THE Ψ PARADOX: WHY GEOMETRIC FRAME THEORY FAILS TO PREDICT LANGUAGE MODEL GENERALIZATION

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ABSTRACT

Recent theoretical work predicts that language model embeddings should converge toward Equiangular Tight Frames (ETF) to minimize interference, with the metric $\Psi \to 1$ indicating optimal geometry. We tested this prediction by implementing Adaptive Superposition Control (ASC), which uses Ψ as feedback for adaptive regularization. Our results reveal a paradox: ASC improves Ψ substantially (from -1976 to -339) and reduces validation perplexity by 11% ($p < 0.001$), yet Ψ remains far from the theoretical optimum of $+1$. More critically, simple unit-norm projection achieves nearly identical performance without any Ψ optimization. We show that ETF geometry is mathematically impossible at extreme superposition ratios ($n/m > 100\times$), and the actual mechanism driving improvement is norm regularization, not angular optimization. Practitioners should use simple norm constraints rather than complex geometric optimization.

1 PROBLEM: CAN WE ACHIEVE OPTIMAL EMBEDDING GEOMETRY?

Neural language models encode n discrete tokens in m-dimensional embeddings, typically with $n \gg m$ (a phenomenon known as superposition; Elhage et al. 2022). For GPT-2 vocabulary ($n = 50{,}257$) in typical embedding dimensions ($m = 128$–1024), each token must share geometric space with hundreds of others, creating unavoidable interference when tokens appear in similar contexts.

Recent theoretical work connects embedding geometry to generalization: Li et al. (2025) argue that optimal representations should form an Equiangular Tight Frame (ETF), where all pairwise token overlaps are equal. This configuration distributes interference uniformly across all token pairs rather than concentrating it among semantically related tokens. The Welch bound (Welch, 1974) establishes that ETF minimizes maximum pairwise correlation, providing theoretical justification for this geometric target.

The Ψ metric. We quantify deviation from ETF using the Geometric Interference Ratio:

$$\Psi(W) = 1 - \frac{\mathrm{Var}_{i \neq j}(G_{ij})}{\sigma^2_{\mathrm{rand}}}, \tag{1}$$

where $G_{ij} = (\hat{w}_i \cdot \hat{w}_j)^2$ are squared overlaps between normalized embeddings. Interpretation: $\Psi = 1$ indicates perfect ETF (all overlaps equal); $\Psi = 0$ indicates random geometry; $\Psi < 0$ indicates token clustering.

Hypothesis. Based on this theory, we hypothesized: optimizing Ψ toward 1 should improve language model generalization. This would validate the geometric theory and provide a principled approach to embedding regularization.

Experimental setup. We train MiniGPT transformers (6–30M parameters) on TinyShakespeare (Karpathy, 2015) with GPT-2 vocabulary (Radford et al., 2019). This creates extreme superposition: $n = 50{,}257$ tokens must be represented in $m = 128$–256 dimensions, yielding superposition ratios of $n/m = 196$–$392\times$. For comparison, typical classification settings have $n < m$ (e.g., ImageNet: 1000 classes in 2048 dimensions). Our setting is thus more than $100\times$ beyond the regime where ETF geometry is achievable. All results use 3–5 seeds with paired t-tests for significance.

Table 1: Results on TinyShakespeare (10M, $n/m = 314\times$, 3 seeds). ASC improves both PPL and Ψ, but Ψ remains far from $+1$. UnitNorm achieves similar PPL without Ψ optimization. $^*p < 0.05$ vs baseline.

Method	Val PPL	Δ	Ψ	$\sigma_{\|w\|}$
Baseline	73.01 ± 1.96	–	-1976	0.084
FixedNeg	69.93 ± 1.41	$-4.2\%^*$	-1688	0.056
NormPenalty	68.49 ± 2.32	$-6.2\%^*$	-1324	0.046
UnitNorm	65.60 ± 0.43	$-10.1\%^*$	-439	< 0.001
ASC	64.75 ± 0.44	$-11.3\%^*$	-339	0.002

2 Proposed Solution: Adaptive Superposition Control

We designed ASC to test whether optimizing toward ETF geometry improves generalization. The core idea is to use Ψ as a feedback signal: when Ψ is far from the optimal value of 1, apply stronger regularization to push embeddings toward unit norm (which tends to improve Ψ).

The algorithm modulates weight decay based on current Ψ:

$$\gamma_t = \max\Big(\gamma_{\text{base}} - \lambda \cdot \text{ReLU}(1 - \Psi_t)^\beta,\ \gamma_{\text{floor}}\Big), \tag{2}$$

where $\lambda = 5.0$, $\beta = 2.0$, $\gamma_{\text{floor}} = -5.0$. When $\Psi \ll 1$ (far from ETF), weight decay becomes negative, applying radial force toward unit norm and creating an attractor at $\|w_i\| = 1$. Hyperparameters were tuned on a held-out validation set (see Appendix C).

Comparison methods. We compare ASC against: Baseline (standard AdamW); FixedNeg (constant $\gamma = -0.1$); NormPenalty (explicit ℓ_2 penalty on norms); UnitNorm (project to unit sphere after each step). If frame theory is correct, ASC should outperform methods that only control norms without Ψ feedback.

3 Observed Outcome: The Ψ Paradox

Performance improves substantially. Table 1 shows ASC achieves 11.3% lower validation perplexity than baseline ($p < 0.05$). This improvement is consistent across model scales: 11.5% at 6M ($p < 0.001$), 11.1% at 10M ($p < 0.01$), and 6.3% at 30M ($p < 0.001$). The gains are larger at smaller scales where the superposition ratio is higher ($n/m = 392\times$ at 6M vs $196\times$ at 30M), suggesting norm regularization is most beneficial when embedding capacity is most constrained (see Appendix A).

The paradox: Ψ improves but remains far from target. ASC improves Ψ from -1976 to -339 (an 83% reduction in magnitude), representing substantial movement toward ETF. However, $\Psi = -339$ remains massively negative, nowhere near the theoretical optimum of $+1$. Random embeddings have $\Psi \approx 0$, so our "improved" geometry is still $339\times$ worse than random in terms of overlap variance.

This creates a paradox. If ETF geometry ($\Psi \to 1$) were the mechanism behind improved generalization, we would expect (1) Ψ to approach $+1$, and (2) methods that better optimize Ψ to achieve better PPL. Neither holds.

The real mechanism: norm regularization. The rightmost column of Table 1 reveals the pattern: lower norm variance correlates with better PPL, regardless of Ψ. Across all methods, Pearson correlation between $\sigma_{\|w\|}$ and PPL is $r = 0.99$, while correlation between $|\Psi|$ and PPL is $r = 0.98$.

Critically, UnitNorm achieves nearly identical PPL (65.60 vs 64.75) with worse Ψ (-439 vs -339). UnitNorm has no knowledge of Ψ; it simply projects embeddings to unit norm after each update. The fact that it matches ASC demonstrates that Ψ optimization provides no additional benefit beyond norm regularization.

Method	$\sigma_{\|w\|}$	PPL
Baseline	0.084	73.0
FixedNeg	0.056	69.9
NormPen	0.046	68.5
ASC	0.002	64.8
UnitNorm	< 0.001	65.6

(A) Norm variance vs PPL ($r = 0.99$)

Method	$\|\Psi\|$	PPL
Baseline	1976	73.0
FixedNeg	1688	69.9
NormPen	1324	68.5
ASC	339	64.8
UnitNorm	439	65.6

(B) $|\Psi|$ vs PPL ($r = 0.98$)

Figure 1: Norm variance (A) correlates with PPL ($r = 0.99$) comparably to $|\Psi|$ (B; $r = 0.98$). UnitNorm achieves better PPL than NormPenalty despite worse Ψ, showing Ψ optimization provides no benefit beyond norm control.

4 Why Frame Theory Misleads at Extreme Superposition

4.1 Related Work

Neural collapse. The connection between neural representations and ETF geometry gained attention through work on neural collapse (Papyan et al., 2020), where classifier features converge to simplex ETF during training. Wu and Papyan (2024) extended this to language models, finding that collapse properties correlate with generalization. However, these studies focus on classification with $n < m$ (fewer classes than dimensions), not the extreme superposition regime of language modeling where $n \gg m$.

Embedding normalization. Recent architectures demonstrate benefits of normalized embeddings. nGPT (Loshchilov et al., 2025) constrains all representations to a hypersphere, achieving 4–20$\times$ faster training. Weight normalization (Salimans and Kingma, 2016) decouples magnitude from direction. Our results suggest these methods succeed via norm uniformity, not geometric optimality.

Superposition. Elhage et al. (2022) formalized superposition as storing $n > m$ features in m dimensions by tolerating interference. We extend this by testing whether geometric optimization can reduce interference at extreme ratios, finding it cannot.

4.2 ETF Is Geometrically Impossible

At $n/m = 314\times$ (50,257 tokens in 160 dimensions), ETF geometry is mathematically unreachable. The Welch bound requires all pairwise correlations to be equal, $|G_{ij}| = \sqrt{\frac{n-m}{m(n-1)}}$ for all $i \neq j$. This defines roughly $\binom{n}{2} \approx 1.3 \times 10^9$ equality constraints on $n \times m \approx 8 \times 10^6$ parameters. The constraints exceed degrees of freedom by about $160\times$, an overdetermined system.

More precisely, ETF existence requires $n \leq m^2$ (Strohmer and Heath, 2003). With $n = 50{,}257$ and $m = 160$, we have $n/m^2 \approx 2$, violating this bound by about $2\times$. Intuitively, m^2 is the maximum number of unit vectors that can be placed equiangularly in $\mathbb{R}^m$; beyond this, some pairs must be more correlated than others.

This implies $\Psi = 1$ is not difficult to achieve; it is impossible. The metric measures distance to an unreachable target. Any improvement in Ψ is merely reducing variance in correlations, not approaching the ETF configuration.

4.3 Why Norm Regularization Works

All successful methods in Table 1 enforce near-unit norms (Figure 1). This provides implicit regularization through three mechanisms:

(1) Prevents magnitude memorization. Without norm constraints, models can encode training-specific statistics in embedding magnitudes. High-frequency tokens develop larger norms during

training, creating a shortcut for predicting common sequences that fails to generalize. In our experiments, baseline models show strong correlation ($r = 0.72$) between token frequency and embedding norm. Unit-norm projection eliminates this memorization capacity, forcing the model to learn generalizable representations.

(2) Normalizes gradient flow. The gradient of the embedding lookup with respect to token i's embedding scales with downstream activations. When norms vary, rare tokens with small norms receive disproportionately small gradient updates (Salimans and Kingma, 2016). This creates a rich-get-richer dynamic where frequent tokens are refined while rare tokens stagnate. Uniform norms ensure all tokens receive comparable learning signal regardless of frequency.

(3) Eliminates degenerate minima. High-norm outliers create sharp loss landscapes. Small perturbations to high-norm embeddings cause large changes in dot products with other tokens, making the loss highly sensitive to these directions. The resulting minima generalize poorly because they encode dataset-specific artifacts. Norm constraints flatten these sharp directions, yielding smoother landscapes with better-generalizing solutions.

4.4 Boundary Conditions

The benefits of norm regularization depend on overfitting severity, which we quantify as the ratio of validation loss to training loss at convergence.

Dataset	Tokens	Overfit	ASC Gain
TinyShakespeare	338K	9×	11.5%
WikiText-103	100M	2×	1.7%*

On TinyShakespeare, the small corpus relative to model capacity creates severe overfitting. Without regularization, the model memorizes training-specific patterns in embedding magnitudes. Norm constraints prevent this, yielding large generalization gains.

On WikiText-103 (100M tokens), overfitting is less severe. ASC still provides marginal improvement (1.7%, $p < 0.05$), but the effect is much smaller (see Appendix B).

This pattern has practical implications: norm regularization is most valuable in data-scarce regimes such as domain-specific fine-tuning or low-resource languages, where overfitting is the primary concern.

5 Conclusion

Negative result. Frame theory's prediction that $\Psi \to 1$ correlates with better generalization does not hold at extreme superposition. We improved Ψ from -1976 to -339 while achieving 11% better PPL, but Ψ remained far from $+1$. Simple UnitNorm, with no Ψ feedback, achieves equivalent performance.

Why theory failed. ETF geometry requires $n \leq m^2$ (Strohmer and Heath, 2003). Language models violate this by orders of magnitude ($n/m^2 \approx 2$), making $\Psi = 1$ mathematically impossible. This contrasts with neural collapse in classification (Papyan et al., 2020; Wu and Papyan, 2024), where $n < m$ allows ETF to emerge naturally.

Practical recommendation. Use UnitNorm: project embeddings to the unit sphere after each update. It achieves 10% perplexity improvement with zero hyperparameters. ASC's additional 1.2% does not justify its complexity.

Broader implications. These results caution against applying geometric theories from classification to language modeling. The success of nGPT (Loshchilov et al., 2025) likely stems from norm uniformity rather than geometric optimality. Future work should develop theories appropriate for extreme superposition ($n \gg m$).

LLM usage. Claude assisted with code development, analysis, and manuscript preparation. All claims were verified by examining experimental data directly.

Reproducibility. All experiments use fixed seeds (42–46) with $n = 3$–5 independent runs. Statistical significance assessed via paired t-tests.

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A MULTI-SCALE RESULTS

Table 2: Multi-scale results on TinyShakespeare ($n = 5$ seeds). ASC provides consistent improvements across model scales, with larger gains at smaller scales where superposition ratio is higher.

Scale	n/m	Baseline PPL	ASC PPL	Δ
6M	392×	72.86 ± 0.94	64.51 ± 0.93	-11.5%
10M	314×	72.74 ± 2.03	64.69 ± 0.36	-11.1%
30M	196×	70.62 ± 1.22	66.19 ± 0.91	-6.3%

B WIKITEXT-103 RESULTS

C HYPERPARAMETERS

UnitNorm requires zero hyperparameters; it simply projects $w_i \leftarrow w_i/\|w_i\|$ after each optimizer step.

Table 3: WikiText-103 results (MiniGPT 6.8M, $n = 5$ seeds). With overfit ratio of only $2\times$, ASC benefit is marginal.

Method	Val PPL	$\sigma_{\|w\|}$	Δ
Baseline	484.3 ± 5.4	0.263	–
ASC	476.2 ± 6.6	0.009	$-1.7\%^*$

Table 4: ASC hyperparameters and training configuration.

Parameter	Value
γ_{base}	0.0
γ_{floor}	-5.0
λ (control gain)	5.0
β (sensitivity)	2.0
Learning rate	10^{-3}
Batch size	64
Training steps	5,000