a CM value of E_λ that embeds in $\mathbb{Z}_p$.

Thus, since

$$T_{\frac{p-1}{2},1} \equiv \frac{p\lambda}{8} \sum_{i=0}^{(p-1)/2} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \left(6(H_{2i}-H_i)+\left(\frac{\left(\frac{\lambda}{64}\right)^{p-1}-1}{p}\right)\right) \pmod{p^2},$$

we have the desired congruence $d_{\frac{p-1}{2}} \equiv 0 \pmod{p}$ whenever we have supercongruences for $a_{\frac{p-1}{2}}$. □

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