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Positive Stabilization and Observer Design for Positive Singular Systems

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***Abstract*— This paper considers the class of positive singular systems and provides a new approach to achieve stabilization and observer design with positivity constraints. First, the connection between singular systems and its equivalent input derivative systems is established and their positivity with stability properties are analyzed. Then, an algebraic transformation is introduced, which allows to eliminate the derivative inputs and to obtain an equivalent standard system. A careful mathematical derivation was performed to obtain closed-form expressions for the coefficient matrices of the resulting transformed system. Consequently, the design of positive stabilization of positive singular systems was made possible using the equivalent standard state space representation. Finally, the design of positive observer for positive singular system is provided. It is shown that a similar procedure is required to eliminate the input derivatives that appear in the output equation allowing the observer to be designed. Both positive stabilization and positive observer designs are formulated and solved in terms of LMIs.**

I. INTRODUCTION

The class of singular systems, which is also known as descriptor systems or generalized state space systems appears naturally in diverse applications. This class of systems constitutes of dynamic and algebraic subsystems due to the constraint imposed on state variables. The theoretical analysis of singular systems is well-established and it has been the subject of research for many years. Analysis and control of singular systems have been investigated by many researchers and a few of them set the foundation of the major results which are captured in survey papers and books [8]-[14]. In spite of this development, the problem of stabilization and observer design for the class of positive singular systems have not been fully investigated. The class of positive systems have impressive stability and robustness properties [15], [16]. The response of positive systems to positive initial conditions and inputs remain in the positive orthant of state space. One can use available stabilization and observer design of positive systems [17]-[21] and apply them to the class of positive singular systems, if possible. The main goal of the paper is to realize this possibility using a new approach.

The proposed approach is motivated by the fact that singular systems have direct connection to the class of systems known as dynamic systems with derivative inputs. Such systems have been treated using polynomial matrix representation and alternative tools from realization theory, which is inconvenient for higher order derivatives [1]-[6]. The approach of this paper is based on SVD representation of singular systems and subsequent Shuffle algorithm to define positive singular systems by its equivalent positive input derivative system. Due to the presence of input derivatives, we provide an algebraic transformation to eliminate the derivative inputs to reduce the system to a standard state space representation. Consequently, we can perform positive stabilization using LMI-based procedure. This is the main contribution of the paper. Subsequently, we also provide the design of positive observer for positive singular systems. We show that a similar procedure is required to eliminate the input derivatives that appear in the output equation. This allows to define the positive observer structure and apply LMI-based procedure to design positive observer.

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II. PRELIMINARY RESULTS ON SINGULAR SYSTEMS AND MOTIVATION

Consider the linear singular system described by

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are state, input, and output vectors, respectively. The matrix $E \in \mathbb{R}^{n\times n}$ is singular with $\text{rank}(E) = r < n$, and the matrices A, B, and C are of appropriate dimensions. For the unforced singular system $u(t) = 0$, we define the generalized spectral abscissa for the pair $\{E, A\}$ as $\alpha(E, A) = \max \Re(\lambda)$, where $\lambda \in \{s : \det(sE - A) = 0\}$. If $E = I$, then $\alpha(I, A) = \alpha(A)$ becomes the conventional spectral abscissa for standard systems.

Definition 1. The singular system (1) is called,

1) Regular if $\det(sE - A) \neq 0$ for some $s \in \mathbb{C}$.
2) Impulse free if deg $\det(sE - A) = \text{rank}(E) = r$.
3) Stable if the roots of $\det(sE - A) = 0$ have negative real parts.
4) Admissible if it is regular, impulse free, and stable.

Based on the above definition, the following results from [10]-[12] can be stated.

Lemma 1. *Consider the singular system (1),(2) and assume that it is regular. Then, there exist two nonsingular matrices*

L and R such that

$$\begin{aligned}\bar{E} &= LER = \textit{block diag}(I_{n_1}, N)\\ \bar{A} &= LAR = \textit{block diag}(A_1, I_{n_2})\\ \bar{B} &= LB = \begin{bmatrix} B_1^T & B_2^T \end{bmatrix}^T\\ \bar{C} &= CR = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\end{aligned} \tag{3}$$

where $n_1 = \deg(\det(sE - A)) \leq r$, $n_2 = n - n_1$, $A_1 \in \mathbb{R}^{n_1 \times n_1}$, *and* $N \in \mathbb{R}^{n_2 \times n_2}$ *is a nilpotent matrix with index* μ *(i.e.* $N^{\mu} = 0$, $N^{\mu-1} \neq 0$*).*

Furthermore,

1) *The pair* $\{E, A\}$ *is impulse free if and only if* $N = 0$.
2) *The pair* $\{E, A\}$ *is stable if and only if* $\alpha(A_1) < 0$.
3) *The pair* $\{E, A\}$ *is admissible if and only if* $N = 0$ *and* $\alpha(A_1) < 0$.

Theorem 1. *The pair* $\{E, A\}$ *associated with the system (1) is admissible if and only if there exists a matrix* P *such that*

$$E^T P = P^T E \succeq 0 \tag{4}$$

$$P^T A + A^T P \prec 0 \tag{5}$$

The conditions in (4), (5) are non-strict LMIs due to the equality (4), which cause numerical issue and cannot be directly solved with LMIs. Therefore, it is more desirable to have a strict LMI by integrating (4) in (5). This was done with the aid of generalized Lyapunov equation (GLE), which was done in connection to stability of singular system (1), (2). The following Theorems were introduced in [22]-[24] to facilitate the process of stability and stabilization of singular systems.

Theorem 2. *The pair* $\{E, A\}$ *associated with the system (1) is admissible if and only if there exist matrices* $P \succ 0$ *and* Q *such that*

$$(PE + SQ)^T A + A^T (PE + SQ) \prec 0 \tag{6}$$

where $S \in \mathbb{R}^{r \times (n-r)}$ *is any full rank matrix satisfying* $E^T S = 0$*. Furthermore, (6) can equivalently be written in terms of* $\{E^T, A^T\}$ *by replacing* E *with* E^T *and* A *with* A^T *in (6), and the side constraint* $ES = 0$*.*

With the aid of the above Theorem the stabilization of singular system can be achieved as follows.

Theorem 3. *Consider the singular system (1) with* $\text{rank}(E) = n$*. Then there exists a feedback control law* $u = Kx$ *such that the closed-loop system* $E\dot{x}(t) = (A+BK)x(t)$ *is admissible if and only if there exist matrices* $P \succ 0$, Q *and* V *such that*

$$W^T(P, Q)A^T + AW(P, Q) + BV + V^T B^T \prec 0 \tag{7}$$

provided that $W(P, Q) = PE^T + SQ$ *is a nonsingular matrix with any* S *satisfying* $ES = 0$*. Furthermore, the feedback gain matrix* K *is obtained by*

$$K = VW^{-1} \tag{8}$$

An extension of Theorem 2, which avoids the side constraint, was given by the LMI

$$A(PE^T + VQU^T) + (PE^T + VQU^T)^T A^T \prec 0 \tag{9}$$

where V and U are $n \times (n - r)$ matrices obtained from the basis of null spaces E and E^T respectively. Additional results in this direction were made in [25], [26]. However, this requires extra mathematical derivation.

In spite of the effort and progress in this direction, the applicability of the reported results for positive stabilization and observer design of positive singular systems is not trivial and requires further investigation. Here, we provide a new design strategy that can be employed to solve this problem.

III. Singular Systems and Input Derivative Representations

The restricted equivalent form of singular system (3) specified in Lemma 1 can be represented by slow and fast subsystems respectively as

$$\begin{aligned}\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t)\\ y_1(t) &= C_1 x_1(t)\end{aligned} \tag{10}$$

and

$$\begin{aligned}N\dot{x}_2(t) &= x_2(t) + B_2 u(t)\\ y_2(t) &= C_2 x_2(t)\end{aligned} \tag{11}$$

It is not difficult to derive the transfer function of the system $G(s) = G_1(s) + G_2(s)$, where $G_1(s) = C_1(sI_{n_1} - A_1)^{-1}B_1$ and $G_2(s) = C_2(sN - I_{n_2})^{-1}B_2$ are associated with slow and fast subsystems, respectively. Using $G_2(s)$, one can extract $X_2(s)$ and by taking the inverse Laplace transform to obtain

$$x_2(t) = -\sum_{i=0}^{\mu-1} N^i B_2 u^{(i)}(t), u^{(0)} = u \tag{12}$$

assuming $x_2(0) = 0$. Differentiating (12) and combining it with slow subsystem results in the overall system

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{i=0}^{\mu} \tilde{B}_i u^{(i)}(t) \tag{13}$$

$$y(t) = \tilde{C}x(t) \tag{14}$$

where

$$\tilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0 \\ -N^{i-1}B_2 \end{bmatrix}, i > 1$$

$$\tilde{C} = [C_1 \quad C_2]$$

Thus, the singular system (1),(2) is represented by input derivative system (13),(14).

The singular system (1),(2) can also be decomposed to dynamic and static parts by applying singular value decomposition (SVD) on the matrix E. Subsequently, one can use the so-called Shuffle algorithm [7] to derive an equivalent standard system with input derivatives. Performing SVD on the matrix E, the singular system can be rewritten as

$$\hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \tag{15}$$

$$y(t) = \hat{C}\hat{x}(t) \tag{16}$$

where

$$\hat{A} = U^T AV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \hat{B} = U^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$\hat{E} = U^T EV = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{C} = CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

with $\Sigma_r = \text{diag}\{\sigma_i, i = 1, \ldots, r\}$ defining the nonzero singular values and orthogonal pair of matrices $\{U, V\}$.

The matrix Σ_r can be replaced by I_r with an additional transformation step. It can be shown that the pair $\{E, A\}$ or equivalently $\{\hat{E}, \hat{A}\}$ is impulse free if and only if A_{22} is nonsingular and in addition, the pair is admissible if and only if $\alpha(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 0$.

The transformation to SVD form (15),(16) is numerically reliable specially for large size system and its structure prepares the system for initial step of Shuffle algorithm by defining

$$\hat{E} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{17}$$

where $E_1 = [\Sigma_r \quad 0], A_1 = [A_{11} \quad A_{12}]$, and $A_2 = [A_{21} \quad A_{22}]$. Therefore, we have

$$E_1\dot{x} = A_1 x + B_1 u \tag{18}$$

$$0 = A_2 x + B_2 u \tag{19}$$

Taking the derivative of (19) and combine it with (18) yields

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} \dot{u} \tag{20}$$

If $\begin{bmatrix} E_1^T & A_2^T \end{bmatrix}^T$ is nonsingular, multiply both sides of (20) from left by its inverse and define the input derivative system. If $\begin{bmatrix} E_1^T & A_2^T \end{bmatrix}^T$ is singular, then it is required to continue the steps of Shuffle algorithm until a regular pencil after $\mu - 1$ steps is obtained where $\begin{bmatrix} E_{\mu-1}^T & A_{\mu-1}^T \end{bmatrix}^T$ becomes nonsingular. Thus, the singular system (1),(2) represented by SVD form (15), (16) is transformed to the input derivative system as follows

$$\dot{x}(t) = \widetilde{A}x(t) + \sum_{i=0}^{\mu-1} \widetilde{B}_i u^{(i)}(t) \tag{21}$$

$$y(t) = \widetilde{C}x(t) \tag{22}$$

It is important to point out that the input derivative representation of singular system (13), (14) derived based on slow and fast subsystems differs from (21), (22), which was obtained based on dynamic and static part of singular systems. However, we used the same notations for system parameters to define the structure of input derivative systems and avoid extra symbols. One can also conclude that the difference between the number of finite modes in (10), n_1, and the number of dynamic modes in (18), r, is the number of infinite dynamic modes or impulsive modes, which is the rank $N = r - n_1$. This means that for a singular system with only finite dynamic modes we have $n_1 = r = \text{rank}(E)$. In this case, the singular system becomes impulse free or index one.

A third approach to transform a singular system to input derivative representation is based on the Drazin inverse [27], which requires extensive mathematical derivations. Here, it is sufficient to concentrate on SVD coordinate form, which is numerically preferable.

IV. Positive Singular Systems and Algebraic Transformation

A. Positive Singular Systems

Let us consider the singular system (1), (2) and analyze its positivity and stability.

Definition 2. The singular system (1), (2) is called weakly positive if and only if $E \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$ and $A \in \mathbb{M}_n$ is a Metzler matrix i.e. $a_{ij} \geq 0$ for $i \neq j$.

This definition is a natural generalization of positivity as defined for standard positive systems. Unfortunately, it does not guarantee the strong positivity of singular systems, which will be elaborated subsequently.

Definition 3. The singular system (1), (2) is internally positive if and only if its equivalent input derivative system (21), (22) is internally positive i.e. for every consistent initial condition $x(0) \in \mathbb{R}_+^n$ and every non-negative input $u(t) \in \mathbb{R}_+^m$, such that $u^{(i)}(t) \in \mathbb{R}_+^m$ for $i = 0, 1, \ldots, \mu - 1$; $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ for $t > 0$ where $u^{(i)}(t) = d^i u(t)/dt$.

Theorem 4. *The singular system (1), (2) is internally positive if and only if its equivalent input derivative system (21), (22) satisfies $\widetilde{A} \in \mathbb{M}_n$ i.e. a Metzler matrix, $\widetilde{B}_i \in \mathbb{R}_+^{n\times m}$, and $\widetilde{C} \in \mathbb{R}_+^{p\times n}$ for all $i = 0, 1, ..., \mu - 1$. Furthermore, it is asymptotically stable if and only if $\widetilde{A}$ is a stable Metzler matrix satisfying one of the following equivalent conditions:*

1) *There exists a positive definite diagonal matrix P such that $\widetilde{A}^T P + P\widetilde{A} \prec 0$.*
2) *There exists a positive vector v such that $\widetilde{A}v < 0$.*

Proof: The proof of this Theorem follows directly from the proof of standard positive systems when applied to (21), (22).

We constructed several examples, which were weakly positive and satisfied the strong positivity based on Definition 3 and Theorem 4. There were also cases that weakly positive system did not satisfy the positivity condition of Theorem 4.

In fact, even if the singular system is not weakly positive, it may lead to positive singular systems after transformation to input derivative representation. The following simple example shows this case. Consider the singular system,

$$\begin{bmatrix}0 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\dot{x} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}x + \begin{bmatrix}0 & 0\\ -1 & 0\\ 0 & 1\end{bmatrix}u$$

which is not weakly positive. However, applying the Shuffle algorithm and using (17)-(20) we obtain

$$\widetilde{A} = \begin{bmatrix}-1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \quad \widetilde{B}_0 = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}, \quad \widetilde{B}_1 = \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

Since $\widetilde{A} \in \mathbb{M}_n$ is a Metzler matrix, $\widetilde{B}_0 \in \mathbb{R}_+^{3\times 2}$ and $\widetilde{B}_1 \in \mathbb{R}_+^{3\times 2}$, the singular system is strongly positive or simply positive.

B. Elimination of Input Derivative by Algebraic Transformation

Let us concentrate on input derivative representation of singular system (21), (22) and define the number of input derivative by $\ell = \mu - 1$. Starting with $\ell = 1$, we have

$$\dot{x}(t) = \widetilde{A}x(t) + \widetilde{B}_0 u(t) + \widetilde{B}_1 \dot{u}(t) \tag{23}$$

$$y(t) = \widetilde{C}x(t) \tag{24}$$

Although traditional approach using polynomial matrix representation and subsequent strict system equivalent can be used to transform (23) to standard form, it is not convenient to apply it for $\ell > 1$. A more transparent and direct way to eliminate the derivative inputs is by algebraic transformation. To apply the process for (23), we define $z(t) = x(t) - \widetilde{B}_1 u(t)$ and obtain

$$\dot{z}(t) = \widetilde{A}z(t) + (\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1)u(t) \tag{25}$$

$$y(t) = \widetilde{C}z(t) + \widetilde{C}\widetilde{B}_1 u(t) \tag{26}$$

For $\ell = 2$, we need to perform the algebraic transformation twice to obtain

$$\dot{z}(t) = \widetilde{A}z(t) + (\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1 + \widetilde{A}^2\widetilde{B}_2)u(t) \tag{27}$$

$$y(t) = \widetilde{C}z(t) + \widetilde{C}(\widetilde{B}_1 + \widetilde{A}\widetilde{B}_2)u(t) + \widetilde{C}\widetilde{B}_2\dot{u}(t) \tag{28}$$

Suppose (23) is the input derivative representation of positive singular system. Then (25),(26) remain positive if $\widetilde{A}\widetilde{B}_1 > 0$. Similarly, the system (27),(28) remain positive if $\widetilde{A}\widetilde{B}_1 \geq 0, \widetilde{A}\widetilde{B}_2 \geq 0$, and $\widetilde{A}^2\widetilde{B}_2 \geq 0$. Continuing the elimination process of input derivatives for $\ell > 2$, one can arrive at the following closed-form expression for ℓ derivative inputs,

$$\dot{z}(t) = \widetilde{A}z(t) + \widetilde{\widetilde{B}}_0 u(t) \tag{29}$$

$$y(t) = \widetilde{C}z(t) + \widetilde{C}\sum_{j=1}^{\ell}\widetilde{\widetilde{B}}_j u^{(j-1)}(t) \tag{30}$$

where

$$\widetilde{\widetilde{B}}_j = \sum_{i=j}^{\ell}\widetilde{A}^{i-j}\widetilde{B}_i, \quad j = 0, 1, \ldots, \ell$$

with

$$\widetilde{\widetilde{B}}_0 = \sum_{i=0}^{\ell}\widetilde{A}^{i}\widetilde{B}_i, \quad i = 0, 1, \ldots, \ell$$

It is important to point out that the elimination of input derivatives in state equation shifts the derivative terms in the output equation except for $\ell = 1$. This does not cause any issue when stabilization by state feedback is applied to (29),(30).

Using the equivalent representation of input derivative system (29),(30) we can state the following Theorem.

Theorem 5. *The equivalent system (29),(30) of positive input derivative system (21),(22) is positive if* $\widetilde{A} \in \mathbb{M}_n, \widetilde{\widetilde{B}}_j \in \mathbb{R}_+^{n\times m}$*, and* $\widetilde{C} \in \mathbb{R}_+^{p\times n}$.

Remark 1. The condition $\widetilde{\widetilde{B}}_j \in \mathbb{R}_+^{n\times m}$ is not required for the special subset of Metzler matrices $\widetilde{A} \in \mathbb{R}_+^{n\times n} \subset \mathbb{M}_n$. In this case, the condition $\widetilde{\widetilde{B}}_j \in \mathbb{R}_+^{n\times m}$ in the above Theorem can be replaced by $\widetilde{B}_i \in \mathbb{R}_+^{n\times m}$.

C. Controllability and Observability

The controllability and observability of singular system (1), (2) has been extensively studied and reported in [6], [9]. In this section we summarize the controllability and observability conditions associated with the equivalent representations developed in previous sections.

Lemma 2. *The singular system (1), (2) is controllable if and only if one of the following equivalent conditions is satisfied*

(a) $\rho[sE - A \quad B] = n$ *for all* $s \in \mathbb{C}$ *and* $\rho[E \quad B] = n$.

(b) $\rho[\widetilde{\widetilde{B}}_0 \quad \widetilde{A}\widetilde{\widetilde{B}}_0 \quad \ldots \quad \widetilde{A}^{n-1}\widetilde{\widetilde{B}}_0; \widetilde{\widetilde{B}}_1, \widetilde{\widetilde{B}}_2, \ldots \widetilde{\widetilde{B}}_\ell] = n$ *with respect to (29), (30).*

Lemma 3. *The singular system (1), (2) is observable if and only if one of the following equivalent conditions is satisfied*

(a) $\rho\begin{bmatrix}sE - A\\ C\end{bmatrix} = n$ *for all* $s \in \mathbb{C}$ *and* $\rho\begin{bmatrix}E\\ C\end{bmatrix} = n$.

(b) $\rho\begin{bmatrix}\widetilde{C}\\ \widetilde{C}\widetilde{A}\\ \vdots\\ \widetilde{C}\widetilde{A}^{n-1}\end{bmatrix} = n$ *with respect to (29), (30).*

V. Stabilization and Observer Design for Positive Singular Systems

A. Positive Stabilization

The algebraic transformation of input derivative system (29),(30) allows us to solve positive stabilization problem for unstable positive singular systems (1),(2).

Theorem 6. *Let the singular system (1),(2) with its equivalent input derivative system (8), (9) be positive and controllable. Furthermore, assume that its algebraic transformation be represented by (29), (30) with the parameters* $\{\widetilde{A}, \widetilde{\widetilde{B}}_0, \widetilde{\widetilde{B}}_j, \widetilde{C}\}$*, where* $\ell = \mu - 1$*. Then, the state feedback* *control law $u = K_z z$ stabilizes (29) and the closed-loop system becomes*

$$\dot{z}(t) = A_z z(t) \tag{31}$$

where

$$A_z = \widetilde{A} + \widetilde{\widetilde{B}}_0 K_z, \quad \widetilde{\widetilde{B}}_0 = \sum_{i=0}^{\ell} \widetilde{A}^i \widetilde{B}_i \tag{32}$$

if and only if the following LMI has a feasible solution

$$W\widetilde{A}^T + Y^T \widetilde{\widetilde{B}}_0{}^T + \widetilde{A}W + \widetilde{\widetilde{B}}_0 Y \prec 0 \tag{33}$$

$$(\widetilde{A}W + \widetilde{\widetilde{B}}_0 Y)_{ij} \geq 0, \quad i \neq j \tag{34}$$

whereby K_z is obtained by $K_z = YW^{-1}$ and $W \succ 0$ is a diagonal positive definite matrix.
Moreover, the feedback gain matrix K_x associated with the control law $u = v + Kx$ that stabilizes the positive singular system (1) is given by

$$K_x = K_z[I + \sum_{i=j}^{\ell} \widetilde{\widetilde{B}}_j K_z A_z{}^{j-1}]^{-1} \tag{35}$$

where x and z are related by

$$x = [I + \sum_{i=j}^{\ell} \widetilde{\widetilde{B}}_j K_z A_z{}^{j-1}]z, \quad j = 1, \ldots, \ell$$

Proof: The proof of this theorem is constructive by applying positive stabilization of regular system (29), (30) using LMI [**?**], [21]. Here, one should apply the Lyapunov stability condition with respect to (32) and the side constraint (34).

B. Observer Design

In this section we first develop the observer equation for singular system (1), (2) using its equivalent system (21), (22) without making the positivity assumption. Let us elaborate the procedure with $\ell = 1, 2, \ldots$ as we did in the development of state feedback design. For $\ell = 1$, we have (23), (24) and using $z = x - \widetilde{B}_1 u$, we eliminated $\dot{u}$ in (23) to obtain (25), (26). Defining $\widetilde{y} = y - \widetilde{C}\widetilde{B}_1 u$, we get

$$\dot{z}(t) = \widetilde{A}z(t) + (\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1)u(t) \tag{36}$$

$$\widetilde{y}(t) = \widetilde{C}z(t) \tag{37}$$

An observer for (36), (37) can be constructed as

$$\begin{aligned}\dot{\hat{z}}(t) &= (\widetilde{A} - L\widetilde{C})\hat{z}(t) + L\widetilde{y}(t) + (\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1)u(t) \\ &= (\widetilde{A} - L\widetilde{C})\hat{z}(t) + Ly(t) + [\widetilde{B}_0 + (\widetilde{A} - L\widetilde{C})\widetilde{B}_1]u(t)\end{aligned} \tag{38}$$

$$\hat{x}(t) = \hat{z}(t) + \widetilde{B}_1 u(t) \tag{39}$$

For $\ell = 2$, we performed the algebraic transformation twice to eliminate $\dot{u}$ and $\ddot{u}$ and obtained (27), (28). Defining $\widetilde{y} = y - \widetilde{C}(\widetilde{B}_1 + \widetilde{A}\widetilde{B}_2)u - \widetilde{C}\widetilde{B}_2\dot{u}$, we get

$$\dot{z}(t) = \widetilde{A}z(t) + (\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1 + \widetilde{A}^2\widetilde{B}_2)u(t) \tag{40}$$

$$\widetilde{y}(t) = \widetilde{C}z(t) \tag{41}$$

An observer for (40), (41) can be constructed as

$$\dot{\hat{z}}(t) = (\widetilde{A} - L\widetilde{C})\hat{z}(t) + L\widetilde{y}(t) + (\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1 + \widetilde{A}^2\widetilde{B}_2)u(t) \tag{42}$$

Since the derivative of the input appears in the state equation by $\widetilde{y}$, we define $\hat{w} = \hat{z} + L\widetilde{C}\widetilde{B}_2$ and rewrite (42) in terms of $\hat{w}$ as

$$\begin{aligned}\dot{\hat{w}}(t) = (\widetilde{A} - L\widetilde{C})\hat{w}(t) + Ly(t) + [\widetilde{B}_0 \\ + (\widetilde{A} - L\widetilde{C})\widetilde{B}_1 + (\widetilde{A} - L\widetilde{C})^2\widetilde{B}_2]u(t)\end{aligned} \tag{43}$$

$$\hat{x}(t) = \hat{w}(t) + [\widetilde{B}_1 + (\widetilde{A} - L\widetilde{C})\widetilde{B}_2]u(t) + \widetilde{B}_2\dot{u}(t) \tag{44}$$

Continuing the process for $\ell > 2$, we can state the following result for ℓ derivative inputs.

Theorem 7. *Let the singular system (1), (2) or equivalently (29), (30) be observable with the pair $\{\widetilde{A}, \widetilde{C}\}$. Then it has an observer of the following form*

$$\dot{\hat{z}}(t) = (\widetilde{A} - L\widetilde{C})\hat{z}(t) + Ly(t) + [\sum_{i=0}^{\ell}(\widetilde{A} - L\widetilde{C})^i\widetilde{B}_i]u(t) \tag{45}$$

$$\hat{x}(t) = \hat{z}(t) + \sum_{i=j}^{\ell}(\widetilde{A} - L\widetilde{C})^{i-j}\widetilde{B}_i u^{j-1}(t), \quad j = 1, \ldots, \ell \tag{46}$$

It is obvious that an observer can be designed by proper selection of the gain L such that $\widetilde{A} - L\widetilde{C}$ is asymptotically stable. Different eigenvalue assignment techniques can be applied to achieve this goal. However, when the singular system is positive, then a positive observer is required to be designed. Due to the structural constraints of positivity on the system and observer, the design task becomes challenging. The equivalent algebraic transformation of input derivative representation of positive singular system allows one to design the positive observer.

C. Positive Observer Design

The input derivative representation (21), (22) of singular system (1), (2) and its algebraic transformation (29), (30) are defined as positive singular system in section IV-A. The above derivation established the structure of observer for singular system in Theorem 7. Thus, it is necessary to define (45), (46) as a positive system as well.

Definition 4. The observer structure (45), (46) is positive if and only if for any positive initial condition $\hat{z}(0) \in \mathbb{R}_+^n$ or equivalently $\hat{x}(0) \in \mathbb{R}_+^n$, and positive input $u(t) \in \mathbb{R}_+^m$, the state and output response remain positive i.e. $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$.

Lemma 4. *Let the singular system (1), (2) represented by the input derivative system (21), (22) and its algebraic transformation (29), (30) be positive and observable. Then, (45), (46) is a positive observer for the positive singular system if and only if $\widetilde{A} - L\widetilde{C}$ is Metzler and Hurwitz stable, $L\widetilde{C} \geq 0$, and $\sum_{i=0}^{\ell}(\widetilde{A} - L\widetilde{C})^i\widetilde{B}_i \geq 0$.*

Proof of the above Lemma follows immediately from standard positive systems.

Conventionally, if the observer is used to estimate the state of systems, it is sufficient to consider unforced systems. Thus, we have the following result for the design of positive observer for positive singular systems with the aid of [19].

Theorem 8. *Let the unforced singular system $E\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ or its equivalent representation form (21), (22) i.e. $\dot{x}(t) = \widetilde{A}x(t)$, $y(t) = \widetilde{C}x(t)$ be positive and observable. Then*

$$\dot{\hat{z}}(t) = (\widetilde{A} - L\widetilde{C})\hat{z} + Ly(t)$$
$$\hat{x}(t) = \hat{z}(t)$$

from (45), (46) is a positive observer for positive singular system if and only if one of the following equivalent conditions is satisfied:

(i) *There exists a gain matrix $L \in \mathbb{R}^{n\times p}$ such that $\widetilde{A}-L\widetilde{C}$ is a stable Metzler matrix and $L\widetilde{C} \geq 0$.*

(ii) *The following LMI in the variables $P \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times p}$ is feasible*

$$\widetilde{A}^T P + P\widetilde{A} - \widetilde{C}^T X^T - X\widetilde{C} \prec 0,$$
$$\widetilde{A}^T P - \widetilde{C}^T X^T + I \geq 0,$$
$$X\widetilde{C} \geq 0,$$
$$P \succ 0.$$

where P is a positive definite diagonal matrix.

Furthermore, the observer gain matrix can be obtained by

$$L = P^{-1}X$$

Proof: The proof for (i) follows by combining the system and observer state equation which requires $L\widetilde{C} \geq 0$ to allow the augmented system matrix to be Metzler. The stability of $\widetilde{A} - L\widetilde{C}$ is verified through the error dynamic. To prove (ii) one can use $(\widetilde{A}-L\widetilde{C})^T P + P(\widetilde{A}-L\widetilde{C}) \prec 0$ and by using the change of variable $PL = X$, the first feasibility condition follows. The second line of (ii) follows from the structural constraint of $\widetilde{A} - L\widetilde{C}$ to be Metzler.

VI. Illustrative Examples

Example 1: Consider a simple second order singular system

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\dot{x} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}x + \begin{bmatrix}1\\ 1\end{bmatrix}u, \quad y = \begin{bmatrix}0 & 1\end{bmatrix}x$$

which is unstable with eigenvalues at 1.

Applying the Shuffle Algorithm one step with the aid of (17)-(20), we get the equivalent input derivative positive system

$$\dot{x} = \underbrace{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}}_{\widetilde{A}}x + \underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{\widetilde{B}_0}u + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{\widetilde{B}_1}\dot{u}.$$

After algebraic transformation and eliminating the input derivative, we obtain (25) where $\tilde{\widetilde{B}}_0 = \widetilde{B}_0 + \widetilde{A}\widetilde{B}_1 = \begin{bmatrix}2 & 0\end{bmatrix}^T$. The state feedback control law $u = K_z z$ results in stable closed-loop system $\dot{z} = A_z z$, where $A_z = \widetilde{A} + \tilde{\widetilde{B}}_0 K_z$ with $K_z = \begin{bmatrix}-2 & 0\end{bmatrix}$. Finally, K_x for the original singular system can be obtained from (35) as $K_x = K_z(I + \widetilde{B}_1 K_z)^{-1} = \begin{bmatrix}-2 & 0\end{bmatrix}$. To check the positivity and stability of the closed-loop singular system, we let $u = v + K_x x$ and after application of shuffle algorithm one step, we get the positive system

$$\dot{x} = \begin{bmatrix}-1 & 1\\ 2 & -2\end{bmatrix}x + \begin{bmatrix}1\\ 0\end{bmatrix}v + \begin{bmatrix}0\\ 1\end{bmatrix}\dot{v}$$

The stability of the closed-loop singular system can be verified by $\det(sE - A_c) = 0$, where $A_c = A + BK_x$ with eigenvalues of -3. Note that other feasible solution is possible, for example if $K_z = \begin{bmatrix}-2 & -1/4\end{bmatrix}$, then $K_x = \begin{bmatrix}-8/3 & -1/3\end{bmatrix}$. Designing a positive observer for the above singular system was not possible. However, if A is replaced by

$$A = \begin{bmatrix}-2 & 1\\ -1 & -1\end{bmatrix}$$

one can apply the steps of transformation to obtain positive system

$$\dot{x} = \underbrace{\begin{bmatrix}-2 & 1\\ 2 & -1\end{bmatrix}}_{\widetilde{A}}x, \quad y = \underbrace{\begin{bmatrix}0 & 1\end{bmatrix}}_{\widetilde{C}}x.$$

and design the positive observer using Theorem 8. This leads to a feasible solution $L = \begin{bmatrix}0.3 & 0.5\end{bmatrix}^T$, which specifies $\dot{\hat{z}} = (\widetilde{A} - L\widetilde{C})\hat{z} + Ly$, $\hat{x} = \hat{z}$.

Example 2: A non-positive singular system is considered. It is possible to show that positive stabilization can be achieved by applying Theorem 6.

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\dot{x} = \begin{bmatrix}-2 & 0 & 2 & -0.1\\ 0 & -2 & 0 & 3\\ 1 & 0 & 1 & -0.4\\ 0 & 1 & 0 & -1\end{bmatrix}x + \begin{bmatrix}1 & 0\\ 0 & 1\\ 4 & 0\\ 0 & 2\end{bmatrix}u$$

Since the roots of $\det(sE - A) = 0$ are $\{-4, 1\}$, the singular system is unstable. After the steps of transformation to the equivalent input derivative system and application of algebraic transformation, one can solve the modified LMI (40),(41) to obtain K_x.

The equivalent input derivative system is obtained as

$$\widetilde{A} = \begin{bmatrix}-2 & 0 & 2 & -0.1\\ 0 & -2 & 0 & 3\\ 2 & -0.8 & -2 & 1.3\\ 0 & -2 & 0 & 3\end{bmatrix},$$

$$\widetilde{B}_0 = \begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0.4\\ 0 & 1\end{bmatrix}, \quad \widetilde{B}_1 = \begin{bmatrix}0 & 0\\ 0 & 0\\ -0.4 & 0.8\\ 0 & 2\end{bmatrix}$$

After algebraic transformation, we obtain

$$\dot{z} = \widetilde{A} + \underbrace{\left(\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1\right)}_{\tilde{\widetilde{B}}_0} u$$

and its stabilizer can be found by applying Theorem 6 with

$$u = K_z z, \quad \dot{z} = \underbrace{\left(\widetilde{A} + \tilde{\widetilde{B}}_0 K_z\right)}_{A_z} z$$

$$K_z = YW^{-1} = \begin{bmatrix} -0.0833 & 0.0222 & 0.3333 & -0.1111 \\ 0 & 0.1667 & 0 & -0.3333 \end{bmatrix}$$

where

$$Y = \begin{bmatrix} -0.11 & 0.0569 & 1.15 & -0.6267 \\ 0 & 0.4267 & 0 & -1.88 \end{bmatrix}$$

and $W = \text{diag}\{1.32, 2.56, 3.45, 5.64\}$. Finally, we obtain,

$$K_x = K_z \left(I + \widetilde{B}_1 K_z\right)^{-1} = \begin{bmatrix} 0.25 & 0 & -1 & 0.2 \\ 0 & 0.5 & 0 & -1 \end{bmatrix}$$

This feedback gain matrix stabilizes the singular system and the closed-loop system matrix $A_x = A + BK_x$ becomes stable and Metzler.

Example 3: Consider the third-order positive singular system in Section IV, which has been represented by an equivalent regular system with one input derivative using the shuffle algorithm. The system is unstable with eigenvalues $\{0, -1\}$. Applying the algebraic transformation and using LMI without the positivity constraint, a positive stabilizer can be obtained for the original singular systems by

$$K_x = \begin{bmatrix} -0.3860 & -1.3271 & -0.2185 \\ -0.7580 & -1.6097 & -1.0901 \end{bmatrix}$$

Thus, $A_x = A + BK_x$, which has stable roots $(-0.2402, -3.4551)$.

VII. CONCLUSION

This paper introduced a new approach for stabilization and observer design for positive singular systems. After a summary of available results on singular systems, the positivity of singular system was established in terms of its equivalent input derivative representation. Then, an algebraic transformation was introduced to eliminate the derivative inputs. Consequently, it was possible to achieve positive stabilization and observer design of positive singular system using the equivalent standard state space representation.

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