Gaussian Approximation for Two-Timescale Linear Stochastic Approximation

Bogdan Butyrin[1, *], Artemy Rubtsov[1, *], Alexey Naumov[1], Vladimir Ulyanov[1, 2], Sergey Samsonov[1]

[1] HSE University, [2] Lomonosov Moscow State University
[*]Equal contribution: bbutyrin@hse.ru, asrubtsov@hse.ru

Abstract

In this paper, we establish non-asymptotic bounds for accuracy of normal approximation for linear two-timescale stochastic approximation (TTSA) algorithms driven by martingale difference or Markov noise. Focusing on both the last iterate and Polyak–Ruppert averaging regimes, we derive bounds for normal approximation in terms of the convex distance between probability distributions. Our analysis reveals a non-trivial interaction between the fast and slow timescales: the normal approximation rate for the last iterate improves as the timescale separation increases, while it decreases in the Polyak–Ruppert averaged setting. We also provide the high-order moment bounds for the error of linear TTSA algorithm, which may be of independent interest. Finally, we demonstrate that our theoretical results are directly applicable to reinforcement learning algorithms such as GTD and TDC.

1 Introduction

Stochastic approximation (SA) methods play an important role in the field of machine learning, especially due to their role in solving reinforcement learning (RL) problems (Sutton and Barto 2018). Recent studies cover both asymptotic (Nemirovskij and Yudin 1983; Polyak and Juditsky 1992) and non-asymptotic (Moulines and Bach 2011) properties of SA estimates. In particular, two-timescale stochastic approximation (TTSA) algorithms (Borkar 1997) refer to the class of methods that update two interdependent variables with separate step size sequences, one typically decreasing faster than the other. This class of methods is especially important in RL, where policy evaluation in the off-policy setting requires TTSA methods such as the Gradient Temporal Difference (GTD) method (Sutton, Maei, and Szepesvári 2008).

An important question for SA algorithms is related to the accuracy of Gaussian approximation (GAR) of the constructed estimates. Classical results on GAR for SA algorithms, such as (Polyak and Juditsky 1992; Konda and Tsitsiklis 2004), are asymptotic and do not provide convergence rates. At the same time, the latter results play an important role in statistical inference for optimization (Fan 2019), as they pave the way for non-asymptotic analysis of various procedures for constructing confidence intervals. We focus on the linear two-timescale SA problem, that is, we aim to find a solution $(\theta^\star, w^\star)$ that solves the system of linear equations:

$$A_{11}\theta + A_{12}w = b_1 \ , \quad A_{21}\theta + A_{22}w = b_2 \ , \qquad (1)$$

assuming that the solution $(\theta^\star, w^\star)$ is unique and is given by

$$\theta^\star = \Delta^{-1}(b_1 - A_{12}A_{22}^{-1}b_2), \quad w^\star = A_{22}^{-1}(b_2 - A_{21}\theta^\star) \ ,$$

with $\Delta := A_{11} - A_{12}A_{22}^{-1}A_{21}$. We consider the setting, where the underlying matrices A_{ij} and vectors b_i, $i, j \in \{1, 2\}$, are not accessible. Instead, following (Borkar 1997), we assume that the learner has access to a sequence of random variables $\{X_k\}_{k\in\mathbb{N}}$ taking values in a measurable space $(\mathsf{X}, \mathcal{X})$, and vector/matrix-valued functions $\mathbf{b}_i(x)$, $\mathbf{A}_{ij}(x)$, $i, j \in \{1, 2\}$, which serves as stochastic estimates of b_i and A_{ij}, respectively. The corresponding recurrence runs as

$$\begin{aligned} \theta_{k+1} &= \theta_k + \beta_k\{\mathbf{b}_1^{k+1} - \mathbf{A}_{11}^{k+1}\theta_k - \mathbf{A}_{12}^{k+1}w_k\} \ , \\ w_{k+1} &= w_k + \gamma_k\{\mathbf{b}_2^{k+1} - \mathbf{A}_{21}^{k+1}\theta_k - \mathbf{A}_{22}^{k+1}w_k\} \ , \end{aligned} \qquad (2)$$

where $\theta_k \in \mathbb{R}^{d_\theta}$, $w_k \in \mathbb{R}^{d_w}$, and $\mathbf{b}_i^k$, $\mathbf{A}_{ij}^k$ are shorthand notations for $\mathbf{b}_i(X_k)$ and $\mathbf{A}_{ij}(X_k)$, respectively. The scalars $\gamma_k, \beta_k > 0$ in (2) are step sizes, and the underlying SA scheme is said to have two timescales as the step sizes satisfy $\lim_{k\to\infty} \beta_k/\gamma_k < 1$ such that w_k is updated at a faster timescale. In our paper we consider $\beta_k = c_{0,\beta}(k + k_0)^{-b}$ and $\gamma_k = c_{0,\gamma}(k + k_0)^{-a}$ with exponents a and b satisfying $1/2 < a < b < 1$. When $\{X_k\}_{k\in\mathbb{N}}$ are i.i.d., and under appropriate technical assumptions on the parameters of (2), it is known (see e.g. (Konda and Tsitsiklis 2004)), that the asymptotic normality of the "slow" timescale θ_k holds:

$$\beta_k^{-1/2}(\theta_k - \theta^\star) \to \mathcal{N}(0, \Sigma_\theta) \ , \qquad (3)$$

with some covariance Σ_θ. The authors in (Mokkadem, Pelletier et al. 2006) generalized this result for the averaged iterates of non-linear SA:

$$\bar{\theta}_n := n^{-1}\textstyle\sum_{k=1}^n \theta_k \ , \quad \bar{w}_n := n^{-1}\sum_{k=1}^n w_k \ . \qquad (4)$$

The latter estimates correspond to the Polyak-Ruppert averaging procedure introduced in (Ruppert 1988; Polyak and Juditsky 1992), a popular technique for stabilization of the SA algorithms. The authors of the recent paper (Kong et al. 2025) obtained the non-asymptotic convergence rates for the averaged iterates $\bar{\theta}_n$ and $\bar{w}_n$ in Wasserstein distance of order 1, using the vector-valued versions of the Berry-Essen theorem for martingale-difference sequences due to (Srikant 2024). In this paper, we not only generalize these results for the setting of Markov noise, but also establish the corresponding convergence rates for the last iterate θ_k. The main contributions of this paper are the following:

- We derive non-asymptotic bounds for the accuracy of normal approximation for the Polyak–Ruppert-averaged TTSA $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ and last iterate $\beta_n^{-1/2}(\theta_n - \theta^\star)$ in terms of convex distance under martingale-difference noise assumptions. Our results indicate that the normal approximation for the last iterate improves as the timescale separation increases and achieves a convergence rate of order up to $n^{-1/4}$, up to $\log n$ factors. We show that the Polyak–Ruppert averaged TTSA iterates achieve the same rate of normal approximation, but require that the timescales β_k and γ_k coincide up to a constant factor. While our analysis for the Polyak–Ruppert averaged TTSA generalizes recent results due to (Kong et al. 2025), we provide, to the best of our knowledge, the first fully non-asymptotic analysis of the normal approximation rates for the last iterate of TTSA.
- We generalize the obtained results for normal approximation for the averaged TTSA and the last iterate to the setting of Markov noise. Our results show a convergence rate of order up to $n^{-1/6}$, up to logarithmic factors, with the same conclusion regarding timescale separation as in the martingale noise case. This is the first result on the normal approximation rate for TTSA with Markov noise.

Notations. For a matrix $A \in \mathbb{R}^{d\times d}$ we denote by $\|A\|$ its operator norm. For symmetric positive-definite matrix $Q = Q^\top \succ 0$, $Q \in \mathbb{R}^{d\times d}$ and $x \in \mathbb{R}^d$ we define the corresponding norm $\|x\|_Q = \sqrt{x^\top Q x}$, and define the respective matrix Q-norm of the matrix $B \in \mathbb{R}^{d\times d}$ by $\|B\|_Q = \sup_{x\neq 0} \|Bx\|_Q/\|x\|_Q$. For sequences a_n and b_n, we write $a_n \lesssim_{\log_n} b_n$ if there exist $c, \alpha > 0$ (not depending upon n), such that $a_n \leq c(1 + \log n)^\alpha b_n$. In the present text, the following abbreviations are used: "w.r.t." stands for "with respect to", "i.i.d. " - for "independent and identically distributed", "GAR" - for "Gaussian Approximation".

Related works Classical results in the stochastic approximation (Borkar 2008) study the asymptotic properties of the single timescale SA algorithms, with the properties of averaged estimated studied in (Polyak and Juditsky 1992). Two-timescale SA schemes were studied in (Borkar 1997; Tadić 2004; Tadic 2006) in terms of almost sure convergence. Asymptotic convergence rates of linear two-timescale SA were studied in (Konda and Tsitsiklis 2004), where the authors showed that asymptotically $\mathbb{E}[\|\theta_k - \theta^\star\|^2] = \mathcal{O}(\beta_k)$ and $\mathbb{E}[\|w_k - w^\star\|^2] = \mathcal{O}(\gamma_k)$.

Non-asymptotic error bounds for TTSA were first developed in (Dalal et al. 2018; Dalal, Szorenyi, and Thoppe 2020) under the martingale noise assumptions and additional projections used in the update scheme (2). These results were further improved in (Kaledin et al. 2020) for linear TTSA problems. (Haque, Khodadadian, and Maguluri 2023) refined the results of (Kaledin et al. 2020) obtaining the MSE bounds $\mathbb{E}[\|\theta_k - \theta^\star\|^2]$ and $\mathbb{E}[\|w_k - w^\star\|^2]$ with the leading terms given by $\beta_k \operatorname{Tr} \Sigma_\theta$ and $\gamma_k \operatorname{Tr} \Sigma_w$, where the covariances Σ_θ and Σ_w aligns with the CLT in (3). (Kwon et al. 2024) considered the version of (2) with constant step sizes and studied convergence to equilibrium for the corresponding Markov chain. Non-linear TTSA has been considered in (Doan 2024) under strong monotonicity assumptions, focusing on obtaining the MSE rate of order $\mathcal{O}(1/k)$ for k-th iterate.

Central limit theorem for TTSA iterates has been established in (Mokkadem, Pelletier et al. 2006), where the asymptotic version of the CLT was proved both for the last iterates (θ_k, w_k) and their Polyak–Ruppert averaged counterparts $(\bar{\theta}_n, \bar{w}_n)$. (Hu, Doshi, and Eun 2024) established an asymptotic CLT for general TTSA under Markov noise and controlled Markov chain dynamics, without quantifying the convergence rate. (Kong et al. 2025) studied the CLT for averaged iterates $(\bar{\theta}_n, \bar{w}_n)$ and provided a non-asymptotic CLT with the convergence rate studied in terms of Wasserstein distance of order 1.

2 Gaussian Approximation for SA algorithms

We outline a general scheme for proving the normal approximation. We consider vector-valued nonlinear statistics $T(X_1, \ldots, X_n) \in \mathbb{R}^d$, which can be represented in the form

$$T = W + D \,, \tag{5}$$

where W is a linear statistic of the random variables $X_1, \ldots, X_n$, and D is a small perturbation. This approach is well studied when $X_1, \ldots, X_n$ are i.i.d. random variables (Chen and Shao 2007; Shao and Zhang 2022) or form a martingale-difference sequence (Shorack 2017). The case of Markov random variables can be reduced to the setting of martingale-difference sequences through the Poisson equation (Douc et al. 2018, Chapter 21). We consider the decomposition (5) and assume, without loss of generality, that $\mathbb{E}[WW^\top] = \mathrm{I}_d$. To measure the approximation quality, a common approach is to use the supremum of the difference between measures taken over some subclass $\mathcal{H} \subseteq \operatorname{Conv}(\mathbb{R}^d)$ of the collection of convex sets $\operatorname{Conv}(\mathbb{R}^d)$. Specifically, for probability measures μ, ν on $\mathbb{R}^d$, we write

$$\mathsf{d}_{\mathcal{H}}(\mu, \nu) = \sup_{B\in\mathcal{H}} |\mu(B) - \nu(B)| \ .$$

Examples of $\mathcal{H}$ include the class of all convex sets, half-spaces, rectangles, ellipsoids, etc. The choice of different collections of sets $\mathcal{H}$ may be motivated by the needs of a particular application and may introduce differences in the dependence of the results on the problem dimension d. Indeed, even this dimensional dependence for linear statistics W can vary; see (Bentkus 2003) and (Kojevnikov and Song 2022) for the respective results for i.i.d. sequences and martingale differences. In this paper, we focus on the convex distance ρ^{Conv}, defined as

$$\rho^{\mathrm{Conv}}(\mu, \nu) = \sup_{B\in\operatorname{Conv}(\mathbb{R}^d)} |\mu(B) - \nu(B)| \,,$$

and rely on the following proposition to reduce the problem of Gaussian approximation for the nonlinear statistic $W + D$ to that for the linear statistic W:

Proposition 1 (Proposition 2 in (Sheshukova et al. 2025)). *Let ν be a standard Gaussian measure in $\mathbb{R}^d$. Then for any* *random vectors W, D taking values in $\mathbb{R}^d$, and any $p \geq 1$,*

$$\rho^{\mathrm{Conv}}(W + D, \nu) \leq \rho^{\mathrm{Conv}}(W, \nu) + 2c_d^{p/(p+1)} \mathbb{E}^{1/(p+1)} \left[\|D\|^p \right],$$

where c_d is the isoperimetric constant of class $\mathrm{Conv}(\mathbb{R}^d)$.

Similar results can be derived for other classes of sets $\mathcal{H}$, with the constant c_d depending on the isoperimetric properties of the specific class $\mathcal{H}$; see, e.g., (Klivans, O'Donnell, and Servedio 2008). Proposition 1 shows that the estimation of $\rho^{\mathrm{Conv}}(W + D, \mathcal{N}(0, \mathrm{I}))$ can be reduced to:

1. Estimating $\rho^{\mathrm{Conv}}(W, \mathcal{N}(0, \mathrm{I}))$;
2. Estimating moments $\mathbb{E}[\|D\|^p]$ for some $p \geq 1$.

To bound $\rho^{\mathrm{Conv}}(W, \mathcal{N}(0, \mathrm{I}))$, one can apply a Berry–Esseen bound for the appropriate linear statistic, e.g., (Shao and Zhang 2022) for i.i.d. random variables or (Srikant 2024; Samsonov et al. 2025; Wu, Wei, and Rinaldo 2025) for the martingale-difference setting. The most involved part of the proof is the proper estimation of $\mathbb{E}[\|D\|^p]$.

3 GAR for TTSA with Martingale noise

Assumptions and definitions. We investigate the linear TTSA algorithm given by the equivalent form of (2):

$$\theta_{k+1} = \theta_k + \beta_k (b_1 - A_{11}\theta_k - A_{12} w_k + V_{k+1}), \quad (6)$$
$$w_{k+1} = w_k + \gamma_k (b_2 - A_{21}\theta_k - A_{22} w_k + W_{k+1}) \,. \quad (7)$$

In this recurrence, the noise terms V_{k+1}, W_{k+1} are given by:

$$\begin{aligned} V_{k+1} &= \boldsymbol{\varepsilon}_V^{k+1} - \tilde{\mathbf{A}}_{11}^{k+1}(\theta_k - \theta^\star) - \tilde{\mathbf{A}}_{12}^{k+1}(w_k - w^\star), \\ W_{k+1} &= \boldsymbol{\varepsilon}_W^{k+1} - \tilde{\mathbf{A}}_{21}^{k+1}(\theta_k - \theta^\star) - \tilde{\mathbf{A}}_{22}^{k+1}(w_k - w^\star), \end{aligned} \quad (8)$$

where we used the notation $\tilde{\mathbf{A}}_{ij}^{k+1} := \mathbf{A}_{ij}^{k+1} - A_{ij}$ for $i, j \in \{1, 2\}$, and the random vectors $\boldsymbol{\varepsilon}_V^{k+1}, \boldsymbol{\varepsilon}_W^{k+1}$ are given by

$$\begin{aligned} \boldsymbol{\varepsilon}_V^{k+1} &= \mathbf{b}_1^{k+1} - \mathbf{A}_{11}^{k+1}\theta^\star - \mathbf{A}_{12}^{k+1} w^\star \,, \\ \boldsymbol{\varepsilon}_W^{k+1} &= \mathbf{b}_2^{k+1} - \mathbf{A}_{21}^{k+1}\theta^\star - \mathbf{A}_{22}^{k+1} w^\star \,. \end{aligned} \quad (9)$$

We consider a setting where the random elements V_{k+1} and W_{k+1} form a martingale-difference w.r.t. filtration $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$, $\mathcal{F}_0$ is trivial. We first consider the martingale noise setting. This setting covers the i.i.d. setting from (Konda and Tsitsiklis 2004) and also serves as a basis for subsequent analysis of the Markov noise setting.

A 1. *The noise terms are zero-mean given $\mathcal{F}_k$, i.e., $\mathbb{E}^{\mathcal{F}_k}[V_{k+1}] = 0$, and $\mathbb{E}^{\mathcal{F}_k}[W_{k+1}] = 0$.*

Next, for a given $p \geq 2$, we impose the following moment bound on V_{k+1}, W_{k+1}:

A 2 (p). *There exist constants $m_W, m_V > 0$ such that for any $k \in \mathbb{N}$:*

$$\mathbb{E}^{1/p}[\|V_{k+1}\|^p] \leq m_V(1 + \mathbb{E}^{1/p}[\|\theta_k - \theta^*\|^p] + \mathbb{E}^{1/p}[\|w_k - w^*\|^p])$$
$$\mathbb{E}^{1/p}[\|W_{k+1}\|^p] \leq m_W(1 + \mathbb{E}^{1/p}[\|\theta_k - \theta^*\|^p] + \mathbb{E}^{1/p}[\|w_k - w^*\|^p])$$

The assumption A 2(p) appears in a similar form with $p = 2$ in (Kaledin et al. 2020, Assumption A4). Since our results require to control high-order moments of the TTSA iterates θ_k and w_k, it is natural to require that p-th moment of V_{k+1} and W_{k+1} are finite. Next, we present an assumption on the quadratic characteristic of V_k and W_k:

A3. *Noise variables $\boldsymbol{\varepsilon}_V^{k+1}$ and $\boldsymbol{\varepsilon}_W^{k+1}$ defined in* (9) *have zero conditional expectation given $\mathcal{F}_k$, that is, $\mathbb{E}^{\mathcal{F}_k}\left[\boldsymbol{\varepsilon}_V^{k+1}\right] = 0$ and $\mathbb{E}^{\mathcal{F}_k}\left[\boldsymbol{\varepsilon}_W^{k+1}\right] = 0$. Moreover, there exist matrices Σ_V, Σ_W, Σ_{VW} such that for any $k > 0$:*

$$\mathbb{E}^{\mathcal{F}_k}\left[\boldsymbol{\varepsilon}_V^{k+1}\{\boldsymbol{\varepsilon}_V^{k+1}\}^\top\right] = \Sigma_V \,, \mathbb{E}^{\mathcal{F}_k}\left[\boldsymbol{\varepsilon}_W^{k+1}\{\boldsymbol{\varepsilon}_W^{k+1}\}^\top\right] = \Sigma_W \,,$$
$$\mathbb{E}^{\mathcal{F}_k}\left[\boldsymbol{\varepsilon}_V^{k+1}\{\boldsymbol{\varepsilon}_W^{k+1}\}^\top\right] = \Sigma_{VW} \,.$$

This assumption relaxes the one stated in (Kong et al. 2025), where the authors required the quadratic characteristic of the entire vectors V_{k+1} and W_{k+1} to be constant. However, this assumption is unlikely to hold due to the structure of these vectors outlined in (8). We also impose the following conditions on the problem matrices:

A4. *Matrices $-A_{22}$ and $-\Delta = -\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)$ are Hurwitz.*

A 4 is common for the analysis of both the linear two-timescale SA, see (Konda and Tsitsiklis 2004), and single-timescale SA, see (Durmus et al. 2025; Mou et al. 2020). A 4 implies, due to the Lyapunov lemma (stated in the supplement paper for completeness), that there exist matrices $Q_{22}^\top = Q_{22} \succ 0$, $Q_\Delta^\top = Q_\Delta \succ 0$, such that

$$\begin{aligned} \|\mathrm{I} - \gamma_k A_{22}\|_{Q_{22}} &\leq 1 - a_{22}\gamma_k, \quad a_{22} := \tfrac{1}{4\|Q_{22}\|} \,, \\ \|\mathrm{I} - \beta_k \Delta\|_{Q_\Delta} &\leq 1 - a_\Delta \beta_k, \quad a_\Delta := \tfrac{1}{4\|Q_\Delta\|} \,, \end{aligned} \quad (10)$$

provided that the step sizes γ_k and β_k are small enough. Precisely, for $p \geq 2$, we impose the following assumption A 5(p) on the step sizes:

A5 (p). *Step sizes $(\gamma_k)_{k \geq 1}$, $(\beta_k)_{k \geq 1}$ are non-increasing sequences of the form*

$$\beta_k = c_{0,\beta}(k + k_0)^{-b}, \quad \gamma_k = c_{0,\gamma}(k + k_0)^{-a} \,,$$

where $1/2 < a < b < 1$, fraction $c_{0,\beta}/c_{0,\gamma}$ is small enough, and constant k_0 satisfies the bound $k_0 \geq C_{A5} p^{4/b}$, where the constant C_{A5} does not depend upon p.

In the subsequent main results, we set the parameter p of order $\log(n)$. Hence, the parameter k_0 will depend on the total number of iterations to be performed. The same effect appears in the single-timescale SA algorithms (Durmus et al. 2025; Wu et al. 2024). This effect is unavoidable at least in the setting of the constant step size algorithms, see (Durmus et al. 2021a, Theorem 1).

A6. *There exist constants $\mathrm{C}_{\mathbf{A}}, \mathrm{C}_{\mathbf{b}} > 0$ such that*

$$\sup_{x \in \mathsf{X}} \|\mathbf{A}_{ij}(x)\| \vee \|\mathbf{A}_{ij}(x) - A_{ij}\| \leq \mathrm{C}_{\mathbf{A}} \,, \ \forall\, i, j \in \{1, 2\} \,,$$
$$\sup_{x \in \mathsf{X}} \|\mathbf{b}_i(x)\| \vee \|\mathbf{b}_i(x) - b_i\| \leq \mathrm{C}_{\mathbf{b}} \,, \ \forall\, i \in \{1, 2\} \,.$$

We expect that A 6 can be replaced with an appropriate moment condition, at least in a setting where the noise variables V_k and W_k form a martingale difference. At the same time, our further generalizations to the Markov noise setting inherently rely on the boundedness of $\mathbf{A}_{ij}(x)$ and $\mathbf{b}_i(x)$.

3.1 Moment bounds for Martingale TTSA

Given the assumptions A 1 - A 6, we present the classical reformulation of the two-timescale SA scheme (6)-(7), which is due to (Konda and Tsitsiklis 2004), see also (Kaledin et al. 2020). We define recursively the following sequence of matrices $\{L_k\}_{k\in\mathbb{N}}$, with $L_0 = 0$, and

$$L_{k+1} := \big(L_k - \gamma_k A_{22} L_k + \beta_k A_{22}^{-1} A_{21} U_k\big) \times \big(\mathrm{I} - \beta_k U_k\big)^{-1} \;,\; U_k := \Delta - A_{12} L_k \;. \tag{11}$$

and define $\mathrm{L}_\infty = a_\Delta \lambda_{\mathsf{max}}(Q_\Delta)/(\lambda_{\mathsf{min}}(Q_{22}) 2\|A_{12}\|)$. As shown in (Kaledin et al. 2020, Lemma 18), under A5 above recursion on L_k is well-defined, and every L_k satisfies the relation $\|L_k\| \leq \mathrm{L}_\infty$. In addition, define the matrices:

$$B_{11}^k := \Delta - A_{12} L_k \;, \quad D_k := L_{k+1} + A_{22}^{-1} A_{21} \;,$$
$$B_{22}^k := (\beta_k/\gamma_k)\big(L_{k+1} + A_{22}^{-1} A_{21}\big) A_{12} + A_{22} \;.$$

In a similar vein as performing Gaussian elimination, we obtain a simplified two-timescale SA recursions:

Proposition 2 (Observation 1 in (Kaledin et al. 2020)). *Consider the following change of variables:*

$$\tilde{\theta}_k := \theta_k - \theta^\star, \quad \tilde{w}_k = w_k - w^\star + D_{k-1}\tilde{\theta}_k. \tag{12}$$

Then the two-timescale SA (6)-(7) *is equivalent to:*

$$\begin{aligned} \tilde{\theta}_{k+1} &= (\mathrm{I} - \beta_k B_{11}^k)\tilde{\theta}_k - \beta_k A_{12}\tilde{w}_k - \beta_k V_{k+1} \;, \\ \tilde{w}_{k+1} &= (\mathrm{I} - \gamma_k B_{22}^k)\tilde{w}_k - \beta_k D_k V_{k+1} - \gamma_k W_{k+1} \;. \end{aligned} \tag{13}$$

Our further analysis, both for martingale and Markov noise, will essentially rely on the decoupled TTSA updates (13). We refer to this dynamics as to the "decoupled" one, since the update of the scale $\tilde{w}_{k+1}$ no longer depends directly on $\tilde{\theta}_k$, only through the noise variables V_{k+1} and W_{k+1}. Now we aim to upper bound the quantities

$$M_{k,p}^{\tilde{w}} := \mathbb{E}^{1/p}[\|\tilde{w}_k\|^p], \quad M_{k,p}^{\tilde{\theta}} := \mathbb{E}^{1/p}[\|\tilde{\theta}_k\|^p] \;. \tag{14}$$

Similarly to (10), we show in the supplement paper, that

$$\begin{aligned} \|\mathrm{I} - \beta_k B_{11}^k\|_{Q_\Delta} &\leq 1 - (1/2)\beta_k a_\Delta \;, \\ \|\mathrm{I} - \gamma_k B_{22}^k\|_{Q_{22}} &\leq 1 - (1/2)\gamma_k a_{22} \;. \end{aligned} \tag{15}$$

The result (15) together with the structure of the updates (13) enables us to expand the recurrence and to show that the error component, associated with the initial error $\theta_0 - \theta^\star$ and $w_0 - w^\star$ decay at the exponential rate. Precisely, the following bound holds:

Proposition 3. *Let $p \geq 2$ and assume A1,A2(p), A3, A4, A 5(p), and A6. Then for any $k \in \mathbb{N}$ it holds*

$$M_{k+1,p}^{\tilde{\theta}} \lesssim \textstyle\prod_{j=0}^k \big(1 - \beta_j a_\Delta/8\big) + p^2 \beta_k^{1/2} \;, \tag{16}$$

$$M_{k+1,p}^{\tilde{w}} \lesssim \textstyle\prod_{j=0}^k \big(1 - \gamma_j a_{22}/8\big) + p^3 \gamma_k^{1/2} \;, \tag{17}$$

where $\lesssim$ stands for inequality up to constants not depending upon k and p.

Discussion. Proposition 3 provides, to best of our knowledge, the first high-order moment bounds in the linear TTSA with martingale noise. The scaling of the r.h.s. with $\beta_k^{1/2}$ for $M_{k+1,p}^{\tilde{\theta}}$ and $\gamma_k^{1/2}$ for $M_{k+1,p}^{\tilde{w}}$ coincides with the one previously obtained for the particular case $p = 2$ in (Kaledin et al. 2020). Similar asymptotic results were previously obtained in (Konda and Tsitsiklis 2004). We expect that the dependence of the r.h.s. of (16) and (17) upon p can be improved based on applying the Pinelis version of Rosenthal inequality (Pinelis 1994, Theorem 4.1) instead of Burkholder's inequality (Osekowski 2012, Theorem 8.6), that was used in the current proof, yet we expect that this approach introduces additional technical difficulties.

3.2 GAR for Polyak-Ruppert averaged TTSA

Based on the results of the previous section, we can now quantify the Gaussian approximation rates for $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ for the Polyak-Ruppert averaged estimator $\bar{\theta}_n$ from (4). Now we present the key decomposition:

$$\Delta(\theta_k - \theta^\star) = \frac{\theta_k - \theta_{k+1}}{\beta_k} - \frac{A_{12} A_{22}^{-1}(w_k - w_{k+1})}{\gamma_k} + (V_{k+1} - A_{12} A_{22}^{-1} W_{k+1}) \;. \tag{18}$$

The proof of the above identity is given in the supplement paper. Taking sum in (18) for $k = 1$ to n, and using the definition of V_{k+1}, W_{k+1} in (8), we get:

$$\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star) = \tfrac{1}{\sqrt{n}} \textstyle\sum_{k=1}^n \psi_{k+1} + R_n^{\mathrm{pr}} \;, \tag{19}$$

where we set $\psi_{k+1} = \boldsymbol{\varepsilon}_V^{k+1} - A_{12} A_{22}^{-1} \boldsymbol{\varepsilon}_W^{k+1}$, and R_n^{pr} is a residual term defined in the supplement paper. Assumption A3 implies that the variance $\mathrm{Var}[\boldsymbol{\varepsilon}_V^{k+1} - A_{12} A_{22}^{-1} \boldsymbol{\varepsilon}_W^{k+1}]$ is constant for any k, so we can define

$$\boldsymbol{\Sigma}_\varepsilon := \mathrm{Var}[\boldsymbol{\varepsilon}_V^1 - A_{12} A_{22}^{-1} \boldsymbol{\varepsilon}_W^1] \in \mathbb{R}^{d_\theta \times d_\theta} \;. \tag{20}$$

The following theorem holds:

Theorem 1. *Assume A1,A2($\log n$), A3, A4, A5($\log n$), and A6. Then, it holds that*

$$\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, \boldsymbol{\Sigma}_\varepsilon)\big) \lesssim_{\log n} \frac{1}{n^{a/2}} + \frac{1}{n^{(1-b)/2}} \;.$$

Proof sketch. We apply Proposition 1 to the decomposition (19) and obtain, with $\nu \sim \mathcal{N}(0, \boldsymbol{\Sigma}_\varepsilon)$, that

$$\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star), \nu\big) \leq \underbrace{\rho^{\mathrm{Conv}}\big(n^{-1/2} \sum_{k=1}^n \psi_{k+1}, \nu\big)}_{T_1} + \underbrace{2 c_d^{p/p+1} \mathbb{E}^{1/(p+1)}\big[\|\boldsymbol{\Sigma}_\varepsilon^{-1/2} R_n^{\mathrm{pr}}\|^p\big]}_{T_2} \;.$$

Due to A 1 and A 6, sequence $\{\psi_{k+1}\}_{k\in\mathbb{N}}$ is a bounded martingale-difference sequence w.r.t. $\mathcal{F}_k$ with constant quadratic characteristic. Hence, T_1 can be estimated applying a slight modification of (Wu, Wei, and Rinaldo 2025, Theorem 1). It remains to bound the moments of T_2, which is done using Proposition 3.

Discussion. In the theorem above, the coefficients before the terms depend upon the initial errors $\|\theta_0 - \theta^\star\|$, $\|w_0 - w^\star\|$, and upon the factors $1/(1-a)$ and $1/(1-b)$. That is why the result in its current form does not apply directly if $b = 1$. We expect that the result holds in this case as well, perhaps at a price of introducing additional logarithmic factors. The same remark applies to Theorem 2-Theorem 4 stated below.

Since $1/2 < a < b < 1$, the bound of Theorem 1 is optimized when setting $a = 1/2 + 1/\log n$ and $b = a + 1/\log n$, yielding the final rate of convergence of order

$$\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, \mathbf{\Sigma}_\varepsilon)\big) \lesssim_{\log n} n^{-1/4} \; . \quad (21)$$

The result of (21) improves upon the previously established results of (Kong et al. 2025). The authors of that paper obtained a rate of $n^{-1/4}$, up to $\log n$ factors, in terms of Wasserstein distance. This implies convergence rate $n^{-1/8}$ in the convex distance, which is slower than (21). The choice of a and b in (21) corresponds to nearly the same scales for β_k and γ_k, effectively reducing the problem to a single-scale LSA. The obtained $n^{-1/4}$ rate aligns with the one established for this problem with i.i.d. noise in (Samsonov et al. 2024).

3.3 GAR for the last iterate.

In this section, we derive the normal approximation rates for the last iterate $\beta_n^{-1/2}\tilde{\theta}_{n+1}$. Following (Konda and Tsitsiklis 2004) and using (13), equations for $\tilde{\theta}_k$ and $\tilde{w}_k$ writes as

$$\tilde{\theta}_{k+1} = (\mathrm{I} - \beta_k\Delta)\tilde{\theta}_k - \beta_k A_{12}\tilde{w}_k - \beta_k V_{k+1} + \beta_k \delta_k^{(1)} \; ,$$
$$\tilde{w}_{k+1} = (\mathrm{I} - \gamma_k A_{22})\tilde{w}_k - \beta_k D_k V_{k+1} - \gamma_k W_{k+1} - \beta_k \delta_k^{(2)} \; ,$$

where we set

$$\delta_k^{(1)} = A_{12} L_k \tilde{\theta}_k \; , \quad \delta_k^{(2)} = -(L_{k+1} + A_{22}^{-1} A_{21}) A_{12} \tilde{w}_k \; .$$

Throughout the analysis we use the following convention:

$$G_{m:k}^{(1)} := \textstyle\prod_{i=m}^k (\mathrm{I} - \beta_i \Delta), \; G_{m:k}^{(2)} := \prod_{i=m}^k (\mathrm{I} - \gamma_i A_{22}) \; .$$

Enrolling the above recurrence and following (Konda and Tsitsiklis 2004), we get from the previous recurrence that

$$\tilde{\theta}_{n+1} = -\textstyle\sum_{j=0}^n \beta_j G_{j+1:n}^{(1)} \psi_{j+1} + R_n^{\mathrm{last}} \; , \quad (22)$$

where R_n^{last} is a remainder term defined in the supplement paper. The leading term in representation (22) is a linear statistics of $\varepsilon_V, \varepsilon_W$ which are martingale difference sequences with constant quadratic characteristics due to A3. Now we define

$$\mathbf{\Sigma}_n^{\mathrm{last}} = \mathrm{Var}\big[\textstyle\sum_{j=0}^n \beta_j G_{j+1:n}^{(1)} \psi_{j+1}\big] \; .$$

It is known that $\beta_n^{-1}\mathbf{\Sigma}_n^{\mathrm{last}}$ converges to a fixed matrix $\mathbf{\Sigma}_\infty^{\mathrm{last}}$ which is a solution of the Ricatti equation

$$\mathbf{\Sigma}_\infty^{\mathrm{last}} = \beta_0(\Delta\mathbf{\Sigma}_\infty^{\mathrm{last}} + \mathbf{\Sigma}_\infty^{\mathrm{last}}\Delta^\top - \mathbf{\Sigma}_\varepsilon) \; ,$$

where $\mathbf{\Sigma}_\varepsilon$ is defined in (20). Moreover, the convergence rate is proportional to β_n, i.e.

$$\|\beta_n^{-1}\mathbf{\Sigma}_n^{\mathrm{last}} - \mathbf{\Sigma}_\infty^{\mathrm{last}}\| \lesssim n^{-b} \; .$$

The proof of the above result is given in the supplement paper. The following assumption guarantees that the covariance matrix $\beta_n^{-1}\mathbf{\Sigma}_n^{\mathrm{last}}$ is non-degenerate, which is important for the further applications of Proposition 1.

A7. *Step size exponents a, b satisfy $2b > 1 + a$. Moreover, assume that the total number of iterations n satisfies $n^b \geq C_{A7}$, where C_{A7} does not depend on a, b, and can be traced following the supplement paper.*

Theorem 2. *Assume A1,A2($\log n$), A3, A4, A5($\log n$), A6, A 7. Then, it holds that*

$$\rho^{\mathrm{Conv}}\big(\beta_n^{-1/2}\tilde{\theta}_{n+1}, \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{last}})\big) \lesssim_{\log n} n^{b/2} \prod_{j=0}^n (1 - \frac{a_\Delta}{8}\beta_j) + \frac{1}{n^{(3b-a-2)/2}} \; . \quad (23)$$

Discussion The proof of Theorem 2 is similar to the one of Theorem 1, but relies on the decomposition (22) instead of (19) used in the averaged setting. Additional technical difficulties arises when controlling the moments of the term R_n^{last}. Bounding the latter term requires additional constraint $2b > 1 + a$ imposed in A7.

Since $1/2 < a < b < 1$, the bound of Theorem 2 is optimized when setting $a = 1/2 + 1/\log n$ and $b = 1 - 1/\log n$, yielding the final rate

$$\rho^{\mathrm{Conv}}\big(\beta_n^{-1/2}\tilde{\theta}_{n+1}, \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{last}})\big) \lesssim_{\log n} n^{-1/4} \; ,$$

provided that n is large enough. To the best of our knowledge, this is the first result concerning the Gaussian approximation rate for the TTSA last iterate.

Note that Theorem 2 reveals phenomenon, which is completely different from what was previously observed for the Polyak-Ruppert averaged iterates in Theorem 1. Indeed, the right-hand side of the bound (23) contains the term $n^{-(3b-a-2)/2}$, which favors separation between β_k and γ_k, and vanishes when the scale exponents are close.

4 GAR for TTSA with Markov noise

In this section we generalize the results obtained in Section 3 to the more practical scenario when $\{X_k\}_{k\in\mathbb{N}}$ form a Markov chain. Namely, we impose the following assumption:

B 1. *The sequence $\{X_k\}_{k\in\mathbb{N}}$ is a Markov chain taking values in a Polish space $(\mathsf{X}, \mathcal{X})$ with the Markov kernel P. Moreover, P admits π as a unique invariant distribution and is uniformly geometrically ergodic, that is, there exists $t_{\mathrm{mix}} \in \mathbb{N}$, such that for any $k \in \mathbb{N}$, it holds that*

$$\Delta(\mathrm{P}^k) := \sup_{x,x' \in \mathsf{X}} \mathsf{d}_{\mathrm{tv}}(\mathrm{P}^k(x,\cdot), \mathrm{P}^k(x',\cdot)) \leq (1/4)^{\lceil k/t_{\mathrm{mix}} \rceil} \; .$$

Moreover, for all $k \in \mathbb{N}$ and $i, j \in \{1, 2\}$ it holds that

$$\mathbb{E}_\pi[\mathbf{A}_{ij}^k] = A_{ij} \text{ and } \mathbb{E}_\pi[\mathbf{b}_i^k] = b_i \; .$$

Parameter t_{mix} in B 1 is referred to as a *mixing time*, see e.g. (Paulin 2015), and controls the rate of convergence of the iterates P^k to π as k increases.

4.1 Moment bounds for TTSA with Markov noise

First, we introduce a counterpart to A 5 that is needed to derive moment bounds for the setting of Markov noise.

B 2 (p). *$(\gamma_k)_{k\geq 1}$, $(\beta_k)_{k\geq 1}$ are non-increasing sequences of the form*

$$\beta_k = c_{0,\beta}(k + k_0)^{-b}, \quad \gamma_k = c_{0,\gamma}(k + k_0)^{-a} \; ,$$

where $1/2 < a < b < 1$, fraction $c_{0,\beta}/c_{0,\gamma}$ is small enough, and constant k_0 satisfies the bound $k_0 \geq C_{B\,2} p^{4/b}$, where the constant $C_{B\,2}$ does not depend upon p.

The proof of moment bounds is more involved compared to the martingale noise case. Following the decomposition outlined in (Kaledin et al. 2020), we first represent the noise variables (V_{k+1}, W_{k+1}) as a sum of their martingale $(V^{(0)}_{k+1}, W^{(0)}_{k+1})$ and Markovian components $(V^{(1)}_{k+1}, W^{(1)}_{k+1})$ in a way that

$$V_{k+1} = V^{(0)}_{k+1} + V^{(1)}_{k+1} \,, \quad W_{k+1} = W^{(0)}_{k+1} + W^{(1)}_{k+1} \,.$$

Here $\mathbb{E}^{\mathcal{F}_k}\left[V^{(0)}_{k+1}\right] = 0$ and $\mathbb{E}^{\mathcal{F}_k}\left[W^{(0)}_{k+1}\right] = 0$. This representation is obtained using the decomposition associated with the Poisson equation, see (Douc et al. 2018, Chapter 21) and additional summation by parts. Then we define a pair of coupled recursions, which form exact counterparts of (13):

$$\tilde{\theta}^{(i)}_{k+1} = (\mathrm{I} - \beta_k B^k_{11})\tilde{\theta}^{(i)}_k - \beta_k A_{12}\tilde{w}^{(i)}_k - \beta_k V^{(i)}_{k+1} \,,$$
$$\tilde{w}^{(i)}_{k+1} = (\mathrm{I} - \gamma_k B^k_{22})\tilde{w}^{(i)}_k - \beta_k D_k V^{(i)}_{k+1} - \gamma_k W^{(i)}_{k+1} \,,$$

where $i \in \{0, 1\}$. Then it is easy to see that $\tilde{\theta}_k = \tilde{\theta}^{(0)}_k + \tilde{\theta}^{(1)}_k$ and $\tilde{w}_k = \tilde{w}^{(0)}_k + \tilde{w}^{(1)}_k$. Precise expressions for $\tilde{\theta}^{(i)}_k, \tilde{w}^{(i)}_k, V^{(i)}_k, W^{(i)}_k$ can be found in the supplement paper.

Proposition 4. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Thus, it holds for any $k \geq 0$ that*

$$M^{\tilde{\theta}}_{k+1,p} \lesssim \textstyle\prod_{j=0}^{k}(1 - \frac{a_\Delta \beta_j}{8}) + p^2\sqrt{\beta_k} \,,$$
$$M^{\tilde{w}}_{k+1,p} \lesssim \textstyle\prod_{j=0}^{k}(1 - \frac{a_{22}\gamma_j}{8}) + p^3\sqrt{\gamma_k} \,.$$

Proof sketch. The idea of the proof is to bound martingale and Markov parts separately using the techniques from Section 3. Note that Proposition 4 directly mimics the similar result obtained under the martingale noise setting in Proposition 3. The only difference is that the constants hidden under $\lesssim$ additionally depends upon the parameter t_{mix}.

4.2 GAR for Polyak-Ruppert averaged TTSA

To proceed with Gaussian approximation for Polyak-Ruppert averaging, we use the decomposition (19) to transform the linear statistic $\sum_{k=1}^{n}\psi_{k+1}$ to a sum of martingale-increments. This transformation is done through the Poisson equation, see (Douc et al. 2018, Chapter 21). Under A 6, function $\psi(x) = \boldsymbol{\varepsilon}_V(x) - A_{12}A_{22}^{-1}\boldsymbol{\varepsilon}_W(x)$ is a.s. bounded, which implies that there exists a function $\boldsymbol{g}^\psi : \mathsf{X} \to \mathbb{R}^{d_\theta}$, such that

$$\boldsymbol{g}^\psi(x) - \mathrm{P}\boldsymbol{g}^\psi(x) = \psi(x) \,.$$

We set $\boldsymbol{g}^\psi_{k+1} := \boldsymbol{g}^\psi(X_{k+1})$ and define

$$\mathrm{M}_k = \boldsymbol{g}^\psi_{k+1} - \mathrm{P}\boldsymbol{g}^\psi_k \,,$$

which form a martingale-increment w.r.t. $\mathcal{F}_k$. Then we can rewrite (19) as

$$\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star) = \tfrac{1}{\sqrt{n}}\textstyle\sum_{k=1}^{n}\mathrm{M}_k + R^{\mathrm{pr,m}}_n \,, \qquad (24)$$

where $R^{\mathrm{pr,m}}_n$ is a residual term defined in the supplement. Under B 1 there exists a matrix $\boldsymbol{\Sigma}^{\mathrm{mark}}_\infty \in \mathbb{R}^{d_\theta \times d_\theta}$ such that

$$n^{-1/2}\textstyle\sum_{k=1}^{n}\{\psi_{k+1} - \pi(\psi)\} \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Sigma}^{\mathrm{mark}}_\infty) \,. \qquad (25)$$

Due to (Douc et al. 2018, Theorem 21.2.5), we get that

$$\mathrm{Var}[\mathrm{M}_k] = \boldsymbol{\Sigma}^{\mathrm{mark}}_\infty \,.$$

Now we state the counterpart to Theorem 1:

Theorem 3. *Assume A4, A6, B 1, B 2($\log n$). Then it holds that*

$$\rho^{\mathrm{Conv}}(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, \boldsymbol{\Sigma}^{mark}_\infty)) \qquad (26)$$
$$\lesssim_{\log n} \frac{1}{n^{1/4}} + \frac{1}{n^{(1-b)/2}} + \frac{1}{n^{a-\frac{1}{2}}} + \sqrt{n}\prod_{j=0}^{n-1}(1 - \frac{a_\Delta \beta_j}{16}) \,.$$

Proof sketch. The proof of Theorem 3 consists of two main parts. First, we derive a Gaussian approximation rate for the linear statistic $\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathrm{M}_k$ using an appropriate martingale CLT. It is especially non-trivial, since $\mathbb{E}^{\mathcal{F}_k}\left[\mathrm{M}_k\{\mathrm{M}_k\}^\top\right]$ is not constant. We circumvent this problem using an appropriate modification of the argument due to (Fan 2019). Next, we estimate the moments of $R^{\mathrm{pr,m}}_n$ using the techniques established in Proposition 3 for $\tilde{\theta}^{(0)}_k, \tilde{w}^{(0)}_k$ and then combining this with a separate bounds for the Markov part $\tilde{\theta}^{(1)}_k, \tilde{w}^{(1)}_k$.

Discussion. It is easy to see that, given that $b > a$, the right-hand side of (26) is optimized when setting $a = 2/3$ and $b = 2/3 + 1/(\log n)$. This yields the final rate of order $n^{-1/6}$ up to logarithmic factors:

$$\rho^{\mathrm{Conv}}(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, \boldsymbol{\Sigma}^{\mathrm{mark}}_\infty)) \lesssim_{\log n} n^{-1/6} \,. \qquad (27)$$

To the best of our knowledge, (27) provides the first result concerning the Gaussian approximation rates for the TTSA problems with Markov noise. The suggested step size schedule mimics the one predicted by Theorem 1 and essentially reduces the TTSA scheme to a single-timescale one.

4.3 GAR for last iterate of TTSA

We start this section by introducing a counterpart to (22) based on the idea of the decomposition (24) for Polyak-Ruppert averaging:

$$\beta_n^{-1/2}\tilde{\theta}_{n+1} = -\textstyle\sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}\mathrm{M}_j + R^{\mathrm{last,m}}_n \,, \qquad (28)$$

where $R^{\mathrm{last,m}}_n$ is a residual term that is given in the supplement paper. Note that the leading term in representation (28) is martingale difference sequence. Now we define

$$\boldsymbol{\Sigma}^{\mathrm{last,m}}_n = \mathrm{Var}\big[\textstyle\sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}\mathrm{M}_j\big] \,.$$

It is known that $\beta_n^{-1}\boldsymbol{\Sigma}^{\mathrm{last,m}}_n$ converges to a fixed matrix $\boldsymbol{\Sigma}^{\mathrm{last,m}}_\infty$ which is a solution of the Ricatti equation

$$\boldsymbol{\Sigma}^{\mathrm{last,m}}_\infty = \beta_0(\Delta\boldsymbol{\Sigma}^{\mathrm{last,m}}_\infty + \boldsymbol{\Sigma}^{\mathrm{last,m}}_\infty\Delta^\top - \boldsymbol{\Sigma}^{\mathrm{mark}}_\infty) \,,$$

where $\boldsymbol{\Sigma}^{\mathrm{mark}}_\infty$ is defined in (25). Moreover, the convergence rate is proportional to β_n, i.e.

$$\|\beta_n^{-1}\boldsymbol{\Sigma}^{\mathrm{last,m}}_n - \boldsymbol{\Sigma}^{\mathrm{last,m}}_\infty\| \lesssim n^{-b} \,.$$

The proof of the above result is given in the supplement paper. Now we formulate a counterpart to A7:

B 3. *Step size exponents a, b satisfy $2b > 1 + a$. Moreover, assume that the total number of iterations n satisfies $n^b \geq C_{B\,3}$, where $C_{B\,3}$ does not depend on a, b, and can be traced from the supplement paper.*

Theorem 4. *Assume A4, A6, B 1, B 2($\log n$), B 3. Then it holds that*

$$\rho^{\mathrm{Conv}}(\beta_n^{-1/2}\tilde{\theta}_{n+1}, \mathcal{N}(0, \boldsymbol{\Sigma}^{\mathrm{last,m}}_\infty)) \qquad (29)$$
$$\lesssim_{\log n} n^{b/2}\prod_{j=0}^{n}(1 - \frac{a_\Delta}{8}\beta_j) + \frac{1}{n^{\frac{b}{2}-\frac{1}{4}}} + \frac{1}{n^{a-\frac{b}{2}}} + \frac{1}{n^{\frac{3b-a-2}{2}}} \,.$$

Proof sketch. The proof of Theorem 4 uses the same machinery of Gaussian approximation for non-linear statistics based on representation (28). In this setting control of the moments of the term $R_n^{\mathrm{last,m}}$ is a delicated problem, which requires the additional constraint $2b > 1 + a$ imposed in B 3.

Discussion. It is easy to see that, given that $b \geq a$, the right-hand side of (29) is optimized when setting $a = 2/3$ and $b = 1 - 1/(\log n)$, and yields the final rate in terms of n of order up to $n^{-1/6}$ up to logarithmic factors:

$$\rho^{\mathrm{Conv}}(\beta_n^{-1/2}\tilde{\theta}_{n+1}, \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{last,m}})) \lesssim_{\log n} n^{-1/6} .$$

This rate, to the best of our knowledge, is the first one obtained for the last iterate of TTSA with Markov noise.

5 Applications to TDC and GTD

In this section, we show that the results derived in Section 3 and Section 4 apply to the Gradient Temporal Difference (GTD) (Sutton, Maei, and Szepesvári 2008) and Temporal Difference with Gradient Correction (TDC) (Sutton et al. 2009) methods. These methods address the problem of classical TD learning, which is based on single-timescale stochastic approximation and is known to fail in off-policy RL settings where data are drawn from a *behavior policy* different from the target policy (Baird 1995; Tsitsiklis and Van Roy 1997). We consider a discounted MDP (Markov Decision Process) given by a tuple $(\mathcal{S}, \mathcal{A}, \mathrm{P}, r, \lambda)$. Here $\mathcal{S}$ and $\mathcal{A}$ denote state and action spaces, which are assumed to be complete separable metric spaces with their Borel σ-algebras $\mathcal{B}(\mathcal{S})$ and $\mathcal{B}(\mathcal{A})$, and $\lambda \in (0, 1)$ is a discount factor. Let $\mathrm{P}(\cdot|s, a)$ be a state-action transition kernel, which determines the probability of moving from (s, a) to a set $B \in \mathcal{B}(\mathcal{S})$. Reward function $r \colon \mathcal{S} \times \mathcal{A} \to [0, 1]$ is deterministic. A *Policy* $\pi(\cdot|s)$ is the distribution over action space $\mathcal{A}$ corresponding to agent's action preferences in state $s \in \mathcal{S}$. We aim to estimate *value function*

$$V^\pi(s) = \mathbb{E}\big[\textstyle\sum_{k=0}^\infty \lambda^k r(S_k, A_k)|S_0 = s\big] ,$$

where $A_k \sim \pi(\cdot|s_k)$, and $S_{k+1} \sim \mathrm{P}(\cdot|S_k, A_k)$. Define the transition kernel under π,

$$\mathrm{P}_\pi(B|s) = \textstyle\int_{\mathcal{A}} \mathrm{P}(B|s, a)\pi(\mathrm{d}a|s) . \tag{30}$$

We consider the *linear function approximation* for $V^\pi(s)$, defined for $s \in \mathcal{S}$, $\theta \in \mathbb{R}^d$, and a feature mapping $\varphi \colon \mathcal{S} \to \mathbb{R}^d$ as $V_\theta^\pi(s) = \varphi^\top(s)\theta$. Our goal is to find a parameter $\theta^\star$, which defines the best linear approximation to V^π. We denote by μ the invariant distribution over the state space $\mathcal{S}$ induced by $\mathrm{P}^\pi(\cdot|s)$ in (30). Consider the following assumptions on the generative mechanism and on the feature mapping $\varphi(\cdot)$:

TD 1. *Tuples (s_k, a_k, s'_k) are generated i.i.d.with $s_k \sim \mu$, $a_k \sim \pi(\cdot|s_k)$, $s'_k \sim \mathrm{P}(\cdot|s_k, a_k)$.*

TD 2. *Feature mapping $\varphi(\cdot)$ satisfies $\sup_{s\in\mathcal{S}} \|\varphi(s)\| \leq 1$.*

As an alternative to the generative model setting **TD** 1, our analysis covers the Markov noise setting:

TD 3. *Suppose that we obtain a Markovian sample trajectory $\{(s_k, a_k, r_k)\}_{k=0}^\infty$ which is generated when a stationary behavior policy π is employed. Assume that the Markov kernel P_π admits a unique invariant distribution μ and is uniformly geometrically ergodic, that is, there exist $t_{\mathrm{mix}} \in \mathbb{N}$, such that for any $k \in \mathbb{N}$, it holds that*

$$\sup_{s,s'\in\mathcal{S}} \mathsf{d}_{\mathrm{tv}}(\mathrm{P}_\pi^k(\cdot|s), \mathrm{P}_\pi^k(\cdot|s')) \leq (1/4)^{\lceil k/t_{\mathrm{mix}}\rceil} .$$

We introduce the k-th step TD error for the linear setting:

$$\delta_k = r_k + \lambda\theta_k^\top\varphi_{k+1} - \theta_k^\top\varphi_k ,$$

where we have defined

$$\varphi_k = \varphi(s_k) , \quad r_k = r(s_k, a_k) .$$

Generalized Temporal Difference learning. The GTD(0) algorithm is defined by the following recurrence for $k \geq 1$:

$$\begin{cases} \theta_{k+1} = \theta_k + \beta_k(\varphi_k - \lambda\varphi_{k+1})(\varphi_k)^\top w_k , \quad \theta_0 \in \mathbb{R}^d , \\ w_{k+1} = w_k + \gamma_k(\delta_k\varphi_k - w_k) , \quad w_0 = 0 . \end{cases} \tag{31}$$

It is clear that the GTD(0) recurrence (31) is a particular case of the linear TTSA given in (6)-(7).

Temporal-difference learning with gradient correction. The TDC algorithm employs dual updates for the primary parameter vector θ_k and the auxiliary weight vector w_k. Its update rule is given by

$$\begin{cases} \theta_{k+1} = \theta_k + \beta_k\delta_k\varphi_k - \beta_k\lambda\varphi_{k+1}(\varphi_k^\top w_k) , \\ w_{k+1} = w_k + \gamma_k(\delta_k - \varphi_k^\top w_k)\varphi_k . \end{cases} \tag{32}$$

It is possible to check that both updates schemes (31) and (32) satisfy the general assumptions A 1-A 4 and A 6 under **TD** 1 and **TD** 2. Similar, B 1 holds under **TD** 3. Thus, all the results from Section 3 and Section 4 applies for both algorithms. We provide details in the supplemental paper.

6 Conclusion

In this paper, we provided, to the best of our knowledge, the first rate of normal approximation for the last iterate and Polyak-Ruppert averaged TTSA iterates in a sense of convex distance, covering both the martingale-difference and Markov noise settings. A natural further research direction is to consider the problem of constructing confidence intervals for the TTSA solution $(\theta^\star, w^\star)$ based on bootstrap approach or asymptotic covariance matrix estimation, and perform a fully non-asymptotic analysis of the suggested procedure. Another important direction is the construction of lower bounds to ensure tightness of the rates obtained in Theorem 1-4.

Acknowledgment

The work was supported by the grant for research centers in the field of AI provided by the Ministry of Economic Development of the Russian Federation in accordance with the agreement 000000C313925P4E0002 and the agreement with HSE University № 139-15-2025-009.

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A Martingale limit theorems

Let $\{X_k\}_{k=1}^n$ be a martingale difference process in $\mathbb{R}^d$ with respect to the natural filtration $\{\mathcal{F}_k\}_{k=0}^n$, $\mathcal{F}_k = \sigma(X_s : s \leq k)$. From now on we introduce the following expressions

$$V_k = \sum_{j=1}^k \mathbb{E}^{\mathcal{F}_{j-1}}[X_j X_j^\top] \; , \; \Sigma_k = \frac{1}{k}\sum_{j=1}^k \mathbb{E}[X_j X_j^\top] \; , \tag{33}$$

where $\mathbb{E}^{\mathcal{F}}[\cdot]$ stands for the conditional expectation w.r.t. a sigma-algebra $\mathcal{F}$.

In order to derive a modification of (Wu, Wei, and Rinaldo 2025, Corollary 2) we first state (Nourdin, Peccati, and Yang 2022, Proposition A.1) that controls convex distance in terms of Wasserstein one:

Lemma 1 (Proposition A.1 in (Nourdin, Peccati, and Yang 2022))**.** *Fix $d \geq 1$, and let $\eta \sim \mathcal{N}(0, \Sigma)$ denote a d-dimensional centered Gaussian vector with invertible covariance matrix Σ. Then, for any d-dimensional random vector F one has that*

$$\rho^{\mathrm{Conv}}(F, \eta) \lesssim \Gamma(\Sigma)^{1/2} d_W(F, \eta)^{1/2} \; ,$$

where $d_W(\cdot, \cdot)$ stands for the Wasserstein distance and $\Gamma(\Sigma)$ is the isoperimetric constant defined by

$$\Gamma(\Sigma) := \sup_{Q \in \mathrm{Conv}(\mathbb{R}^d), \varepsilon > 0} \frac{\mathbb{P}(\eta \in Q^\varepsilon) - \mathbb{P}(\eta \in Q)}{\varepsilon} \; ,$$

where Q^ε indicates the set of all elements of $\mathbb{R}^d$ whose Euclidean distance from Q does not exceed ε.

Remark 1. *Following (Nourdin, Peccati, and Yang 2022, Remark A.2) one can check that for the absolute constants $0 < c < C < \infty$ it holds that*

$$c\sqrt{\|\Sigma\|_{Fr}} \leq \Gamma(\Sigma) \leq C\sqrt{\|\Sigma\|_{Fr}} \; ,$$

where $\|\cdot\|_{Fr}$ stands for the Frobenius norm.

Now we give a slight modification of the result proven in (Wu, Wei, and Rinaldo 2025) that can be obtained applying Lemma 1:

Lemma 2 (modified Corollary 2 in (Wu, Wei, and Rinaldo 2025))**.** *Let $\{X_k\}_{k=1}^n$ be a martingale difference process in with respect to the filtration $\{\mathcal{F}_k\}_{k=0}^n$. Assume that*

$$V_n = n\Sigma_n \text{ a.s. } ,$$

and for any $1 \leq k \leq n$, $\mathbf{A} \in \mathbb{R}^{d \times d}$ it holds that

$$\mathbb{E}^{\mathcal{F}_{k-1}}[\|\mathbf{A}X_k\|^2 \|X_k\|] \leq M \mathbb{E}^{\mathcal{F}_{k-1}}[\|\mathbf{A}X_k\|^2] \; .$$

Then for every $\Sigma \in \mathbb{S}_+^d$ it can be guaranteed that

$$\rho^{\mathrm{Conv}}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^n X_k, \mathcal{N}(0, \Sigma_n)\right) \lesssim \Gamma(\Sigma_n)^{1/2}[1 + M(2 + \log(dn\|\Sigma_n\|))^+]^{1/2} \frac{\sqrt{d\log n}}{n^{1/4}} + \Gamma(\Sigma_n)^{1/2} \frac{\{\mathrm{Tr}(\Sigma_n)\}^{1/4}}{n^{1/4}} \; .$$

where Σ_n is defined in (33)*.*

The following lemma states the upper bound for the convex distance for any bounded martingale difference sequence:

Lemma 3. *Let $0 < \kappa < \infty$ and $\{X_k\}_{k=1}^n$ be a martingale difference process in $\mathbb{R}^d$ with respect to the filtration $\{\mathcal{F}_k\}_{k=0}^n$, $\mathcal{F}_k = \sigma(X_s : s \leq k)$. Assume that*

$$\max_{1 \leq i \leq n} \|X_i\| \leq \kappa \text{ almost surely } ,$$

and there exist constants $C_1, C_2 > 0$ such that for all $t > 0$ it holds that:

$$\mathbb{P}[\|V_n - n\Sigma_n\| \geq nt] \leq C_1 \exp\bigl(-C_2 nt^2\bigr) \; , \tag{34}$$

where V_n, Σ_n are given in (33)*. Then for any $p \geq 1$ it holds that*

$$\begin{aligned}
&\rho^{\mathrm{Conv}}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^n X_k, \mathcal{N}(0, \Sigma_n)\right) \lesssim \\
&\quad \Gamma(\Sigma_n)^{1/2}\left\{1 + \kappa\left(2 + \log\frac{nd^2p^{1/2}\|\Sigma_n\|(2\kappa^2 + \|\Sigma_n\| + C_2^{-1/2})}{\kappa^2}\right)^+\right\}^{1/2} \frac{\sqrt{d\log\bigl(ndp^{1/2}\frac{2\kappa^2 + \|\Sigma_n\| + C_2^{-1/2}}{\kappa^2}\bigr)}}{n^{1/4}} \\
&\quad + \Gamma(\Sigma_n)^{1/2}\frac{\kappa^{1/2}}{n^{1/4}} + p^{3/4} d \frac{c_d^{\frac{2p}{2p+1}}}{\|n\Sigma_n\|^{\frac{2p}{2p+1}}}\left\{\left(\frac{n\log n}{C_2}\right)^{\frac{p}{2(2p+1)}} + C_1^{\frac{1}{2p+1}}\left(\frac{2\kappa^2 + \|\Sigma_n\| + (\frac{\log n}{nC_2})^{1/2}}{\kappa}\right)^{\frac{2p}{2p+1}}\right\} ,
\end{aligned}$$

where $\Gamma(\cdot)$ is introduced in Lemma 1, c_d is defined in Proposition 1.

Proof. We adapt the arguments from (Fan 2019) and (Wu, Wei, and Rinaldo 2025, p. 35) for the multidimensional case. Consider the following stopping time

$$\tau = \max\{0 \le k \le n : V_k \preceq n\Sigma_n + tn\mathrm{I}\} ,$$

where $t \in \mathbb{R}_+$ is a parameter we will choose later. Now introduce

$$m = \left\lceil \frac{1}{\kappa^2} \operatorname{Tr}(n\Sigma_n + tn\mathrm{I} - V_\tau) \right\rceil , \; N = \left\lceil \frac{\operatorname{Tr}(n\Sigma_n + tn\mathrm{I})}{\kappa^2} \right\rceil + n .$$

For the further analyses, we observe that

$$n \le N \le nd \frac{2\kappa^2 + \|\Sigma_n\| + t}{\kappa^2} .$$

Our goal is to construct the sequence $\{X'_i\}_{i=1}^N$ that has a constant quadratic characteristic equal to $n\Sigma_n$. To proceed, consider the spectral decomposition of $n\Sigma_n - V_\tau$:

$$n\Sigma_n - V_\tau = \sum_{j=1}^d \lambda_j u_j u_j^\top .$$

Now we set for $i \in \{1, 2, \ldots, N\}$

$$X'_i = \begin{cases} X_i , & 1 \le i \le \tau , \\ \frac{1}{\sqrt{m}} \sum_{j=1}^d (\lambda_j)^{1/2} \varepsilon_{ij} u_j , & \tau + 1 \le i \le \tau + m , \\ 0 , & \tau + m + 1 \le i \le N , \end{cases}$$

where ε_{ij} are i.i.d. Rademacher random variables. For the natural fitration $\mathcal{F}'_i = \sigma(X'_s : s \le i)$ one can check that $\mathbb{E}^{\mathcal{F}'_{i-1}}[X'_i] = 0$ almost surely. Moreover, $\|X'_i\| \le \kappa$ and $\mathbb{E}^{\mathcal{F}_{i-1}}[X'_i X_i'^\top] = \frac{1}{m}(n\Sigma_n - V_\tau)$ a.s. by the definition of κ and m. Thus, we obtain by the construction

$$\sum_{i=1}^N \mathbb{E}^{\mathcal{F}_{i-1}}[X'_i X_i'^\top] = V_\tau + m \cdot \frac{1}{m}(n\Sigma_n - V_\tau) = n\Sigma_n .$$

Now we apply Proposition 1 and get

$$\rho^{\mathrm{Conv}}\big(\tfrac{1}{\sqrt{n}} S_n,\ \mathcal{N}(0, \Sigma_n)\big) \le \rho^{\mathrm{Conv}}\big(\tfrac{1}{\sqrt{N}} S'_N,\ \mathcal{N}(0, \Sigma_n)\big) + 2c_d^{2p/(2p+1)} \|n\Sigma_n\|^{-\frac{2p}{2p+1}} \big(\mathbb{E}\big[\|S_n - S'_N\|^{2p}\big]\big)^{1/(2p+1)} ,$$

where we have set

$$S_n = \sum_{j=1}^n X_j , \; S'_N = \sum_{j=1}^N X'_j .$$

Since X' satisfies the assumptions of Lemma 2 with $M := \kappa$ and, moreover, $\operatorname{Tr}(\Sigma_n) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i^\top X_i] \le \kappa^2$, we get using Lemma 2:

$$\begin{aligned} \rho^{\mathrm{Conv}}(\tfrac{1}{\sqrt{N}} S'_N, \mathcal{N}(0, \Sigma_n)) &\lesssim \Gamma(\Sigma_n)^{1/2} \bigg\{ 1 + \kappa \bigg(2 + \log(dN\|\Sigma_n\|) \bigg)^+ \bigg\}^{1/2} \frac{\sqrt{d \log N}}{N^{1/4}} + \Gamma(\Sigma_n)^{1/2} \frac{\kappa^{1/2}}{N^{1/4}} , \\ &\lesssim \Gamma(\Sigma_n)^{1/2} \bigg\{ 1 + \kappa \bigg(2 + \log \frac{nd^2 \|\Sigma_n\| (2\kappa^2 + \|\Sigma_n\| + t)}{\kappa^2} \bigg)^+ \bigg\}^{1/2} \frac{\sqrt{d \log(nd \frac{2\kappa^2 + \|\Sigma_n\| + t}{\kappa^2})}}{n^{1/4}} + \Gamma(\Sigma_n)^{1/2} \frac{\kappa^{1/2}}{n^{1/4}} , \end{aligned}$$

To control the moments of $S_n - S'_N$ we consider two events:

$$\Omega_1 = \{\omega : \|V_n - n\Sigma_n\| \le tn\} , \quad \Omega_2 = \{\omega : \|V_n - n\Sigma_n\| > tn\} .$$

Hence,

$$\mathbb{E}[\|S_n - S'_N\|^{2p}] = \underbrace{\mathbb{E}[\|S_n - S'_N\|^{2p} \mathbf{1}\{\Omega_1\}]}_{\mathcal{T}_1} + \underbrace{\mathbb{E}[\|S_n - S'_N\|^{2p} \mathbf{1}\{\Omega_2\}]}_{\mathcal{T}_2} . \tag{35}$$

Now we bound $\mathcal{T}_1$, $\mathcal{T}_2$ separately.

Bound for $\mathcal{T}_1$. On the event Ω_1 it holds that $V_n \preceq n\Sigma_n + tn\mathrm{I}$. Thus, $\tau = n$ and

$$S_n - S'_N = \frac{1}{\sqrt{m}} \sum_{i=n+1}^{n+m} \sum_{j=1}^{d} (\lambda_j)^{1/2} \varepsilon_{ij} u_j = \frac{1}{\sqrt{m}} \sum_{j=1}^{d} \Big(\sum_{i=n+1}^{n+m} \varepsilon_{ij} \Big) (\lambda_j)^{1/2} u_j \ .$$

Note that $\|V_n - n\Sigma_n\| \leq tn$ yields $\lambda_j \leq tn$ for any j. Therefore, we obtain

$$\|S_n - S'_N\|^2 \leq \frac{tn}{m} \sum_{j=1}^{d} \Big(\sum_{i=n+1}^{n+m} \varepsilon_{ij} \Big)^2 \ .$$

Thus, applying Minkowski's inequality combined with Khintchine inequality (Vershynin 2018, Theorem 2.7.5), we get

$$\mathcal{T}_1 \leq \mathbb{E}\Big[\Big| \frac{tn}{m} \sum_{j=1}^{d} \Big(\sum_{i=n+1}^{n+m} \varepsilon_{ij} \Big)^2 \Big|^p \Big] = \mathbb{E}\Big[\frac{(tn)^p}{m^p} \mathbb{E}^{\mathcal{F}_n} \Big[\Big| \sum_{j=1}^{d} \Big(\sum_{i=n+1}^{n+m} \varepsilon_{ij} \Big)^2 \Big|^p \Big] \Big] \leq (dtn)^p \mathbb{E}\Big[\frac{1}{m^p} \mathbb{E}^{\mathcal{F}_n} \Big[\Big| \sum_{i=n+1}^{n+m} \varepsilon_{i1} \Big|^{2p} \Big] \Big] \lesssim (2tnpd)^p \ .$$

Bound for $\mathcal{T}_2$. First, we use (34) and get

$$\mathbb{P}[\Omega_2] \leq C_1 \exp\big(-C_2 n t^2\big) \ .$$

Note that

$$\|S_n - S'_N\| \leq 2N\kappa \leq 2nd \frac{2\kappa^2 + \|\Sigma_n\| + t}{\kappa} \ .$$

Thus, we obtain that

$$\mathcal{T}_2 \leq 2^{2p} d^{2p} n^{2p} \Big(\frac{2\kappa^2 + \|\Sigma_n\| + t}{\kappa} \Big)^{2p} \mathbb{P}[\Omega_2] \leq C_1 2^{2p} d^{2p} \Big(\frac{2\kappa^2 + \|\Sigma_n\| + t}{\kappa} \Big)^{2p} n^{2p} \exp\big(-C_2 n t^2\big) \ .$$

Choose $t = (\frac{2p \log n}{n C_2})^{1/2}$. Thus,

$$\mathcal{T}_2 \leq C_1 (2d)^{2p} (2p)^p \Big(\frac{2\kappa^2 + \|\Sigma_n\| + (\frac{\log n}{n C_2})^{1/2}}{\kappa} \Big)^{2p} \ ,$$

and

$$\mathcal{T}_1 \lesssim (8 C_2^{-1} p^3 d^2 n \log n)^{p/2} \ .$$

Now we substitute the latter inequalities into (35) and get applying Minkowski's inequality:

$$\mathbb{E}^{\frac{1}{2p+1}} [\|S_n - S'_N\|^{2p}] \lesssim p^{3/4} d \Big(\Big(\frac{n \log n}{C_2} \Big)^{\frac{p}{2(2p+1)}} + C_1^{\frac{1}{2p+1}} \Big(\frac{2\kappa^2 + \|\Sigma_n\| + (\frac{\log n}{n C_2})^{1/2}}{\kappa} \Big)^{\frac{2p}{2p+1}} \Big) \ ,$$

and the proof follows. □

B High-order bounds on the error moments

We follow the decoupling idea of (Konda and Tsitsiklis 2004) and perform the change of variables in the recurrence (6)-(7), which is similar to the Gaussian elimination. Using Proposition 2, we obtain, with $\tilde{\theta}_k$ and $\tilde{w}_k$ defined in (12), that the two-timescale SA (6)-(7) reduces to the system of updates:

$$\begin{cases} \tilde{\theta}_{k+1} &= (\mathrm{I} - \beta_k B_{11}^k) \tilde{\theta}_k - \beta_k A_{12} \tilde{w}_k - \beta_k V_{k+1} \ , \\ \tilde{w}_{k+1} &= (\mathrm{I} - \gamma_k B_{22}^k) \tilde{w}_k - \beta_k D_k V_{k+1} - \gamma_k W_{k+1} \ . \end{cases}$$

Recall that the sequence of matrices D_k has a form

$$D_k = L_{k+1} + A_{22}^{-1} A_{21} \ ,$$

where L_k are defined in (11). The following proposition shows that norms of matrices D_k are bounded. Moreover, L_k converges to 0 under A5(2). This result is due to (Kaledin et al. 2020).

Lemma 4 (Lemma 19 in (Kaledin et al. 2020)). *Assume A4 and A5(2). Then for any $k \in \mathbb{N}$,*

$$\|L_k\| \le \ell_\infty \frac{\beta_k}{\gamma_k} \;, \quad \|D_k\| \le c_\infty \;,$$

where the value of the constant ℓ_∞ can be found in (Kaledin et al. 2020) and c_∞ has form

$$c_\infty = \ell_\infty r_{\mathrm{step}} + \|A_{22}^{-1}A_{21}\| \;, \quad r_{\mathrm{step}} = c_{0,\beta}/c_{0,\gamma} \;. \tag{36}$$

Let us note the important properties of our steps. Since $a < b$, the ratio β_i/γ_i decreases as i increases, hence $\beta_i/\gamma_i \le \beta_0/\gamma_0$ for all $i \in \mathbb{N}$. Moreover, $k_0^{a-b} < 1$, therefore $\beta_0/\gamma_0 \le c_{0,\beta}/c_{0,\gamma} = r_{\mathrm{step}}$. To proceed with the p-th moment bounds for $\tilde{w}_{k+1}$ and $\tilde{\theta}_{k+1}$, we introduce the random vectors

$$\xi_{k+1} = \gamma_k W_{k+1} + \beta_k D_k V_{k+1} \;.$$

Our next lemma allows to bound moments of $V_{k+1}, W_{k+1}, \xi_{k+1}$ in terms of $M_{k,p}^{\tilde{w}}$ and $M_{k,p}^{\tilde{\theta}}$ introduced in (14).

Lemma 5. *Assume A2(p), A4, and A5(2). Then it holds that*

$$\begin{aligned}
\mathbb{E}^{1/p}[\|V_{k+1}\|^p] &\le \widetilde{m}_V(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}) \;, \\
\mathbb{E}^{1/p}[\|W_{k+1}\|^p] &\le \widetilde{m}_W(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}) \;, \\
\mathbb{E}^{1/p}[\|\xi_{k+1}\|^p] &\le \widetilde{m}\gamma_k\big(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}\big) \;,
\end{aligned}$$

where we have defined

$$\widetilde{m}_V = m_V(1 + c_\infty) \;, \quad \widetilde{m}_W = m_W(1 + c_\infty) \;, \quad \widetilde{m} = r_{\mathrm{step}}\widetilde{m}_V c_\infty + \widetilde{m}_W \;. \tag{37}$$

and r_{step} is defined in (36).

Proof. Since $w_k - w^* = \tilde{w}_k - D_{k-1}\tilde{\theta}_k$, we get applying Lemma 4:

$$\mathbb{E}^{1/p}[\|w_k - w^\star\|^p] \le \mathbb{E}^{1/p}[\|\tilde{w}_k\|^p] + c_\infty \mathbb{E}^{1/p}[\|\tilde{\theta}_k\|] \;.$$

Combining the above bound with A2(p), we obtain

$$\mathbb{E}^{1/p}[\|V_{k+1}\|^p] \le m_V(1 + \mathbb{E}^{1/p}[\|\theta_k - \theta^\star\|^p] + \mathbb{E}^{1/p}[\|w_k - w^\star\|^p]) \le \widetilde{m}_V(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}) \;,$$

and

$$\mathbb{E}^{1/p}[\|W_{k+1}\|^p] \le \widetilde{m}_W(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}) \;.$$

Similarly,

$$\begin{aligned}
\mathbb{E}^{1/p}[\|\xi_{k+1}\|^p] &\le \gamma_k \mathbb{E}^{1/p}[\|W_{k+1}\|^p] + \beta_k c_\infty \mathbb{E}^{1/p}[\|V_{k+1}\|^p] \\
&\le \gamma_k \widetilde{m}_W\big(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}\big) + \beta_k \widetilde{m}_V c_\infty\big(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}\big) \\
&\le \widetilde{m}\gamma_k\big(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}\big),
\end{aligned}$$

where $\widetilde{m}$ is defined in (37). □

B.1 Bounding the products of deterministic matrices

Now we state and prove the results regarding the stability of matrix products. The key element of the proof is the Hurwitz stability assumption A4. Below we state and prove the Lyapunov stability lemma:

Lemma 6. *Let $-A$ be a Hurwitz matrix. Then there exists a unique matrix $Q = Q^\top \succ 0$, satisfying the Lyapunov equation $A^\top Q + QA = \mathrm{I}$. Moreover, setting*

$$a = \tfrac{1}{2\|Q\|} \;, \quad \textit{and} \quad \alpha_\infty = \tfrac{1}{2\|Q\|\|A\|_Q^2} \;,$$

it holds for any $\alpha \in [0, \alpha_\infty]$ that

$$\|\mathrm{I} - \alpha A\|_Q^2 \le 1 - \alpha a \;.$$

Proof. The fact that there exists a unique matrix Q, such that the following Lyapunov equation holds:

$$A^\top Q + QA = \mathrm{I} \;,$$

follows directly from (Poznyak 2008, Lemma 9.1, p. 140). In order to show the second part of the statement, we note that for any non-zero vector $x \in \mathbb{R}^d$, we have

$$\begin{aligned}\frac{x^\top(\mathrm{I}-\alpha A)^\top Q(\mathrm{I}-\alpha A)x}{x^\top Q x} &= 1 - \alpha\frac{x^\top(A^\top Q + QA)x}{x^\top Q x} + \alpha^2\frac{x^\top A^\top Q A x}{x^\top Q x} \\ &= 1 - \alpha\frac{x^\top x}{x^\top Q x} + \alpha^2\,\frac{x^\top A^\top Q A x}{x^\top Q x} \\ &\le 1 - \frac{\alpha}{\|Q\|} + \alpha^2\,\|A\|_Q^2 \\ &\le 1 - \alpha a \;,\end{aligned}$$

where used the fact that $\alpha \le \alpha_\infty$. □

Note that Lemma 6 implies the existence of matrices Q_{22} and Q_Δ, such that

$$A_{22}^\top Q_{22} + Q_{22}A_{22} = \mathrm{I}, \quad Q_\Delta \Delta + \Delta^\top Q_\Delta = \mathrm{I} \;.$$

This ensures the contraction in the respective matrix Q-norm: provided that $\gamma_k \in [0, 1/(2\|A_{22}\|_{Q_{22}}^2\|Q_{22}\|)], \beta_k \in [0, 1/(2\|A_\Delta\|_{Q_\Delta}^2\|Q_\Delta\|)]$, it holds, that

$$\begin{aligned}\|\mathrm{I}-\gamma_k A_{22}\|_{Q_{22}} &\le 1 - a_{22}\gamma_k, \quad a_{22} := \tfrac{1}{2\|Q_{22}\|} \;, \\ \|\mathrm{I}-\beta_k \Delta\|_{Q_\Delta} &\le 1 - a_\Delta\beta_k, \quad a_\Delta := \tfrac{1}{2\|Q_\Delta\|} \;.\end{aligned}$$

We now define a few constants related to the matrices Q_Δ, Q_{22}:

$$\kappa_\Delta := \frac{\lambda_{\mathsf{max}}(Q_\Delta)}{\lambda_{\mathsf{min}}(Q_\Delta)}, \quad \kappa_{22} := \frac{\lambda_{\mathsf{max}}(Q_{22})}{\lambda_{\mathsf{min}}(Q_{22})}.$$

Next we show that the factors $\mathrm{I} - \beta_k B_{11}^k$ and $\mathrm{I} - \gamma_k B_{22}^k$ in the transformed recursion (13) are also contractive in the same matrix norms induced by Q_Δ and Q_{22}, respectively.

Lemma 7. *Assume A4 and A5(2). Then it holds that*

$$\|\mathrm{I}-\beta_k B_{11}^k\|_{Q_\Delta} \le 1 - (1/2)\beta_k a_\Delta \;, \quad \|\mathrm{I}-\gamma_k B_{22}^k\|_{Q_{22}} \le 1 - (1/2)\gamma_k a_{22} \;. \tag{38}$$

Proof. Using (10), we observe that

$$\begin{aligned}\|\mathrm{I}-\beta_k B_{11}^k\|_{Q_\Delta} &= \|\mathrm{I}-\beta_k\Delta + \beta_k A_{12}L_k\|_{Q_\Delta} \le \|\mathrm{I}-\beta_k\Delta\|_{Q_\Delta} + \beta_k\|A_{12}L_k\|_{Q_\Delta} \\ &\le (1-\beta_k a_\Delta) + \beta_k\sqrt{\kappa_\Delta}\|A_{12}\|\,\|L_k\| \le (1-\beta_k a_\Delta) + \beta_k\sqrt{\kappa_\Delta}\|A_{12}\| r_{\mathrm{step}}\ell_\infty\end{aligned}$$

Using $r_{\mathrm{step}} \le a_\Delta/(2\|A_{12}\|\sqrt{\kappa_\Delta}\ell_\infty)$, the above inequality yields the first part of (38). Similarly, using (10), we get that

$$\begin{aligned}\|\mathrm{I}-\gamma_k B_{22}^k\|_{Q_{22}} = \|\mathrm{I}-\gamma_k A_{22} - \beta_k D_k A_{12}\|_{Q_{22}} &\le \|\mathrm{I}-\gamma_k A_{22}\|_{Q_{22}} + r_{\mathrm{step}}\gamma_k\sqrt{\kappa_{22}}c_\infty\|A_{12}\| \\ &\le (1-\gamma_k a_{22}) + r_{\mathrm{step}}\gamma_k\sqrt{\kappa_{22}}c_\infty\|A_{12}\|.\end{aligned}$$

Recalling that $r_{\mathrm{step}} \le a_{22}/(2\|A_{12}\|\sqrt{\kappa_{22}}c_\infty)$, the second part of (38) follows. □

Throughout our analysis we use the following notations:

$$\begin{aligned}\Gamma_{m:n}^{(1)} &:= \prod_{i=m}^{n}(\mathrm{I}-\beta_i B_{11}^i), \quad \Gamma_{m:n}^{(2)} := \prod_{i=m}^{n}(\mathrm{I}-\gamma_i B_{22}^i), \\ P_{m:n}^{(1)} &:= \prod_{i=m}^{n}(1-(1/2)\beta_i a_\Delta), \quad P_{m:n}^{(2)} := \prod_{i=m}^{n}(1-(1/2)\gamma_i a_{22}).\end{aligned} \tag{39}$$

As a convention, we define $\Gamma_{m:n}^{(1)} = \mathrm{I}$ and $\Gamma_{m:n}^{(2)} = \mathrm{I}$ if $m > n$.

Corollary 1. *Under the assumptions of Lemma 7, it holds for any $n, m \ge 0$, that*

$$\|\Gamma_{m:n}^{(1)}\| \le \sqrt{\kappa_\Delta}P_{m:n}^{(1)} \;.$$

Similarly, we have

$$\|\Gamma_{m:n}^{(2)}\| \le \sqrt{\kappa_{22}}P_{m:n}^{(2)} \;. \tag{40}$$

We shall prove that

$$M^{\tilde{\theta}}_{k+1,p} \le \mathrm{C}^{\tilde{\theta}}_0 \prod_{j=0}^{k} \big(1 - \beta_j a_{\Delta}/8\big) + \mathrm{C}_{\mathrm{slow}}\, p^2 \beta_k^{1/2} \;,$$

$$M^{\tilde{w}}_{k+1,p} \le \mathrm{C}^{\tilde{w}}_0 \prod_{j=0}^{k} \big(1 - \gamma_j a_{22}/8\big) + \mathrm{C}_{\mathrm{fast}}\, p^3 \gamma_k^{1/2} \;,$$

First, we introduce the constants

$$\mathrm{C}^{\tilde{\theta}}_0 = \{C^{\tilde{\theta}}_0\}^{1/2} \;, \quad \mathrm{C}^{\tilde{w}}_0 = \{C^{\tilde{w}}_0\}^{1/2} \;, \quad \mathrm{C}_{\mathrm{slow}} = \{24 C^{\tilde{\theta}}_1\}^{1/2} \;, \quad \mathrm{C}_{\mathrm{fast}} = \{C^{\tilde{w}}_1 + 24 a_{22}^{-1} C^{\tilde{w}}_2 (2 C^{\tilde{\theta}}_0 + 2\, \mathrm{C}^2_{\mathrm{slow}}\, \beta_0)\}^{1/2} \;.$$

where $C^{\tilde{w}}_0$, $C^{\tilde{w}}_1$, $C^{\tilde{w}}_2$ and $C^{\tilde{\theta}}_0$, $C^{\tilde{\theta}}_1$ are defined in (45) and (50) respectively. In order to prove Proposition 3, we employ the following scheme. We first consider the moments "fast" scale $M^{\tilde{w}}_{k+1,p}$ and upper bound them in terms of the moments of "slow" time scale $M^{\tilde{\theta}}_{j,p}$ with $j \in \{1, \ldots, p\}$. This is formalized in the following proposition:

Proposition 5. *Let $p \ge 2$ and assume A1,A2(p), A4, A5(p). Then for any $k \in \mathbb{N}$ it holds that*

$$\big(M^{\tilde{w}}_{k+1,p}\big)^2 \le C^{\tilde{w}}_0 P^{(2)}_{0:k} + p^2 C^{\tilde{w}}_1 \gamma_k + p^2 C^{\tilde{w}}_2 \sum_{j=0}^{k} \gamma_j^2 P^{(2)}_{j+1:k} \big(M^{\tilde{\theta}}_{j,p}\big)^2, \tag{41}$$

where the constants $C^{\tilde{w}}_0, C^{\tilde{w}}_1, C^{\tilde{w}}_2$ are given in (45)*.*

Now we derive the following recursive bounds for the moments of "slow" time scale $M^{\tilde{\theta}}_{k,p}$:

Proposition 6. *Let $p \ge 2$ and assume A1,A2(p), A4, A5(p). Then for any $k \in \mathbb{N}$ it holds that*

$$\big(M^{\tilde{\theta}}_{k+1,p}\big)^2 \le C^{\tilde{\theta}}_0 P^{(1)}_{0:k} + p^4 C^{\tilde{\theta}}_1 \beta_k + p^4 C^{\tilde{\theta}}_2 \sum_{j=0}^{k} \beta_j^2 P^{(1)}_{j+1:k} \big(M^{\tilde{\theta}}_{j,p}\big)^2, \tag{42}$$

where the constants $C^{\tilde{\theta}}_0, C^{\tilde{\theta}}_1, C^{\tilde{\theta}}_2$ are given in (50)*.*

Proof of Proposition 3.
(I) Proof of the bound (16)**.** Now we aim to solve the recurrence (42) and prove the upper bound (16). Towards this, we consider the recurrence $\widetilde{U}_k$, which is driven by the right-hand side of (42):

$$\widetilde{U}_{k+1} = C^{\tilde{\theta}}_0 P^{(1)}_{0:k} + p^4 C^{\tilde{\theta}}_1 \beta_k + p^4 C^{\tilde{\theta}}_2 \sum_{j=0}^{k} \beta_j^2 P^{(1)}_{j+1:k} \widetilde{U}_j, \quad \widetilde{U}_0 = C^{\tilde{\theta}}_0 \;, \tag{43}$$

and $C^{\tilde{\theta}}_0$ is defined in (50). The constructed sequence $\widetilde{U}_k$ provides an upper bound for the moments, that is,

$$(M^{\tilde{\theta}}_{k+1,p})^2 \le \widetilde{U}_{k+1} \;. \tag{44}$$

To verify (44), observe that $(M^{\tilde{\theta}}_{0,p})^2 \le \widetilde{U}_0$ by definition of $\widetilde{U}_0$ in (43). By induction, assuming the validity of (44) for all $j \le k$, we establish its correctness for $k+1$. Indeed, using Proposition 6, we get

$$\begin{aligned}\big(M^{\tilde{\theta}}_{k+1,p}\big)^2 &\le C^{\tilde{\theta}}_0 P^{(1)}_{0:k} + p^4 C^{\tilde{\theta}}_1 \beta_k + p^4 C^{\tilde{\theta}}_2 \sum_{j=0}^{k} \beta_j^2 P^{(1)}_{j+1:k} \big(M^{\tilde{\theta}}_{j,p}\big)^2 \\ &\le C^{\tilde{\theta}}_0 P^{(1)}_{0:k} + p^4 C^{\tilde{\theta}}_1 \beta_k + p^4 C^{\tilde{\theta}}_2 \sum_{j=0}^{k} \beta_j^2 P^{(1)}_{j+1:k} \widetilde{U}_k = \widetilde{U}_{k+1} \;.\end{aligned}$$

Using the definition of the product $P^{(1)}_{j+1:k}$ in (39), we observe that

$$\widetilde{U}_{k+1} = (1 - \beta_k a_{\Delta}/2 + p^4 C^{\tilde{\theta}}_2 \beta_k^2) \widetilde{U}_k + p^4 C^{\tilde{\theta}}_1 (\beta_k - \beta_{k-1} + \beta_{k-1}\beta_k a_{\Delta}/2) \;.$$

Since $\beta_k \le a_{\Delta}/(4p^4 C^{\tilde{\theta}}_2)$ due to A5, and $\beta_{k-1} \le 2\beta_k$, we have

$$\widetilde{U}_{k+1} \le (1 - \beta_k a_{\Delta}/4) \widetilde{U}_k + p^4 C^{\tilde{\theta}}_1 a_{\Delta} \beta_k^2 \;.$$

Enrolling the above recurrence, we get

$$\widetilde{U}_{k+1} \le C_0^{\tilde{\theta}} \prod_{j=0}^{k} \big(1 - \beta_j \frac{a_\Delta}{4}\big) + p^4 C_1^{\tilde{\theta}} a_\Delta \sum_{j=0}^{k} \beta_j^2 \prod_{i=j+1}^{k} \big(1 - \beta_i \frac{a_\Delta}{4}\big) \ .$$

Applying Lemma 31-(ii), the bound (44), and the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, we get (16).
(II) Proof of the bound (17)**.** Substituting (16) into Proposition 5 we obtain that

$$\begin{aligned}
\big(M_{k+1,p}^{\tilde{w}}\big)^2 &\le C_0^{\tilde{w}} P_{0:k}^{(2)} + p^2 C_1^{\tilde{w}} \gamma_k + p^2 C_2^{\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 P_{j+1:k}^{(2)} \big(M_{j,p}^{\tilde{\theta}}\big)^2 \\
&\le C_0^{\tilde{w}} P_{0:k}^{(2)} + p^2 C_1^{\tilde{w}} \gamma_k + p^2 C_2^{\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 P_{j+1:k}^{(2)} \big(2 C_0^{\tilde{\theta}} \prod_{i=0}^{j-1} \big(1 - \beta_i a_\Delta/4\big) + 2p^4 \,\mathrm{C}_{\mathrm{slow}}^2 \,\beta_j\big) \\
&\le C_0^{\tilde{w}} P_{0:k}^{(2)} + p^2 C_1^{\tilde{w}} \gamma_k + p^2 C_2^{\tilde{w}} (2 C_0^{\tilde{\theta}} + 2p^4 \,\mathrm{C}_{\mathrm{slow}}^2 \,\beta_0) 24 \gamma_k a_{22}^{-1} \\
&\le C_0^{\tilde{w}} P_{0:k}^{(2)} + p^6 \gamma_k \underbrace{\big(C_1^{\tilde{w}} + 24 a_{22}^{-1} C_2^{\tilde{w}} (2 C_0^{\tilde{\theta}} + 2 \,\mathrm{C}_{\mathrm{slow}}^2 \,\beta_0)\big)}_{\mathrm{C}_{\mathrm{fast}}^2} \ .
\end{aligned}$$

The inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ completes the proof. ☐

Now we derive a moment bound for $w_{k+1} - w^\star$ using Proposition 3.

Lemma 8. *Let $p \ge 2$. Assume A1,A2(p), A3, A4, A5(p), and A6. Then it holds for $k \in \mathbb{N}$ that*

$$\mathbb{E}^{1/p}[\|w_{k+1} - w^\star\|^p] \le \widehat{\mathrm{C}}_0^{\tilde{w}} \prod_{j=0}^{k} (1 - \beta_j \frac{a_\Delta}{8}) + p^3 \widehat{\mathrm{C}}_{\mathrm{fast}} \gamma_k^{1/2} \ ,$$

where we have set

$$\widehat{\mathrm{C}}_0^{\tilde{w}} = c_\infty \,\mathrm{C}_0^{\tilde{\theta}} + \mathrm{C}_0^{\tilde{w}} \ , \ \widehat{\mathrm{C}}_{\mathrm{fast}} = \mathrm{C}_{\mathrm{fast}} + r_{\mathrm{step}}^{1/2} c_\infty \,\mathrm{C}_{\mathrm{slow}} \ .$$

Proof. Recall that $w_{k+1} - w^\star = \tilde{w}_{k+1} - D_k(\theta_{k+1} - \theta^\star)$. Since A5 guarantees that $\beta_k/\gamma_k \le 2a_{22}/a_\Delta$, we have $(1 - \gamma_k a_{22}/8) \le (1 - \beta_k a_\Delta/8)$. Hence, applying Lemma 4, Proposition 3 together with Minkowski's inequality we get

$$\mathbb{E}^{1/p}[\|w_{k+1} - w^*\|^p] \le M_{k+1,p}^{\tilde{w}} + c_\infty M_{k+1,p}^{\tilde{\theta}} \le (c_\infty \,\mathrm{C}_0^{\tilde{\theta}} + \mathrm{C}_0^{\tilde{w}}) \prod_{j=0}^{k} (1 - \beta_j \frac{a_\Delta}{8}) + p^3 \gamma_k^{1/2} (c_\infty \,\mathrm{C}_{\mathrm{slow}} \, r_{\mathrm{step}}^{1/2} + \mathrm{C}_{\mathrm{fast}}) \ ,$$

and the proof is complete. ☐

Now we prove Proposition 5 and Proposition 6.

Proof of Proposition 5. We first introduce the constants:

$$C_0^{\tilde{w}} = 2\kappa_{22} \|\tilde{w}_0\|^2 \ , \quad C_1^{\tilde{w}} = \frac{72}{a_{22}} \kappa_{22} \widetilde{m}^2 \ , \quad C_2^{\tilde{w}} = 6 \kappa_{22} \widetilde{m}^2 \ . \tag{45}$$

The recursion (13) implies that

$$\tilde{w}_{k+1} = \prod_{j=0}^{k} (\mathrm{I} - \gamma_j B_{22}^j) \tilde{w}_0 - \sum_{j=0}^{k} \prod_{i=j+1}^{k} (\mathrm{I} - \gamma_i B_{22}^i)(\gamma_j W_{j+1} + \beta_j D_j V_{j+1}) = \Gamma_{0:k}^{(2)} \tilde{w}_0 - \sum_{j=0}^{k} \Gamma_{j+1:k}^{(2)} \xi_{j+1} \ , \tag{46}$$

where $\Gamma_{j+1:k}^{(2)}$ is defined in (39). Using Minkowski inequality and (40), we get

$$\big(M_{k+1,p}^{\tilde{w}}\big)^2 = \mathbb{E}^{2/p}[\|\tilde{w}_{k+1}\|^p] \le 2\kappa_{22} \big(P_{0:k}^{(2)}\big)^2 \|\tilde{w}_0\|^2 + 2\mathbb{E}^{2/p}\big[\| \sum_{j=0}^{k} \Gamma_{j+1:k}^{(2)} \xi_{j+1} \|^p\big].$$

Now we proceed with the second term. Applying Burholder's inequality (Osekowski 2012, Theorem 8.6), we obtain that

$$\mathbb{E}^{2/p}\big[\big\|\sum_{j=0}^{k}\Gamma^{(2)}_{j+1:k}\xi_{j+1}\big\|^p\big] \le p^2\mathbb{E}^{2/p}\big[\big(\sum_{j=0}^{k}\|\Gamma^{(2)}_{j+1:k}\xi_{j+1}\|^2\big)^{p/2}\big] \le p^2\sum_{j=0}^{k}\|\,\Gamma^{(2)}_{j+1:k}\|^2\mathbb{E}^{2/p}\big[\|\xi_{j+1}\|^p\big]$$
$$\le 3p^2\widetilde{m}^2\kappa_{22}\sum_{j=0}^{k}\gamma_j^2\big(P^{(2)}_{j+1:k}\big)^2\big(1+\big(M^{\tilde{\theta}}_{j,p}\big)^2+\big(M^{\tilde{w}}_{j,p}\big)^2\big)\;,$$

where in the last step we have used Lemma 5 and (40). Finally, we get

$$\big(M^{\tilde{w}}_{k+1,p}\big)^2 \le C^{\tilde{w}\prime}_0\big(P^{(2)}_{0:k}\big)^2 + p^2C^{\tilde{w}\prime}_1\sum_{j=0}^{k}\gamma_j^2\big(P^{(2)}_{j+1:k}\big)^2\big(1+\big(M^{\tilde{\theta}}_{j,p}\big)^2+\big(M^{\tilde{w}}_{j,p}\big)^2\big)\;, \tag{47}$$

where $C^{\tilde{w}\prime}_0 = 2\kappa_{22}\|\tilde{w}_0\|^2, \quad C^{\tilde{w}\prime}_1 = 6\kappa_{22}\widetilde{m}^2$. Define the sequence U_k by the following recurrence:

$$U_{k+1} = C^{\tilde{w}\prime}_0\big(P^{(2)}_{0:k}\big)^2 + p^2C^{\tilde{w}\prime}_1\sum_{j=0}^{k}\gamma_j^2\big(P^{(2)}_{j+1:k}\big)^2\big(1+\big(M^{\tilde{\theta}}_{j,p}\big)^2+U_j\big)\;,\quad U_0 = C^{\tilde{w}\prime}_0\;. \tag{48}$$

The constructed sequence U_{k+1} provides an upper bound for the moments, that is,

$$(M^{\tilde{w}}_{k+1,p})^2 \le U_{k+1}\;. \tag{49}$$

To verify (49), observe that $(M^{\tilde{w}}_{0,p})^2 \le U_0$ by definition of U_0 in (48). By induction, assuming the validity of (49) for all $j \le k$, we establish its correctness for $k+1$. Indeed, using (47), we get

$$\big(M^{\tilde{w}}_{k+1,p}\big)^2 \le C^{\tilde{w}\prime}_0\big(P^{(2)}_{0:k}\big)^2 + p^2C^{\tilde{w}\prime}_1\sum_{j=0}^{k}\gamma_j^2\big(P^{(2)}_{j+1:k}\big)^2\big(1+\big(M^{\tilde{\theta}}_{j,p}\big)^2+U_j\big) = U_{k+1}\;.$$

Using the definition of the product $P^{(2)}_{j+1:k}$, we observe that the sequence $\big(U_k\big)_{k\ge 0}$ satisfies the following recursion:

$$U_{k+1} = (1-a_{22}\gamma_k/2)^2U_k + p^2C^{\tilde{w}\prime}_1\gamma_k^2\big(1+\big(M^{\tilde{\theta}}_{k,p}\big)^2+U_k\big),\quad U_0 = C^{\tilde{w}\prime}_0.$$

Since $\gamma_k \le a_{22}/(2p^2C^{\tilde{w}\prime}_1 + a_{22}^2/2)$, we have

$$U_{k+1} \le (1-a_{22}\gamma_k/2)U_k + p^2C^{\tilde{w}\prime}_1\gamma_k^2\big(1+\big(M^{\tilde{\theta}}_{k,p}\big)^2\big)$$

which implies

$$U_{k+1} \le C^{\tilde{w}\prime}_0P^{(2)}_{0:k} + p^2C^{\tilde{w}\prime}_1\sum_{j=0}^{k}\gamma_j^2P^{(2)}_{j+1:k}\big(1+\big(M^{\tilde{\theta}}_{j,p}\big)^2\big)\;.$$

Applying Lemma 31-(ii) we get (41). □

Proof of Proposition 6. First we introduce the constants

$$\begin{aligned}
C^{\tilde{\theta}}_0 &= 4\kappa_\Delta\|\tilde{\theta}_0\|^2 + 4\kappa_\Delta\kappa_{22}\|A_{12}\|^2r_{\text{step}}^2(\mathrm{C}^P_\gamma)^2\|\tilde{w}_0\|^2\;,\\
C^{\tilde{\theta}}_1 &= 144(a_\Delta)^{-1}\big(\widetilde{m}_V^2\kappa_\Delta + (\mathrm{C}^P_\gamma)^2\kappa_\Delta\kappa_{22}\widetilde{m}^2\|A_{12}\|^2\big)\big(1+C^{\tilde{w}}_0+\gamma_0C^{\tilde{w}}_1\big)\;,\\
C^{\tilde{\theta}}_2 &= 12\big(\widetilde{m}_V^2\kappa_\Delta + (\mathrm{C}^P_\gamma)^2\kappa_\Delta\kappa_{22}\widetilde{m}^2\|A_{12}\|^2\big)\big(C^{\tilde{w}}_2\,\mathrm{C}^P_\gamma\,\gamma_0+1\big)\;.
\end{aligned} \tag{50}$$

Expanding the recursion (13), we get with $\Gamma^{(1)}_{j+1:k}$ defined in (39), that

$$\widetilde{\theta}_{k+1} = \Gamma^{(1)}_{0:k}\tilde{\theta}_0 - \sum_{j=0}^{k}\beta_j\Gamma^{(1)}_{j+1:k}A_{12}\tilde{w}_j - \sum_{j=0}^{k}\beta_j\Gamma^{(1)}_{j+1:k}V_{j+1}\;. \tag{51}$$

Next, we substitute $\widetilde{w}_j$ from (46):

$$\begin{aligned}
\sum_{j=0}^{k}\beta_j\Gamma^{(1)}_{j+1:k}A_{12}\widetilde{w}_j &= \sum_{j=0}^{k}\beta_j\Gamma^{(1)}_{j+1:k}A_{12}\big(\Gamma^{(2)}_{0:j-1}\widetilde{w}_0 - \sum_{i=0}^{j-1}\Gamma^{(2)}_{i+1:j-1}\xi_{i+1}\big)\\
&= \sum_{j=0}^{k}\beta_j\Gamma^{(1)}_{j+1:k}A_{12}\Gamma^{(2)}_{0:j-1}\widetilde{w}_0 - \sum_{i=0}^{k-1}\big(\sum_{j=i+1}^{k}\beta_j\Gamma^{(1)}_{j+1:k}A_{12}\Gamma^{(2)}_{i+1:j-1}\big)\xi_{i+1}\;.
\end{aligned}$$

Define the quantity

$$T_{m:n} = \sum_{\ell=m}^{n} \beta_\ell \Gamma^{(1)}_{\ell+1:n} A_{12} \Gamma^{(2)}_{m:\ell-1} \;,\; m \leq n \;.$$

Thus, with $P^{(1)}_{k:j}$, $P^{(2)}_{k:j}$ defined in (39), it holds that

$$\|T_{m:n}\| \leq \sqrt{\kappa_\Delta \kappa_{22}} \|A_{12}\| \sum_{\ell=m}^{n} \beta_\ell P^{(1)}_{\ell+1:n} P^{(2)}_{m:\ell-1} \;. \tag{52}$$

Now we rewrite (51) as follows:

$$\tilde{\theta}_{k+1} = \Gamma^{(1)}_{0:k} \tilde{\theta}_0 - T_{0:k} \tilde{w}_0 + \sum_{j=0}^{k-1} T_{j+1:k} \xi_{j+1} - \sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} V_{j+1} \;.$$

Applying Minkowski inequality and (40), we obtain that

$$\big(M^{\tilde{\theta}}_{k+1,p}\big)^2 \leq \underbrace{4\kappa_\Delta \big(P^{(1)}_{0:k}\big)^2 \|\tilde{\theta}_0\|^2}_{\mathcal{R}_1} + \underbrace{4\|T_{0:k}\|^2 \|\tilde{w}_0\|^2}_{\mathcal{R}_2} + \underbrace{4\mathbb{E}^{2/p}\big[\big\| \sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} V_{j+1} \big\|^p\big]}_{\mathcal{R}_3} + \underbrace{4\mathbb{E}^{2/p}\big[\big\| \sum_{j=0}^{k-1} T_{j+1:k} \xi_{j+1} \big\|^p\big]}_{\mathcal{R}_4} \;.$$

Next, we get the upper bounds for $\mathcal{R}_i$ separately. Applying Lemma 32 with $j+1=0$ and using $\beta_j \leq r_{\mathrm{step}} \gamma_j$, one can get:

$$\begin{aligned}
\mathcal{R}_2 &\leq 4\kappa_\Delta \kappa_{22} \|A_{12}\|^2 \big(\sum_{j=0}^{k} \beta_j P^{(1)}_{j+1:k} P^{(2)}_{0:j-1} \big)^2 \|\tilde{w}_0\|^2 \leq 4\kappa_\Delta \kappa_{22} \|A_{12}\|^2 r_{\mathrm{step}}^2 \big(\sum_{j=0}^{k} \gamma_j P^{(1)}_{j+1:k} P^{(2)}_{0:j-1} \big)^2 \|\tilde{w}_0\|^2 \\
&\leq 4\kappa_\Delta \kappa_{22} \|A_{12}\|^2 r_{\mathrm{step}}^2 (\mathrm{C}^P_\gamma)^2 \big(P^{(1)}_{0:k}\big)^2 \|\tilde{w}_0\|^2 \;.
\end{aligned}$$

Applying Lemma 5 and Burholder's inequality, we obtain that

$$\begin{aligned}
\mathcal{R}_3 &\leq 4p^2 \mathbb{E}^{2/p}\big[\big(\sum_{j=0}^{k} \beta_j^2 \|\Gamma^{(1)}_{j+1:k} V_{j+1}\|^2 \big)^{p/2} \big] \leq 4p^2 \sum_{j=0}^{k} \beta_j^2 \| \Gamma^{(1)}_{j+1:k}\|^2 \mathbb{E}^{2/p}\big[\|V_{j+1}\|^p\big] \\
&\leq 12 p^2 \widetilde{m}_V^2 \kappa_\Delta \sum_{j=0}^{k} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \big(1 + \big(M^{\tilde{\theta}}_{j,p}\big)^2 + \big(M^{\tilde{w}}_{j,p}\big)^2\big) \;.
\end{aligned}$$

In order to derive a bound for $\mathcal{R}_4$, we apply Lemma 5, (52) and Burholder's inequality:

$$\begin{aligned}
\mathcal{R}_4 &\leq 4p^2 \mathbb{E}^{2/p}\big[\big(\sum_{j=0}^{k-1} \|T_{j+1:k} \xi_{j+1}\|^2 \big)^{p/2} \big] \leq 4p^2 \sum_{j=0}^{k-1} \|T_{j+1:k}\|^2 \big(\mathbb{E}^{2/p}[\|\xi_{j+1}\|^p] \big) \\
&\leq 12 p^2 \kappa_\Delta \kappa_{22} \widetilde{m}^2 \|A_{12}\|^2 \sum_{j=0}^{k-1} \gamma_j^2 \big(\sum_{i=j+1}^{k} \beta_i P^{(1)}_{i+1:k} P^{(2)}_{j+1:i-1} \big)^2 \big(1 + \big(M^{\tilde{\theta}}_{j,p}\big)^2 + \big(M^{\tilde{w}}_{j,p}\big)^2\big) \\
&\leq 12 p^2 \kappa_\Delta \kappa_{22} \widetilde{m}^2 \|A_{12}\|^2 \sum_{j=0}^{k-1} \beta_j^2 \big(\sum_{i=j+1}^{k} \gamma_j P^{(1)}_{i+1:k} P^{(2)}_{j+1:i-1} \big)^2 \big(1 + \big(M^{\tilde{\theta}}_{j,p}\big)^2 + \big(M^{\tilde{w}}_{j,p}\big)^2\big) \\
&\overset{(a)}{\leq} 12 p^2 (\mathrm{C}^P_\gamma)^2 \kappa_\Delta \kappa_{22} \widetilde{m}^2 \|A_{12}\|^2 \sum_{j=0}^{k-1} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \big(1 + \big(M^{\tilde{\theta}}_{j,p}\big)^2 + \big(M^{\tilde{w}}_{j,p}\big)^2\big) \;,
\end{aligned}$$

where the inequality (a) follows from Lemma 32. Combining the above bounds, we get

$$\begin{aligned}
\big(M^{\tilde{\theta}}_{k+1,p}\big)^2 &\leq \big(4\kappa_\Delta \|\tilde{\theta}_0\|^2 + 4\kappa_\Delta \kappa_{22} \|A_{12}\|^2 r_{\mathrm{step}}^2 (\mathrm{C}^P_\gamma)^2 \|\tilde{w}_0\|^2\big) \big(P^{(1)}_{0:k}\big)^2 \\
&\quad + 12 p^2 \big(\widetilde{m}_V^2 \kappa_\Delta + (\mathrm{C}^P_\gamma)^2 \kappa_\Delta \kappa_{22} \widetilde{m}^2 \|A_{12}\|^2 \big) \sum_{j=0}^{k-1} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \big(1 + \big(M^{\tilde{\theta}}_{j,p}\big)^2 + \big(M^{\tilde{w}}_{j,p}\big)^2\big) \;,
\end{aligned} \tag{53}$$

Moreover, applying Proposition 5 and Lemma 31-(ii) we bound last term in (53) as follows

$$\begin{aligned}
\sum_{j=0}^{k} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \big(M^{\tilde{w}}_{j,p}\big)^2 &\le \sum_{j=0}^{k} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \big(C^{\tilde{w}}_0 P^{(2)}_{0:j-1} + p^2 C^{\tilde{w}}_1 \gamma_j + p^2 C^{\tilde{w}}_2 \sum_{i=0}^{j-1} \gamma_i^2 P^{(2)}_{i+1:j-1} \big(M^{\tilde{\theta}}_{i,p}\big)^2\big) \\
&\le p^2 C^{\tilde{w}}_0 \sum_{j=0}^{k} \beta_j^2 P^{(1)}_{j+1:k} + p^2 \gamma_0 C^{\tilde{w}}_1 \sum_{j=0}^{k} \beta_j^2 P^{(1)}_{j+1:k} + p^2 C^{\tilde{w}}_2 \sum_{j=0}^{k} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \sum_{i=0}^{j-1} \gamma_i^2 P^{(2)}_{i+1:j-1} \big(M^{\tilde{\theta}}_{i,p}\big)^2 \\
&\le p^2 \big(C^{\tilde{w}}_0 + \gamma_0 C^{\tilde{w}}_1\big) \frac{12}{a_\Delta} \beta_k + p^2 C^{\tilde{w}}_2 \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \beta_j^2 \gamma_i^2 P^{(2)}_{i+1:j-1} P^{(1)}_{j+1:k} \big(M^{\tilde{\theta}}_{i,p}\big)^2 .
\end{aligned}$$

Using $\beta_j^2 \le \beta_i^2$ for $j \ge i+1$ and $\gamma_i^2 \le \gamma_0 \gamma_i$, we get

$$\begin{aligned}
\sum_{j=0}^{k} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \big(M^{\tilde{w}}_{j,p}\big)^2 &\le p^2 \big(C^{\tilde{w}}_0 + \gamma_0 C^{\tilde{w}}_1\big) \frac{12}{a_\Delta} \beta_k + p^2 C^{\tilde{w}}_2 \gamma_0 \sum_{i=0}^{k-1} \beta_i^2 \big(\sum_{j=i+1}^{k} \gamma_i P^{(2)}_{i+1:j-1} P^{(1)}_{j+1:k}\big) \big(M^{\tilde{\theta}}_{i,p}\big)^2 \\
&\overset{(a)}{\le} p^2 \big(C^{\tilde{w}}_0 + \gamma_0 C^{\tilde{w}}_1\big) \frac{12}{a_\Delta} \beta_k + p^2 C^{\tilde{w}}_2 \gamma_0 \, \mathrm{C}^P_\gamma \sum_{i=0}^{k-1} \beta_i^2 P^{(1)}_{i+1:k} \big(M^{\tilde{\theta}}_{i,p}\big)^2 ,
\end{aligned}$$

where (a) follows from Lemma 32. Substituting the above inequalities into (53) we obtain (42). □

C CLT for the Polyak-Ruppert averaged estimator

We preface the proof of Theorem 1 with a key decomposition isolating a linear statistics of $\boldsymbol{\varepsilon}_V^{k+1}, \boldsymbol{\varepsilon}_W^{k+1}$, which form a martingale difference sequences w.r.t. the natural filtration $\mathcal{F}_k = \sigma(X_s : s \le k)$.

Lemma 9. *The following decomposition holds:*

$$\Delta(\theta_k - \theta^*) = \beta_k^{-1}(\theta_k - \theta_{k+1}) - \gamma_k^{-1} A_{12} A_{22}^{-1}(w_k - w_{k+1}) + (V_{k+1} - A_{12} A_{22}^{-1} W_{k+1}) . \tag{54}$$

Moreover, it holds that

$$\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\boldsymbol{\varepsilon}_V^{k+1} - A_{12} A_{22}^{-1} \boldsymbol{\varepsilon}_W^{k+1}) + R_n^{\mathrm{pr}} , \tag{55}$$

where the residual term $R_n^{\mathrm{pr}} = Y_1 + Y_2 + Y_3$, and Y_1, Y_2, Y_3 are given by

$$\begin{aligned}
Y_1 &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \beta_k^{-1}(\theta_k - \theta_{k+1}) , \\
Y_2 &= -\frac{1}{\sqrt{n}} \sum_{k=1}^{n} A_{12} A_{22}^{-1} \gamma_k^{-1} (w_k - w_{k+1}) , \\
Y_3 &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \{ \underbrace{(A_{12} A_{22}^{-1} \tilde{\mathbf{A}}_{21}^{k+1} - \tilde{\mathbf{A}}_{11}^{k+1})}_{R^\theta_{k+1}} \tilde{\theta}_{k+1} + \underbrace{(A_{12} A_{22}^{-1} \tilde{\mathbf{A}}_{22}^{k+1} - \tilde{\mathbf{A}}_{12}^{k+1})}_{R^w_{k+1}} (w_{k+1} - w^\star) \} .
\end{aligned}$$

Proof. First, we prove the representation (54). Equation (7) implies:

$$w_k = (\gamma_k A_{22})^{-1}(w_k - w_{k+1}) + A_{22}^{-1}(b_2 - A_{21}\theta_k + W_{k+1}) . \tag{56}$$

Substituting (56) into the slow-time-scale variable's recursion (6) we obtain that

$$\begin{aligned}
\theta_{k+1} &= \theta_k + \beta_k(b_1 - A_{11}\theta_k + V_{k+1}) - \beta_k A_{12}(\gamma_k A_{22})^{-1}(w_k - w_{k+1}) - \beta_k A_{12} A_{22}^{-1}(b_2 - A_{21}\theta_k + W_{k+1}) \qquad (57) \\
&= (\mathrm{I} - \beta_k \Delta)\theta_k - \frac{\beta_k}{\gamma_k} A_{12} A_{22}^{-1}(w_k - w_{k+1}) + \beta_k(b_1 - A_{12} A_{22}^{-1} b_2) + \beta_k(V_{k+1} - A_{12} A_{22}^{-1} W_{k+1}) ,
\end{aligned}$$

Recall that (θ^*, w^*) is the solution of the system (1). This implies that $b_1 - A_{12}A_{22}^{-1}b_2 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\theta^* = \Delta\theta^*$. Substituting this equality into (57), we obtain the formula (54). To establish (55), we sum (54) over $k = 1, \ldots, n$ using the expressions for V_{k+1}, W_{k+1} (8) and unrolling the corresponding recurrence. □

To proceed with Theorem 1, we first formulate the moment bounds for Y_1, Y_2, Y_3.

Lemma 10. *Let $p \geq 2$. Assume A1,A2(p), A3, A4, A5(p), and A6. Then for any $k \in \mathbb{N}$ it holds that*

$$\mathbb{E}^{1/p}[\|Y_1\|^p] \leq \frac{\mathrm{C}_1^{Y_1}}{\sqrt{n}} + \mathrm{C}_2^{Y_1}(1+k_0)^{b+1}\frac{(n+k_0)^{b/2}}{\sqrt{n}} \;,$$

where we have set

$$\mathrm{C}_1^{Y_1} = \mathrm{C}_0^{\tilde{\theta}} \;,\; \mathrm{C}_2^{Y_1} = 5\,\mathrm{C}_{\mathrm{slow}}\, c_{0,\beta}^{-1/2} + c_{0,\beta}^{-2} C_0^{\tilde{\theta}}(c_{0,\beta} + \frac{8}{a_\Delta(1-b)}) + c_{0,\beta}^{-1}(\mathrm{C}_0^{\tilde{\theta}} + \mathrm{C}_{\mathrm{slow}}\, c_{0,\beta}^{1/2}) \;.$$

Lemma 11. *Let $p \geq 2$. Assume A1,A2(p), A3, A4, A5(p), and A6. Then for any $k \in \mathbb{N}$ it holds*

$$\mathbb{E}^{1/p}[\|Y_2\|] \leq \frac{\mathrm{C}_1^{Y_2}}{\sqrt{n}} + \mathrm{C}_2^{Y_2}(1+k_0)^{b+1}\frac{(n+k_0)^{a/2}}{\sqrt{n}} \;,$$

where we have set

$$\mathrm{C}_1^{Y_2} = \|A_{12}A_{22}^{-1}\|\widehat{\mathrm{C}}_0^{\tilde{w}}\, r_{\mathrm{step}} \;,\; \mathrm{C}_2^{Y_2} = 5\|A_{12}A_{22}^{-1}\|c_{0,\gamma}^{-1/2}\widehat{\mathrm{C}}_{\mathrm{fast}} + \|A_{12}A_{22}^{-1}\|\left(\frac{r_{\mathrm{step}}}{c_{0,\gamma}c_{0,\beta}}(c_{0,\beta} + \frac{8}{a_\Delta(1-b)}) + c_{0,\gamma}^{-1}(\widehat{\mathrm{C}}_0^{\tilde{w}} + c_{0,\gamma}^{1/2}\widehat{\mathrm{C}}_{\mathrm{fast}})\right) \;.$$

Lemma 12. *Let $p \geq 2$. Assume A1,A2(p), A3, A4, A5(p), and A6. Therefore, it holds that*

$$\mathbb{E}^{1/p}\big[\|Y_3\|^p\big] \leq \mathrm{C}_1^{Y_3}\,\frac{p^4}{n^{a/2}} + \mathrm{C}_2^{Y_3}\,\frac{k_0^{b/2}p}{\sqrt{n}} \;,$$

where X is defined in (19) *and we have set*

$$\mathrm{C}_1^{Y_3} = 2q_R\,\mathrm{C}_{\mathrm{slow}}\left(\frac{c_{0,\beta}}{1-b}\right)^{1/2} + 2q_R\widehat{\mathrm{C}}_{\mathrm{fast}}\left(\frac{c_{0,\gamma}}{1-a}\right)^{1/2}, \quad \mathrm{C}_2^{Y_3} = 2c_{0,\beta}^{-1/2}q_R\left(1 + \frac{2\sqrt{2}}{\sqrt{a_\Delta(1-b)}}\right)(\mathrm{C}_0^{\tilde{\theta}} + \widehat{\mathrm{C}}_0^{\tilde{w}}) \;,$$

and

$$q_R = \mathrm{C}_{\mathbf{A}} + \|A_{12}A_{22}^{-1}\|\,\mathrm{C}_{\mathbf{A}} \;. \tag{58}$$

Proof of Theorem 1. Our proof starts from the error decomposition (19), which allows us to write

$$\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\boldsymbol{\varepsilon}_V^{k+1} - A_{12}A_{22}^{-1}\boldsymbol{\varepsilon}_W^{k+1}) + R_n^{\mathrm{pr}} \;,$$

where the term R_n is given in Lemma 9. Note that the term

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\boldsymbol{\varepsilon}_V^{k+1} - A_{12}A_{22}^{-1}\boldsymbol{\varepsilon}_W^{k+1})$$

is a linear statistic of the random variables $\psi_{k+1} = \boldsymbol{\varepsilon}_V^{k+1} - A_{12}A_{22}^{-1}\boldsymbol{\varepsilon}_W^{k+1}$, while R_n^{pr} is a "remainder" term, which moments are small, as we show below. Under A1, $\{\psi_{k+1}\}_{k\in\mathbb{N}}$ form a martingale-difference w.r.t. $\mathcal{F}_k$. Since the convex distance is invariant to non-degenerate linear transformations,

$$\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, \boldsymbol{\Sigma}_\varepsilon)\big) = \rho^{\mathrm{Conv}}\big(\sqrt{n}\boldsymbol{\Sigma}_\varepsilon^{-1/2}\Delta(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, \mathrm{I})\big) \;.$$

Set $p = \log n$. To control this term we apply Proposition 1 and obtain

$$\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star),\ \mathcal{N}(0, \boldsymbol{\Sigma}_\varepsilon)\big) \leq \underbrace{\rho^{\mathrm{Conv}}\big(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\psi_{k+1},\ \mathcal{N}(0, \boldsymbol{\Sigma}_\varepsilon)\big)}_{T_1} + \underbrace{2c_{d_\theta}^{p/p+1}\mathbb{E}^{1/(p+1)}\big[\|\boldsymbol{\Sigma}_\varepsilon^{-1/2}R_n^{\mathrm{pr}}\|^p\big]}_{T_2} \;, \tag{59}$$

where c_d is the isoperimetric constant of the convex sets, see Proposition 1 for detailed discussion. Hence, now it remains to estimate the normal approximation rate for T_1, and to control T_2. To proceed with T_1, we use the martingale CLT (Wu, Wei, and Rinaldo 2025, Theorem 1). For completeness, we state this result in the current paper, see Lemma 2. It is important to acknowledge that this result requires that ψ_{k+1} are a.s. bounded and have constant quadratic characteristic. Both assumptions hold in our setting, since, due to A6,

$$\|\psi_{k+1}\| \leq (1 + \|A_{12}A_{22}^{-1}\|)(\mathrm{C}_{\mathbf{b}} + \mathrm{C}_{\mathbf{A}}(\|\theta^\star\| + \|w^\star\|) =: \Psi \;,$$

and assumption A3 implies that

$$\mathbb{E}^{\mathcal{F}_k}\left[\psi_{k+1}\psi_{k+1}^\top\right] = \Sigma_V + A_{12}A_{22}^{-1}\Sigma_W(A_{12}A_{22}^{-1})^\top + \Sigma_{VW}(A_{12}A_{22}^{-1})^\top + A_{12}A_{22}^{-1}\Sigma_{VW}^\top =: \mathbf{\Sigma}_\varepsilon \; .$$

Hence, applying Lemma 2, we get:

$$\rho^{\mathrm{Conv}}\big(\frac{1}{\sqrt{n}}\sum_{k=1}^n \psi_{k+1},\, \mathcal{N}(0,\mathbf{\Sigma}_\varepsilon)\big) \lesssim [1+(2+\log(d_\theta n\|\mathbf{\Sigma}_\varepsilon\|))^+]^{1/2}\frac{\sqrt{d_\theta \log n}}{n^{1/4}} \; .$$

Now we proceed with the term T_2 defined in (59). We use the representation $R_n^{\mathrm{pr}} = Y_1 + Y_2 + Y_3$ and Lemmas 10 to 12 to control the moments of each Y_i. Combining these lemmas,

$$\begin{aligned}
\mathbb{E}^{1/p}\big[\|R_n^{\mathrm{pr}}\|^p\big] &\le \mathbb{E}^{1/p}\big[\|Y_1\|^p\big] + \mathbb{E}^{1/p}\big[\|Y_2\|^p\big] + \mathbb{E}^{1/p}\big[\|Y_3\|^p\big] \\
&\le \mathrm{C}_1^{Y_1}\frac{1}{\sqrt{n}} + \mathrm{C}_2^{Y_1} k_0^{b+1}\frac{(n+k_0)^{b/2}}{\sqrt{n}} + \mathrm{C}_1^{Y_2}\frac{1}{\sqrt{n}} + \mathrm{C}_2^{Y_2}(1+k_0)^{b+1}\frac{(n+k_0)^{a/2}}{\sqrt{n}} + \mathrm{C}_1^{Y_3}\frac{p^4}{n^{a/2}} + \mathrm{C}_2^{Y_3}\frac{k_0^{b/2}p}{\sqrt{n}} \\
&\lesssim p^{4+4/b}\left(\frac{1}{n^{a/2}} + \frac{1}{n^{(1-b)/2}}\right) ,
\end{aligned}$$

where we use inequality $(1+k_0)^{b+1} \lesssim p^{4+4/b}$. Combining the above inequalities, we obtain

$$\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star),\, \mathcal{N}(0,\mathbf{\Sigma}_\varepsilon)\big) \lesssim [1+(2+\log(d_\theta n\|\mathbf{\Sigma}_\varepsilon\|))^+]^{1/2}\frac{\sqrt{d_\theta\log n}}{n^{1/4}} + c_{d_\theta} p^{\frac{p}{p+1}(4+4/b)}\big(\frac{1}{n^{a/2}} + \frac{1}{n^{(1-b)/2}}\big)^{\frac{p}{p+1}}$$

Note that $(n^\alpha)^{\frac{\log n}{1+\log n}} \lesssim n^\alpha$ for all $\alpha \in \mathbb{R}$ and for all $a, b > 0$ and $q \in (0,1)$ it holds that $(a+b)^q \le a^q + b^q$. Hence, we get

$$\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star),\, \mathcal{N}(0,\mathbf{\Sigma}_\varepsilon)\big) \lesssim \{\log n\}^{4+4/b}\left(\frac{1}{n^{a/2}} + \frac{1}{n^{(1-b)/2}}\right) .$$

□

We finish this section by giving the proofs for Lemmas 10-12.

Proof of Lemma 10. Observe that

$$Y_1 = \frac{1}{\sqrt{n}}\sum_{k=1}^n \beta_k^{-1}(\theta_k - \theta_{k+1}) = \frac{1}{\sqrt{n}}\beta_1^{-1}\tilde{\theta}_1 - \frac{1}{\sqrt{n}}\beta_n^{-1}\tilde{\theta}_{n+1} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}(\beta_{k+1}^{-1} - \beta_k^{-1})\tilde{\theta}_{k+1} \; .$$

Applying Minkowski's inequality and an elementary inequality $\beta_{k+1}^{-1} - \beta_k^{-1} \le (k\beta_k)^{-1}$, we obtain that

$$\mathbb{E}^{1/p}[\|Y_1\|^p] \le \frac{1}{\sqrt{n}}\beta_1^{-1}\mathbb{E}^{1/p}[\|\tilde{\theta}_1\|^p] + \frac{1}{\sqrt{n}}\beta_n^{-1}\mathbb{E}^{1/p}[\|\tilde{\theta}_{n+1}\|^p] + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}(k\beta_k)^{-1}\mathbb{E}^{1/p}[\|\tilde{\theta}_{k+1}\|^p] \; .$$

Using Proposition 3 we get:

$$\beta_n^{-1}\mathbb{E}^{1/p}[\|\tilde{\theta}_{n+1}\|^p] \le \beta_n^{-1}\,\mathrm{C}_0^{\tilde{\theta}}\prod_{j=0}^n\big(1-\beta_j\frac{a_\Delta}{8}\big) + p^2\,\mathrm{C}_{\mathrm{slow}}\,\beta_n^{-1/2} \overset{(i)}{\le} \mathrm{C}_0^{\tilde{\theta}} + p^2\,\mathrm{C}_{\mathrm{slow}}\,\beta_n^{-1/2} ,$$

where in (i) we have used the inequality $\prod_{j=0}^n\big(1-\beta_j\frac{a_\Delta}{8}\big) \le \beta_n$. Next, we observe that

$$\begin{aligned}
\beta_1^{-1}\mathbb{E}^{1/p}[\|\tilde{\theta}_1\|^p] &\le \frac{p^2(1+k_0)^b}{c_{0,\beta}}\{\mathrm{C}_0^{\tilde{\theta}} + \mathrm{C}_{\mathrm{slow}}\,c_{0,\beta}^{1/2}\} , \\
\sum_{k=1}^{n-1}(k\beta_k)^{-1}\mathbb{E}^{1/p}[\|\tilde{\theta}_{k+1}\|^p] &\le \mathrm{C}_0^{\tilde{\theta}}\sum_{k=1}^{n-1}(k\beta_k)^{-1}\prod_{j=0}^k\big(1-\beta_j\frac{a_\Delta}{8}\big) + p^2\,\mathrm{C}_{\mathrm{slow}}\sum_{k=1}^{n-1}(k\beta_k)^{-1}\beta_k^{1/2} ,
\end{aligned}$$

Now we derive an upper bound for the r.h.s. of the latter inequality. Applying Lemma 31-(iii) and $(k\beta_k)^{-1} \le c_{0,\beta}^{-1}k_0$, we get:

$$\mathrm{C}_0^{\tilde{\theta}}\sum_{k=1}^{n-1}(k\beta_k)^{-1}\prod_{j=0}^k\big(1-\beta_j\frac{a_\Delta}{8}\big) \le \mathrm{C}_0^{\tilde{\theta}}\frac{k_0}{c_{0,\beta}}\sum_{k=1}^n\prod_{j=0}^k\big(1-\beta_j\frac{a_\Delta}{8}\big) \le \mathrm{C}_0^{\tilde{\theta}}\frac{k_0^{b+1}}{c_{0,\beta}^2}\big(c_{0,\beta} + \frac{8}{a_\Delta(1-b)}\big) \; .$$

Bound for the second term can be obtained from the straightforward computations:

$$p^2\,\mathrm{C}_{\mathrm{slow}}\sum_{k=1}^{n-1}(k\beta_k)^{-1}\beta_k^{1/2} = \frac{p^2\,\mathrm{C}_{\mathrm{slow}}}{c_{0,\beta}^{1/2}}\sum_{k=1}^{n-1}\frac{(k+k_0)^{b/2}}{k} \le \frac{p^2\,\mathrm{C}_{\mathrm{slow}}}{c_{0,\beta}^{1/2}}(1+k_0)\frac{2}{b}(n+k_0)^{b/2} \le \frac{4p^2\,\mathrm{C}_{\mathrm{slow}}}{c_{0,\beta}^{1/2}}(1+k_0)(n+k_0)^{b/2} \; .$$

The proof follows from gathering the above inequalities. □

Proof of Lemma 11. Observe that

$$Y_2 = A_{12}A_{22}^{-1}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\gamma_k^{-1}(w_k - w_{k+1}) = A_{12}A_{22}^{-1}\bigg(\frac{1}{\sqrt{n}}\gamma_1^{-1}(w_1 - w^*) - \frac{1}{\sqrt{n}}\gamma_n^{-1}(w_{n+1} - w^*) + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}(\gamma_{k+1}^{-1} - \gamma_k^{-1})(w_{k+1} - w^*)\bigg) .$$

Applying Minkowski's inequality and $\gamma_{k+1}^{-1} - \gamma_k^{-1} \leq (k\gamma_k)^{-1}$, we obtain that

$$\mathbb{E}^{1/p}[\|Y_2\|^p] \leq \|A_{12}A_{22}^{-1}\|\bigg(\frac{1}{\sqrt{n}}\gamma_1^{-1}\mathbb{E}^{1/p}[\|w_1 - w^*\|^p] + \frac{1}{\sqrt{n}}\gamma_n^{-1}\mathbb{E}^{1/p}[\|w_{n+1} - w^*\|^p] + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}(k\gamma_k)^{-1}\mathbb{E}^{1/p}[\|w_{k+1} - w^*\|^p]\bigg) .$$

Using Lemma 8 we get:

$$\gamma_n^{-1}\mathbb{E}^{1/p}[\|w_{n+1} - w^*\|^p] \leq \widehat{\mathrm{C}}_0^{\tilde{w}}\gamma_n^{-1}\prod_{j=0}^{n}(1 - \beta_j\frac{a_\Delta}{8}) + p^3\widehat{\mathrm{C}}_{\mathrm{fast}}\frac{(n+k_0)^{a/2}}{c_{0,\gamma}^{1/2}} \leq \widehat{\mathrm{C}}_0^{\tilde{w}}r_{\mathrm{step}} + p^3\widehat{\mathrm{C}}_{\mathrm{fast}}\frac{(n+k_0)^{a/2}}{c_{0,\gamma}^{1/2}} ,$$

where the last inequality follows from the inequalities $\gamma_n^{-1} \leq \beta_n^{-1}r_{\mathrm{step}}$ and $\prod_{j=0}^{n}\big(1 - \beta_j\frac{a_\Delta}{8}\big) \leq \beta_n$. Note that

$$\gamma_1^{-1}\mathbb{E}^{1/p}[\|w_1 - w^\star\|^p] \leq p^3\frac{(1+k_0)^a}{c_{0,\gamma}}(\widehat{\mathrm{C}}_0^{\tilde{w}} + \widehat{\mathrm{C}}_{\mathrm{fast}}c_{0,\gamma}^{1/2}) ,$$

$$\sum_{k=1}^{n-1}(k\gamma_k)^{-1}\mathbb{E}^{1/p}[\|w_{k+1} - w^*\|^p] \leq \widehat{\mathrm{C}}_0^{\tilde{w}}\sum_{k=1}^{n-1}(k\gamma_k)^{-1}\prod_{j=0}^{k}\big(1 - \beta_j\frac{a_\Delta}{8}\big) + p^3\widehat{\mathrm{C}}_{\mathrm{fast}}\sum_{k=1}^{n-1}(k\gamma_k)^{-1}\gamma_k^{1/2} ,$$

We bound the r.h.s of the latter inequality using Lemma 31-(iii):

$$\sum_{k=1}^{n-1}(k\gamma_k)^{-1}\prod_{j=0}^{k}\big(1 - \beta_j\frac{a_\Delta}{8}\big) \leq \frac{k_0^{b+1}r_{\mathrm{step}}}{c_{0,\gamma}c_{0,\beta}}(c_{0,\beta} + \frac{8}{a_\Delta(1-b)}) .$$

Bound for the second term can be obtained from the straightforward computations:

$$p^3\widehat{\mathrm{C}}_{\mathrm{fast}}\sum_{k=1}^{n-1}(k\gamma_k)^{-1}\beta_k^{1/2} = \frac{p^3\widehat{\mathrm{C}}_{\mathrm{fast}}}{c_{0,\gamma}^{1/2}}\sum_{k=1}^{n-1}\frac{(k+k_0)^{a/2}}{k} \leq \frac{p^3\widehat{\mathrm{C}}_{\mathrm{fast}}}{c_{0,\gamma}^{1/2}}(1+k_0)\frac{2}{a}(n+k_0)^{a/2} \leq \frac{4p^3\widehat{\mathrm{C}}_{\mathrm{fast}}}{c_{0,\gamma}^{1/2}}\mathrm{C}_{\mathrm{slow}}(1+k_0)(n+k_0)^{a/2} .$$

The proof follows from gathering the above bounds. □

Proof of Lemma 12. Note that since V_{k+1}, W_{k+1} and ε_V, ε_W are martingale difference sequences, $R_{k+1}^{\theta}\tilde{\theta}_k + R_{k+1}^{w}(w_k - w^\star)$ is a martingale difference sequence. Therefore, Burkholder's inequality (Osekowski 2012, Theorem 8.1) implies that

$$\begin{aligned}\mathbb{E}^{1/p}\Big[\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(R_{k+1}^{\theta}\tilde{\theta}_{k+1} + R_{k+1}^{w}(w_{k+1} - w^\star))\|^p\Big] &\leq p\Big(\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}^{2/p}[\|R_{k+1}^{\theta}\tilde{\theta}_{k+1} + R_{k+1}^{w}(w_{k+1} - w^\star)\|^p]\Big)^{1/2} \\ &\leq p\Big(\frac{2}{n}\sum_{k=1}^{n}(\mathbb{E}^{2/p}[\|R_{k+1}^{\theta}\tilde{\theta}_{k+1}\|^p] + \mathbb{E}^{2/p}[\|R_{k+1}^{w}(w_{k+1} - w^\star)\|^p])\Big)^{1/2}\end{aligned}$$

Using A 6 easy to see that $\|R_{k+1}^{\theta}\| \leq q_R$ and $\|R_{k+1}^{w}\| \leq q_R$, where q_R is defined in (58). Hence, Proposition 3 and Lemma 31-(iii) imply that:

$$\begin{aligned}\sum_{k=1}^{n}(\mathbb{E}^{2/p}[\|R_{k+1}^{\theta}\tilde{\theta}_{k+1}\|^p] &\leq 2q_R^2\sum_{k=1}^{n}\big\{\{\mathrm{C}_0^{\tilde{\theta}}\}^2\prod_{j=0}^{k}\big(1 - \beta_j\frac{a_\Delta}{8}\big) + p^4\{\mathrm{C}_{\mathrm{slow}}\}^2\beta_{k+1}\big\} \\ &\leq 2q_R^2\{\mathrm{C}_0^{\tilde{\theta}}\}^2\frac{k_0^b}{c_{0,\beta}}(1 + \frac{8}{a_\Delta(1-b)}) + 2q_R^2p^4\{\mathrm{C}_{\mathrm{slow}}\}^2\frac{c_{0,\beta}n^{1-b}}{1-b} .\end{aligned}$$

Similarly, one can get using Lemma 8:

$$\begin{aligned}\sum_{k=1}^{n}(\mathbb{E}^{2/p}[\|R_{k+1}^{\theta}(w_{k+1} - w^\star)\|^p] &\leq 2q_R^2\sum_{k=1}^{n}\big\{(\widehat{\mathrm{C}}_0^{\tilde{w}})^2\prod_{j=0}^{k}\big(1 - \beta_j\frac{a_\Delta}{8}\big) + p^6\{\widehat{\mathrm{C}}_{\mathrm{fast}}\}^2\gamma_{k+1}\big\} \\ &\leq 2q_R^2(\widehat{\mathrm{C}}_0^{\tilde{w}})^2\frac{k_0^b}{c_{0,\beta}}(1 + \frac{8}{a_\Delta(1-b)}) + 2q_R^2p^6\{\widehat{\mathrm{C}}_{\mathrm{fast}}\}^2\frac{c_{0,\gamma}n^{1-a}}{1-a} .\end{aligned}$$

The proof follows from gathering similar terms. □

D CLT for the Last iteration estimator

The proof of Theorem 2 is conceptually similar to Theorem 1 but is complicated by two additional factors. First, the covariance of the linear statistic formed by martingale differences now depends on n, which complicates the identification of the limiting covariance matrix. Second, the analysis of the remaining terms becomes more involved.

Before proceeding to the main part of the proof, we state an auxiliary lemma that decomposes the approximation error at the final iteration into linear and nonlinear components.

Lemma 13. *The following decomposition holds:*

$$\tilde{\theta}_{n+1} = \sum_{j=0}^{n} \beta_j G^{(1)}_{j+1:n}\big(A_{12}A_{22}^{-1}\boldsymbol{\varepsilon}_W^{j+1} - \boldsymbol{\varepsilon}_V^{j+1}\big) + R_n^{\mathrm{last}} \;,$$

where the residual term R_n^{last} is defined in (64).

Proof. Following (Konda and Tsitsiklis 2004) and using (13), the equations for $\tilde{\theta}_{n+1}$ and $\tilde{w}_{n+1}$ can be rewritten as follows:

$$\begin{aligned}
\tilde{\theta}_{n+1} &= (\mathrm{I} - \beta_n\Delta)\tilde{\theta}_n - \beta_n A_{12}\tilde{w}_n - \beta_n V_{n+1} + \beta_n\delta_n^{(1)} \;,\\
\tilde{w}_{n+1} &= (\mathrm{I} - \gamma_n A_{22})\tilde{w}_n - \beta_n D_n V_{n+1} - \gamma_n W_{n+1} + \beta_n\delta_n^{(2)} \;,
\end{aligned}$$

where we have set

$$\delta_n^{(1)} = A_{12}L_n\tilde{\theta}_n \;, \quad \delta_n^{(2)} = -(L_{n+1} + A_{22}^{-1}A_{21})A_{12}\tilde{w}_n \;.$$

Throughout the analysis we use the following convention:

$$G^{(1)}_{m:n} := \prod_{i=m}^{n}(\mathrm{I} - \beta_i\Delta), \quad G^{(2)}_{m:n} := \prod_{i=m}^{n}(\mathrm{I} - \gamma_i A_{22}) \;.$$

Hence, $\tilde{\theta}_{n+1}$ and $\tilde{w}_{n+1}$ can be rewritten as follows:

$$\tilde{\theta}_{n+1} = G^{(1)}_{0:n}\tilde{\theta}_0 - \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}\tilde{w}_j - \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}V_{j+1} + \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}\delta_j^{(1)} \;, \tag{60}$$

$$\tilde{w}_{n+1} = G^{(2)}_{0:n}\tilde{w}_0 - \sum_{j=0}^{n}\beta_j G^{(2)}_{j+1:n}D_jV_{j+1} - \sum_{j=0}^{n}\gamma_j G^{(2)}_{j+1:n}W_{j+1} + \sum_{j=0}^{k}\beta_j G^{(2)}_{j+1:n}\delta_j^{(2)} \;. \tag{61}$$

We substitute the right-hand side of (61) into (60) and obtain:

$$\begin{aligned}
\tilde{\theta}_{n+1} = G^{(1)}_{0:n}\tilde{\theta}_0 - \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}G^{(2)}_{0:j-1}\tilde{w}_0 &+ \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}\;\delta_j^{(1)} + S_n^{(1)} + S_n^{(2)} + S_n^{(3)}\\
&+ \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}\big(A_{12}A_{22}^{-1}W_{j+1} - V_{j+1}\big) \;,
\end{aligned}$$

where

$$S_n^{(1)} = -\sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}\sum_{i=0}^{j-1}\beta_i G^{(2)}_{i+1:j-1}\delta_i^{(2)} \;, \tag{62}$$

$$S_n^{(2)} = \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}\sum_{i=0}^{j-1}\beta_i G^{(2)}_{i+1:j-1}D_iV_{i+1} \;,$$

$$S_n^{(3)} = \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}\sum_{i=0}^{j-1}\gamma_i G^{(2)}_{i+1:j-1}W_{i+1} - \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}A_{22}^{-1}W_{j+1} \;.$$

Recall that $\psi_{j+1} = \boldsymbol{\varepsilon}_V^{j+1} - A_{12}A_{22}^{-1}\boldsymbol{\varepsilon}_W^{j+1}$. Substituting V_{j+1}, W_{j+1} from (8) we get

$$\tilde{\theta}_{n+1} = -\sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}\psi_{j+1} + R_n^{\mathrm{last}} \;, \tag{63}$$

where we have set

$$R_n^{\mathrm{last}} = G_{0:n}^{(1)}\tilde{\theta}_0 - \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} G_{0:j-1}^{(2)} \tilde{w}_0 + \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)}\ \delta_j^{(1)} + S_n^{(1)} + S_n^{(2)} + S_n^{(3)} \\ + \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \big(\overbrace{\{\tilde{\mathbf{A}}_{11}^{j+1} - A_{12}A_{22}^{-1}\tilde{\mathbf{A}}_{21}^{j+1}\}\tilde{\theta}_j}^{R_{j+1}^{\theta}} + \overbrace{\{\tilde{\mathbf{A}}_{12}^{j+1} - A_{12}A_{22}^{-1}\tilde{\mathbf{A}}_{21}^{j+1}\}(w_j - w^{\star})}^{R_{j+1}^{w}} \big)}_{H_n} \tag{64}$$

□

Now we proceed with Theorem 2 applying the decomposition that is proven above together with Lemmas 14-18 which imply a moment bound for R_n^{last}.

Lemma 14. *Let $p \geq 2$. Assume A1, A2(p), A3, A4, A5, A6, A7. Therefore, it holds that*

$$\mathbb{E}^{1/p}\big[\|\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \delta_j^{(1)}\|^p\big] \lesssim (2b-a-1)^{-1} P_{0:n}^{(1)} + p^4 (2b-a-1)^{-1} \beta_n^{\frac{2b-a/2-1}{b}} .$$

Lemma 15. *Let $p \geq 2$. Assume A1, A2(p), A3, A4, A5, A6. Therefore, it holds that*

$$\mathbb{E}^{1/p}\big[\|H_n\|^p\big] \lesssim p(2b-1)^{-1/2} \prod_{j=0}^{n} (1 - \frac{a_\Delta}{8}\beta_j) + p^4 \beta_n^{\frac{b+a}{2b}} .$$

Lemma 16. *Let $p \geq 2$. Assume A1, A2(p), A3, A4, A5, A6. Therefore, it holds that*

$$\mathbb{E}^{1/p}\big[\|S_n^{(1)}\|^p\big] \lesssim P_{0:n}^{(1)} + p^4 \beta_n^{(3b-2a)/2b} ,$$

where $S_n^{(1)}$ is defined in (62).

Lemma 17. *Let $p \geq 2$. Assume A1, A2(p), A3, A4, A5, A6. Therefore, it holds that*

$$\mathbb{E}^{1/p}\big[\|S_n^{(2)}\|^p\big] \lesssim p^4 \beta_n^{\frac{3b-2a}{2b}} ,$$

where $S_n^{(2)}$ is defined in (62).

Lemma 18. *Let $p \geq 2$. Assume A1, A2(p), A3, A4, A5, A6. Therefore, it holds that*

$$\mathbb{E}^{1/p}\big[\|S_n^{(3)}\|^p\big] \lesssim p^4 \beta_n^{\frac{2b-a}{2b}} ,$$

where $S_n^{(3)}$ is defined in (62).

Proof of Theorem 2. Our proof starts from the error decomposition (63), which allows us to write

$$\beta_n^{-1/2}\tilde{\theta}_{n+1} = -\beta_n^{-1/2} \sum_{j=0}^{k} \beta_j G_{j+1:n}^{(1)} \psi_{j+1} + \beta_n^{-1/2} R_n^{\mathrm{last}} ,$$

where the term R_n^{last} is given in Lemma 9. Assumption A3 implies that

$$\mathbf{\Sigma}_n^{\mathrm{last}} := \sum_{j=0}^{n} \beta_j^2 G_{j+1:n}^{(1)} \mathbf{\Sigma}_\varepsilon \big(G_{j+1:n}^{(1)}\big)^\top .$$

As established in Proposition 9, the sequence $\mathbf{\Sigma}_n^{\mathrm{last}}$ converges to the matrix $\mathbf{\Sigma}_\infty^{\mathrm{last}}$. Since the convex distance is invariant to non-degenerate linear transformations, we get

$$\rho^{\mathrm{Conv}}\big(\beta_n^{-1/2}\tilde{\theta}_{n+1}, \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{last}})\big) = \rho^{\mathrm{Conv}}\big(\beta_n^{-1/2}(\mathbf{\Sigma}_\infty^{\mathrm{last}})^{-1/2}\tilde{\theta}_{n+1}, \mathcal{N}(0, \mathrm{I})\big) .$$

Hence, to control this term we apply Proposition 1 and triangle inequality for the convex distance and obtain

$$\rho^{\mathrm{Conv}}\big(\beta_n^{-1/2}\tilde{\theta}_{n+1}\ , \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{last}})\big) \leq 2c_d^{p/p+1} \mathbb{E}^{1/(p+1)}\big[\|\big(\mathbf{\Sigma}_\infty^{\mathrm{last}}\big)^{-1/2} \beta_n^{-1/2} R_n^{\mathrm{last}}\|^p\big] \\ + \rho^{\mathrm{Conv}}\big(-\beta_n^{-1/2} \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \psi_{j+1}\big)\ , \mathcal{N}(0, \beta_n^{-1}\mathbf{\Sigma}_n^{\mathrm{last}})\big) \\ + \rho^{\mathrm{Conv}}\big(\mathcal{N}(0, \beta_n^{-1}\mathbf{\Sigma}_n^{\mathrm{last}})\ , \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{last}})\big)\big) , \tag{65}$$

To handle the second term in (65) we apply Lemma 2 and get

$$\rho^{\mathrm{Conv}}\big(-\beta_n^{-1/2}\sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}\psi_{j+1}\big)\ ,\mathcal{N}(0,\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last}})\big) \lesssim [1+(2+\log(d_\theta\|\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last}}\|))^+]^{1/2}\frac{\sqrt{d_\theta\log n}}{n^{1/4}} + \frac{d_\theta^{1/4}\|\frac{1}{n}\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last}}\|^{1/4}}{n^{1/4}}$$

$$\overset{(a)}{\lesssim} [1+(2+\log(d_\theta\|\boldsymbol{\Sigma}_\infty^{\mathrm{last}}\|))^+]^{1/2}\frac{\sqrt{d_\theta\log n}}{n^{1/4}}\ ,$$

where in (a) we have used A7, i.e. $\frac{1}{2}\|\boldsymbol{\Sigma}_\infty^{\mathrm{last}}\| \le \|\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last}}\| \lesssim \|\boldsymbol{\Sigma}_\infty^{\mathrm{last}}\|$. The bound for the third term in (65) follows from (Devroye, Mehrabian, and Reddad 2018, Theorem 1.1) and Proposition 9:

$$\rho^{\mathrm{Conv}}(\mathcal{N}(0,\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last}}),\mathcal{N}(0,\boldsymbol{\Sigma}_\infty^{\mathrm{last}})) \lesssim \|\{\boldsymbol{\Sigma}_\infty^{\mathrm{last}}\}^{-1/2}(\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last}})\{\boldsymbol{\Sigma}_\infty^{\mathrm{last}}\}^{-1/2} - \mathrm{I}\|_{\mathrm{Fr}} \lesssim \frac{\sqrt{d}}{n^b\lambda_{\min}(\boldsymbol{\Sigma}_\infty^{\mathrm{last}})}\ .$$

Now we proceed with the first term in (65). The moment bound for R_n^{last} follows from Lemmas 14-18 that we have stated above. Combining these lemmas together with the inequality $\beta_n^{\frac{2b-a/2-1}{b}} > \beta_n^{\frac{2b-a}{2b}} > \beta_n^{\frac{3b-2a}{2b}}$ and $\beta_n^{\frac{2b-a}{2b}} \ge \beta_n^{\frac{b+a}{2b}}$, we obtain

$$\mathbb{E}^{1/p}[\|R_n^{\mathrm{last}}\|^p] \lesssim \frac{p}{2b-1}\prod_{j=0}^{n}(1-\frac{a_\Delta}{8}\beta_j) + p^4\big(\frac{1}{2b-a-1}\beta_n^{\frac{2b-a/2-1}{b}} + \beta_n^{\frac{b+a}{2b}} + \beta_n^{\frac{3b-2a}{2b}} + \beta_n^{\frac{2b-a}{2b}}\big)$$

$$\lesssim \frac{p}{2b-1}\prod_{j=0}^{n}(1-\frac{a_\Delta}{8}\beta_j) + \frac{p^4}{2b-a-1}\beta_n^{\frac{2b-a/2-1}{b}}\ .$$

Now we set $p=\log n$. Next, the bound for the first term in (65) follows from $(n^{-\alpha})^{\frac{\log n}{1+\log n}} \lesssim n^{-\alpha}$ and $(\sum a_i)^q \le \sum a_i^q$ for $a_i>0$ and $q\in(0,1)$:

$$\mathbb{E}^{1/(p+1)}\big[\|\big(\boldsymbol{\Sigma}_\infty^{\mathrm{last}}\big)^{-1/2}\beta_n^{-1/2}R_n^{\mathrm{last}}\|^p\big] \lesssim \beta_n^{-1/2}\frac{\log n}{2b-1}\prod_{j=0}^{n}(1-\frac{a_\Delta}{8}\beta_j) + \beta_n^{\frac{3b-a-2}{2b}}\frac{\log^4 n}{2b-a-1}\ .$$

Gathering previous bounds we obtain

$$\rho^{\mathrm{Conv}}\big(\beta_n^{-1/2}\tilde{\theta}_{n+1},\mathcal{N}(0,\boldsymbol{\Sigma}_\infty^{\mathrm{last}})\big) \lesssim n^{b/2}\frac{\log n}{2b-1}\prod_{j=0}^{n}(1-\frac{a_\Delta}{8}\beta_j) + \frac{\log^4 n}{(2b-a-1)n^{\frac{3b-a-2}{2}}}\ .$$

□

Now we give proofs for Lemmas 14-18.

Proof of Lemma 14. Recall that

$$\delta_j^{(1)} = A_{12}L_j\big\{\Gamma_{0:j-1}^{(1)}\tilde{\theta}_0 - \sum_{i=0}^{j-1}\beta_i\Gamma_{i+1:j-1}^{(1)}A_{12}\tilde{w}_i - \sum_{i=0}^{j-1}\beta_i\Gamma_{i+1:j-1}^{(1)}V_{i+1}\big\}\ .$$

Therefore, easy to see that

$$\sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}\delta_j^{(1)} = \underbrace{\sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}A_{12}L_j\Gamma_{0:j-1}^{(1)}\tilde{\theta}_0}_{T_1} - \underbrace{\sum_{i=0}^{n-1}\sum_{j=i+1}^{n}\beta_j G_{j+1:n}^{(1)}A_{12}L_j\beta_i\Gamma_{i+1:j-1}^{(1)}A_{12}\tilde{w}_i}_{T_2}$$

$$- \underbrace{\sum_{i=0}^{n-1}\sum_{j=i+1}^{n}\beta_j G_{j+1:n}^{(1)}A_{12}L_j\beta_i\Gamma_{i+1:j-1}^{(1)}V_{i+1}}_{T_3}\ .$$

First, we derive a bound for T_1 using Minkowski's inequality

$$\mathbb{E}^{1/p}\big[\|T_1\|^p\big] \le \kappa_\Delta\ell_\infty\|A_{12}\|\|\tilde{\theta}_0\|(1-\frac{a_\Delta}{2}\beta_0)^{-1}P_{0:n}^{(1)}\sum_{j=0}^{n}\frac{\beta_j^2}{\gamma_j} \lesssim (2b-a-1)^{-1}P_{0:n}^{(1)}\ ,$$

where the last transition employs A7 and integral bound

$$\sum_{j=0}^{n}\frac{\beta_j^2}{\gamma_j} = \frac{c_{0,\beta}^2}{c_{0,\gamma}}\sum_{j=0}^{n}(j+k_0)^{a-2b} \leq \frac{c_{0,\beta}^2}{c_{0,\gamma}}\frac{k_0^{a+1-2b}}{2b-a-1} \leq \frac{c_{0,\beta}^2}{c_{0,\gamma}(2b-a-1)} \;.$$

We conclude that $\mathbb{E}^{1/p}\big[\|T_1\|^p\big] \lesssim (2b-a-1)^{-1}P_{0:n}^{(1)}$. Since V_{i+1} is a martingale difference sequence, Burkholder's inequality (Osekowski 2012, Therorem 8.1) implies

$$\mathbb{E}^{2/p}[\|T_3\|^p] \leq p^2\sum_{i=0}^{k-1}\beta_i^2\big\|\sum_{j=i+1}^{n}\beta_j G_{j+1:n}^{(1)}A_{12}L_j\Gamma_{i+1:j-1}^{(1)}\big\|^2\mathbb{E}^{2/p}[\|V_{i+1}\|^p] \;.$$

On the other hand, bounding the term $\sum_{j=i+1}^{n}\beta_j^2/\gamma_j$ via integral estimates gives

$$\begin{aligned}\|\sum_{j=i+1}^{n}\beta_j G_{j+1:n}^{(1)}A_{12}L_j\Gamma_{i+1:j-1}^{(1)}\| &\leq \frac{\kappa_\Delta\|A_{12}\|\ell_\infty}{1-\frac{a_\Delta}{2}\beta_0}P_{i+1:n}^{(1)}\sum_{j=i+1}^{n}\frac{\beta_j^2}{\gamma_j} \leq \frac{\kappa_\Delta\|A_{12}\|\ell_\infty c_{0,\beta}^2(i+k_0)^{a+1-2b}}{(1-\frac{a_\Delta}{2}\beta_0)c_{0,\gamma}(2b-a-1)}P_{i+1:n}^{(1)}\\ &\lesssim \beta_i^{\frac{2b-a-1}{b}}(2b-a-1)^{-1}P_{i+1:n}^{(1)} \;.\end{aligned}$$

Note that $\mathbb{E}^{1/p}[\|V_{i+1}\|^p] \lesssim p^3$ due to Proposition 3. Thus, Lemma 31-(ii) implies the following bound

$$\mathbb{E}^{2/p}[\|T_3\|^p] \lesssim p^8(2b-a-1)^{-2}\sum_{i=0}^{n-1}\beta_i^{2+\frac{2(2b-a-1)}{b}}P_{i+1:n}^{(1)} \lesssim (2b-a-1)^{-2}p^8\beta_n^{1+\frac{2(2b-a-1)}{b}} \;.$$

Now we will get a bound for $\mathbb{E}^{1/p}\big[\|T_2\|^p\big]$. Minkowski's inequality and Proposition 3 imply that

$$\begin{aligned}\mathbb{E}^{1/p}\big[\|T_2\|^p\big] &\leq \frac{\kappa_\Delta\ell_\infty\|A_{12}\|^2}{1-\frac{a_\Delta}{2}\beta_0}\sum_{i=0}^{n-1}\beta_i P_{i+1:n}^{(1)}\sum_{j=i+1}^{n}\frac{\beta_j^2}{\gamma_j}(C_0^{\tilde{w}}P_{0:i-1}^{(2)} + \widetilde{\mathrm{C}}_{\mathrm{fast}}p^3\sqrt{\gamma_{i-1}})\\ &\lesssim (2b-a-1)^{-1}\sum_{i=0}^{n-1}\beta_i^{1+\frac{2b-a-1}{b}}P_{i+1:n}^{(1)}(P_{0:i-1}^{(2)} + p^3\sqrt{\gamma_{i-1}}) \;.\end{aligned}$$

Using A5 and Lemma 32-(ii) we get

$$\sum_{i=0}^{n-1}\beta_i^{1+\frac{2b-a-1}{b}}P_{i+1:n}^{(1)}P_{0:i-1}^{(2)} \leq \sum_{i=0}^{n-1}\beta_0 P_{i+1:n}^{(1)}P_{0:i-1}^{(2)} \lesssim P_{0:k}^{(1)} \;.$$

and

$$\sum_{i=0}^{n-1}\beta_i^{1+\frac{2b-a-1}{b}}P_{i+1:n}^{(1)}\sqrt{\gamma_{i-1}} \lesssim \sum_{i=0}^{k-1}\beta_i^{1+\frac{2b-a/2-1}{b}}P_{i+1:n}^{(1)} \lesssim \beta_n^{\frac{2b-a/2-1}{b}} \;.$$

Hence,

$$\mathbb{E}^{1/p}\big[\|T_2\|^p\big] \lesssim (2b-a-1)^{-1}\big(p^3\beta_n^{\frac{2b-a/2-1}{b}} + P_{0:n}^{(1)}\big) \;.$$

We finish the proof applying Minkowski's inequality

$$\mathbb{E}^{1/p}\big[\|\sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}\delta_j^{(1)}\|^p\big] \leq \sum_{i=1}^{3}\mathbb{E}^{1/p}\big[\|T_i\|^p\big] \lesssim (2b-a-1)^{-1}\big(P_{0:n}^{(1)} + p^4\beta_n^{\frac{2b-a-1}{b}}\big) \;.$$

□

Proof of Lemma 15. Note that since V_{k+1}, W_{k+1} and $\varepsilon_V, \varepsilon_W$ are martingale difference sequences, $R_{k+1}^{\theta}\tilde{\theta}_{k+1} + R_{k+1}^{w}(w_k - w^\star)$ is a martingale difference sequence. Recall that

$$H_n = \sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}R_{j+1}^{\theta}\tilde{\theta} + \sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}R_{j+1}^{w}(w_j - w^\star) \;.$$

Note that $\|R^{\theta}_{k+1}\| \leq q_R$ and $\|R^{w}_{k+1}\| \leq q_R$, where q_R is defined in (58). Burkholder's inequality (Osekowski 2012, Theorem 8.1) implies that

$$\mathbb{E}^{1/p}[\|H_n\|^p] \leq p\bigg(\sum_{j=0}^{n} \beta_j^2 P^{(1)}_{j+1:n} \mathbb{E}^{2/p}[\|R^{\theta}_{j+1}\tilde{\theta}_j\|^p]\bigg)^{1/2} + p\bigg(\sum_{j=0}^{n} \beta_j^2 P^{(1)}_{j+1:n} \mathbb{E}^{2/p}[\|R^{\theta}_{j+1}(w_j - w^\star)\|^p]\bigg)^{1/2} .$$

Hence, we get applying Proposition 3 and Lemma 31-(ii):

$$\begin{aligned}\sum_{j=0}^{n} \beta_j^2 P^{(1)}_{j+1:n} \mathbb{E}^{2/p}[\|R^{\theta}_{j+1}\tilde{\theta}_j\|^p] &\lesssim \big(q_R\, \mathrm{C}^{\tilde{\theta}}_0\big)^2 \sum_{j=0}^{n} \beta_j^2 P^{(1)}_{j+1:n} \prod_{t=0}^{j-1}(1 - \frac{a_\Delta}{4}\beta_t) + p^4\big(q_R C_{\mathrm{slow}}\big)^2 \sum_{j=0}^{n} \beta_j^3 P^{(1)}_{j+1:n} \\ &\lesssim (2b-1)^{-1} \prod_{j=0}^{n}(1 - \frac{a_\Delta}{4}\beta_j) + p^4 \beta_n^2 .\end{aligned}$$

Now we use Lemma 8 and obtain the similar bound

$$\begin{aligned}\sum_{j=0}^{n} \beta_j^2 P^{(1)}_{j+1:n} (\mathbb{E}^{1/p}\big[\|R^{w}_{j+1}(w_j - w^\star)\|^p\big])^2 &\lesssim \big(q_R \widehat{\mathrm{C}}^{\tilde{w}}_0\big)^2 \sum_{j=0}^{n} \beta_j^2 P^{(1)}_{j+1:n} \prod_{t=0}^{j-1}(1 - \frac{a_\Delta}{4}\beta_t) + +p^6\big(q_R \widehat{\mathrm{C}}_{\mathrm{fast}}\big)^2 \sum_{j=0}^{n} \beta_j^{2+a/b} P^{(1)}_{j+1:n} \\ &\lesssim (2b-1)^{-1} \prod_{j=0}^{n}(1 - \frac{a_\Delta}{4}\beta_j) + p^6 \beta_n^{\frac{a+b}{b}} .\end{aligned}$$

Square-root operation followed by dominant-term selection proves the claim. □

Proof of Lemma 16. First, rewrite $\delta_i^{(2)}$ as follows:

$$\delta_i^{(2)} = -(L_{i+1} + A_{22}^{-1}A_{21})A_{12}\big(\Gamma^{(2)}_{0:i-1}\tilde{w}_0 - \sum_{t=0}^{i-1} \Gamma^{(2)}_{t+1:i-1}\xi_{t+1}\big) .$$

Thus, we get

$$\begin{aligned}S_n^{(1)} = &-\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{j-1} \beta_j G^{(1)}_{j+1:n} A_{12} \beta_i G^{(2)}_{i+1:j-1}(L_{i+1} + A_{22}^{-1}A_{21})A_{12}\Gamma^{(2)}_{0:i-1}\tilde{w}_0}_{T_1} \\ &+\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{j-1}\sum_{t=0}^{i-1} \beta_j G^{(1)}_{j+1:n} A_{12} \beta_i G^{(2)}_{i+1:j-1}(L_{i+1} + A_{22}^{-1}A_{21})A_{12}\Gamma^{(2)}_{t+1:i-1}\xi_{t+1}}_{T_2} .\end{aligned}$$

First, we derive a bound for T_1

$$\mathbb{E}^{1/p}\big[\|T_1\|^p\big] = \|T_1\| \leq \kappa_{22}\sqrt{\kappa_\Delta}\|A_{12}\|(\ell_\infty + \|A_{22}^{-1}A_{21}\|)\|A_{12}\|\|\tilde{w}_0\| \sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} \beta_i\beta_j P^{(1)}_{j+1:n} P^{(2)}_{i+1:j-1} P^{(2)}_{0:i-1} ,$$

and using Lemma 32-(ii) and $\beta_j \leq \beta_i$ when $j < i$ we get

$$\begin{aligned}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} \beta_i\beta_j P^{(1)}_{j+1:n} P^{(2)}_{i+1:j-1} P^{(2)}_{0:i-1} &\leq \sum_{i=0}^{n-1} \beta_i P^{(2)}_{0:i-1} \sum_{j=i+1}^{n-1} \beta_i P^{(1)}_{j+1:n} P^{(2)}_{i+1:j-1} \leq \mathrm{C}^{P}_{\beta} \sum_{i=0}^{n-1} \beta_i \gamma_i^{(b-a)/a} P^{(1)}_{i+1:n} P^{(2)}_{0:i-1} \\ &\lesssim \mathrm{C}^{P}_{\beta} \sum_{i=1}^{n-1} \beta_0 P^{(1)}_{i+1:n} P^{(2)}_{1:i-1} \lesssim P^{(1)}_{0:n} .\end{aligned}$$

Collecting these results yields the bound $\mathbb{E}^{1/p}\big[\|T_1\|^p\big] \lesssim P^{(1)}_{0:n}$. To write a bound for T_2 we change the order of summation

$$T_2 = \sum_{t=0}^{n-2} \underbrace{\big(\sum_{i=t+1}^{n-1}\sum_{j=i+1}^{n} \beta_j G^{(1)}_{j+1:n} A_{12} \beta_i G^{(2)}_{i+1:j-1}(L_{i+1} + A_{22}^{-1}A_{21})A_{12}\Gamma^{(2)}_{t+1:i-1}\big)}_{U_t} \xi_{t+1} ,$$

and combine Burkholder's inequality (Osekowski 2012, Theorem 8.1) with $\mathbb{E}^{1/p}[\|\xi_{t+1}\|^p] \lesssim p^3\gamma_t$:

$$\mathbb{E}^{1/p}\big[\|T_2\|^p\big] \lesssim p^4\bigg(\sum_{t=0}^{k-2}\|U_t\|^2\gamma_t^2\bigg)^{1/2} .$$

Now we use Lemma 32-(ii) and get

$$\|U_t\| \lesssim \sum_{i=t+1}^{n-1}\sum_{j=i+1}^{n}\beta_j\beta_i P^{(1)}_{j+1:n}P^{(2)}_{i+1:j-1}P^{(2)}_{t+1:i-1} \lesssim \sum_{i=t+1}^{n-1}\beta_i\gamma_i^{(b-a)/a}P^{(1)}_{i+1:n}P^{(2)}_{t+1:i-1} \lesssim P^{(1)}_{t+1:n}\gamma_t^{2(b-a)/a} .$$

Note that due to Lemma 31-(ii)

$$\sum_{t=0}^{n-2}\gamma_t^{2+4(b-a)/a}P^{(1)}_{t+1:n} \lesssim \sum_{t=0}^{n-2}\beta_t^{(4b-2a)/b}P^{(1)}_{t+1:n} \lesssim \beta_n^{(3b-2a)/b} .$$

The latter inequality yields the required bound. □

Proof of Lemma 17. First, recall that

$$S^{(2)}_n = \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}\sum_{i=0}^{j-1}\beta_i G^{(2)}_{i+1:j-1}D_iV_{i+1} .$$

We change the order of summation and get

$$S^{(2)}_n = \sum_{i=0}^{n-1}\beta_i\big(\sum_{j=i+1}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}G^{(2)}_{i+1:j-1}\big)D_iV_{i+1} .$$

Burkholder's inequality (Osekowski 2012, Theorem 8.1) immediately implies that

$$\mathbb{E}^{1/p}\big[\|S^{(2)}_n\|^p\big] \le p\bigg(\sum_{i=0}^{n-1}\beta_i^2c_\infty^2\Big\|\sum_{j=i+1}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}G^{(2)}_{i+1:j-1}\Big\|^2\mathbb{E}^{2/p}[\|V_{i+1}\|^p]\bigg)^{1/2} .$$

Now we use Lemma 32-(ii) and get

$$\Big\|\sum_{j=i+1}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}G^{(2)}_{i+1:j-1}\Big\| \lesssim \sum_{j=i+1}^{n}\beta_j P^{(1)}_{j+1:n}P^{(2)}_{i+1:j-1} \lesssim \gamma_i^{(b-a)/a}P^{(1)}_{i+1:n} ,$$

and the desirable result follows from Lemma 31-(ii):

$$\mathbb{E}^{1/p}\big[\|S^{(2)}_n\|^p\big] \lesssim p\bigg(p^6\sum_{i=0}^{n-1}\beta_i^2\gamma_i^{2(b-a)/a}P^{(1)}_{i+1:n}\bigg)^{1/2} \lesssim p^4\bigg(\sum_{i=0}^{n-1}\beta_i^{2+2(b-a)/b}P^{(1)}_{i+1:n}\bigg)^{1/2} \lesssim p^4\beta_n^{\frac{3b-2a}{2b}} .$$

□

Proof of Lemma 18. Recall that

$$S^{(3)}_n = \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}\sum_{i=0}^{j-1}\gamma_i G^{(2)}_{i+1:j-1}W_{i+1} - \sum_{j=0}^{n}\beta_j G^{(1)}_{j+1:n}A_{12}A_{22}^{-1}W_{j+1} .$$

Changing the order of summation, rewrite $S^{(3)}_n$ as follows

$$S^{(3)}_n = \sum_{i=0}^{n}\beta_i G^{(1)}_{i+1:n}\big(\frac{\gamma_i}{\beta_i}\sum_{j=i+1}^{n}\beta_j\big(G^{(1)}_{i+1:j}\big)^{-1}A_{12}G^{(2)}_{i+1:j-1} - A_{12}A_{22}^{-1}\big)W_{i+1} . \tag{66}$$

We can rewrite the term inside the brackets in (66) as

$$\begin{aligned} S^{(3)}_n = \sum_{i=0}^{n}\beta_i G^{(1)}_{i+1:n}\bigg\{&\underbrace{\sum_{j=i+1}^{n}\gamma_j\big(\frac{\gamma_i\beta_j}{\beta_i\gamma_j}\big(G^{(1)}_{i+1:j}\big)^{-1} - \mathrm{I}\big)A_{12}G^{(2)}_{i+1:j-1}}_{Z_i^{(1)}} + \underbrace{A_{12}\big(\sum_{j=i+1}^{n}\gamma_j G^{(2)}_{i+1:j-1} - \int_0^{\sum_{j=i+1}^n\gamma_j}\exp(-A_{22}t)\,dt\big)}_{Z_i^{(2)}} \\ &\underbrace{-A_{12}A_{22}^{-1}\exp\big(-\sum_{j=i+1}^{n}\gamma_jA_{22}\big)}_{Z_i^{(3)}}\bigg\}W_{i+1} . \end{aligned}$$

Consider the real valued positive sequence $\{\epsilon_k\}$ defined by the equations:

$$\frac{\beta_{k+1}}{\gamma_{k+1}} = \frac{\beta_k}{\gamma_k}\big(1 - \epsilon_k \gamma_k\big) \ .$$

As shown in (Konda and Tsitsiklis 2004), the following estimates hold:

$$\|Z_i^{(1)}\| \lesssim \beta_i/\gamma_i + \epsilon_i, \quad \|Z_i^{(2)}\| \lesssim \gamma_i \ .$$

(Godunov 1997) implies that:

$$\|Z_i^{(3)}\| \leq \sqrt{\kappa_{22}} \exp\big(- \frac{1}{2\|Q_{22}\|} \sum_{j=i+1}^{n} \gamma_j \big) \ .$$

Using A5, easy to see that:

$$\epsilon_k = \gamma_k^{-1} - \beta_{k+1}\beta_k^{-1}\gamma_{k+1}^{-1} \leq \gamma_{k+1}^{-1}\beta_k^{-1}(\beta_k - \beta_{k+1}) \lesssim \beta_k^{-a/b}\beta_k^{-1}\beta_k^2 = \beta_k^{1-a/b} \ .$$

Thus, $\|Z_i^{(1)}\| \lesssim \beta_i^{1-a/b}$. Now we use $\mathbb{E}^{1/p}[\|W_{i+1}\|^p] \lesssim p^3$ together with Burkholder's inequality (Osekowski 2012, Theorem 8.1) and get:

$$\begin{aligned}\mathbb{E}^{2/p}[\|S_n^{(3)}\|^p] &\lesssim p^8 \sum_{i=0}^{n} \beta_i^2 P_{i+1:n}^{(1)} \big(\|Z_i^{(1)}\|^2 + \|Z_i^{(2)}\|^2 + \|Z_i^{(3)}\|^2\big) \\ &\lesssim p^8 \left(\sum_{i=0}^{n} \beta_i^4/\gamma_i^2 P_{i+1:n}^{(1)} + \sum_{i=0}^{n} \beta_i^2 \beta_i^{2-\frac{2a}{b}} P_{i+1:n}^{(1)} + \sum_{i=0}^{n} \beta_i^2 \gamma_i^2 P_{i+1:n}^{(1)} + \sum_{i=0}^{n} \beta_i^2 \exp\big(- \frac{1}{2}\|Q_{22}\|^{-1} \sum_{j=i+1}^{n} \gamma_j \big) P_{i+1:n}^{(1)} \right) .\end{aligned}$$

Rewriting γ_i in terms of β_i and applying Lemma 31-(ii) together with A5 we get:

$$\mathbb{E}^{2/p}[\|S_n^{(3)}\|^p] \lesssim p^8 \big(\beta_n^{\frac{3b-2a}{b}} + \beta_n^{\frac{3b-2a}{b}} + \beta_n^{\frac{b+2a}{b}} + \beta_n^{\frac{2b-a}{b}} \big) \lesssim p^8 \beta_n^{\frac{2b-a}{b}} \ .$$

□

E Markov noise

E.1 High-order moment bounds

We preface this section with a brief reminder of notation used in the Markov chains literature. For a Markov kernel P on $(\mathsf{X}, \mathcal{X})$, and a measurable function $f : \mathsf{X} \to \mathbb{R}$, we set

$$\mathrm{P} f(x) = \int_{\mathsf{X}} f(y) \mathrm{P}(x, \mathrm{d}y) \ .$$

Define also total variation distance $\mathsf{d}_{\mathrm{tv}}(\mu, \nu)$ for probability measures μ, ν:

$$\mathsf{d}_{\mathrm{tv}}(\mu, \nu) = \sup_{\|f\|_\infty \leq 1} |\mu(f) - \nu(f)| \ .$$

B 1 ensures that P is uniformly geometrically ergodic and, moreover, for all k it holds that

$$\Delta(\mathrm{P}^k) := \sup_{x, x' \in \mathsf{X}} \mathsf{d}_{\mathrm{tv}}(\mathrm{P}^k(x, \cdot), \mathrm{P}^k(x', \cdot)) \leq (1/4)^{\lceil k/t_{\mathrm{mix}} \rceil} \ , \tag{67}$$

where $t_{\mathrm{mix}} \in \mathbb{N}$ is the mixing time that controls the rate of convergence to the stationary distribution.

We proceed with the proof based on the Poison decomposition, following (Kaledin et al. 2020). Note that under B 1 the Poisson equation, associated with P, that

$$\boldsymbol{g}^f(x) - \mathrm{P}\boldsymbol{g}^f(x) = f(x) - \pi(f) \ , \quad x \in \mathsf{X} \ , \tag{68}$$

has a unique solution for any bounded measurable f, which is given by the formula

$$\boldsymbol{g}^f(x) = \sum_{k=0}^{\infty} \{\mathrm{P}^k f(x) - \pi(f)\} \ .$$

Moreover, using B 1 and the inequality (67), one can show that $\boldsymbol{g}^f$ is also bounded with

$$\|\boldsymbol{g}^f\|_\infty \leq \sum_{k=0}^{+\infty} \sup_{x \in \mathsf{X}} \|\mathrm{P}^k f(x) - \pi(f)\|_\infty \leq 2\|f\|_\infty \sum_{k=0}^{+\infty} (1/4)^{\lfloor k/t_{\mathrm{mix}} \rfloor} \leq (8/3) t_{\mathrm{mix}} \|f\|_\infty \ . \tag{69}$$

Throughout this chapter, we use a shorthand notation

$$\boldsymbol{g}_k^f := \boldsymbol{g}^f(X_k) \; . \tag{70}$$

We use the above notations for the solution to Poisson equation with different vector- and matrix-valued functions in the equation (68). To proceed with the proof, we follow the idea of (Kaledin et al. 2020), where the authors have obtained similar results for the 2nd moment bounds. The main idea is to decompose the TTSA updates θ_k and w_k into a sum of two coupled TTSA recursions. Namely, $\theta_k = \theta_k^{(0)} + \theta_k^{(1)}$ and $w_k = w_k^{(0)} + w_k^{(1)}$, where

$$\begin{cases} \theta_{k+1}^{(0)} & = \theta_k^{(0)} + \beta_k(b_1 - A_{11}\theta_k^{(0)} - A_{12}w_k^{(0)} + V_{k+1}^{(0)}) \; , \theta_0^{(0)} = \theta_0 \; , \\ w_{k+1}^{(0)} & = w_k^{(0)} + \gamma_k(b_2 - A_{21}\theta_k^{(0)} - A_{22}w_k^{(0)} + W_{k+1}^{(0)}) \; , w_0^{(0)} = w_0 \; , \end{cases} \tag{71}$$

and

$$\begin{cases} \theta_{k+1}^{(1)} & = \theta_k^{(1)} - \beta_k(A_{11}\theta_k^{(1)} + A_{12}w_k^{(1)} - V_{k+1}^{(1)}) \; , \theta_0^{(1)} = 0 \; , \\ w_{k+1}^{(1)} & = w_k^{(1)} - \gamma_k(A_{21}\theta_k^{(1)} + A_{22}w_k^{(1)} - W_{k+1}^{(1)}) \; , w_0^{(1)} = 0 \; . \end{cases}$$

In the above recursions the noise variables $V_k^{(0)}, V_k^{(1)}, W_k^{(0)}, W_k^{(1)}$ are defined as follows:

$$\begin{aligned} V_{k+1}^{(0)} &= \{\boldsymbol{g}_{k+1}^{\boldsymbol{\varepsilon}_V} - \mathrm{P}\boldsymbol{g}_k^{\boldsymbol{\varepsilon}_V}\} - \{\boldsymbol{g}_{k+1}^{\mathbf{A}_{11}} - \mathrm{P}\boldsymbol{g}_k^{\mathbf{A}_{11}}\}(\theta_k - \theta^\star) - \{\boldsymbol{g}_{k+1}^{\mathbf{A}_{12}} - \mathrm{P}\boldsymbol{g}_k^{\mathbf{A}_{12}}\}(w_k - w^\star) \; , \\ W_{k+1}^{(0)} &= \{\boldsymbol{g}_{k+1}^{\boldsymbol{\varepsilon}_W} - \mathrm{P}\boldsymbol{g}_k^{\boldsymbol{\varepsilon}_W}\} - \{\boldsymbol{g}_{k+1}^{\mathbf{A}_{21}} - \mathrm{P}\boldsymbol{g}_k^{\mathbf{A}_{21}}\}(\theta_k - \theta^\star) - \{\boldsymbol{g}_{k+1}^{\mathbf{A}_{22}} - \mathrm{P}\boldsymbol{g}_k^{\mathbf{A}_{22}}\}(w_k - w^\star) \; , \\ V_{k+1}^{(1)} &= \{\mathrm{P}\boldsymbol{g}_k^{\boldsymbol{\varepsilon}_V} - \mathrm{P}\boldsymbol{g}_{k+1}^{\boldsymbol{\varepsilon}_V}\} + \{\mathrm{P}\boldsymbol{g}_{k+1}^{\mathbf{A}_{11}} - \mathrm{P}\boldsymbol{g}_k^{\mathbf{A}_{11}}\}(\theta_k - \theta^\star) + \{\mathrm{P}\boldsymbol{g}_{k+1}^{\mathbf{A}_{12}} - \mathrm{P}\boldsymbol{g}_k^{\mathbf{A}_{12}}\}(w_k - w^\star) \; , \\ W_{k+1}^{(1)} &= \{\mathrm{P}\boldsymbol{g}_k^{\boldsymbol{\varepsilon}_W} - \mathrm{P}\boldsymbol{g}_{k+1}^{\boldsymbol{\varepsilon}_W}\} + \{\mathrm{P}\boldsymbol{g}_{k+1}^{\mathbf{A}_{21}} - \mathrm{P}\boldsymbol{g}_k^{\mathbf{A}_{21}}\}(\theta_k - \theta^\star) + \{\mathrm{P}\boldsymbol{g}_{k+1}^{\mathbf{A}_{22}} - \mathrm{P}\boldsymbol{g}_k^{\mathbf{A}_{22}}\}(w_k - w^\star) \; . \end{aligned}$$

It is easy to see that $\mathbb{E}^{\mathcal{F}_k}\left[V_{k+1}^{(0)}\right] = 0$ and $\mathbb{E}^{\mathcal{F}_k}\left[W_{k+1}^{(0)}\right] = 0$ $\mathbb{P}$-a.s. Similar to (12), we do a change of variables and define

$$\begin{cases} \tilde{\theta}_k^{(0)} & = \theta_k^{(0)} - \theta^\star \; , \\ \tilde{w}_k^{(0)} & = w_k^{(0)} - w^\star + D_{k-1}\tilde{\theta}_k^{(0)} \; , \end{cases} \qquad \begin{cases} \tilde{\theta}_k^{(1)} & = \theta_k^{(1)} \; , \\ \tilde{w}_k^{(1)} & = w_k^{(1)} + D_{k-1}\tilde{\theta}_k^{(1)} \; . \end{cases} \tag{72}$$

It is easy to notice that $\tilde{\theta}_k = \tilde{\theta}_k^{(0)} + \tilde{\theta}_k^{(1)}$ and $\tilde{w}_k = \tilde{w}_k^{(0)} + \tilde{w}_k^{(1)}$. Introduce the following notation

$$\xi_{k+1}^{(i)} = \gamma_k W_{k+1}^{(i)} + \beta_k D_k V_{k+1}^{(i)} \; , \; i \in \{0, 1\} \; .$$

Now we prove the lemma, which is a direct counterpart to A2, previously obtained under a martingale noise assumption.

Lemma 19. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Then, for $i \in \{0, 1\}$ and any $k \in \mathbb{N} \cup \{0\}$ it holds that*

$$\begin{aligned} \mathbb{E}^{1/p}[\|V_{k+1}^{(i)}\|^p] &\lesssim 1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}} \; , \\ \mathbb{E}^{1/p}[\|W_{k+1}^{(i)}\|^p] &\lesssim 1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}} \; , \\ \mathbb{E}^{1/p}[\|\xi_{k+1}^{(0)}\|^p] &\lesssim \gamma_k(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}) \; . \end{aligned}$$

Proof. A6 implies that

$$\|\boldsymbol{\varepsilon}_V\| \leq \mathrm{C}_{\mathbf{b}} + \mathrm{C}_{\mathbf{A}}\,\|\theta^\star\| + \mathrm{C}_{\mathbf{A}}\,\|w^\star\| \text{ and } \|\boldsymbol{\varepsilon}_W\| \leq \mathrm{C}_{\mathbf{b}} + \mathrm{C}_{\mathbf{A}}\,\|\theta^\star\| + \mathrm{C}_{\mathbf{A}}\,\|w^\star\| \; .$$

It remains to note that, due to construction of $\tilde{w}_k$ in (12), it holds that $\mathbb{E}^{1/p}[\|w_k - w^\star\|^p] \leq M_{k,p}^{\tilde{w}} + c_\infty M_{k,p}^{\tilde{\theta}}$. Then it remains to gather similar terms and apply (69). □

To prove Proposition 4 we first state the counterparts to Proposition 5 and Proposition 6:

Proposition 7. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Then it holds that*

$$(M_{k+1,p}^{\tilde{w}})^2 \lesssim P_{0:k}^{(2)} + p^2\gamma_k + p^2\sum_{j=0}^{k}\gamma_j^2 P_{j+1:k}^{(2)}(M_{j,p}^{\tilde{\theta}})^2 \; . \tag{73}$$

Proposition 8. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Then it holds that*

$$(M_{k+1,p}^{\tilde{\theta}})^2 \lesssim P_{0:k}^{(1)} + p^4\beta_k + p^4\sum_{j=0}^{k}\beta_j^2 P_{j+1:k}^{(1)}(M_{j,p}^{\tilde{\theta}})^2 \; .$$

Proof of Proposition 4.. First, we proceed with the bound for $M^{\tilde{\theta}}_{k,p}$. Using Proposition 8 we get

$$(M^{\tilde{\theta}}_{k+1,p})^2 \lesssim P^{(1)}_{0:k} + p^4\beta_k + p^4 \sum_{j=0}^{k} \beta_j^2 P^{(1)}_{j+1:k} (M^{\tilde{\theta}}_{j,p})^2 \ .$$

Hence, there exists a constant $C_{\tilde{\theta}}$ such that

$$(M^{\tilde{\theta}}_{k+1,p})^2 \le C_{\tilde{\theta}} P^{(1)}_{0:k} + p^4 C_{\tilde{\theta}}\beta_k + p^4 C_{\tilde{\theta}} \sum_{j=0}^{k} \beta_j^2 P^{(1)}_{j+1:k} (M^{\tilde{\theta}}_{j,p})^2 \ .$$

Denote the right hand side of the latter inequality by $U_{k+1}, U_0 = C_{\tilde{\theta}}$. Thus, since $(M^{\tilde{\theta}}_{j,p})^2 \le U_j$ for all $j \ge 0$, we get

$$U_{k+1} \le (1 - \frac{a_\Delta \beta_k}{2}) U_k + p^4 C_{\tilde{\theta}} \beta_k - p^4 C_{\tilde{\theta}} \beta_{k-1} (1 - \frac{a_\Delta \beta_k}{2}) + p^4 \beta_k^2 C_{\tilde{\theta}} U_k \overset{(a)}{\le} (1 - \frac{a_\Delta \beta_k}{4}) U_k + p^4 a_\Delta C_{\tilde{\theta}} \beta_k^2 \ ,$$

where in (a) we used B 2. Hence, enrolling the latter recursion and applying Lemma 31, it is easy to get

$$(M^{\tilde{\theta}}_{k+1,p})^2 \le U_{k+1} \lesssim \prod_{j=0}^{k} (1 - \frac{a_\Delta \beta_j}{4}) + p^4 \beta_k \ , \tag{74}$$

and,

$$M^{\tilde{\theta}}_{k+1,p} \lesssim \prod_{j=0}^{k} (1 - \frac{a_\Delta \beta_j}{8}) + p^2 \sqrt{\beta_k} \ ,$$

Now we substitute (74) to (73), apply Lemma 31 and get

$$\begin{aligned}(M^{\tilde{w}}_{k+1,p})^2 &\lesssim P^{(2)}_{0:k} + p^2\gamma_k + p^2 \sum_{j=0}^{k} \gamma_j^2 P^{(2)}_{j+1:k} \prod_{t=0}^{j-1} (1 - \frac{a_\Delta}{4}\beta_t) + p^6 \sum_{j=0}^{k} \gamma_j^2 \beta_{j-1} P^{(1)}_{j+1:k} \ , \\ &\lesssim P^{(2)}_{0:k} + p^2\gamma_k + p^2\beta_k + p^6\beta_k \ ,\end{aligned}$$

and the proof follows. □

Thus, the following bound holds for the initial fast scale:

Lemma 20. *Let $p \ge 2$. Assume A4, A6, B 1, B 2(p). Then it holds for all $k \in \mathbb{N}$ that*

$$\mathbb{E}^{1/p}[\|\hat{w}_{k+1}\|^p] \lesssim \prod_{j=0}^{k} (1 - \frac{a_\Delta}{8}\beta_j) + p^3 \gamma_k \ , \text{ where } \hat{w}_{k+1} = w_{k+1} - w^\star \ .$$

Proof. Note that B 2 guarantees that $\frac{a_\Delta}{8}\beta_j \le \frac{a_{22}}{8}\gamma_j$. Thus, the proof follows from Proposition 4. □

For completeness we also provide the following technical lemma, which is proved in (Kaledin et al. 2020).

Lemma 21 (Lemma 11 in (Kaledin et al. 2020))**.** *Let $(a_j)_{j\ge 0}$ be a sequence of d_θ-dimensional vectors and $(b_j)_{j\ge 0}$ be a sequence of d_w-dimensional vectors. Then it holds that*

$$\sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} (a_j - a_{j+1}) = \beta_0 \Gamma^{(1)}_{1:k} a_0 - \beta_k a_{k+1} + \sum_{j=1}^{k} (\beta_j^2 B_{11}^j \Gamma^{(1)}_{j+1:k} + (\beta_j - \beta_{j-1}) \Gamma^{(1)}_{j:k}) a_j \ ,$$

$$\sum_{j=0}^{k} \gamma_j \Gamma^{(2)}_{j+1:k} (b_j - b_{j+1}) = \gamma_0 \Gamma^{(2)}_{1:k} b_0 - \gamma_k b_{k+1} + \sum_{j=1}^{k} (\gamma_j^2 B_{22}^j \Gamma^{(2)}_{j+1:k} + (\gamma_j - \gamma_{j-1}) \Gamma^{(2)}_{j:k}) b_j \ ,$$

Proof of Proposition 7. First, since $\tilde{w}_{k+1} = \tilde{w}^{(0)}_{k+1} + \tilde{w}^{(1)}_{k+1}$, we get

$$\mathbb{E}^{2/p}[\|\tilde{w}_{k+1}\|^p] \le 2\mathbb{E}^{2/p}[\|\tilde{w}^{(0)}_{k+1}\|^p] + 2\mathbb{E}^{2/p}[\|\tilde{w}^{(1)}_{k+1}\|^p] \ .$$

From now on, we provide bounds for $\mathbb{E}^{2/p}[\|\tilde{w}^{(0)}_{k+1}\|^p]$ and $\mathbb{E}^{2/p}[\|\tilde{w}^{(1)}_{k+1}\|^p]$ separately.

(I) Bound on $\mathbb{E}^{2/p}[\|\tilde{w}_{k+1}^{(0)}\|^p]$.

First, we derive a bound for $\mathbb{E}^{1/p}[\|\tilde{w}_{k+1}^{(0)}\|^p]$. Since $V_{k+1}^{(0)}$ and $W_{k+1}^{(0)}$ are martingale-difference sequences w.r.t. $\mathcal{F}_k$, we obtain, following the lines of Proposition 5, that

$$\mathbb{E}^{2/p}[\|\tilde{w}_{k+1}^{(0)}\|^p] \lesssim \big(P_{0:k}^{(2)}\big)^2 + p^2 \sum_{j=0}^{k} \gamma_j^2 \big(P_{j+1:k}^{(2)}\big)^2 \left(1 + \big(M_{j,p}^{\tilde{\theta}}\big)^2 + \big(M_{j,p}^{\tilde{w}}\big)^2\right) . \tag{75}$$

(II) Bound on $\mathbb{E}^{2/p}[\|\tilde{w}_{k+1}^{(1)}\|^p]$.

Let us introduce the notation

$$E_j = \frac{\beta_j}{\gamma_j} D_j \ .$$

Thus, a counterpart to (46) with initial condition $\tilde{w}_0^{(1)} = 0$, we get

$$\tilde{w}_{k+1}^{(1)} = -\sum_{j=0}^{k} \Gamma_{j+1:k}^{(2)} \gamma_j (W_{j+1}^{(1)} + E_j V_{j+1}^{(1)}) \ . \tag{76}$$

From now on for any vector or matrix sequence $\{S_i\}_{i\geq 0}$ we let $S_i = 0$ if $i < 0$. From now on, using algebraic manipulations and recursion (71), we get, following (Kaledin et al. 2020, Derivation of Eq. (63), p. 39), that for any $j \geq 0$:

$$\begin{aligned} W_{j+1}^{(1)} + E_j V_{j+1}^{(1)} = \Psi_{j+1} - \Psi_j &+ \Phi_j^{(1)}(\tilde{\theta}_{j+1} - \tilde{\theta}_j) + \Phi_j^{(2)}(\tilde{\theta}_j - \tilde{\theta}_{j-1}) \\ &+ \Xi_j^{(1)}(\tilde{w}_{j+1} - \tilde{w}_j) + \Xi_j^{(2)}(\tilde{w}_j - \tilde{w}_{j-1}) \\ &+ \Upsilon_{j+1}^{(1)}\tilde{\theta}_{j+1} - \Upsilon_j^{(1)}\tilde{\theta}_j + \Upsilon_j^{(2)}\tilde{\theta}_j - \Upsilon_{j-1}^{(2)}\tilde{\theta}_{j-1} \\ &+ \Lambda_{j+1}^{(1)}\tilde{w}_{j+1} - \Lambda_j^{(1)}\tilde{w}_j + \Lambda_j^{(2)}\tilde{w}_j - \Lambda_{j-1}^{(2)}\tilde{w}_{j-1} \\ &+ \Pi_j^{\theta}\tilde{\theta}_j + \Pi_j^{w}\tilde{w}_j + (E_{j+1} - E_j)(\mathrm{P}\boldsymbol{g}_{j+1}^{\boldsymbol{\varepsilon}_V}) \ , \end{aligned} \tag{77}$$

where we have defined

$$\begin{aligned} \Psi_j &= -\mathrm{P}\boldsymbol{g}_j^{\boldsymbol{\varepsilon}_W} - E_j(\mathrm{P}\boldsymbol{g}_j^{\boldsymbol{\varepsilon}_V}) \ , \\ \Phi_j^{(1)} &= -\mathrm{P}\boldsymbol{g}_{j+1}^{\mathbf{A}_{21}} \ , \quad \Phi_j^{(2)} = -E_{j-1}(\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{11}}) + (\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{22}})D_{j-2} + E_{j-1}(\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{12}})D_{j-2} \ , \\ \Xi_j^{(1)} &= -\mathrm{P}\boldsymbol{g}_{j+1}^{\mathbf{A}_{22}} \ , \quad \Xi_j^{(2)} = -E_{j-1}(\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{12}}) \ , \\ \Upsilon_j^{(1)} &= \mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{21}} \ , \quad \Upsilon_j^{(2)} = E_j(\mathrm{P}\boldsymbol{g}_{j+1}^{\mathbf{A}_{11}}) - (\mathrm{P}\boldsymbol{g}_{j+1}^{\mathbf{A}_{22}})D_{j-1} - E_j(\mathrm{P}\boldsymbol{g}_{j+1}^{\mathbf{A}_{12}})D_{j-1} \ , \\ \Lambda_j^{(1)} &= \mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{22}} \ , \quad \Lambda_j^{(2)} = E_j \mathrm{P}\boldsymbol{g}_{j+1}^{\mathbf{A}_{12}} \ , \\ \Pi_j^{\theta} &= -(E_j - E_{j-1})(\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{11}}) + (\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{22}})(D_{j-1} - D_{j-2}) + (E_j - E_{j-1})(\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{12}})D_{j-2} + E_j(\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{12}})(D_{j-1} - D_{j-2}) \ , \\ \Pi_j^{w} &= -(E_j - E_{j-1})(\mathrm{P}\boldsymbol{g}_j^{\mathbf{A}_{12}}) \ . \end{aligned}$$

A6, B 1, and Lemma 4 imply that for all $j \geq 0$ and $i \in \{0, 1\}$, it holds that

$$\|\Psi_j\| \vee \|\Phi_j^{(i)}\| \vee \|\Xi_j^{(i)}\| \vee \|\Upsilon_j^{(i)}\| \vee \|\Lambda_j^{(i)}\| \lesssim 1 \ .$$

Now we derive a bound for Π_j^{θ} and Π_j^{w}. First, note that

$$\|D_{t+1} - D_t\| = \|L_{t+1} - L_t\| \lesssim \gamma_{t+1} \ , \tag{78}$$

Now, using Lemma 4 and assumption B 2,

$$\|E_{t+1} - E_t\| = \|\frac{\beta_{t+1}}{\gamma_{t+1}} D_{t+1} - \frac{\beta_t}{\gamma_t} D_t\| \lesssim \frac{\beta_0}{\gamma_0}\gamma_{t+1} + \frac{\beta_t - \beta_{t+1}}{\gamma_t}\|D_t\| \lesssim \gamma_{t+1} \ .$$

Finally, we get that

$$\|\Pi_j^{\theta}\| \vee \|\Pi_j^{w}\| \lesssim \gamma_j \ , \tag{79}$$

Introduce

$$v_{j+1} = \Psi_{j+1} + \Upsilon_{j+1}^{(1)}\tilde{\theta}_{j+1} + \Upsilon_j^{(2)}\tilde{\theta}_j + \Lambda_{j+1}^{(1)}\tilde{w}_{j+1} + \Lambda_j^{(2)}\tilde{w}_j \ .$$

Expanding now the recurrence (76) together with representation (77), we obtain that

$$
\begin{aligned}
\tilde{w}_{k+1}^{(1)} = &-\underbrace{\sum_{j=0}^{k}\gamma_j\Gamma_{j+1:k}^{(2)}(v_{j+1}-v_j)}_{T_1} \underbrace{- \gamma_k\Phi_k^{(1)}(\tilde{\theta}_{k+1}-\tilde{\theta}_k) - \sum_{j=1}^{k}\{\gamma_j\Gamma_{j+1:k}^{(2)}\Phi_j^{(2)} + \gamma_{j-1}\Gamma_{j:k}^{(2)}\Phi_{j-1}^{(1)}\}(\tilde{\theta}_j-\tilde{\theta}_{j-1})}_{T_2} \\
&\underbrace{- \gamma_k\Xi_k^{(1)}(\tilde{w}_{k+1}-\tilde{w}_k) - \sum_{j=1}^{k}\{\gamma_j\Gamma_{j+1:k}^{(2)}\Xi_j^{(2)} + \gamma_{j-1}\Gamma_{j:k}^{(2)}\Xi_{j-1}^{(1)}\}(\tilde{w}_j-\tilde{w}_{j-1})}_{T_3} \\
&\underbrace{- \sum_{j=0}^{k}\gamma_j\Gamma_{j+1:k}^{(2)}\{\Pi_j^{\theta}\tilde{\theta}_j + \Pi_j^{w}\tilde{w}_j + (E_{j+1}-E_j)(\mathrm{P}\boldsymbol{g}_{j+1}^{\boldsymbol{\varepsilon}_V})\}}_{T_4} \; .
\end{aligned}
$$

Now we estimate the terms T_1 to T_4 separately. To proceed with T_1, we use Lemma 21 and obtain

$$
T_1 = -\gamma_0\Gamma_{1:k}^{(2)}v_0 + \gamma_k v_{k+1} - \sum_{j=1}^{k}(\gamma_j^2 B_{11}^j\Gamma_{j+1:k}^{(2)} + (\gamma_j-\gamma_{j-1})\Gamma_{j:k}^{(2)})v_j \; .
$$

Hence,

$$
\mathbb{E}^{1/p}[\|T_1\|^p] \lesssim P_{1:k}^{(2)} + \gamma_k(1 + M_{k+1,p}^{\tilde{\theta}} + M_{k+1,p}^{\tilde{w}} + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}) + \sum_{j=1}^{k}\gamma_j^2 P_{j+1:k}^{(2)}(1 + M_{j+1,p}^{\tilde{\theta}} + M_{j+1,p}^{\tilde{w}} + M_{j,p}^{\tilde{\theta}} + M_{j,p}^{\tilde{w}}) \; . \tag{80}
$$

Using Lemma 19 we get for any $j \geq 0$

$$
M_{j+1,p}^{\tilde{\theta}} \lesssim 1 + M_{j,p}^{\tilde{\theta}} + M_{j,p}^{\tilde{w}} \; , \; M_{j+1,p}^{\tilde{w}} \lesssim 1 + M_{j,p}^{\tilde{\theta}} + M_{j,p}^{\tilde{w}} \; . \tag{81}
$$

Thus, we get combining (81) with (80):

$$
\mathbb{E}^{1/p}[\|T_1\|^p] \lesssim P_{0:k}^{(2)} + \gamma_k(1 + M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}) + \sum_{j=0}^{k}\gamma_j^2 P_{j+1:k}^{(2)}(1 + M_{j,p}^{\tilde{\theta}} + M_{j,p}^{\tilde{w}}) \; .
$$

Then, using Lemma 31-(ii) we get

$$
\begin{aligned}
\mathbb{E}^{2/p}[\|T_1\|^p] &\lesssim (P_{0:k}^{(2)})^2 + \gamma_k^2(1 + (M_{k,p}^{\tilde{\theta}})^2 + (M_{k,p}^{\tilde{w}})^2) + \big\{\sum_{j=0}^{k}\gamma_j^2 P_{j+1:k}^{(2)}(1 + M_{j,p}^{\tilde{\theta}} + M_{j,p}^{\tilde{w}})\big\}^2 \\
&\lesssim (P_{0:k}^{(2)})^2 + \gamma_k^2(1 + (M_{k,p}^{\tilde{\theta}})^2 + (M_{k,p}^{\tilde{w}})^2) + \gamma_k\sum_{j=0}^{k}\gamma_j^2 P_{j+1:k}^{(2)}(1 + (M_{j,p}^{\tilde{\theta}})^2 + (M_{j,p}^{\tilde{w}})^2) \; .
\end{aligned} \tag{82}
$$

To derive bounds for T_2, T_3 we use the definition of $\tilde{\theta}_j$, $\tilde{w}_j$, Lemma 19, and Lemma 4 to obtain that

$$
\mathbb{E}^{1/p}[\|\tilde{\theta}_j - \tilde{\theta}_{j-1}\|^p] \lesssim \gamma_{j-1}(1 + M_{j-1,p}^{\tilde{\theta}} + M_{j-1,p}^{\tilde{w}}) \; , \; \mathbb{E}^{1/p}[\|\tilde{w}_j - \tilde{w}_{j-1}\|^p] \lesssim \gamma_{j-1}(1 + M_{j-1,p}^{\tilde{\theta}} + M_{j-1,p}^{\tilde{w}}) \; , \tag{83}
$$

Thus, the moment bound for $T_2 + T_3$ writes as follows:

$$
\mathbb{E}^{2/p}[\|T_2+T_3\|^p] \lesssim \left\{\sum_{j=0}^{k}\gamma_j^2 P_{j+1:k}(1 + M_{j-1,p}^{\tilde{\theta}} + M_{j-1,p}^{\tilde{w}})\right\}^2 \lesssim \gamma_k\sum_{j=0}^{k}\gamma_j^2 P_{j+1:k}^{(2)}(1 + (M_{j-1,p}^{\tilde{\theta}})^2 + (M_{j-1,p}^{\tilde{w}})^2) \; . \tag{84}
$$

Finally, the term T_4 can be bounded using (79):

$$
\mathbb{E}^{1/p}[\|T_4\|^p] \lesssim \sum_{j=0}^{k}\gamma_j^2 P_{j+1:k}^{(2)}(1 + M_{j,p}^{\tilde{\theta}} + M_{j,p}^{\tilde{w}}) \; . \tag{85}
$$

Then Lemma 31-(ii) implies that

$$\mathbb{E}^{2/p}[\|T_4\|^p] \lesssim \gamma_k \sum_{j=0}^{k} \gamma_j^2 P_{j+1:k}^{(2)} (1 + (M_{j,p}^{\tilde{\theta}})^2 + (M_{j,p}^{\tilde{w}})^2) \; .$$

Gathering the bounds (82), (84), (85) we obtain

$$\mathbb{E}^{2/p}[\|\tilde{w}_{k+1}^{(1)}\|^p] \lesssim (P_{0:k}^{(2)})^2 + \gamma_k^2 (1 + (M_{k,p}^{\tilde{\theta}})^2 + (M_{k,p}^{\tilde{w}})^2) + \gamma_k \sum_{j=0}^{k} \gamma_j^2 P_{j+1:k}^{(2)} (1 + (M_{j,p}^{\tilde{\theta}})^2 + (M_{j,p}^{\tilde{w}})^2) \; . \tag{86}$$

(III) Gathering (I) and (II).
Equations (75) and (86) from the previous paragraphs imply

$$(M_{k+1,p}^{\tilde{w}})^2 \lesssim (P_{0:k}^{(2)})^2 + p^2 \sum_{j=0}^{k} \gamma_j^2 P_{j+1:k}^{(2)} (1 + (M_{j,p}^{\tilde{\theta}})^2 + (M_{j,p}^{\tilde{w}})^2)) \; .$$

Thus, there exists a constant $C_{\tilde{w}} > 0$ such that

$$(M_{k+1,p}^{\tilde{w}})^2 \leq C_{\tilde{w}} (P_{0:k}^{(2)})^2 + p^2 C_{\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 P_{j+1:k}^{(2)} (1 + (M_{j,p}^{\tilde{\theta}})^2 + (M_{j,p}^{\tilde{w}})^2)) \; .$$

Denote the right hand side of the latter inequality by U_{k+1} for $k \geq 0$, $U_0 = C_{\tilde{w}}$. Hence, for all $s \geq 0$ it holds that $(M_{s,p}^{\tilde{w}})^2 \leq U_s$. Thus, we get

$$U_{k+1} \leq (1 - \frac{a_{22}}{2}\gamma_k)^2 U_k + p^2 C_{\tilde{w}} \gamma_k^2 (1 + U_k + (M_{k,p}^{\tilde{\theta}})^2) \; .$$

The conditions on k_0 in B 2 guarantee that

$$U_{k+1} \leq (1 - \frac{a_{22}}{2}\gamma_k) U_k + p^2 C_{\tilde{w}} \gamma_k^2 (1 + (M_{k,p}^{\tilde{\theta}})^2) \; .$$

Enrolling the latter recursion and applying Lemma 31-(ii) we get

$$(M_{k+1,p}^{\tilde{w}})^2 \lesssim P_{0:k}^{(2)} + p^2 \gamma_k + p^2 \sum_{j=0}^{k} \gamma_j^2 P_{j+1:k}^{(2)} (M_{k+1,p}^{\tilde{\theta}})^2 \; .$$

□

Proof of Proposition 8. Expanding the recursion (71) yields, with $\Gamma_{j+1:k}^{(1)}$ defined in (39), that

$$\begin{aligned} \tilde{\theta}_{k+1} &= \Gamma_{0:k}^{(1)} \tilde{\theta}_0 - \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} A_{12} \tilde{w}_j - \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} V_{j+1} \\ &= \Gamma_{0:k}^{(1)} \tilde{\theta}_0 - \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} A_{12} \tilde{w}_j^{(0)} - \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} V_{j+1}^{(0)} - \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} A_{12} \tilde{w}_j^{(1)} - \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} V_{j+1}^{(1)} \; . \end{aligned} \tag{87}$$

Next, we recursively expand $\tilde{w}_j^{(0)}$ using the relation (46):

$$\begin{aligned} \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} A_{12} \tilde{w}_j^{(0)} &= \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} A_{12} \big(\Gamma_{0:j-1}^{(2)} \tilde{w}_0 - \sum_{i=0}^{j-1} \Gamma_{i+1:j-1}^{(2)} \xi_{i+1}^{(0)} \big) \\ &= \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} A_{12} \Gamma_{0:j-1}^{(2)} \tilde{w}_0 - \sum_{i=0}^{k-1} \big(\sum_{j=i+1}^{k} \beta_j \Gamma_{j+1:k}^{(1)} A_{12} \Gamma_{i+1:j-1}^{(2)} \big) \xi_{i+1}^{(0)} \; . \end{aligned}$$

Define, for $m \leq n$, the quantity

$$T_{m:n} = \sum_{\ell=m}^{n} \beta_\ell \Gamma_{\ell+1:n}^{(1)} A_{12} \Gamma_{m:\ell-1}^{(2)} \; ,$$

and note that, with $P^{(1)}_{k:j}$, $P^{(2)}_{k:j}$ defined in (39), it holds that

$$\|T_{m:n}\| \lesssim \sum_{\ell=m}^{n} \beta_\ell P^{(1)}_{\ell+1:n} P^{(2)}_{m:\ell-1} \; . \tag{88}$$

With the above notations, we can rewrite (87) as follows:

$$\tilde{\theta}_{k+1} = \Gamma^{(1)}_{0:k}\tilde{\theta}_0 - T_{0:k}\tilde{w}_0 + \sum_{j=0}^{k-1} T_{j+1:k}\xi^{(0)}_{j+1} - \sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} V^{(0)}_{j+1} - \sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} A_{12}\tilde{w}^{(1)}_j - \sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} V^{(1)}_{j+1} \; .$$

Thus,

$$(M^{\tilde{\theta}}_{k+1,p})^2 \lesssim \underbrace{\mathbb{E}^{2/p}[\|\Gamma^{(1)}_{0:k}\tilde{\theta}_0\|^p]}_{\mathcal{R}_1} + \underbrace{\mathbb{E}^{2/p}[\|T_{0:k}\tilde{w}_0\|^p]}_{\mathcal{R}_2} + \underbrace{\mathbb{E}^{2/p}[\|\sum_{j=0}^{k-1} T_{j+1:k}\xi^{(0)}_{j+1}\|^p]}_{\mathcal{R}_3} + \underbrace{\mathbb{E}^{2/p}[\|\sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} V^{(0)}_{j+1}\|^p]}_{\mathcal{R}_4}$$
$$+ \underbrace{\mathbb{E}^{2/p}[\|\sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} A_{12}\tilde{w}^{(1)}_j\|^p]}_{\mathcal{R}_5} + \underbrace{\mathbb{E}^{2/p}[\|\sum_{j=0}^{k} \beta_j \Gamma^{(1)}_{j+1:k} V^{(1)}_{j+1}\|^p]}_{\mathcal{R}_6}$$

(I) Bounds on $\{\mathcal{R}_i\}_{i=1}^4$.
Easy to see that

$$\mathcal{R}_1 \lesssim (P^{(1)}_{0:k})^2 \; . \tag{89}$$

To proceed with $\mathcal{R}_2$, we apply Lemma 32 with $j+1=0$ and use $\beta_j \le r_{\text{step}}\gamma_j$:

$$\mathcal{R}_2 \lesssim \big(\sum_{j=0}^{k} \beta_j P^{(1)}_{j+1:k} P^{(2)}_{0:j-1}\big)^2 \lesssim \big(\sum_{j=0}^{k} \gamma_j P^{(1)}_{j+1:k} P^{(2)}_{0:j-1}\big)^2 \lesssim \big(P^{(1)}_{0:k}\big)^2 \; . \tag{90}$$

Applying Lemma 5 and Burkholder's inequality, we obtain that

$$\begin{aligned}\mathcal{R}_3 &\lesssim p^2 \mathbb{E}^{2/p}\big[\big(\sum_{j=0}^{k} \beta_j^2 \|\Gamma^{(1)}_{j+1:k} V^{(0)}_{j+1}\|^2\big)^{p/2}\big] \lesssim p^2 \sum_{j=0}^{k} \beta_j^2 \|\Gamma^{(1)}_{j+1:k}\|^2 \mathbb{E}^{2/p}\big[\|V^{(0)}_{j+1}\|^p\big] \\ &\lesssim p^2 \sum_{j=0}^{k} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \big(1 + (M^{\tilde{\theta}}_{j,p})^2 + (M^{\tilde{w}}_{j,p})^2\big) \; .\end{aligned} \tag{91}$$

Applying Lemma 5, (88) and Burkholder's inequality to the $\mathcal{R}_4$, we obtain that

$$\begin{aligned}\mathcal{R}_4 &\lesssim p^2 \mathbb{E}^{2/p}\big[\big(\sum_{j=0}^{k-1} \|T_{j+1:k}\xi^{(0)}_{j+1}\|^2\big)^{p/2}\big] \le p^2 \sum_{j=0}^{k-1} \|T_{j+1:k}\|^2 \big(\mathbb{E}^{2/p}[\|\xi^{(0)}_{j+1}\|^p]\big) \\ &\lesssim p^2 \sum_{j=0}^{k-1} \gamma_j^2 \big(\sum_{i=j+1}^{k} \beta_i P^{(1)}_{i+1:k} P^{(2)}_{j+1:i-1}\big)^2 \big(1 + \big(M^{\tilde{\theta}}_{j,p}\big)^2 + \big(M^{\tilde{w}}_{j,p}\big)^2\big) \\ &\lesssim p^2 \sum_{j=0}^{k-1} \beta_j^2 \big(\sum_{i=j+1}^{k} \gamma_j P^{(1)}_{i+1:k} P^{(2)}_{j+1:i-1}\big)^2 \big(1 + \big(M^{\tilde{\theta}}_{j,p}\big)^2 + \big(M^{\tilde{w}}_{j,p}\big)^2\big) \\ &\overset{(a)}{\lesssim} p^2 \sum_{j=0}^{k-1} \beta_j^2 \big(P^{(1)}_{j+1:k}\big)^2 \big(1 + \big(M^{\tilde{\theta}}_{j,p}\big)^2 + \big(M^{\tilde{w}}_{j,p}\big)^2\big) \; ,\end{aligned} \tag{92}$$

where the inequality (a) follows from Lemma 32.

(II) Bounds on $\mathcal{R}_5$ and $\mathcal{R}_6$.
To proceed with $\mathcal{R}_5$, we combine Minkowski's inequality together with Lemma 31-(ii) and get

$$\mathcal{R}_5 \lesssim \sum_{j=0}^{k} \beta_j P_{j+1:k}^{(1)} \mathbb{E}^{2/p}[\|\tilde{w}_j^{(1)}\|^p] \ .$$

Applying (86) we obtain

$$\begin{aligned}\sum_{j=0}^{k} \beta_j P_{j+1:k}^{(1)} \mathbb{E}^{2/p}[\|\tilde{w}_j^{(1)}\|^p] \lesssim \sum_{j=0}^{k} \beta_j P_{j+1:k}^{(1)} P_{0:j-1}^{(2)} &+ \sum_{j=0}^{k} \beta_j P_{j+1:k}^{(1)} \gamma_{j-1}^2 (1 + (M_{j,p}^{\tilde{w}})^2 + (M_{j,p}^{\tilde{\theta}})^2) \\ &+ \sum_{t=0}^{k-1} \sum_{j=t+1}^{k} \beta_j \gamma_{j-1} \gamma_t^2 P_{j+1:k}^{(1)} P_{t+1:j-1}^{(2)} (1 + (M_{t,p}^{\tilde{w}})^2 + (M_{t,p}^{\tilde{\theta}})^2) \ .\end{aligned}$$

Now we use Lemma 32 and $\beta_j \leq r_{\mathrm{step}} \gamma_j$ together with $\gamma_j^2 \leq \frac{\gamma_0^2}{\beta_0} \beta_j$:

$$\sum_{j=0}^{k} \beta_j P_{j+1:k}^{(1)} \mathbb{E}^{2/p}[\|\tilde{w}_j^{(1)}\|^p] \lesssim P_{0:k}^{(1)} + \sum_{j=0}^{k} \beta_j^2 P_{j+1:k}^{(1)} (1 + (M_{j,p}^{\tilde{w}})^2 + (M_{j,p}^{\tilde{\theta}})^2) \ .$$

Thus,

$$\mathcal{R}_5 \lesssim P_{0:k}^{(1)} + \sum_{j=0}^{k} \beta_j^2 P_{j+1:k}^{(1)} (1 + (M_{j,p}^{\tilde{w}})^2 + (M_{j,p}^{\tilde{\theta}})^2) \ . \tag{93}$$

Set $\hat{w}_k = w_k - w^\star$. Hence

$$\hat{w}_j - \hat{w}_{j+1} = \tilde{w}_{j+1} - \tilde{w}_j - D_{j-1} \tilde{\theta}_j + D_j \tilde{\theta}_{j+1} \ ,$$

and

$$\begin{aligned}V_{j+1}^{(1)} = (\mathrm{P} \boldsymbol{g}_j^{\boldsymbol{\varepsilon}_V} - (\mathrm{P} \boldsymbol{g}_j^{\mathbf{A}_{11}}) \tilde{\theta}_j - (\mathrm{P} \boldsymbol{g}_j^{\mathbf{A}_{12}}) \hat{w}_j) - (\mathrm{P} \boldsymbol{g}_{j+1}^{\boldsymbol{\varepsilon}_V} - (\mathrm{P} \boldsymbol{g}_{j+1}^{\mathbf{A}_{11}}) \tilde{\theta}_{j+1} - (\mathrm{P} \boldsymbol{g}_{j+1}^{\mathbf{A}_{12}}) \hat{w}_{j+1}) \\ + (\mathrm{P} \boldsymbol{g}_{j+1}^{\mathbf{A}_{11}}) (\tilde{\theta}_j - \tilde{\theta}_{j+1}) + (\mathrm{P} \boldsymbol{g}_{j+1}^{\mathbf{A}_{12}}) (\mathrm{I} + D_j)(\tilde{w}_{j+1} - \tilde{w}_j) + (\mathrm{P} \boldsymbol{g}_{j+1}^{\mathbf{A}_{12}}) (D_j - D_{j-1}) \tilde{\theta}_j \ .\end{aligned}$$

Now we derive a couple of auxiliary bounds. First, from the definition of $\hat{w}_j$ and $\tilde{w}_j$ we get

$$\mathbb{E}^{1/p}[\|\hat{w}_j\|^p] \lesssim M_{j,p}^{\tilde{w}} + M_{j,p}^{\tilde{\theta}} \ .$$

Set for simplicity

$$u_j = \mathrm{P} \boldsymbol{g}_j^{\boldsymbol{\varepsilon}_V} - (\mathrm{P} \boldsymbol{g}_j^{\mathbf{A}_{11}}) \tilde{\theta}_j - (\mathrm{P} \boldsymbol{g}_j^{\mathbf{A}_{12}}) \hat{w}_j \ ,$$

and note that

$$\|u_j\| \lesssim 1 + M_{j,p}^{\tilde{\theta}} + M_{j,p}^{\tilde{w}} \ .$$

Thus, using Lemma 21 and Equation (78) we obtain

$$\begin{aligned}\mathbb{E}^{1/p}[\|\sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} V_{j+1}^{(1)}\|^p] \lesssim P_{0:k}^{(1)} + \beta_k (1 + M_{k,p}^{\tilde{w}} + M_{k,p}^{\tilde{\theta}}) + \sum_{j=1}^{k} \beta_j^2 P_{j+1:k}^{(1)} (1 + M_{j,p}^{\tilde{\theta}} + M_{j,p}^{\tilde{w}}) \\ + \sum_{j=1}^{k} \beta_j P_{j+1:k}^{(1)} \mathbb{E}^{1/p}[\|\tilde{\theta}_{j+1} - \tilde{\theta}_j\|^p] + \mathrm{C}_{\mathbf{A}} \sqrt{\kappa_\Delta} \sum_{j=1}^{k} \beta_j P_{j+1:k}^{(1)} \mathbb{E}^{1/p}[\|\tilde{w}_{j+1} - \tilde{w}_j\|^p] \\ + \sum_{j=1}^{k} \beta_j \gamma_j P_{j+1:k}^{(1)} M_{j,p}^{\tilde{\theta}} \ .\end{aligned}$$

Next, applying (81), (83) we get

$$\mathbb{E}^{1/p}[\|\sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} V_{j+1}^{(1)}\|^p] \lesssim P_{0:k}^{(1)} + \beta_k (1 + M_{k,p}^{\tilde{w}} + M_{k,p}^{\tilde{\theta}}) + \sum_{j=0}^{k} \beta_j \gamma_j P_{j+1:k}^{(1)} (1 + M_{j,p}^{\tilde{w}} + M_{j,p}^{\tilde{\theta}}) \ .$$

Hence, Lemma 31-(ii) implies that

$$\begin{aligned}\mathcal{R}_6 &\lesssim (P_{0:k}^{(1)})^2 + \beta_k^2 (1 + (M_{k,p}^{\tilde{w}})^2 + (M_{k,p}^{\tilde{\theta}})^2) + \sum_{j=0}^{k} \beta_j P_{j+1:k}^{(1)} \gamma_j^2 (1 + (M_{j,p}^{\tilde{w}})^2 + (M_{j,p}^{\tilde{\theta}})^2) \\ &\lesssim (P_{0:k}^{(1)})^2 + \beta_k^2 (1 + (M_{k,p}^{\tilde{w}})^2 + (M_{k,p}^{\tilde{\theta}})^2) + \sum_{j=0}^{k} \beta_j^2 P_{j+1:k}^{(1)} (1 + (M_{j,p}^{\tilde{w}})^2 + (M_{j,p}^{\tilde{\theta}})^2) \ .\end{aligned} \tag{94}$$

(III) Gathering (I) and (II).
Gathering the similar terms in (89), (90), (91), (92), (93), (94) we obtain

$$(M_{k+1,p}^{\tilde{\theta}})^2 \lesssim P_{0:k} + p^2 \sum_{j=0}^{k} \beta_j^2 P_{j+1:k}^{(1)} (1 + (M_{k,p}^{\tilde{w}})^2 + (M_{k,p}^{\tilde{\theta}})^2) \ . \tag{95}$$

Applying Proposition 5 and Lemma 31, we obtain

$$\begin{aligned}\sum_{j=0}^{k} \beta_j^2 \big(P_{j+1:k}^{(1)}\big)^2 \big(M_{j,p}^{\tilde{w}}\big)^2 &\lesssim \sum_{j=0}^{k} \beta_j^2 \big(P_{j+1:k}^{(1)}\big)^2 \big(P_{0:j-1}^{(2)} + p^2 \gamma_{j-1} + p^2 \sum_{i=0}^{j-1} \gamma_i^2 P_{i+1:j-1}^{(2)} \big(M_{i,p}^{\tilde{\theta}}\big)^2\big) \\ &\lesssim p^2 \sum_{j=0}^{k} \beta_j^2 P_{j+1:k}^{(1)} + p^2 \sum_{j=0}^{k} \beta_j^2 \big(P_{j+1:k}^{(1)}\big)^2 \sum_{i=0}^{j-1} \gamma_i^2 P_{i+1:j-1}^{(2)} \big(M_{i,p}^{\tilde{\theta}}\big)^2 \\ &\lesssim p^2 \beta_k + p^2 \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \beta_j^2 \gamma_i^2 P_{i+1:j-1}^{(2)} P_{j+1:k}^{(1)} \big(M_{i,p}^{\tilde{\theta}}\big)^2 \ .\end{aligned}$$

Using that $\beta_j^2 \le \beta_i^2$ for $j \ge i+1$ and $\gamma_i^2 \le \gamma_0 \gamma_i$, we get that

$$\sum_{j=0}^{k} \beta_j^2 \big(P_{j+1:k}^{(1)}\big)^2 \big(M_{j,p}^{\tilde{w}}\big)^2 \lesssim p^2 \beta_k + p^2 \sum_{i=0}^{k-1} \beta_i^2 \big(\sum_{j=i+1}^{k} \gamma_i P_{i+1:j-1}^{(2)} P_{j+1:k}^{(1)} \big) \big(M_{i,p}^{\tilde{\theta}}\big)^2 \overset{(a)}{\le} p^2 \beta_k + p^2 \sum_{i=0}^{k-1} \beta_i^2 P_{i+1:k}^{(1)} \big(M_{i,p}^{\tilde{\theta}}\big)^2 \ ,$$

where (a) follows from Lemma 32. The proof follows from substituting the latter ineqaulity into (95). □

E.2 CLT for the Polyak-Ruppert averaged estimator

From equations (8) and (18), we derive the extended version of (19):

$$\begin{aligned}\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star) =& \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\boldsymbol{\varepsilon}_V^{k+1} - A_{12} A_{22}^{-1} \boldsymbol{\varepsilon}_W^{k+1}) \\ &+ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \Big\{ \underbrace{(A_{12} A_{22}^{-1} \tilde{\mathbf{A}}_{21}^{k+1} - \tilde{\mathbf{A}}_{11}^{k+1})}_{\Phi_{k+1}} \tilde{\theta}_k + \underbrace{(A_{12} A_{22}^{-1} \tilde{\mathbf{A}}_{22}^{k+1} - \tilde{\mathbf{A}}_{12}^{k+1})}_{\Psi_{k+1}} (w_k - w^\star) \Big\} \\ &+ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \beta_k^{-1} (\tilde{\theta}_k - \tilde{\theta}_{k+1}) - \frac{1}{\sqrt{n}} \sum_{k=1}^{n} A_{12} A_{22}^{-1} \gamma_k^{-1} (w_k - w_{k+1})\end{aligned}$$

Setting

$$\psi_{j+1} = \boldsymbol{\varepsilon}_V^{j+1} - A_{12} A_{22}^{-1} \boldsymbol{\varepsilon}_W^{j+1} \ ,$$

we derive a decomposition of $\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star)$ using the Poisson equation construction (70):

$$
\begin{aligned}
\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star) = & \underbrace{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\{\boldsymbol{g}_{k+1}^{\psi} - \mathrm{P}\boldsymbol{g}_{k}^{\psi}\}}_{T^{\mathrm{mark}}} \\
& + \underbrace{\frac{1}{\sqrt{n}}\{\mathrm{P}\boldsymbol{g}_1^{\psi} - \mathrm{P}\boldsymbol{g}_{n+1}^{\psi} + (\mathrm{P}\boldsymbol{g}_1^{\Phi})(\mathrm{I}+D_0)\tilde{\theta}_1 - (\mathrm{P}\boldsymbol{g}_{n+1}^{\Phi})(\mathrm{I}+D_n)\tilde{\theta}_{n+1} + (\mathrm{P}\boldsymbol{g}_1^{\Psi})\tilde{w}_1 - (\mathrm{P}\boldsymbol{g}_{n+1}^{\Psi})\tilde{w}_{n+1}\}}_{R_1} \\
& + \underbrace{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\big\{(\boldsymbol{g}_{k+1}^{\Phi} - \mathrm{P}\boldsymbol{g}_{k}^{\Phi})\tilde{\theta}_k + (\boldsymbol{g}_{k+1}^{\Psi} - \mathrm{P}\boldsymbol{g}_{k}^{\Psi})(w_k - w^\star)\big\}}_{R_2} \\
& + \underbrace{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\beta_k^{-1}\{\tilde{\theta}_k - \tilde{\theta}_{k+1}\} - \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\mathrm{P}\boldsymbol{g}_{k+1}^{\Phi})\{\tilde{\theta}_k - \tilde{\theta}_{k+1}\}}_{R_3} \\
& \underbrace{- \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\gamma_k^{-1}A_{12}A_{22}^{-1}\{w_k - w_{k+1}\} - \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\mathrm{P}\boldsymbol{g}_{k+1}^{\Psi})\{w_k - w_{k+1}\}}_{R_4} \;.
\end{aligned} \tag{96}
$$

Note that B 1 implies that

$$
\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\psi_{k+1} \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Sigma}_\infty^{\mathrm{mark}})
$$

for some covariance matrix $\boldsymbol{\Sigma}_\infty^{\mathrm{mark}} \in \mathbb{R}^{d_\theta \times d_\theta}$. Due to the properties of Poisson equation, we have using the notation of (96):

$$
\mathrm{Var}[T^{\mathrm{mark}}] = \boldsymbol{\Sigma}_\infty^{\mathrm{mark}} \;.
$$

Using the decomposition (96) we assume that T^{mark} is a leading term, while

$$
R_n^{\mathrm{pr,m}} = \sum_{i=1}^{4} R_i
$$

corresponds to a residual one. The proof of Theorem 3 is given below and is based on Proposition 1 and Lemma 22.

Lemma 22. *Let $2 \le p \le \log n$. Assume A4, A6, B 1, B 2($\log n$). Then it holds that*

$$
\mathbb{E}^{1/p}\big[\big\|R_n^{\mathrm{pr,m}}\big\|^p\big] \lesssim n^{b-1/2}\prod_{j=0}^{n-1}\big(1 - \frac{a_\Delta}{8}\beta_j\big) + \frac{\log^3(n)}{n^{(1-b)/2}} + \log^3(n)\frac{(1-a)^{-1} + (1-b)^{-1}}{n^{a-1/2}} + \frac{\log^4(n)(1-a)^{-1}}{n^{a/2}} \;.
$$

Proof of Theorem 3.. Note that for all k it holds that

$$
\|\boldsymbol{g}_{k+1}^{\psi} - \mathrm{P}\boldsymbol{g}_{k}^{\psi}\| \le \frac{16}{3}t_{\mathrm{mix}}\sup_{x \in \mathsf{X}}\|\psi(x)\| < \infty.
$$

Inroduce

$$
h(X_k) = \mathbb{E}[(\boldsymbol{g}_{k+1}^{\psi} - \mathrm{P}\boldsymbol{g}_{k}^{\psi})(\boldsymbol{g}_{k+1}^{\psi} - \mathrm{P}\boldsymbol{g}_{k}^{\psi})^\top \mid X_k] - \boldsymbol{\Sigma}_\infty^{\mathrm{mark}} \;.
$$

One can check that $\pi(h) = 0$ using Poisson equation properties, where π is given in B 1. Thus, h satisfies the assumptions of Lemma 29. Hence, we get applying Lemma 3 with $p = 1$:

$$
\rho^{\mathrm{Conv}}\big(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\{\boldsymbol{g}_{k+1}^{\psi} - \mathrm{P}\boldsymbol{g}_{k}^{\psi}\}, \mathcal{N}(0, \boldsymbol{\Sigma}_\infty^{\mathrm{mark}})\big) \lesssim \frac{1 + \log n}{n^{1/4}} \;.
$$

Since for all $q \in (0,1)$ and $a_1, \ldots, a_m > 0$ it holds that $(\sum_{i=1}^m a_i)^q \le \sum_{i=1}^m a_i^q$, Proposition 1, Lemma 22 imply that

$$\begin{aligned}\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{mark}})\big) &\lesssim \rho^{\mathrm{Conv}}\big(\frac{1}{\sqrt{n}}\sum_{k=1}^n \{\boldsymbol{g}_{k+1}^\psi - \mathrm{P}\boldsymbol{g}_{k+1}^\psi\}, \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{mark}})\big) + c_{d_\theta}^{\frac{p}{p+1}}\big(\mathbb{E}^{1/p}[\|R_1 + R_2 + R_3 + R_4\|]\big)^{\frac{p}{p+1}} \\ &\lesssim \frac{1+\log n}{n^{1/4}} + c_{d_\theta}^{\frac{p}{p+1}}\big\{n^{b-1/2}\prod_{j=0}^{n-1}\big(1 - \frac{a_\Delta}{8}\beta_j\big)\big\}^{\frac{p}{p+1}} + c_{d_\theta}^{\frac{p}{p+1}}\big\{\frac{\log^3(n)}{n^{(1-b)/2}}\big\}^{\frac{p}{p+1}} \\ &+ c_{d_\theta}^{\frac{p}{p+1}}\big\{\log^3(n)\frac{(1-a)^{-1} + (1-b)^{-1}}{n^{a-1/2}}\big\}^{\frac{p}{p+1}} + c_{d_\theta}^{\frac{p}{p+1}}\big\{\frac{\log^4(n)(1-a)^{-1}}{n^{a/2}}\big\}^{\frac{p}{p+1}} .\end{aligned}$$

Note that $(n^\alpha)^{\frac{\log n}{1+\log n}} \le n^\alpha \exp(|\alpha|)$ for all $\alpha \in \mathbb{R}$. Thus, substituting $p := \log n$ into the latter inequality we get

$$\begin{aligned}\rho^{\mathrm{Conv}}\big(\sqrt{n}\Delta(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, \mathbf{\Sigma}_\infty^{\mathrm{mark}})\big) &\lesssim \frac{1+\log n}{n^{1/4}} + c_{d_\theta} n^{b-1/2}\prod_{j=0}^{n-1}\big(1 - \frac{a_\Delta}{16}\beta_j\big) + c_{d_\theta}\frac{\log^3(n)}{n^{(1-b)/2}} \\ &+ c_{d_\theta}\log^3(n)\frac{(1-a)^{-1} + (1-b)^{-1}}{n^{a-1/2}} + c_{d_\theta}\frac{\log^4(n)(1-a)^{-1}}{n^{a/2}} ,\end{aligned}$$

and the proof follows. □

Proof of Lemma 22. First, we use Minkowski's inequality

$$\mathbb{E}^{1/p}[\|\textstyle\sum_{i=1}^4 R_i\|^p] \le \sum_{i=1}^4 \mathbb{E}^{1/p}[\|R_i\|^p] .$$

Proposition 4 and B 2 directly imply that $\mathbb{E}^{1/p}[\|R_1\|^p] \lesssim p^6 n^{-1/2} \le n^{-1/2}\log^6(n)$. To proceed with R_2, we note that R_2 is a sum of martingale difference sequence due to the properties of Markov kernel P. Thus, Burkholder's inequality (Osekowski 2012, Theorem 8.1) and Proposition 4 imply that

$$\begin{aligned}\mathbb{E}^{2/p}[\|R_2\|^p] \le \frac{p^2}{n}\sum_{k=1}^n\{\mathbb{E}^{2/p}[\|\tilde{\theta}_k\|] + \mathbb{E}^{2/p}[\|w_k - w^\star\|^p]\} &\lesssim \frac{p^2}{n}\sum_{k=1}^n\{\prod_{j=0}^{k-1}(1 - \frac{a_{22}\gamma_j}{4}) + \prod_{j=0}^{k-1}(1 - \frac{a_\Delta\beta_j}{8}) + p^6\gamma_k\} \\ &\overset{(a)}{\lesssim} \frac{p^2 k_0^b}{n} + \frac{p^8}{n^a(1-a)} \overset{(b)}{\lesssim} \frac{\log^8(n)}{n^a(1-a)} ,\end{aligned}$$

where in (a) we have additionally used Lemma 31-(iii) and (b) holds because B 2$(\log n)$ implies $k_0^b = \mathcal{O}(\log^4 n)$. Now we derive bounds for R_3 and R_4. Rewrite R_3 as follows:

$$R_3 = \frac{1}{\sqrt{n}}\beta_1^{-1}\tilde{\theta}_1 - \frac{1}{\sqrt{n}}\beta_n^{-1}\tilde{\theta}_{n+1} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}(\beta_{k+1}^{-1} - \beta_k^{-1})\tilde{\theta}_{k+1} - \frac{1}{\sqrt{n}}\sum_{k=1}^n(\mathrm{P}\boldsymbol{g}_{k+1}^\Phi)\{\tilde{\theta}_k - \tilde{\theta}_{k+1}\} .$$

Note that since $(1+x)^b \le 1 + bx$ for $b \in [0,1]$, we get $\beta_{k+1}^{-1} - \beta_k^{-1} \le b(k\beta_k)^{-1}$. Therefore, Proposition 4 and Lemma 31-(iii) imply that

$$\sum_{k=1}^{n-1}(\beta_{k+1}^{-1} - \beta_k^{-1})M_{k+1,p}^{\tilde{\theta}} \lesssim \sum_{k=1}^{n-1}(k\beta_k)^{-1}M_{k+1,p}^{\tilde{\theta}} \lesssim \sum_{k=1}^{n-1}\prod_{j=0}^k\big(1 - \beta_j\frac{a_\Delta}{8}\big) + p^2\sum_{k=1}^{n-1}(k\beta_k)^{-1}\beta_k^{1/2} \lesssim k_0^b + p^2 n^{b/2} .$$

Now, using Proposition 4 and $\mathbb{E}^{1/p}[\|\tilde{\theta}_k - \tilde{\theta}_{k+1}\|^p] \lesssim p^3\gamma_j$, we obtain

$$\begin{aligned}\mathbb{E}^{1/p}[\|R_3\|^p] &\lesssim \frac{k_0^b}{\sqrt{n}} + \frac{(n+k_0)^b}{\sqrt{n}}\prod_{j=0}^{n-1}(1 - \frac{a_\Delta\beta_j}{8}) + \frac{p^2(n+k_0)^{b/2}}{\sqrt{n}} + \frac{k_0^b + p^2 n^{b/2}}{\sqrt{n}} + \frac{p^3}{(1-b)n^{b-1/2}} \\ &\lesssim n^{b-1/2}\prod_{j=0}^{n-1}(1 - \frac{a_\Delta\beta_j}{8}) + \frac{\log^2(n)}{n^{(1-b)/2}} + \frac{\log^3(n)}{(1-b)n^{b-1/2}} + \frac{\log^4(n)}{\sqrt{n}} .\end{aligned}$$

To bound R_4 it is sufficient to apply $\mathbb{E}^{1/p}[\|w_k - w^\star\|^p] \lesssim M_{k,p}^{\tilde{\theta}} + M_{k,p}^{\tilde{w}}$. Thus, using $\gamma_k^{-1} \lesssim \beta_k^{-1}$ and $a_\Delta\beta_j \le a_{22}\gamma_j$ to bound the terms with $M_{k,p}^{\tilde{\theta}}$ and $M_{k,p}^{\tilde{w}}$ separately, one can check that

$$\begin{aligned}\mathbb{E}^{1/p}[\|R_4\|^p] &\lesssim n^{b-1/2}\prod_{j=0}^{n-1}\big(1 - \frac{a_\Delta}{8}\beta_j\big) + \frac{\log^2(n)}{n^{(1-b)/2}} + \frac{\log^3(n)}{(1-b)n^{b-1/2}} + \frac{1}{n^{(1-a)/2}} + \frac{1}{(1-a)n^{a-1/2}} \\ &\lesssim n^{b-1/2}\prod_{j=0}^{n-1}\big(1 - \frac{a_\Delta}{8}\beta_j\big) + \frac{\log^3(n)}{n^{(1-b)/2}} + \log^3(n)\frac{(1-a)^{-1} + (1-b)^{-1}}{n^{a-1/2}} + \frac{\log^4(n)}{\sqrt{n}} .\end{aligned}$$

The proof follows from gathering similar terms. $\square$

E.3 CLT for the Last iteration estimator

First, we start from the same decomposition as in the martingale noise setting (63):

$$\begin{aligned}\tilde{\theta}_{n+1} = &- \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \psi_{j+1} + G_{0:n}^{(1)} \tilde{\theta}_0 - \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} G_{0:j-1}^{(2)} \tilde{w}_0 \\ &+ \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)}\ \delta_j^{(1)} + S_n^{(1)} + S_n^{(2)} + S_n^{(3)} \\ &+ \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \big(\overbrace{\{ \tilde{\mathbf{A}}_{11}^{j+1} - A_{12} A_{22}^{-1} \tilde{\mathbf{A}}_{21}^{j+1} \}}^{\Phi_{j+1}} \tilde{\theta}_j + \overbrace{\{ \tilde{\mathbf{A}}_{12}^{j+1} - A_{12} A_{22}^{-1} \tilde{\mathbf{A}}_{21}^{j+1} \}}^{\Psi_{j+1}} (w_j - w^\star) \big) ,\end{aligned} \tag{97}$$

where

$$\begin{aligned}\psi_{j+1} &= \boldsymbol{\varepsilon}_V^{j+1} - A_{12} A_{22}^{-1} \boldsymbol{\varepsilon}_W^{j+1} \ , \ \delta_j^{(1)} = A_{12} L_j \tilde{\theta}_j \ , \ \delta_j^{(2)} = -(L_{j+1} + A_{22}^{-1} A_{21}) A_{12} \tilde{w}_j \ , \\ S_n^{(1)} &= - \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \beta_i G_{i+1:j-1}^{(2)} \delta_i^{(2)} \ , \\ S_n^{(2)} &= \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \beta_i G_{i+1:j-1}^{(2)} D_i V_{i+1} \ , \\ S_n^{(3)} &= \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \gamma_i G_{i+1:j-1}^{(2)} W_{i+1} - \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} A_{22}^{-1} W_{j+1} \ .\end{aligned}$$

Now we apply the Poisson equation technique and obtain from (97):

$$\begin{aligned}\tilde{\theta}_{n+1} = &\underbrace{- \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} (\boldsymbol{g}_{j+1}^{\psi} - \mathrm{P} \boldsymbol{g}_j^{\psi})}_{T_{\text{last}}^{\text{mark}}} + \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} (\mathrm{P} \boldsymbol{g}_{j+1}^{\psi} - \mathrm{P} \boldsymbol{g}_j^{\psi} + \Phi_{j+1} \tilde{\theta}_j + \Psi_{j+1} \hat{w}_j)}_{H_n} + G_{0:n}^{(1)} \tilde{\theta}_0 \\ &- \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} G_{0:j-1}^{(2)} \tilde{w}_0 + \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)}\ \delta_j^{(1)} + S_n^{(1)} + S_n^{(2)} + S_n^{(3)} \ ,\end{aligned}$$

Also define

$$R_n^{\text{last,m}} = G_{0:n}^{(1)} \tilde{\theta}_0 - - \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} G_{0:j-1}^{(2)} \tilde{w}_0 + \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)}\ \delta_j^{(1)} + S_n^{(1)} + S_n^{(2)} + S_n^{(3)} + H_n \ .$$

The proof of Theorem 4 is based on gaussian approximation of $T_{\text{last}}^{\text{mark}}$ and the moment bound for $R_n^{\text{last,m}}$ which follows from Lemmas 23-27 that we state below:

Lemma 23. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p), B 3. Then it holds that*

$$\mathbb{E}^{1/p}[\| \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \delta_j^{(1)} \|^p] \lesssim \frac{p^3}{2b-a-1} P_{0:n}^{(1)} + \frac{p^4}{2b-a-1} \beta_n^{\frac{2b-a/2-1}{b}} \ .$$

Lemma 24. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Then it holds that*

$$\mathbb{E}^{1/p}[\| S_n^{(1)} \|^p] \lesssim P_{0:n}^{(1)} + p^4 \beta_n^{\frac{3b-2a}{2b}} \ .$$

Lemma 25. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Then it holds that*

$$\mathbb{E}^{1/p}[\| S_n^{(2)} \|^p] \lesssim p^3 P_{0:n}^{(1)} + p^4 \beta_n^{\frac{2b-a}{2b}} \ .$$

Lemma 26. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Then it holds that*

$$\mathbb{E}^{1/p}[\| S_n^{(3)} \|^p] \lesssim p^3 P_{0:n}^{(1)} + p^3 \beta_n^{\frac{a}{b}} + p^4 \beta_k^{\frac{2b-a}{2b}} \ .$$

Lemma 27. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Then it holds that*

$$\mathbb{E}^{1/p}[\|H_n\|^p] \lesssim \frac{p^3}{2b-1}\prod_{j=1}^{n}(1-\frac{a_\Delta}{4}\beta_j) + p^4\beta_n^{a/b} \;.$$

Proof of Theorem 4. First, we introduce

$$\psi_{j+1} = \boldsymbol{g}_{j+1}^{\psi} - \mathrm{P}\boldsymbol{g}_j^{\psi} \;,\; R_n^{\mathrm{last}} \quad = G_{0:k}^{(1)}\tilde{\theta}_0 - \sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}A_{12}G_{0:j-1}^{(2)}\tilde{w}_0 + \sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}\delta_j^{(1)} + H_n + S_n^{(1)} + S_n^{(2)} + S_n^{(3)} \;.$$

Set $p = \log n$. Thus, applying Lemma 32-(i) we get:

$$\mathbb{E}^{1/p}[\|R_n^{\mathrm{last}}\|^p] \lesssim P_{0:n}^{(1)} + P_{0:n}^{(1)} + \mathbb{E}^{1/p}[\|\sum_{j=0}^{n}\beta_j G_{j+1:n}^{(1)}\delta_j^{(1)}\|^p] + \mathbb{E}^{1/p}[\|H_n\|^p] + \mathbb{E}^{1/p}[\|S_n^{(1)}\|^p] + \mathbb{E}^{1/p}[\|S_n^{(2)}\|^p] + \mathbb{E}^{1/p}[\|S_n^{(3)}\|^p] \;.$$

Now we use Lemmas 23-27 together with the inequality $\beta_n^{\frac{2b-a/2-1}{b}} > \beta_n^{\frac{2b-a}{2b}}$ and get

$$\mathbb{E}^{1/p}[\|R_n^{\mathrm{last}}\|^p] \lesssim \frac{p^3}{2b-a-1}\prod_{j=0}^{n}(1-\frac{a_\Delta}{4}\beta_j) + p^3\beta_n^{a/b} + \frac{p^4}{2b-a-1}\beta_n^{\frac{2b-a/2-1}{b}}$$

Therefore, Proposition 1 implies that:

$$\begin{aligned}\rho^{\mathrm{Conv}}(\beta_n^{-1/2}\tilde{\theta}_{n+1}, \mathcal{N}(0,\boldsymbol{\Sigma}_\infty^{\mathrm{last,m}})) \lesssim \quad & \rho^{\mathrm{Conv}}(\beta_n^{-1/2}T_{\mathrm{last}}^{\mathrm{mark}}, \mathcal{N}(0,\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last,m}})) \\ & + \rho^{\mathrm{Conv}}(\mathcal{N}(0,\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last,m}}), \mathcal{N}(0,\boldsymbol{\Sigma}_\infty^{\mathrm{last,m}})) \\ & + c_{d_\theta}^{\frac{p}{p+1}}(\mathbb{E}^{1/p}[\|\beta_n^{-1/2}R_n^{\mathrm{last}}\|^p])^{\frac{p}{p+1}} \;.\end{aligned} \tag{98}$$

The bound for the second term follows from (Devroye, Mehrabian, and Reddad 2018, Theorem 1.1) and Proposition 9:

$$\rho^{\mathrm{Conv}}(\mathcal{N}(0,\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last,m}}), \mathcal{N}(0,\boldsymbol{\Sigma}_\infty^{\mathrm{last,m}})) \lesssim \|\{\boldsymbol{\Sigma}_\infty^{\mathrm{last,m}}\}^{-1/2}(\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last,m}})\{\boldsymbol{\Sigma}_\infty^{\mathrm{last,m}}\}^{-1/2} - \mathrm{I}\|_{\mathrm{Fr}} \lesssim \frac{\sqrt{d_\theta}}{n^b\lambda_{\min}(\boldsymbol{\Sigma}_\infty^{\mathrm{last,m}})} \;.$$

Next, the bound for the third term in (98) follows from $(n^{-\alpha})^{\frac{\log n}{1+\log n}} \lesssim n^{-\alpha}$ and $(\sum a_i)^q \leq \sum a_i^q$ for $a_i > 0$ and $q \in (0,1)$:

$$c_{d_\theta}^{\frac{p}{p+1}}(\mathbb{E}^{1/p}[\|\beta_n^{-1/2}R_n^{\mathrm{last}}\|^p])^{\frac{p}{p+1}} \lesssim c_{d_\theta}\frac{\beta_n^{-1/2}\log^3(n)}{2b-a-1}\prod_{j=0}^{n}(1-\frac{a_\Delta}{8}\beta_j) + c_{d_\theta}\beta_n^{\frac{2a-b}{2b}}\log^3(n) + c_{d_\theta}\frac{\log^4(n)}{2b-a-1}\beta_n^{\frac{3b-a-2}{2b}} \;. \tag{99}$$

Now we derive a bound for $\rho^{\mathrm{Conv}}(\beta_n^{-1/2}T_{\mathrm{last}}^{\mathrm{mark}}, \mathcal{N}(0,\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last,m}}))$. Introduce

$$M_i = \beta_n^{-1/2}\beta_i G_{i+1:n}^{(1)}\psi_{i+1} \;.$$

Hence, we get

$$\|M_i\| \lesssim \beta_n^{-1/2}\beta_i P_{i+1:n}^{(1)} \;.$$

Note that B 2 implies that

$$\frac{\beta_{i+1}P_{i+2:n}^{(1)}}{\beta_i P_{i+1:n}^{(1)}} = \frac{1}{(1-\frac{a_\Delta}{2}\beta_{i+1})\frac{\beta_i}{\beta_{i+1}}} \geq \frac{1}{(1-\frac{a_\Delta}{2}\beta_{i+1})(1+\frac{a_\Delta}{16}\beta_{i+1})} \geq \frac{1}{(1-\frac{a_\Delta}{2}\beta_{i+1})(1+\frac{a_\Delta}{2}\beta_{i+1})} > 1 \;.$$

Thus, for all i it holds that $\beta_i P_{i+1:n}^{(1)} \leq \beta_n$ and $\|M_i\| \lesssim \beta_n^{1/2}$. Now we introduce the function

$$h(X_i) = \mathbb{E}^{\mathcal{F}_i}[M_i M_i^\top] \;,\; \mathcal{F}_j = \sigma(X_s : s \leq j) \;.$$

Note that $\|h(X_i)\| \leq \beta_n$. Thus, since

$$\beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last,m}} = \sum_{i=0}^{n}\mathbb{E}[h(X_i)] \;,$$

Lemma 29 implies that

$$\mathbb{P}\left[\|\sum_{i=0}^{n}h(X_i) - \beta_n^{-1}\boldsymbol{\Sigma}_n^{\mathrm{last,m}}\| \geq nt\right] \leq 4\exp(-\frac{nt^2}{80dt_{\mathrm{mix}}\beta_n^2}) \;.$$

Hence, the assumptions of Lemma 3 hold with $C_1 = 4$ and $C_2 = (80 d t_{\text{mix}} \beta_n^2)^{-1}$, which yiels with $\kappa := \beta_n^{1/2}$ and $p := \log n$:

$$
\begin{aligned}
\rho^{\text{Conv}}(\sum_{i=0}^{n} M_i, \beta_n^{-1}\boldsymbol{\Sigma}_n^{\text{last,m}}) &= \rho^{\text{Conv}}\left(\tfrac{1}{\sqrt{n+1}}\sum_{i=0}^{n} M_i, \mathcal{N}(0, \tfrac{1}{n+1}\beta_n^{-1}\boldsymbol{\Sigma}_n^{\text{last,m}})\right) \\
&\overset{(a)}{\lesssim} (\log(n))^{3/4}\{\beta_n^{1/2} n^{1/4}(\log n)^{1/4} + \beta_n^{1/2} + \frac{1}{n\beta_n^{1/2}} + \frac{\beta_n\sqrt{\log n}}{\sqrt{n}}\} \\
&\lesssim \frac{\log n}{n^{b/2-1/4}} \,,
\end{aligned}
$$

where in (a) we have used an elementary inequality $(n^{-\alpha})^{\frac{\log n}{1+\log n}} \lesssim n^{-\alpha}$ together with $\frac{1}{2}\|\boldsymbol{\Sigma}_\infty^{\text{last,m}}\| \leq \|\beta_n^{-1}\boldsymbol{\Sigma}_n^{\text{last,m}}\| \lesssim \|\boldsymbol{\Sigma}_\infty^{\text{last,m}}\|$ which holds due to B 3 and Proposition 9. Now we combine (99) with the latter inequality and get:

$$
\rho^{\text{Conv}}(\beta_n^{-1/2}\tilde{\theta}_{n+1}, \mathcal{N}(0, \boldsymbol{\Sigma}_\infty^{\text{last,m}})) \lesssim \frac{n^{b/2}\log^3(n)}{2b-a-1}\prod_{j=0}^{n}(1 - \frac{a_\Delta}{8}\beta_j) + \frac{\log^3(n)}{n^{a-b/2}} + \frac{\log^4(n)}{(2b-a-1)n^{\frac{3b-a-2}{2}}} + \frac{\log n}{n^{b/2-1/4}} \,.
$$

□

To prove Lemmas 23-27 we formulate an auxiliary result that controls the moments of $\tilde{\theta}_k^{(1)}$, $\tilde{w}_k^{(1)}$:

Lemma 28. *Let $p \geq 2$. Assume A4, A6, B 1, B 2(p). Then it holds for all $k \in \mathbb{N}$ that*

$$
\mathbb{E}^{1/p}[\|\tilde{w}_k^{(1)}\|^p] \lesssim P_{0:k}^{(2)} + p^3\gamma_k \,, \quad \mathbb{E}^{1/p}[\|\tilde{\theta}_k^{(1)}\|^p] \lesssim p^3 P_{0:k}^{(1)} + p^3\gamma_k \,.
$$

Proof. The bound for $\tilde{w}_k^{(1)}$ follows from Equation 86 applying Proposition 4 and Lemma 31-(ii). To proceed with $\tilde{\theta}_k^{(1)}$, we use the decomposition that follows from (72):

$$
\tilde{\theta}_k^{(1)} = -\underbrace{\sum_{i=0}^{k-1}\beta_i\Gamma_{i+1:k-1}^{(1)}A_{12}\tilde{w}_i^{(1)}}_{Z_1} - \underbrace{\sum_{i=0}^{k-1}\beta_i\Gamma_{i+1:k-1}^{(1)}V_{i+1}^{(1)}}_{Z_2} \,.
$$

The bound for Z_1 follows from Minkowski's inequality and the bound for $\tilde{w}_k^{(1)}$:

$$
\mathbb{E}^{1/p}[\|Z_1\|^p] \lesssim \sum_{i=0}^{k-1}\beta_i P_{i+1:k-1}^{(1)} P_{0:i-1}^{(2)} + p^3\sum_{i=0}^{k-1}\beta_i\gamma_i P_{i+1:k-1}^{(1)} \overset{(a)}{\lesssim} P_{0:k}^{(1)} + p^3\gamma_k \,,
$$

where in (a) we additionally used Lemma 31-(ii) and Lemma 32-(ii).

$$
V_{i+1}^{(1)} = (\mathrm{P}\boldsymbol{g}_i^{\boldsymbol{\varepsilon}_V} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{11}})\tilde{\theta}_i - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{12}})\hat{w}_i) - (\mathrm{P}\boldsymbol{g}_{i+1}^{\boldsymbol{\varepsilon}_V} - (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{11}})\tilde{\theta}_{i+1} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{12}})\hat{w}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{11}})(\tilde{\theta}_i - \tilde{\theta}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{12}})(\hat{w}_i - \hat{w}_{i+1}) \,.
$$

Introduce the following notation:

$$
v_i = \mathrm{P}\boldsymbol{g}_i^{\boldsymbol{\varepsilon}_V} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{11}})\tilde{\theta}_i - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{12}})\hat{w}_i \,.
$$

Thus, we rewrite Z_2 using Lemma 21:

$$
\begin{aligned}
Z_2 = \beta_0\Gamma_{1:k-1}^{(1)}v_0 - \beta_k v_k &+ \sum_{i=1}^{k}(\beta_i^2 B_{11}^i\Gamma_{i+1:k-1}^{(1)} + (\beta_i - \beta_{i-1})\Gamma_{i:k-1}^{(1)})v_i \\
&+ \sum_{i=0}^{k-1}\beta_i\Gamma_{i+1:k-1}^{(1)}\{(\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{11}})(\tilde{\theta}_i - \tilde{\theta}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{12}})(\hat{w}_i - \hat{w}_{i+1})\} \,.
\end{aligned}
$$

Therefore, we get applying Minkowski's inequality together with Lemma 31-(ii) and $\gamma_i \lesssim \beta_i^{a/b}$:

$$
\mathbb{E}^{1/p}[\|Z_2\|^p] \lesssim p^3 P_{0:k}^{(1)} + p^3\beta_k + p^3\beta_k^{a/b} \lesssim p^3 P_{0:k}^{(1)} + p^3\gamma_k \,.
$$

The proof follows from gathering bounds for Z_1 and Z_2. □

Proof of Lemma 23. First, we introduce:

$$\delta_j^{(1,i)} = A_{12} L_j \tilde{\theta}_j^{(i)} \ , \ i \in \{0,1\} \ .$$

Thus, we rewrite the initial sum

$$\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \delta_j^{(1)} = \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \delta_j^{(1,0)}}_{Z_1} + \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \delta_j^{(1,1)}}_{Z_2} \ .$$

Z_1 can be bounded repeating the lines of Lemma 14 due to the recurrence properties (72):

$$\mathbb{E}^{1/p}[\|Z_1\|^p] \lesssim (2b-a-1)^{-1} P_{0:n}^{(1)} + p^4 (2b-a-1)^{-1} \beta_n^{\frac{2b-a/2-1}{b}} \ .$$

To estimate $\mathbb{E}^{1/p}[\|Z_2\|^p]$ we first use Lemma 4, Lemma 28 and obtain

$$\mathbb{E}^{1/p}[\|\delta_j^{(1,1)}\|^p] \lesssim \frac{\beta_j}{\gamma_j} \{ p^3 P_{0:j}^{(1)} + p^3 \gamma_j \} \ .$$

Thus,

$$\mathbb{E}^{1/p}[\|Z_2\|^p] \lesssim p^3 P_{0:n}^{(1)} \sum_{j=0}^{k} \frac{\beta_j^2}{\gamma_j} + p^3 \beta_n \lesssim \frac{p^3}{2b-a-1} P_{0:n}^{(1)} + p^3 \beta_n \ .$$

The proof follows from gathering the bounds for Z_1 and Z_2 together with applying B 2. □

Proof of Lemma 24. First, we introduce:

$$\delta_j^{(2,i)} = -(L_{j+1} + A_{22}^{-1} A_{21}) A_{12} \tilde{w}_j^{(i)} \ , \ i \in \{0,1\} \ .$$

Thus, we rewrite the initial sum as follows:

$$S_n^{(1)} = \underbrace{-\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \beta_i G_{i+1:j-1}^{(2)} \delta_i^{(2,0)}}_{Z_1} \underbrace{- \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \beta_i G_{i+1:j-1}^{(2)} \delta_i^{(2,1)}}_{Z_2} \ .$$

One can obtain the bound on Z_1 following the lines of Lemma 16 due to the fact that $\tilde{w}_j^{(0)}$ is a martingale-difference sequence w.r.t. $\mathcal{F}_j = \sigma(X_s : s \le j)$:

$$\mathbb{E}^{1/p}[\|Z_1\|^p] \lesssim P_{0:n}^{(1)} + p^4 \beta_n^{\frac{3b-2a}{2b}} \ .$$

To derive a bound for Z_2 we use Minkowski's inequality and Lemma 28:

$$\begin{aligned}
\mathbb{E}^{1/p}[\|Z_2\|^p] &\lesssim \sum_{j=0}^{n} \sum_{i=0}^{j-1} \beta_j \beta_i P_{j+1:n}^{(1)} P_{i+1:j-1}^{(2)} \{ P_{0:i}^{(2)} + p^3 \gamma_i \} = \sum_{i=0}^{n-1} \beta_i \{ P_{0:i}^{(2)} + p^3 \gamma_i \} \sum_{j=i+1}^{n} \beta_j P_{j+1:n}^{(1)} P_{i+1:j-1}^{(2)} \\
&\overset{(a)}{\lesssim} \sum_{i=0}^{n-1} \beta_i \gamma_i^{\frac{b-a}{a}} \{ P_{0:i}^{(2)} + p^3 \gamma_i \} P_{i+1:n}^{(1)} \overset{(b)}{\lesssim} P_{0:n}^{(1)} + p^3 \beta_n \ ,
\end{aligned}$$

where in (a) and (b) we have used Lemma 32-(ii) together with Lemma 31-(ii). The proof follows from gathering similar terms in the bounds for Z_1 and Z_2. □

Proof of Lemma 25. First, we decompose $S_n^{(2)}$ as follows:

$$S_n^{(2)} = \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \beta_i G_{i+1:j-1}^{(2)} D_i V_{i+1}^{(0)}}_{Z_1} + \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \beta_i G_{i+1:j-1}^{(2)} D_i V_{i+1}^{(1)}}_{Z_2} \ .$$

Since $V_{i+1}^{(0)}$ is a martingale difference sequence w.r.t. $\mathcal{F}_i$, repeating the lines of Lemma 17 one can obtain that

$$\mathbb{E}^{1/p}[\|Z_1\|^p] \lesssim p^4 \beta_n^{\frac{2b-a}{2b}} \ .$$

To proceed with Z_2, recall the decomposition

$$V_{i+1}^{(1)} = (\mathrm{P}\boldsymbol{g}_i^{\boldsymbol{\varepsilon}_V} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{11}})\tilde{\theta}_i - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{12}})\hat{w}_i) - (\mathrm{P}\boldsymbol{g}_{i+1}^{\boldsymbol{\varepsilon}_V} - (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{11}})\tilde{\theta}_{i+1} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{12}})\hat{w}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{11}})(\tilde{\theta}_i - \tilde{\theta}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{12}})(\hat{w}_i - \hat{w}_{i+1}) \ .$$

Introduce the following notation:

$$v_i = \mathrm{P}\boldsymbol{g}_i^{\boldsymbol{\varepsilon}_V} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{11}})\tilde{\theta}_i - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{12}})\hat{w}_i \ , Q_i = \sum_{j=i+1}^{n} \beta_j G_{j+1:k}^{(1)} A_{12} G_{i+1:j-1}^{(2)} D_i \ .$$

Lemma 32-(ii) implies that $\|Q_i\| \lesssim \gamma_i^{(b-a)/a} P_{i+1:k}^{(1)}$. Then we estimate $\|Q_i - Q_{i+1}\|$ using Lemma 32-(i), (ii):

$$\begin{aligned}\|Q_i - Q_{i+1}\| &\lesssim \sum_{j=i+1}^{n} \beta_j P_{j+1:n}^{(1)} P_{i+1:j-1}^{(2)} \|D_i - D_{i+1}\| + \beta_{i+1} P_{i+2:n}^{(1)} + \gamma_i \sum_{j=i+2}^{n} \beta_j P_{j+1:n}^{(1)} P_{i+2:j-1}^{(2)} \\ &\lesssim \gamma_i \gamma_i^{(b-a)/a} P_{i+1:n}^{(1)} + \beta_i P_{i+1:n}^{(1)} + \gamma_i \gamma_i^{(b-a)/a} P_{i+1:n}^{(1)} \lesssim \beta_i P_{i+1:n}^{(1)} \ .\end{aligned}$$

Now we swap the order of summation and rewrite Z_2 as follows:

$$Z_2 = \underbrace{\sum_{i=0}^{n-1} Q_i \beta_i (v_i - v_{i+1})}_{Z_{21}} + \underbrace{\sum_{i=0}^{n-1} Q_i \beta_i \{(\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{11}})(\tilde{\theta}_i - \tilde{\theta}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{12}})(\hat{w}_i - \hat{w}_{i+1})\}}_{Z_{22}} \ .$$

Then we further decompose Z_{21}:

$$Z_{21} = \sum_{i=0}^{n-1} \{(Q_i \beta_i v_i - Q_{i+1} \beta_{i+1} v_{i+1}) + (Q_{i+1} - Q_i)\beta_i v_{i+1} + Q_i(\beta_{i+1} - \beta_i) v_{i+1}\} \ .$$

Now note that for all i it holds that $\mathbb{E}^{1/p}[\|v_i\|^p] \lesssim p^3$ due to Proposition 4. Thus, we get using Lemma 31-(ii):

$$\begin{aligned}\mathbb{E}^{1/p}[\|Z_{21}\|^p] &\lesssim \mathbb{E}^{1/p}[\|Q_0 \beta_0 v_0\|^p] + \mathbb{E}^{1/p}[\|Q_n \beta_n v_n\|^p] + \sum_{i=0}^{n-1} p^3 \{\beta_i^2 + \beta_i^3\} P_{i+1:n}^{(1)} + \sum_{i=0}^{n-1} p^3 \beta_i \gamma_i \gamma_i^{(b-a)/a} P_{i+1:n}^{(1)} \\ &\overset{(a)}{\lesssim} p^3 P_{1:n}^{(1)} + p^3 \beta_n \ ,\end{aligned}$$

where in (a) we have additionally used $\gamma_i \lesssim \beta_i^{a/b}$ and Lemma 31-(ii). Since $\mathbb{E}^{1/p}[\|\tilde{\theta}_i - \tilde{\theta}_{i+1}\|^p] \lesssim p^3 \gamma_i$ and $\mathbb{E}^{1/p}[\|\hat{w}_i - \hat{w}_{i+1}\|^p] \lesssim p^3 \gamma_i$, we obtain applying Lemma 31-(ii):

$$\mathbb{E}^{1/p}[\|Z_{22}\|^p] \lesssim p^3 \sum_{i=0}^{n-1} \beta_i \gamma_i^{(b-a)/a} \gamma_i P_{i+1:n}^{(1)} \lesssim p^3 \beta_n \ .$$

Finally,

$$\mathbb{E}^{1/p}[\|S_n^{(2)}\|^p] \lesssim \mathbb{E}^{1/p}[\|Z_1\|^p] + \mathbb{E}^{1/p}[\|Z_{21}\|^p] + \mathbb{E}^{1/p}[\|Z_{22}\|^p] \lesssim p^3 P_{0:n}^{(1)} + p^4 \beta_n^{\frac{2b-a}{2b}} \ .$$

□

Proof of Lemma 26. First, we decompose $S_k^{(3)}$ into two parts:

$$\begin{aligned}S_k^{(3)} &= \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \gamma_i G_{i+1:j-1}^{(2)} W_{i+1}^{(0)} - \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} A_{22}^{-1} W_{j+1}^{(0)}}_{Z_1} \\ &+ \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \gamma_i G_{i+1:j-1}^{(2)} W_{i+1}^{(1)} - \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} A_{22}^{-1} W_{j+1}^{(1)}}_{Z_2} \ .\end{aligned}$$

Note that Z_1 can be bounded following the lines of Lemma 18. Precisely,

$$\mathbb{E}^{1/p}[\|Z_1\|^p] \lesssim p^4 \beta_n^{\frac{2b-a}{2b}} \ . \tag{100}$$

To proceed with Z_2, we derive a decomposition for $W_{i+1}^{(1)}$:

$$W_{i+1}^{(1)} = (\mathrm{P}\boldsymbol{g}_i^{\boldsymbol{\varepsilon}_W} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{21}})\tilde{\theta}_i - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{22}})\hat{w}_i) - (\mathrm{P}\boldsymbol{g}_{i+1}^{\boldsymbol{\varepsilon}_W} - (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{21}})\tilde{\theta}_{i+1} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{22}})\hat{w}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{21}})(\tilde{\theta}_i - \tilde{\theta}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{22}})(\hat{w}_i - \hat{w}_{i+1}) \; . \tag{101}$$

Introduce the following notation:

$$v_i = \mathrm{P}\boldsymbol{g}_i^{\boldsymbol{\varepsilon}_W} - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{11}})\tilde{\theta}_i - (\mathrm{P}\boldsymbol{g}_i^{\mathbf{A}_{12}})\hat{w}_i \; , Q_i = \sum_{j=i+1}^{k} \beta_j G_{j+1:k}^{(1)} A_{12} G_{i+1:j-1}^{(2)} \; ,$$

$$Z_{21} = \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} \sum_{i=0}^{j-1} \gamma_i G_{i+1:j-1}^{(2)} W_{i+1}^{(1)} \; , \; Z_{22} = \sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} A_{12} A_{22}^{-1} W_{j+1}^{(1)} \; .$$

Lemma 32-(ii) implies that $\|Q_i\| \le \gamma_i^{(b-a)/a} P_{i+1:k}^{(1)}$ and

$$\|Q_i - Q_{i+1}\| \lesssim \beta_i P_{i+1:n}^{(1)} + \gamma_i \gamma_i^{\frac{b-a}{a}} P_{i+1:n}^{(1)} \lesssim \beta_i P_{i+1:n}^{(1)} \; .$$

Now we swap the order of summation and rewrite Z_{21} as follows:

$$Z_{21} = \underbrace{\sum_{i=0}^{n-1} \gamma_i Q_i (v_i - v_{i+1})}_{Z_{211}} + \underbrace{\sum_{i=0}^{n-1} Q_i \beta_i \{(\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{21}})(\tilde{\theta}_i - \tilde{\theta}_{i+1}) + (\mathrm{P}\boldsymbol{g}_{i+1}^{\mathbf{A}_{22}})(\hat{w}_i - \hat{w}_{i+1})\}}_{Z_{212}} \; .$$

Then we further decompose Z_{211}:

$$Z_{211} = \sum_{i=0}^{n-1} \{(Q_i \gamma_i v_i - Q_{i+1}\gamma_{i+1} v_{i+1}) + (Q_{i+1} - Q_i)\gamma_i v_{i+1} + Q_i(\gamma_{i+1} - \gamma_i) v_{i+1}\} \; .$$

Therefore, using Lemma 31-(ii) and $\gamma_i \lesssim \beta_i^{a/b}$, easy to see that:

$$\mathbb{E}^{1/p}[\|Z_{211}\|^p] \lesssim p^3 P_{0:n}^{(1)} + p^3 \gamma_n + p^3 \beta_n^{\frac{a+b}{b}-1} \lesssim p^3 P_{0:n}^{(1)} + p^3 \beta_n^{a/b} \; . \tag{102}$$

To derive a bound for Z_{212} we use Minkowski's inequality and get

$$\mathbb{E}^{1/p}[\|Z_{212}\|^p] \lesssim \sum_{i=0}^{n-1} \beta_i \gamma_i^{\frac{b-a}{a}} p^3 \gamma_i P_{i+1:n}^{(1)} = p^3 \sum_{i=0}^{n-1} \beta_i^{1+\frac{b-a}{b}+\frac{a}{b}} P_{i+1:n}^{(1)} \lesssim p^3 \beta_n \; . \tag{103}$$

Thus, $\mathbb{E}^{1/p}[\|Z_{21}\|^p] \lesssim p^3 P_{0:n}^{(1)} + p^3 \beta_n^{a/b}$. Substituting the decomposition (101) into the expression for Z_{22} one can check applying Lemma 21 that

$$\mathbb{E}^{1/p}[\|Z_{22}\|^p] \lesssim p^3 P_{0:n}^{(1)} + p^3 \beta_n + p^3 \gamma_n \lesssim p^3 P_{0:n}^{(1)} + p^3 \beta_n^{a/b}. \tag{104}$$

The proof follows from gathering the bounds (100), (102), (103), (104). ☐

Proof of Lemma 27. First, we rewrite H_n using the solutions $\boldsymbol{g}_i^{\Phi}$, $\boldsymbol{g}_i^{\Psi}$ of the corresponding Poisson equations:

$$\Phi_{j+1}\tilde{\theta}_j = (\boldsymbol{g}_{j+1}^{\Phi} - \mathrm{P}\boldsymbol{g}_j^{\Phi})\tilde{\theta}_j + \{(\mathrm{P}\boldsymbol{g}_j^{\Phi})\tilde{\theta}_j - (\mathrm{P}\boldsymbol{g}_{j+1}^{\Phi}\tilde{\theta}_{j+1})\} + (\mathrm{P}\boldsymbol{g}_{j+1}^{\Phi})(\tilde{\theta}_{j+1} - \tilde{\theta}_j) \; ,$$
$$\Psi_{j+1}\hat{w}_j = (\mathrm{P}\boldsymbol{g}_{j+1}^{\Psi} - \mathrm{P}\boldsymbol{g}_j^{\Psi})\hat{w}_j + \{(\mathrm{P}\boldsymbol{g}_j^{\Psi})\hat{w}_j - (\mathrm{P}\boldsymbol{g}_{j+1}^{\Psi}\hat{w}_{j+1})\} + (\mathrm{P}\boldsymbol{g}_{j+1}^{\Psi})(\hat{w}_{j+1} - \hat{w}_j) \; ,$$

Now we rewrite H_k as follows

$$H_n = \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \{v_j - v_{j+1}\}}_{H_1} + \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \{(\boldsymbol{g}_{j+1}^{\Phi} - \mathrm{P}\boldsymbol{g}_j^{\Phi})\tilde{\theta}_j + (\boldsymbol{g}_{j+1}^{\Psi} - \mathrm{P}\boldsymbol{g}_j^{\Psi})\hat{w}_j\}}_{H_2}$$
$$+ \underbrace{\sum_{j=0}^{n} \beta_j G_{j+1:n}^{(1)} \{(\mathrm{P}\boldsymbol{g}_{j+1}^{\Phi})(\tilde{\theta}_{j+1} - \tilde{\theta}_j) + (\mathrm{P}\boldsymbol{g}_{j+1}^{\Psi})(\hat{w}_{j+1} - \hat{w}_j)\}}_{H_3} \; ,$$

where we have set

$$v_i = -\mathrm{P}\boldsymbol{g}_j^{\boldsymbol{\varepsilon}_\theta} + (\mathrm{P}\boldsymbol{g}_j^{\Phi})\tilde{\theta}_j + (\mathrm{P}\boldsymbol{g}_j^{\Psi})\hat{w}_j \; .$$

The bound for H_1 follows from Lemma 21 and Lemma 31-(ii):

$$\mathbb{E}^{1/p}[\|H_1\|^p] \lesssim p^3 P_{0:n}^{(1)} + p^3\beta_n + \sum_{j=0}^{n} \beta_j^2 P_{j+1:n}^{(1)} p^3 \lesssim p^3 P_{0:n}^{(1)} + p^3 \beta_n \; . \tag{105}$$

Next, we note that H_2 is a sum of martingale difference sequence w.r.t. the filtration $\mathcal{F}_k = \sigma(X_s : s \leq k)$. Thus, we get applying Burkholders inequality (Osekowski 2012, Theorem 8.1), Proposition 4 and Lemma 31-(ii):

$$\mathbb{E}^{2/p}[\|H_2\|^p] \lesssim p^2 \sum_{j=0}^{n} \beta_j^2 P_{j+1:n}^{(1)} \{P_{0:j}^{(1)} + p^6 \gamma_j\} \lesssim \frac{p^2}{2b-1} P_{0:n}^{(1)} + p^6 \beta_n^{\frac{b+a}{b}} \; . \tag{106}$$

Finally, we derive a bound for H_3 using Minkowski's inequality

$$\mathbb{E}^{1/p}[\|H_3\|^p] \lesssim \sum_{j=0}^{n} p^3 \beta_j \gamma_j P_{j+1:n}^{(1)} \lesssim p^3 \beta_n^{a/b} \; . \tag{107}$$

The proof follows from gathering the bounds (105), (106), (107). □

E.4 Matrix concentration inequality

In this section we state the lemma that derives a McDiarmid-type concentration inequality for matrix-valued functions of an UGE Markov chain.

Lemma 29. *Assume B 1. Let $\{g_i\}_{i=1}^n$ be a family of measurable functions from Z to $\mathbb{R}^{d\times d}$ such that $M = \sup_{Z\in\mathsf{Z}} \|g(Z)\| < \infty$ and $\pi(g_i) = 0$ for any $i \in \{1,\dots,n\}$. Then, for any initial probability ξ on $(\mathsf{Z},\mathcal{Z})$, $n \in \mathbb{N}$, $t \geq 0$, it holds*

$$\mathbb{P}_\xi\left(\|\sum\nolimits_{i=1}^n g_i(Z_i)\| \geq t\right) \leq 4\exp\left\{-\frac{t^2}{80ndt_{\mathrm{mix}}M^2}\right\} \; . \tag{108}$$

Proof. The function $\varphi(z_1,\dots,z_n) := \|\sum_{i=1}^n g_i(z_i)\|$ on Z^n satisfies the bounded differences property:

$$|\varphi(z_1,\dots,z_n) - \varphi(z_1',\dots,z_n')| \leq \sum_{i=1}^n 2M\mathbf{1}\{z_i \neq z_i'\} \; .$$

Hence, since $(1/2)\sup_{z,z'\in\mathsf{Z}} \|\mathrm{P}^{t_{\mathrm{mix}}}(z,\cdot) - \mathrm{P}^{t_{\mathrm{mix}}}(z',\cdot)\|_{\mathsf{TV}} \leq 1/4$ by definition of t_{mix} under B 1, applying (Paulin 2015, Corollary 2.10), we get for $t \geq \mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|]$,

$$\mathbb{P}_\xi\left(\|\sum\nolimits_{i=1}^n g_i(Z_i)\| \geq t\right) \leq 2\exp\left\{-\frac{2(t-\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|])^2}{9(n\cdot 4M^2)t_{\mathrm{mix}}}\right\} = 2\exp\left\{-\frac{(t-\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|])^2}{18nM^2 t_{\mathrm{mix}}}\right\} \; .$$

It remains to upper bound $\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|]$. Note that

$$\mathbb{E}_\xi[\|\sum\nolimits_{i=1}^n g_i(Z_i)\|^2] \leq \mathbb{E}_\xi[\|\sum\nolimits_{i=1}^n g_i(Z_i)\|_{\mathrm{Fr}}^2] = \sum\nolimits_{i=1}^n \mathbb{E}_\xi[\|g_i(Z_i)\|_{\mathrm{Fr}}^2] + 2\sum\nolimits_{k=1}^{n-1}\sum\nolimits_{\ell=1}^{n-k} \mathrm{Tr}\left(\mathbb{E}_\xi[g_k(Z_k)^\top g_{k+\ell}(Z_{k+\ell})]\right) \; ,$$

and, using B 1 and $\pi(g_{k+\ell}) = 0$, we obtain

$$\begin{aligned}\mathrm{Tr}\left(\mathbb{E}_\xi[g_k(Z_k)^\top g_{k+\ell}(Z_{k+\ell})]\right) &= \int_{\mathsf{Z}} \mathrm{Tr}\left\{g_k(z)^\top\left(\mathrm{P}^\ell g_{k+\ell}(z) - \pi(g_{k+\ell})\right)\right\}\xi\mathrm{P}^k(\mathrm{d}z) \\ &\leq 2dM^2\Delta(\mathrm{P}^\ell) \; .\end{aligned}$$

Together with the definition of t_{mix}, this implies

$$\sum\nolimits_{k=1}^{n-1}\sum\nolimits_{\ell=1}^{n-k} \mathrm{Tr}\left(\mathbb{E}_\xi[g_k(Z_k)^\top g_{k+\ell}(Z_{k+\ell})]\right) \leq 2ndM^2 \sum_{\ell=1}^{n-1} \Delta(\mathrm{P}^\ell) \leq (8/3)dM^2 t_{\mathrm{mix}} n \; .$$

Combining the bounds above, we upper bound $\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|]$ as

$$\mathbb{E}_\xi[\|\sum\nolimits_{i=1}^n g_i(Z_i)\|] \leq \left\{\mathbb{E}_\xi[\|\sum\nolimits_{i=1}^n g_i(Z_i)\|^2]\right\}^{1/2} \leq 2\sqrt{dn}M\sqrt{t_{\mathrm{mix}}} =: v_n \; .$$

Plugging this result in (108), we obtain that

$$\mathbb{P}_\xi\left(\|\sum\nolimits_{i=1}^n g_i(Z_i)\| \geq t\right) \leq \begin{cases} 1, & t < v_n, \\ 2\exp\left\{-\frac{(t-v_n)^2}{18v_n^2}\right\}, & t \geq v_n \; . \end{cases} \tag{109}$$

Now it is easy to see that right-hand side of (109) is upper bounded by $4\exp\{-t^2/(20v_n^2)\}$ for any $t \geq 0$, and the statement follows. □

F Limit of matrix sums

Fix $\mathbf{A}, \boldsymbol{\Sigma} \in \mathbb{R}^{d \times d}$. Introduce the following notation:

$$\boldsymbol{\Sigma}_n = \sum_{k=1}^{n} \beta_k^2 G_{k+1:n} \boldsymbol{\Sigma} (G_{k+1:n})^\top ,$$

where $\beta_j = c_{0,\beta}/(n+k_0)^b$ and $G_{m:k} = \prod_{j=m}^{k} (\mathrm{I} - \beta_j \mathbf{A})$.

Proposition 9. *There exists a matrix $\boldsymbol{\Sigma}_\infty$ such that*

$$\lim_{n \to \infty} \{\beta_n^{-1} \boldsymbol{\Sigma}_n\} = \boldsymbol{\Sigma}_\infty .$$

Moreover, there exists a constant C_Σ that depends on the problem parameters such that

$$\|\beta_n^{-1} \boldsymbol{\Sigma}_n - \boldsymbol{\Sigma}_\infty\| \leq \mathrm{C}_\Sigma \, n^{-b} .$$

Proof. First, we define $\boldsymbol{\Sigma}^{(0)}$, $\boldsymbol{\Sigma}^{(1)}$ as a solution of the system of Ricatti equations

$$\begin{aligned} \boldsymbol{\Sigma}^{(0)} &= \beta_0 \{\Delta \boldsymbol{\Sigma}^{(0)} + \boldsymbol{\Sigma}^{(0)} \Delta^\top\} - \beta_0^2 \boldsymbol{\Sigma} , \\ \boldsymbol{\Sigma}^{(0)} - 2\boldsymbol{\Sigma}^{(1)} &= \beta_0 \{\Delta \boldsymbol{\Sigma}^{(0)} + \boldsymbol{\Sigma}^{(0)} \Delta^\top - (\Delta \boldsymbol{\Sigma}^{(1)} + \boldsymbol{\Sigma}^{(1)} \Delta^\top)\} + \beta_0^2 \{\Delta \boldsymbol{\Sigma}^{(0)} \Delta^\top - 2\boldsymbol{\Sigma}\}. \end{aligned} \tag{110}$$

Our goal is to compute $\boldsymbol{\Sigma}_n$. To derive a closed form solution, we observe that $\boldsymbol{\Sigma}_n$ can be iteratively computed as

$$\boldsymbol{\Sigma}_{n+1} = (\mathrm{I} - \beta_{n+1} \Delta) \boldsymbol{\Sigma}_n (\mathrm{I} - \beta_{n+1} \Delta)^\top + \beta_{n+1}^2 \boldsymbol{\Sigma} \tag{111}$$

Now we consider the diminishing step size rule with $\beta_n = \beta_0 n^{-b}$ for $b \in (\frac{1}{2}, 1]$ and $\beta_0 > 0$. Note we have ignored k_0 in the step size selection as we focus on the asymptotic expression with $n \gg 1$. Arranging terms in (111) yields

$$\boldsymbol{\Sigma}_{n+1} - \boldsymbol{\Sigma}_n = -\beta_0 (n+1)^{-b} \big\{\Delta \boldsymbol{\Sigma}_n + \boldsymbol{\Sigma}_n \Delta^\top\big\} + \beta_0^2 (n+1)^{-2b} \big\{\Delta \boldsymbol{\Sigma}_n \Delta^\top + \boldsymbol{\Sigma}\big\}. \tag{112}$$

Set

$$\boldsymbol{\Sigma}_n \equiv n^{-b} \boldsymbol{\Sigma}^{(0)} + n^{-2b} \boldsymbol{\Sigma}^{(1)} + D_n , \tag{113}$$

where D_n is a residual term whose order will be determined later. Note that for any $b > 0$, it holds

$$(n+1)^{-b} = n^{-b} - b n^{-1-b} + \frac{b(b+1)}{2} n^{-2-b} + \mathcal{O}(n^{-3-b}) ,$$

We focus on the case $b = 1$. Applying the above with $b = 1$, we observe that

$$\begin{aligned} \boldsymbol{\Sigma}_{n+1} - \boldsymbol{\Sigma}_n &= \{(n+1)^{-1} - n^{-1}\} \boldsymbol{\Sigma}^{(0)} + \{(n+1)^{-2} - n^{-2}\} \boldsymbol{\Sigma}^{(1)} + D_{n+1} - D_n \\ &= \{-n^{-2} + n^{-3}\} \boldsymbol{\Sigma}^{(0)} - 2n^{-3} \boldsymbol{\Sigma}^{(1)} + D_{n+1} - D_n + \mathcal{O}(n^{-4}) . \end{aligned} \tag{114}$$

On the other hand, observe that the right hand side of (112) can be written as follows

$$\begin{aligned} &- \beta_0 (n+1)^{-1} \{\Delta \boldsymbol{\Sigma}_n + \boldsymbol{\Sigma}_n \Delta^\top\} = -\beta_0 (n^{-1} - n^{-2} + \mathcal{O}(n^{-3})) \{\Delta \boldsymbol{\Sigma}_n + \boldsymbol{\Sigma}_n \Delta^\top\} \\ &= -\beta_0 (n^{-1} - n^{-2}) (n^{-1} \{\Delta \boldsymbol{\Sigma}^{(0)} + \boldsymbol{\Sigma}^{(0)} \Delta^\top\} + n^{-2} \{\Delta \boldsymbol{\Sigma}^{(1)} + \boldsymbol{\Sigma}^{(1)} \Delta^\top\}) + \mathcal{O}(n^{-4} + n^{-1} \|D_n\|) \end{aligned}$$

We also have

$$\begin{aligned} &\beta_0^2 (n+1)^{-2} \{\Delta \boldsymbol{\Sigma}_n \Delta^\top + \boldsymbol{\Sigma}\} = \beta_0^2 (n^{-2} - 2n^{-3}) \{\Delta \boldsymbol{\Sigma}_n \Delta^\top + \boldsymbol{\Sigma}\} + \mathcal{O}(n^{-4}) \\ &= \beta_0^2 (n^{-2} - 2n^{-3}) \{n^{-1} \Delta \boldsymbol{\Sigma}^{(0)} \Delta^\top + n^{-2} \Delta \boldsymbol{\Sigma}^{(1)} \Delta^\top + \boldsymbol{\Sigma}\} + \mathcal{O}(n^{-4} + n^{-2} \|D_n\|) \end{aligned}$$

Matching terms of the same order with (114) shows that

$$\begin{aligned} &(n^{-2}) & \boldsymbol{\Sigma}^{(0)} &= \beta_0 \{\Delta \boldsymbol{\Sigma}^{(0)} + \boldsymbol{\Sigma}^{(0)} \Delta^\top\} - \beta_0^2 \boldsymbol{\Sigma} \\ &(n^{-3}) & \boldsymbol{\Sigma}^{(0)} - 2\boldsymbol{\Sigma}^{(1)} &= \beta_0 \{\Delta \boldsymbol{\Sigma}^{(0)} + \boldsymbol{\Sigma}^{(0)} \Delta^\top - (\Delta \boldsymbol{\Sigma}^{(1)} + \boldsymbol{\Sigma}^{(1)} \Delta^\top)\} + \beta_0^2 \{\Delta \boldsymbol{\Sigma}^{(0)} \Delta^\top - 2\boldsymbol{\Sigma}\}. \end{aligned}$$

We observe that the remaining terms are all in the order of at most $\mathcal{O}(n^{-4})$. As such, we also conclude that the residual term in (113) is of the order at most $D_n = \mathcal{O}(n^{-3})$. In particular, solving the system of Riccati equations (110) yield $\boldsymbol{\Sigma}^{(0)}, \boldsymbol{\Sigma}^{(1)}$, i.e. the asymptotic expression for $\boldsymbol{\Sigma}_n$ is

$$\sum_{k=1}^{n} \beta_k^2 G_{k+1:n} \boldsymbol{\Sigma} (G_{k+1:n})^\top = n^{-1} \boldsymbol{\Sigma}^{(0)} + n^{-2} \boldsymbol{\Sigma}^{(1)} + \mathcal{O}(n^{-3})$$

Note that as the above analysis assumes the asymptotic case when $n \gg 1$, it actually covers the case when $\beta_n = \beta_0 (n+n_0)^{-1}$. The similar computations with $b \in (1/2, 1)$ imply that

$$\mathbf{\Sigma}_n = n^{-b} \mathbf{\Sigma}^{(0)} + n^{-2b} \mathbf{\Sigma}^{(1)} + \mathcal{O}(n^{-1-2b}) \; .$$

Therefore, setting $\mathbf{\Sigma}_\infty = \mathbf{\Sigma}^{(0)} / \beta_0$ we get

$$\| \beta_n^{-1} \mathbf{\Sigma}_n - \mathbf{\Sigma}_\infty \| \leq \frac{\| \mathbf{\Sigma}^{(0)} \| + \| \mathbf{\Sigma}^{(1)} \|}{\beta_0} \, \mathrm{C}_b \, n^{-b} = \mathrm{C}_\Sigma \, n^{-b} \; ,$$

where C_b depends only on b. □

The next lemma controls the minimal eigenvalue of $\mathbf{\Sigma}_n$:

Lemma 30. *Under the assumptions of Proposition 9 it holds for all* $n^b \geq \frac{2\,\mathrm{C}_\Sigma}{\lambda_{\min}(\mathbf{\Sigma}_\infty)}$ *that*

$$\lambda_{\min}(\beta_n^{-1} \mathbf{\Sigma}_n) \geq \frac{\lambda_{\min}(\mathbf{\Sigma}_\infty)}{2} \; .$$

Proof. First, we use Proposition 9 and obtain

$$\| \beta_n^{-1} \mathbf{\Sigma}_n - \mathbf{\Sigma}_\infty^{\mathrm{last}} \| \leq \frac{\mathrm{C}_\Sigma}{n^b} \; .$$

Hence, Lidskiy's inequality implies that

$$\lambda_{\min}(\beta_n^{-1} \mathbf{\Sigma}_n) = \lambda_{\min}(\beta_n^{-1} \mathbf{\Sigma}_n - \mathbf{\Sigma}_\infty + \mathbf{\Sigma}_\infty) \geq \lambda_{\min}(\mathbf{\Sigma}_\infty) - \| \beta_n^{-1} \mathbf{\Sigma}_n - \mathbf{\Sigma}_\infty \| \geq \frac{\lambda_{\min}(\mathbf{\Sigma}_\infty)}{2} \; .$$

□

G Applications

In this section, we verify that the GTD and TDC algorithms satisfy A4. Verification of the remaining assumptions is straightforward and thus omitted. We concentrate on the Markovian setting, as it is more prevalent in practical applications. Recall that the behavior policy π generates a trajectory $\{(s_k, a_k, r_k)\}_{k=0}^{\infty}$, where $a_k \sim \bar{\pi}(\cdot \mid s_k),\; s_{k+1} \sim \mathrm{P}(\cdot | s_k, a_k)$ for all $k \geq 0$ and the corresponding Markov kernel P_π satisfies **TD** 3.

Generalized Temporal Difference learning. The GTD algorithm was first introduced in (Sutton, Maei, and Szepesvári 2008). Recall its update rule:

$$\begin{cases} \theta_{k+1} = \theta_k + \beta_k (\varphi_k - \lambda \varphi_{k+1})(\varphi_k)^\top w_k \;, \quad \theta_0 \in \mathbb{R}^d \;, \\ w_{k+1} = w_k + \gamma_k (\delta_k \varphi_k - w_k) \;, \quad w_0 = 0 \; . \end{cases}$$

The above recursion is a special case of our linear two-timescale SA in (6), (7) with the notations:

$$\begin{aligned} b_1 &= 0 \;, \quad A_{11} = 0 \;, \quad A_{12} = -\mathbb{E}[(\varphi_k - \lambda \varphi_{k+1}) \varphi_k^\top] \;, \\ b_2 &= \mathbb{E}[\varphi_k r_k] \;, \quad A_{21} = -\mathbb{E}[\varphi_k (\lambda \varphi_{k+1} - \varphi_k)^\top] \;, \quad A_{22} = \mathrm{I}_d \;, \\ V_{k+1} &= \big((\varphi_k - \lambda \varphi_{k+1}) \varphi_k^\top - \mathbb{E}[(\varphi_k - \lambda \varphi_{k+1}) \varphi_k^\top] \big) \, w_k, \\ W_{k+1} &= \varphi_k r_k - \mathbb{E}[\varphi_k r_k] + \big((\varphi_k - \lambda \varphi_{k+1}) \varphi_k^\top - \mathbb{E}[(\varphi_k - \lambda \varphi_{k+1}) \varphi_k^\top] \big) \, \theta_k \;, \end{aligned}$$

where the above expectations are taken with respect to the randomness of the policy $\bar{\pi}$. The noise boundedness follows from **TD** 2, while A4 holds since $A_{22} = \mathrm{I}$ and $\Delta = \mathbb{E}[(\varphi_k - \lambda \varphi_{k+1}) \varphi_k^\top] \mathbb{E}[\varphi_k (\varphi_k - \lambda \varphi_{k+1})^\top]$ is positive definite.

Temporal-difference learning with gradient correction. The TDC algorithm was first introduced in (Sutton et al. 2009). Its update rule is:

$$\begin{cases} \theta_{k+1} = \theta_k + \beta_k \delta_k \varphi_k - \beta_k \gamma \varphi_{k+1} (\varphi_k^\top w_k) \;, \\ w_{k+1} = w_k + \gamma_k (\delta_k - \varphi_k^\top w_k) \varphi_k \; . \end{cases}$$

Reformulating these updates as an instance of (6)–(7) yields:

$$\begin{aligned} b_1 &= \mathbb{E}[\varphi_k r_k], \quad A_{11} = \mathbb{E}[\varphi_k (\varphi_k - \lambda \varphi_{k+1})^\top)], \quad A_{12} = \mathbb{E}[\lambda \varphi_{k+1} \varphi_k^\top], \\ b_2 &= \mathbb{E}[\varphi_k r_k], \quad A_{21} = \mathbb{E}[\varphi_k (\varphi_k - \lambda \varphi_{k+1})^\top)], \quad A_{22} = \mathbb{E}[\varphi_k \varphi_k^\top], \\ V_{k+1} &= \{ \mathbb{E}[\varphi_k (\varphi_k - \lambda \varphi_{k+1})^\top)] - \varphi_k (\varphi_k - \lambda \varphi_{k+1})^\top \} \theta_k + \{ \mathbb{E}[\lambda \varphi_{k+1} \varphi_k^\top] - \lambda \varphi_{k+1} \varphi_k^\top \} w_k, \\ W_{k+1} &= \{ \mathbb{E}[\varphi_k (\varphi_k - \lambda \varphi_{k+1})^\top)] - \varphi_k (\varphi_k - \lambda \varphi_{k+1})^\top \} \theta_k + \{ \mathbb{E}[\varphi_k \varphi_k^\top] - \varphi_k \varphi_k^\top \} w_k \; . \end{aligned}$$

The relations $A_{11} = A_{21}$ and $A_{12} = A_{22} - A_{11}^\top$ imply that Δ is positive definite:

$$\Delta = A_{11} - A_{12} A_{22}^{-1} A_{21} = A_{11} - (A_{22} - A_{11}^\top) A_{22}^{-1} A_{11} = A_{11}^\top A_{22}^{-1} A_{11} \; .$$

H Technical lemmas

We begin this section with technical lemmas that allows to upper bound the sums of the form

$$\sum_{j=1}^{k} \alpha_j^q \prod_{\ell=j+1}^{k} (1-\alpha_\ell b) \;.$$

Lemma 31. *The following statement holds:*

(i) *Let $b > 0$ and $(\alpha_k)_{k\geq 0}$ be a non-increasing sequence such that $\alpha_0 \leq 1/b$. Then*

$$\sum_{j=1}^{k} \alpha_j \prod_{l=j+1}^{k} (1-\alpha_l b) = \frac{1}{b}\left\{1 - \prod_{l=1}^{k}(1-\alpha_l b)\right\} \;.$$

(ii) *Let $b > 0$ and $\alpha_k = \frac{c_0}{(k+k_0)^\gamma}$, $\gamma \in (0,1)$, such that $c_0 \leq 1/b$ and $k_0^{1-\gamma} \geq \frac{8\gamma}{bc_0}$. Then for any $q \in (1,4]$ it holds that*

$$\sum_{j=1}^{k} \alpha_j^q \prod_{\ell=j+1}^{k} (1-\alpha_\ell b) \leq \frac{6}{b}\alpha_k^{q-1} \;.$$

Moreover, for any real-valued sequence $(b_j)_{j\geq 0}$ it holds that

$$\left\{\sum_{j=1}^{k} b_j \alpha_j^q \prod_{\ell=j+1}^{k} (1-\alpha_\ell b)\right\}^2 \leq \frac{6}{b}\alpha_k^{q-1} \sum_{j=1}^{k} b_j^2 \alpha_j^q \prod_{\ell=j+1}^{k} (1-\alpha_\ell b) \;. \tag{113}$$

(iii) *Let $b, c_0, k_0 > 0$ and $\alpha_\ell = c_0(\ell+k_0)^{-\gamma}$ for $\gamma \in (1/2,1)$ and $\ell \in \mathbb{N}$. Assume that $bc_0 < 1$ and $k_0^{1-\gamma} \geq \frac{1}{bc_0}$. Then, for any $\ell, n \in \mathbb{N}$, $\ell \leq n$, it holds that*

$$\sum_{k=\ell}^{n} \alpha_\ell \prod_{j=\ell+1}^{k} (1-b\alpha_j) \leq c_0 + \frac{1}{b(1-\gamma)} \;.$$

Proof. Lemma 31-(i) follows from Lemma 24 in (Durmus et al. 2021b). The first part of Lemma 31-(ii) follows from Lemma 33 in (Samsonov et al. 2025) and the second one (113) is a consequence of Jensen's inequality applied to $f(x) = x^2$. Lemma 31-(iii) is elementary. □

Lemma 32. *Assume A5 or B 2. Then it holds for all $j, k \in \mathbb{N}$ that*

(i)

$$\sum_{i=j+1}^{k} \gamma_j P^{(1)}_{i+1:k} P^{(2)}_{j+1:i-1} \leq \mathrm{C}^P_\gamma\, P^{(1)}_{j+1:k} \;, \text{ where } \mathrm{C}^P_\gamma = \frac{24}{a_{22}(1-\frac{c_{0,\beta}a_\Delta}{2})} \;.$$

(ii)

$$\sum_{i=j+1}^{k} \beta_j P^{(1)}_{i+1:k} P^{(2)}_{j+1:i-1} \leq \mathrm{C}^P_\beta\, \gamma_j^{(b-a)/a} P^{(1)}_{j+1:k} \;, \text{ where } \mathrm{C}^P_\beta = \frac{24c_{0,\beta}}{a_{22}c_{0,\gamma}^{b/a}(1-\frac{c_{0,\beta}a_\Delta}{2})} \;.$$

Proof. First, we prove (i). Since the step size was chosen such that $\beta_k/\gamma_k \leq r_{\mathrm{step}} \leq a_{22}/(2a_\Delta)$ we have

$$\frac{\left(1-\frac{a_{22}}{2}\gamma_\ell\right)}{\left(1-\frac{a_\Delta}{2}\beta_\ell\right)} \leq \left(1-\frac{a_{22}}{4}\gamma_\ell\right) \text{ and } \frac{P^{(2)}_{j+1:i-1}}{P^{(1)}_{j+1:i-1}} \leq \prod_{\ell=j+1}^{i-1}(1-\frac{a_{22}}{4}\gamma_\ell) \;.$$

Now proposition follows from Lemma 31-(iii):

$$\begin{aligned}\sum_{i=j+1}^{k} \gamma_j P^{(1)}_{i+1:k} P^{(2)}_{j+1:i-1} &= (1-\frac{\beta_j a_\Delta}{2})^{-1} P^{(1)}_{j+1:k} \sum_{i=j+1}^{k} \gamma_j \frac{P^{(2)}_{j+1:i-1}}{P^{(1)}_{j+1:i-1}} \leq (1-\frac{\beta_0 a_\Delta}{2})^{-1} P^{(1)}_{j+1:k} \sum_{i=j+1}^{k} \gamma_j \prod_{\ell=j+1}^{i-1}(1-\frac{a_{22}}{4}\gamma_\ell) \\ &\leq \frac{24}{a_{22}(1-\frac{c_{0,\beta}a_\Delta}{2})} P^{(1)}_{j+1:k} = \mathrm{C}^P_\gamma\, P^{(1)}_{j+1:k} \;.\end{aligned}$$

To proceed with (ii), we note that $\beta_j = \frac{c_{0,\beta}}{c_{0,\gamma}^{b/a}} \gamma_j^{1+(b-a)/a}$. Hence, we get using the technique similar to Lemma 32 combined with Lemma 31-(iii):

$$\sum_{i=j+1}^{k} \beta_j P_{i+1:k}^{(1)} P_{j+1:i-1}^{(2)} = (1 - \frac{\beta_j a_\Delta}{2})^{-1} P_{j+1:k}^{(1)} \sum_{i=j+1}^{k} \beta_j \frac{P_{j+1:i-1}^{(2)}}{P_{j+1:i-1}^{(1)}} \leq \frac{c_{0,\beta}}{c_{0,\gamma}^{b/a}(1 - \frac{\beta_0 a_\Delta}{2})} P_{j+1:k}^{(1)} \sum_{i=j+1}^{k} \gamma_j^{b/a} \prod_{\ell=j+1}^{i-1} (1 - \frac{a_{22}}{4} \gamma_\ell)$$
$$\leq \frac{24 c_{0,\beta}}{a_{22} c_{0,\gamma}^{b/a} (1 - \frac{c_{0,\beta} a_\Delta}{2})} \gamma_j^{(b-a)/a} .$$

□

Lemma 33 (Lemma 36 in (Samsonov et al. 2025))**.** *For any* $A > 0$*, any* $1 \leq i \leq n-1$*, and* $\gamma \in (1/2, 1)$ *it holds*

$$\sum_{j=i}^{n-1} \exp\left\{-A(j^{1-\gamma} - i^{1-\gamma})\right\} \leq \begin{cases} 1 + \exp\{\frac{1}{1-\gamma}\} \frac{1}{A^{1/(1-\gamma)}(1-\gamma)} \Gamma(\frac{1}{1-\gamma}) , & \text{if } A i^{1-\gamma} \leq \frac{1}{1-\gamma} ; \\ 1 + \frac{1}{A(1-\gamma)^2} i^{\gamma} , & \text{if } A i^{1-\gamma} > \frac{1}{1-\gamma} . \end{cases}$$