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Counter-cyclical Margins for Option Portfolios

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A R T I C L E I N F O

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A B S T R A C T

We propose a counter-cyclical initial margin model for option portfolios. Our model explores the intrinsic netting within a given portfolio of European options and outputs a constant upper bound of the maximum possible loss. This feature would allow option clearinghouses and regulators to gauge the tightest margin levels that are stable. We compare our model with the scenario-based SPAN model and the sensitivity-based SIMM model in terms of the netting efficiency and the procyclical property. Using the SPX options and the interest rate swaptions as examples, we quantify the minimum amount of additional margins needed to make them fully counter-cyclical. We then show how to strike a balance between risk-sensitivity and counter-cyclicality if needed by mixing our model flexibly with a prevailing risk-sensitive margin model.

1. Introduction

This paper studies the upper bounds of the maximum loss of an option portfolio and discusses the potential of using them to set stable initial margins during the Margin Period of Risk (MPOR). Whether a derivative contract trades through an exchange or bilaterally over-the-counter (OTC), the trade needs to be cleared and settled in the subsequent one to ten days (the MPOR). The clearing counterparties require their clearing members to post collateral in the form of initial margins to fend against potential loss during the settlement period, and initial margins are typically designed to be risk sensitive. The prevailing methodologies of setting initial margins for derivatives typically gain their risk sensitivity either through historical simulation[1], or through a set of pre-defined scenarios, or through the sensitivities of the product with respect to the underlying and the volatility. Examples include the Standard Portfolio Analysis of Risk (SPAN) method practiced by Chicago Merchandise of Exchange (CME)[2]; the System for Theoretical Analysis and Numerical Simulation (STANS) adopted by the Options Clearing Corporation (OCC)[3]; and the Standard Initial Margin Model (SIMM) proposed by the International Swaps and Derivatives Association (ISDA)[4].

Risk-sensitive margin requirements tend to be procyclical in the sense that they can amplify shocks Murphy et al. (2014). Volatility spikes lead to margin calls on clearing members and market stress such as elevated cost of funding is likely

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[1] In the form of Value-at-Risk (VaR) or Expected Shortfall (ES) using data from a historical window.

[2] See for more details at: https://www.cmegroup.com/clearing/span-methodology.html.

[3] See for more details at: https://www.theocc.com/Risk-Management/Margin-Methodology.

[4] See for more details at: https://www.isda.org/a/osMTE/ISDA-SIMM-v2.2-PUBLIC.pdf.

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to be corrected with the increase of volatility. Clearing members therefore need to post additional collateral precisely at the time when it becomes most difficult to raise funding. As the procyclicality in margin requirements threatens financial stability Constâncio (2016); European Systemic Risk Board (2017), global regulators propose countermeasures such as adding additional margin buffers, placing higher weights on stressed scenarios, and lengthening the historical look back window (Duffie, 2018; European Commission, 2013; Murphy et al., 2016). When examining these countermeasures, a key insight pointed by Glasserman and Wu (Glasserman and Wu, 2018a) is that the buffer required to offset procyclicality corresponds to the unconditional quantile of price changes. Its magnitude depends on the tail heaviness of the unconditional distribution, which depends on the persistence and burstiness of the volatility dynamic.

For linear derivatives, such as US 10 year interest rate swap and North America Investment Grade CDX, investigations in Glasserman and Wu (2018a) show that the stable margin level could be 35% and 65% higher respectively than the average conditional or risk-sensitive margin levels computed from the 5-day 99.5% VaR metric. As of the first half of 2021, the outstanding notional of OTC interest rate swaps alone stands at 372.4 trillion US dollars globally, with 120.9 trillion for USD-denominated contracts and 94 trillion of EUR-denominated contracts[5]. Given the sheer size of linear derivatives alone, the consequences and implications of additional collateralization are already far-reaching including how loss and defaults spread across a collateralized financial network Ghamami et al. (2021), and how clearinghouses can strike the balance between higher fees and better default protection versus decreased market volume (Capponi and Cheng, 2018).

The situation is more complex for options whose payoffs are *nonlinear* functions of the underlying. Although the size of global options market is relatively smaller than that of the linear derivatives, the degree of margin procyclicality is by no means small or simple. Taking the sensitivity margin approach on the OTC swaption (options on interest rate swaps) as an example, Glasserman and Wu (Glasserman and Wu, 2018b) show that, in a separate study, although the Standard Initial Margin Model (SIMM) includes features to reduce procyclicality, sensitivity-based margin requirements are still exposed to procyclicality through the dependence of price sensitivities on current market conditions, and this indirect exposure of greeks (delta, gamma, vega, etc) to market volatility changes may not be obvious in advance. To access the magnitude of stable margin for options, one needs to characterize the unconditional tail heaviness of the price changes of options, which depends nonlinearly on the dynamics of both the price change of the underlying and the change of the implied volatility. This will likely to be probabilistically difficult even for a single option.

A large portfolio of options, however, might present room for netting among payoffs if they share the same underlying. The observation is that the option books of many retail brokers such as Robinhood and TD Ameritrade typically contain from hundreds of thousands to millions of positions from individual customer accounts, and at especially at turbulant times, a significant portion of their option books concentrate on just a few underlying names in the front expiries[6]. If one develops constant bounds of a portfolio's maximum potential loss and they are indeed tight enough after netting, clearing members whose businesses are option heavy can use them to quickly assess their margin requirements at book level before clearing through OCC. This assessment would be distribution- and model-free, therefore insensitive to risk and volatility.

We therefore seek to develop a strategy-based netting algorithm[7] to identify the best possible internal netting for a given option portfolio and evaluate the feasibility of using it to set margins that are high enough to reduce credit exposure yet stable enough to avoid procyclical effects. The idea of netting is not new and goes back to studies such as the tradeoff between multilateral netting across dealers versus bilateral netting across asset classes Cont and Kokholm (2014). The essence of the method developed in this paper centers around how to optimally divide a given option portfolio into pre-defined simple strategies, where the margin requirements for individual simple strategies are specified by the counterparty, to achieve maximum netting efficiency inside the portfolio. The main issues with the existing strategy-based margin models are that they restrict the option portfolio to be balanced (Section 3 discusses the detail on balanced versus unbalanced portfolios), are computationally complex, and tend to overestimate the potential loss under normal market conditions.

In the following, we present the idea and contributions of our approach, begining with the maximum possible loss. We assume individual options in a portfolio share the same maturity T and use $f(S_T)$ to denote the portfolio payoff at T, where S_T is the underlying asset's price at the maturity. The portfolio value at time $t \in [0, T]$ is thus $P_t = \mathbb{E}[f(S_T)|\mathcal{F}_t]$. Let Δt be the MPOR, the portfolio loss is then the greater one between zero and the negative of the portfolio value at $t + \Delta t$, i.e., $(-P_{t+\Delta t})^+$, with $(x)^+ = x$, if $x \geq 0$, and $(x)^+ = 0$ otherwise. Different risk-sensitive models present different estimation methodologies of this conditional expectation and different margin definitions based on the possible losses. For example, the SPAN by CME estimates the possible loss via simulating several possible scenarios of the underlying price changes or/and the volatility changes and identifies the maximum possible loss among these scenarios as the margin. The SIMM by the ISDA estimates the possible loss by approximating the change of option value via Greeks and identifies an approximation of the 99th percentile loss as the margin. We define the maximum portfolio loss of a given option portfolio as $\bar{m} := \max_{S_T} -f(S_T)$. With a constant interest rate $r > 0$, it is easy to show that portfolio loss $(-P_{t+\Delta t})^+$ at any time before maturity $t \in [0, T - \Delta t]$ is bounded above by $\bar{m}^+$:

$$(-P_{t+\Delta t})^+ = (\mathbb{E}[-e^{-r(T-t-\Delta t)} f(S_T)|\mathcal{F}_{t+\Delta t}])^+ \leq (\mathbb{E}[e^{-r(T-t-\Delta t)}\bar{m}|\mathcal{F}_{t+\Delta t}])^+ \leq (\mathbb{E}[\bar{m}|\mathcal{F}_{t+\Delta t}])^+ = \bar{m}^+.$$

[5] The BIS OTC derivatives statistics. https://stats.bis.org/statx/srs/table/d7.

[6] See "Staff Report on Equity and Options Market Structure Conditions in Early 2021" for more details at: https://www.sec.gov/files/staff-report-equity-options-market-struction-conditions-early-2021.pdf.

[7] Also known as Regulation T. See for more details at: http://www.federalreserve.gov/bankinforeg/reglisting.htm.

Note that this maximum is a supremum since the loss approaches $\bar{m}^{+}$ when time $(t + \Delta t)$ approaches the maturity T, and the underlying price approaches the one corresponding to the worst case. We thus aim to eliminate the inefficient netting in terms of $\bar{m}^{+}$.

A key component of our approach is the notion of base offsets which contain two to six legs of pre-defined simple option strategies. We prove the proposed base offsets are sufficient to capture the hedging properties inside any balanced option portfolio. It is a combinatorial problem to search for the optimal division of a given option portfolio into offsets so that the maximum possible loss is minimized. We apply an integer programming model with base offsets to provide an exact calculation of the upper bound of the potential loss for a balanced option portfolio. Numerical studies on real historical data show that the problem can be solved in less than one second for twenty different strike prices using ILOG CPLEX 12.6. For problems of smaller size involving base offsets up to four legs, we further show that our formulation is equivalent to its linear relaxation and can be solved in polynomial time.

We then discuss the potential of using the maximum possible loss to set portfolio margins within a MPOR from the clearning member perspective. We examine two cases. The first case is the exchange-traded products such as the SPX options. The clearing counterparties such as CME typically use historical Value-at-Risk(VaR) method or the SPAN method to set portfololio margins. The second case is the OTC products such as the interest rate swaptions where the current market standard is the SIMM method proposed by ISDA. We compare the maximum possible loss prescribed by the proposed model with the margin levels set by three prevailing risk-sensitive models: historical VaR, SPAN and SIMM. As expected, the risk-sensitive models require less conservative margin levels than our estimation of the maximum possible loss. However, the difference is relatively acceptable after exploring the best possible payoff netting using base offsets. The benefit of our approach is that margin requirements set by the minimized maximum possible loss is independence of market conditions. They are therefore free from the procyclicalities inherent in risk-sensitive margin models.

If margin requirement does not need to be the most aggressive one, one can mix our approach with a risk-sensitive model in the spirit of European Commission (2013) to allow a given degree of risk sensitivity. We compare the performances of the mixture models with those of the risk-based models in terms of both margin level and procyclicality with SPX data in turbulent times. With the benchmark provided by our model, we present that the risk-sensitive margin level runs up and approaches the objective maximum possible loss when the market becomes turbulent. We can think of the difference between the risk-sensitive margin level and the supremum as the result of the optimistic sentiment, which overlooks the possible risk in good times. Furthermore, the margin run-ups happen when market participants are faced up with the existence of risk. Similar to the phenomenon pointed out by Antoniou et al. (2016) in stock markets, unsophisticated trading in risky opportunities would be more prevalent when the sentiment is optimistic, whereas market participants making such trades take less risk during pessimistic periods. Our strategy-based model would be on the alert guard against optimistic sentiment. By incorporating the risk-sensitive model with our model, we aim to reduce margin run-ups in turbulent times to stabilize the market.

This paper is related to the literature on margin requirement of derivatives and the netting efficiency. Lopez et al. (2017) propose a new methodology to estimate the margin requirement by a derivatives central counterparty (CCP), by introducing the interdependence among different market participants' profits and losses to a risk-sensitive model. Capponi and Cheng (2018) model the decision problem faced by a profit-maximizing clearinghouse, which sets fee and margin requirements for heterogeneous participants who may default. Duffie and Zhu (2011), Cont and Kokholm (2014) and Garratt and Zimmerman (2020) investigate how the introduction of centralized netting affects the netting efficiency, since the exposures to different counterparties cannot be netted. We focus on how to do the netting based on the special properties inside the option portfolio, by one specific counterparty to calculate exposures for each individual netting set.

The rest of this paper is as follows. In Section 2, we review the literature on the strategy-based margin calculation approach for option portfolios. In Section 3, we introduce the concept of base offsets and prove that the margin reduced by the hedging property inside any balanced option portfolio can be ascribed to base offsets. We also set up our strategy-based netting model and investigate its properties in Section 3. In Section 4, we apply our model to the historical data of SPX options and swaptions to illustrate the difference between our model and the risk-sensitive models. We also incorporate these two methodologies to present the mixed margin calculation models. We conclude our paper in Section 5. All the proofs are placed in the online supplement to make the paper concise.

2. Strategy-based Portfolio Margin

We define an options "strategy" as a portfolio of European options with the same expiration date. The component positions can have different strikes and different notional amounts and can take either the call or put sides. Two component options are considered two different "types" as long as either the strike or the side is different. We refer to the total number of option types in a strategy as the number of "legs". We call an options strategy an *offset* strategy if the portfolio risk is smaller than that of its component sum.

The essence of the strategy-based approach is to define the margin of an arbitrary option portfolio q based on the best combination of the recognized offsets strategies allowed by the margin counterparty. Specifically, if we denote $O_1, \ldots, O_k$ as the recognized offset strategies whose margins have been defined in the manual and m_j as the margin of O_j, then the strategy-based approach is to identify the best partition of this arbitrary portfolio such that

Table 1
Different MPL results corresponding to possible partitions with the strategy-based approach of size two.

possible partition	maximum possible loss
$O_1 + O_2 + O_3 + O_4$	$0 + 0 + K_2 +$ infinity $=$ infinity
$O_5 + O_3 + O_4$	$0 + K_2 +$ infinity $=$ infinity
$O_1 + O_3 + O_7$	$0 + K_2 + (K_2 - K_1) = 2K_2 - K_1$
$O_6 + O_2 + O_4$	$(K_2 - K_1) + 0 +$ infinity $=$ infinity
$O_6 + O_7$	$(K_2 - K_1) + (K_2 - K_1) = 2(K_2 - K_1)$

i) $q = x_1 O_1 + \ldots + x_k O_k$ where $x_1, \ldots, x_k$ are nonnegative integer multipliers;
ii) The resulting margin $m_1 x_1 + \ldots + m_k x_k$ attains its minimum.

We define the "size" of this strategy-based model, s, as the maximum number of legs among all the offset strategies $\{O_i, i = 1, \ldots, k\}$ involved in this problem. For example, if each offset strategy O_i has only one option leg (one type), then the maximum number of legs among all $O_1, \ldots, O_k$ is one. The size of this problem will be one. A strategy-based model of greater size s could yield a lower margin requirement with the same risk protection since the greater the size s, the greater the potential offsetting, thus making better use of the hedging properties inside the portfolio q.

Next, we illustrate the strategy-based approach and the meaning of size s by calculating the maximum possible loss (MPL), corresponding to full risk protection, for a portfolio q containing four different option positions: one long put at strike K_1, one short put at strike K_2, one short call at strike K_1 and one long call at strike K_2. In particular, denote $P_1,\ P_2,\ P_3$ and P_4 as these four individual positions, assuming equal notional for each position.

If we consider the strategy-based approach of size $s = 1$, involved offsets can contain only one option type, and thus are these four individual options themselves: $O_1 = P_1, O_2 = P_2, O_3 = P_3$ and $O_4 = P_4$. This case is in fact corresponding to the "gross margin" rule in the market, meaning no netting or offsetting is allowed. There is only one possible partition $q = O_1 + O_2 + O_3 + O_4$, with the resulting MPL approaching infinity. If alternatively, we margin the long box portfolio under the "net margin" rule with the strategy-based approach of size $s = 2$, the pairing strategies are taken into consideration. There are six possible pairing strategies with four individual option types, $P_i + P_j$, $i \neq j \in \{1, \cdots, 4\}$. Among them, new offsets are recognized as $O_5 = P_1 + P_2$, $O_6 = P_1 + P_3$ and $O_7 = P_2 + P_4$. The possible partitions of this portfolio q into combinations of offsets O_i, $i = 1, \cdots, 7$, with their resulting MPLs are shown in Table 1. The MPL of this portfolio becomes $2(K_2 - K_1)$, since the best partition is a bull put spread O_6 and a bear call spread O_7.

Finally, if we consider all the strategies with up to four legs, this portfolio itself should be recognized as a box spread $O_8 = P_1 + P_2 + P_3 + P_4$, which is an options strategy with four legs. Then, the estimated maximum possible loss of this portfolio is reduced to $K_2 - K_1$, with the partition $q = O_8$. This simple case study demonstrates that i) overlooking the offsets inside an option portfolio would lead to an overestimation of the margin and ii) recognizing strategies with more legs would reduce the degree of overestimation of the margin.

Rudd and Schroeder (1982) show that the strategy-based approach of size two can be solved in polynomial time by a reformulation. Matsypura and Timkovsky (2013) devote their paper to identify and take advantage of offsets with three or four legs.[8]

To formulate a mathematical model for the strategy-based approach, we adopt the vector model by Matsypura and Timkovsky (2013) to express the offsets. In particular, for the same expiration date on an underlying asset, a set of call and put options with d different strike prices, K_j, $j = 1, \ldots, d$, is considered, and an options strategy of dimension d can be then expressed by an integer vector,

$$v = (v_1, v_2, \ldots, v_d, v_{d+1}, v_{d+2}, \ldots, v_{2d})^T, \tag{1}$$

whose components are associated with positions in options. More specifically, v_j or v_{j+d}, $j = 1, \ldots, d$, is the number of option contracts in the jth call or put option, respectively, with the exercise price K_j. We assume that all the option contracts have a unit notional. Furthermore, a nonzero component can be either positive or negative, which corresponds to a long or short leg, respectively. When vector v involves only one nonzero component, it corresponds to a naked option. When vector v involves multiple nonzero components and has a market risk, which is less than the total position risk of its components, it corresponds to an offset.

With this expression, Matsypura and Timkovsky (2013) formulate such a problem as the account margin minimization (AMM) problem: $\min\{m^T x : \ Ox = q,\ x \in \mathbb{Z}_+^k\}$.

The optimal value of this model is defined as the estimated market risk of the portfolio q. An AMM problem of size s is modeled as an integer programming model where the coefficient matrix O is a collection of recognized offsets with up to s legs. Each column of O, O_j, is a $2d$ dimensional integer vector taking the form of a specific v in (1). Recall that a combination of different offsets may generate a new offset with more legs. For example, for k different options, there might

[8] The definition of size in this paper may be different from that in Matsypura and Timkovsky (2013). Although they consider the margin calculation for multi-leg option strategies, the strategies in the corresponding coefficient matrix in Matsypura and Timkovsky (2013) have at most four legs.

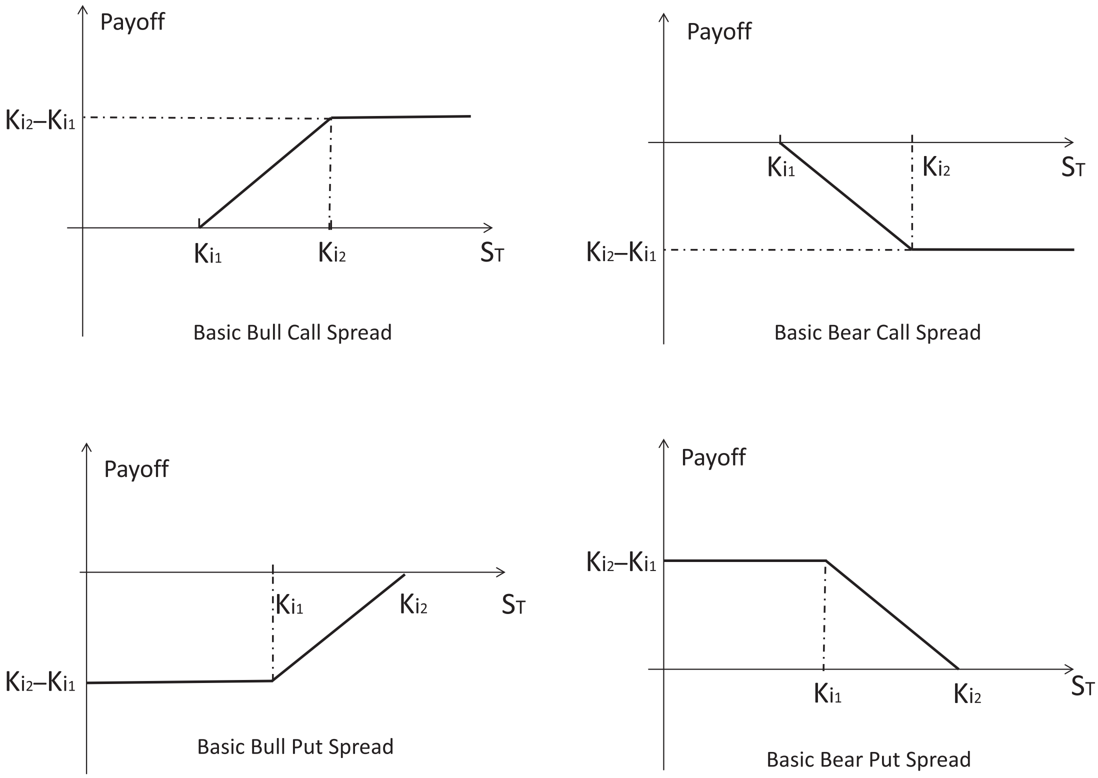


Fig. 1. Payoff functions of basic spreads defined in Matsypura and Timkovsky (2013).

be $C_k^1 + C_k^2$ offsets with up to two legs and $C_k^1 + C_k^2 + C_k^3$ offsets with up to three legs. As a result, both the dimension of the coefficient matrix O and the decision variable x could be exponentially increasing with s. Therefore, the problem would become a high-dimensional integer problem when s is large. Most importantly, the identification and construction of the offsets included in the coefficient matrix O are crucial to a strategy-based approach's success.

3. Base Offsets

As mentioned above, redundant offsets involved in the AMM model's coefficient matrix introduce a high complexity that blocks the development of fast algorithms. Furthermore, the state-of-the-art model is of size four and does not guarantee an exact estimation of the market risk (which implies possible overestimation). In this section, we propose a new concept of offsets to ensure an exact estimation of market risk and facilitate a more efficient margin calculation.

We assume that all the options are written on the same underlying asset in this paper and denote the price of this underlying asset at maturity by S_T. For any options strategy v defined in Equation (1), we denote $f_v(S_T)$ as its payoff function. Thus, the maximum possible loss (MPL) is defined as

$$m(v) := (-\min_{S_T} f_v(S_T))^+, \tag{2}$$

and the number of leg of v is the number of non-zero entries of vector v, i.e.,

$$leg(v) := ||v||_0, \tag{3}$$

where $||v||_0$ denotes the l_0 norm of v.

3.1. Balanced Portfolio

We focus on the options strategies generated by basic spreads, which, according to Definition 2 in Matsypura and Timkovsky (2013), are defined as options strategies with two non-zero components, 1 and -1, and both of the two components are on the same side, either call or put. A basic spread is a basic call/put spread if the two components are on the call/put side. A basic spread is a basic bull/bear spread if the component with the lower strike price is associated with a long/short position. The width of a basic spread is defined as the difference between the strike prices of the two non-zero components. For example, the basic spreads shown in Figure 1 are of width $K_{i_2} - K_{i_1}$. Without loss of generality, we assume from now that the d strike prices in options strategy v satisfy $K_i = i$, $i = 1, \cdots, d$.[9]

For each kind of the four basic spreads depicted in Figure 1, there are $(d-1)$ basic spreads of width one.

1. Basic bull call spreads:

$$B_1 := e_1 - e_2,\ B_2 := e_2 - e_3, \dots,\ B_{d-1} := e_{d-1} - e_d, \tag{4}$$

where $e_i \in \mathbb{Z}^{2d}$ is a vector with i-th element being one and all other elements being zero;

2. Basic bull put spreads:

$$B_d := e_{d+1} - e_{d+2},\ B_{d+1} := e_{d+2} - e_{d+3}, \dots,\ B_{2(d-1)} := e_{2d-1} - e_{2d}; \tag{5}$$

3. Basic bear call spreads:

$$B_{2d-1} := -B_1,\ B_{2d} := -B_2, \cdots, B_{3(d-1)} := -B_{d-1}; \tag{6}$$

4. Basic bear put spreads:

$$B_{3d-2} := -B_d,\ B_{3d-1} := -B_{d+1}, \dots,\ B_{4(d-1)} := -B_{2(d-1)}. \tag{7}$$

We first identify the entire set of options strategy which can be generated by these four kinds of basic spreads as follows.

Definition 3.1. Options strategy $v = (v_1, \dots, v_{2d})^T \in \mathbb{Z}^{2d}$ is called a balanced options strategy if and only if $\sum_{i=1}^{d} v_i = \sum_{i=d+1}^{2d} v_i = 0$.

In other words, a balanced options strategy has the same number of long and short positions in calls/puts, respectively. We will focus on the balanced option strategies in this paper, since an unbalanced options strategy may have infinite MPL under the definition in (2), which makes the corresponding optimization problem ill-posed. In addition, we prove that an options strategy can be expressed as an integer combination of basic spreads if and only if it is balanced (Readers can refer to the proof of Proposition 3.1 in the online supplement). In fact, we will proceed to prove that an options strategy is balanced if and only if it can be represented as a non-negative integer combination of the basic spreads of width one.

Proposition 3.1. *Options strategy $v = (v_1, \dots, v_{2d})^T \in \mathbb{Z}^{2d}$ is a balanced options strategy if and only if there exist $k_i \in \mathbb{Z}_+$, $i = 1, \cdots, 4(d-1)$, such that $v = \sum_{i=1}^{4(d-1)} k_i B_i$.*

According to Proposition 3.1, we can always partition a balanced options strategy into a non-negative combination of the basic spreads of width one, which form a special class of options strategies. The strategy-based model of size two, for the balanced option strategies, is then to identify the best candidate with the lowest resulting MPL among all these feasible partitions into basic spreads, which have two legs. However, as discussed in Section 2, the combinations of these basic spreads may generate new offsets with more legs. A strategy-based model will have lower MPL results if we consider offsets with more legs. We thus testify whether a combination of different strategies can generate a new offset by introducing the following concept of dominance among option strategies.

Definition 3.2. An options strategy v is dominated by a portfolio of options strategies $\{b^1, \dots, b^n\}$, $n \geq 1$, if and only if the strategy v can be expressed into a non-negative combination of these strategies b^i, $i = 1, \cdots, n$, with the resulting MPL no higher than the MPL of v, i.e., there exists some non-negative integer vector $z = (z_1, z_2, \dots, z_n)^T$ such that $z_1 b^1 + \cdots + z_n b^n = v,\ z_1 m(b^1) + \cdots z_n m(b^n) \leq m(v)$.

If the strategy v is dominated by a portfolio of options strategies V, v is not a new offset given the option strategies and thus unable to lower the MPL results of a strategy-based model with strategies V. It is obvious that any options strategy is dominated by a portfolio of options strategies that contains itself. Moreover, if an options strategy v is dominated by a set of options strategies, V_1, which is in turn dominated by another set of options strategies, V_2, then v is also dominated by V_2. We state this transitive property in the following proposition.

Proposition 3.2. *(Transitivity) For options strategies v, $v^1, \dots, v^t$, $b^1, \dots, b^n$, if there exist $u \in \{1, \dots, t\}$, non-negative integers z_i, $i = 1, \dots, t$, and k_j, $j = 1, \dots, n$, such that*

$$v = \sum_{i=1}^{t} z_i v^i,\ \ m(v) \geq \sum_{i=1}^{t} z_i m(v^i),\ \ v^u = \sum_{j=1}^{n} k_j b^j,\ \ m(v^u) \geq \sum_{j=1}^{n} k_j m(b^j). \tag{8}$$

Then, v is dominated by $\{v^i, i = 1, \dots, u-1, u+1, \dots, t\} \cup \{b^1, \dots, b^n\}$.

[9] We normalize the difference between adjacent strike prices to be one to simplify the notations, the result remains the same for cases where the difference $K_{i+1} - K_i$ is a constant for all $i = 1, \cdots, d-1$. It is common in the market to find that the strike prices of listed options on one same asset have a common difference. If not, one can also easily add the strike prices to make the difference a constant.

If we can identify a set of options strategies that dominate all balanced option strategies, the MPL results calculated by a strategy-based model with such a set of strategies will never overestimate the market risk of balanced option strategies. Thus, the formation of these base strategies is crucial, as they form a complete base for the maximum possible loss calculation.

In other words, we aim at identifying a "complete base", which can not only generate all the balanced strategies by non-negative linear combinations but also dominate all these balanced strategies in terms of the maximum possible loss. We prove in Proposition 3.1 that any balanced options strategy can be expressed as a non-negative combination of the four kinds of basic spreads of width one. However, the set of these basic spreads, which have two legs, cannot dominate all the balanced strategies (Readers can refer to the illustrative example in Section 2). To form a complete base, one needs to include offsets with more legs. A trivial construction is to include all the non-negative combinations of these basic spreads since the set of all these combinations is precisely the set of all the balanced strategies and, thus, a dominance. However, there will be too many possible combinations that are redundant in forming a complete base. We thus would like to remove these redundant combinations to construct a *minimum* complete base. In order to achieve this goal of forming a minimum complete base, we define the *no-hedging combinations*.

Definition 3.3. Any combination of strategies, $\sum_{i=1}^{n} k_i b_i$, with $k_1, \ldots, k_n \in \mathbb{Z}_+$, satisfying one of the following two conditions is called a *no-hedging combination*: i) All the payoff functions of b_i, $i = 1, \ldots, n$, are non-negative; ii) There exists one point $S_T^0 \in [0, \infty)$ such that the payoff functions of b_i, $i = 1, \ldots, n$, all achieve their non-positive minimal values at S_T^0.

Lemma 3.1. *If an options strategy v can be expressed into a no-hedging combination of strategies $b_1, \ldots, b_n$, then the strategy v is dominated by strategies $b_1, \ldots, b_n$.*

The intuition of this fact is relatively straightforward, as i) it's unnecessary to consider any offset when all these strategies have zero market risks, and ii) if there is one market condition where all these strategies have their worst performances, then it is impossible to offset each other when we consider full risk protection with the maximum possible loss. In other words, a no-hedging combination of option strategies cannot generate a new offset, since there is no hedging property among these option strategies. According to the transitive property stated in Proposition 3.2, the balanced strategies dominated by no-hedging combinations of basic spreads are also dominated by basic spreads. The no-hedging combinations of basic spreads are thus redundant in the sense of forming a complete base.

We next look into combinations of basic spreads to remove the no-hedging combinations and form a minimum complete base. First, the combination of basic spreads of the same kind is always a no-hedging combination. For example, all the basic bear call spreads achieve their maximum possible losses at a sufficiently large terminal price S_T and thus cannot hedge each other when we consider full risk protection. Moreover, both the maximum possible gain and loss of any basic spread of width one are no larger than one. That is to say, for example, if we use basic bull call spreads to offset a basic bear call spread, all of which are of width one, we only need at most one basic bull call spread. Otherwise, the combination would become a no-hedging combination, since all the payoff functions of the other basic bull call spreads and the combination of the basic bull and bear call spreads are non-negative. Thus, we conjecture that the possible "base" options strategies should involve at most one of each of the four kinds of basic spreads of width one, and a collection of such "base" options strategies should be sufficient to capture the hedge properties inside balanced options strategies. Following this direction, we now introduce a new concept of *base offsets* in the following definition.

Definition 3.4. Base offsets consist of the following seven forms:

Form 1) B_i, $i = 1, ..., 4(d-1)$;
Form 2) $B_{i+2(d-1)} + B_{i-1}$, where $i \in \{2, ..., d-1\}$;
Form 3) $B_{i+2d-3} + B_i + B_{i+3(d-1)}$, where $i \in \{2, ..., d-1\}$;
Form 4) $B_{i+d-2} + B_{i+3(d-1)}$, where $i \in \{2, ..., d-1\}$;
Form 5) $B_{i+d-1} + B_{i-1} + B_{i+3d-4}$, where $i \in \{2, ..., d-1\}$;
Form 6) $B_{i+2(d-1)} + B_{i+d-1}$, where $i \in \{1, ..., d-1\}$;
Form 7) $\pm(B_i + B_{i+d} + B_{i+2d-1} + B_{i+3(d-1)})$, where $i \in \{1, ..., d-2\}$.

For an illustration purpose, we present the payoff functions for Forms 2-7 of the base offsets in Figure 2. In particular, the base offsets include the basic spreads of width one and the combinations of these basic spreads, which are not no-hedging combinations. Next, we briefly explain the intuition of how we construct the base offsets.

As discussed above, a base offset should involve at most one of the four kinds of basic spread. We first consider the six possible combinations of any two out of the four basic spreads of width one. From the payoffs of these basic spreads in Figure 1 and the definition of the no-hedging combination, it is easy to find there are three combinations out of the six that can generate new offsets: the combination of basic bull and bear call spreads, the combination of basic bull and bear put spreads, and the combination of basic bear call and bull put spreads. The basic bull call spread can hedge any basic bear call spread with a higher strike price. Meanwhile, according to the definition of basic bear call spreads in (6), the basic bear call spread with the strike price K_i is $B_{2d-2+i} = -B_i$, where B_i is the basic bull call spread with the strike price K_i, $i = 1, \cdots, d-1$. The combination of a basic bear call spread B_{2d-2+i} and a basic bull call spread B_j with the difference of the strike prices of these two spreads larger than one, i.e., $2 \le j+1 < i \le d-1$, can be expressed as a no-hedging combination

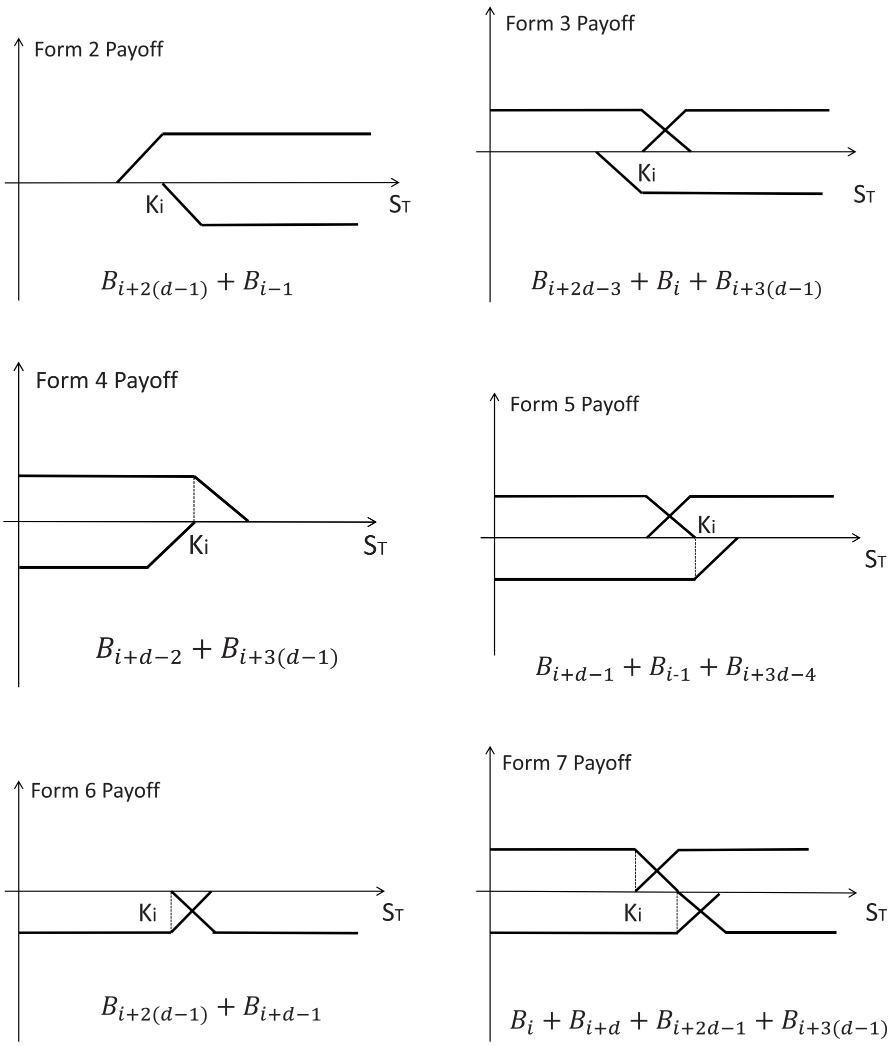


Fig. 2. Payoff functions of Forms 2 to 7 of base offsets.

of $B_j + B_{2d-2+j+1}$, $B_{j+1} + B_{2d-2+j+2}, \dots, B_{i-1} + B_{2d-2+i}$. As a result, we include the combinations of the basic bull call B_{i-1} and the basic bear call B_{2d-2+i}, $i = 2, 3, \cdots, d-1$, as Form 2 base offsets. Similarly, we also include the combinations of the basic bull and bear put spreads and the combinations of the bear call and bull put spreads as Form 4 and Form 6 base offsets.

Next, we consider a combination of a basic bull call spread B_j and a basic bear call spread B_{2d-2+i} with a lower strike price. Because the bull call spread B_j cannot offset the possible loss of the bear call spread B_{2d-2+i} when the terminal price lies inside the interval $[1 + K_j, K_i]$, we add a bear put spread to form a box spread with the bull call spread and thus hedge the bear call spread. As a result, we include the combinations of a box spread and a bear call spread as Form 3 base offsets. Similarly, we include the combinations of a box spread and a bull put spread as Form 5 base offsets. Because a Form 3 (5) base offset can furthermore offset a basic bull put (bear call) spread, we document such combinations by Form 7 base offsets.

After removing all the no-hedging combinations, it can be verified from the definition that there are $11d - 17$ base offsets for d different strike prices. Note that the total number of balanced strategies for d different strike prices is $C_d^1 + C_d^2 + \cdots + C_d^d = 2^d - 1$. We next prove that these $11d - 17$ base offsets form a complete base for the MPL calculation of all the $(2^d - 1)$ balanced strategies with the help of Proposition 3.2 and Lemma 3.1. As a result, to study the MPL of a balanced options strategy, we only need to study these base offsets.

Theorem 3.1. *Any balanced options strategy can be expressed into a no-hedging combination of base offsets.*

Let us illustrate the base offsets and the no-hedging combination introduced above by two cases with $d = 3$. A bull put spread $\hat{v} = (0, 0, 0, 3, -3, 0)^T$ is a simple balanced options strategy. It can be expressed into a no-hedging combination of the basic spreads (Form 1 base offset):

$$\hat{v} = 3 \cdot (0, 0, 0, 1, -1, 0)^T = 3 \cdot B_3,$$

since all the payoff functions of these three bull put spreads B_3 achieve their minimums at $S_T \in [0, K_1]$. A balanced strategy $\bar{v} = (2, -3, 1, -2, 4, -2)^T$, can be expressed into a no-hedging combination

$$\bar{v} = (0, 1, -1, 0, 0, 0)^T + 2 \cdot (1, -2, 1, -1, 2, -1)^T,$$

where the first resulting vector is the Form 1 base offset B_2 and the second one is a Form 7 base offset. The Form 1 base offset B_2 is a basic bull call spread with a non-negative payoff function, while the payoff of a Form 7 base offset is always zero. Therefore, the combination of these two base offsets is a no-hedging combination.

One may notice that there are many different ways to express the strategy $\bar{v}$ into combinations of base offsets. Theorem 3.1 guarantees the existence of the no-hedging combination of base offsets for each balanced strategy. To identify the no-hedging combination out of the possible expressions, we introduce the netting model for MPL as follows.

As we proved in Proposition 3.1, any balanced strategy v can be expressed as a portfolio of Form 1 base offsets. Therefore, any balanced strategy v can be expressed as a portfolio of base offsets, $v = \sum_{i=1}^{n} A_i q_i$, $q = (q_1, \ldots, q_n)^T \in \mathbb{Z}_+^n$, where A_i, $i = 1, \ldots, n$ are the proposed base offsets (with up to six legs). Denote the MPL of the base offset A_i by m_i, $i = 1, \ldots, n$. For any base offsets portfolio q, if we can find another portfolio a with the same payoff curve and lower MPL, then a makes a better use of the offsets than q. Among all these portfolios with lower MPLs, we choose the best candidate a^* by solving the following netting model of size six[10],

$$\min_a \{m^T a : \ Aa = Aq, \ a \in \mathbb{Z}_+^n\}. \tag{NM}$$

Then, the optimal value, $m^T a^*$, is the MPL of the base offsets portfolio q or the options strategy v calculated by model (NM).

Corollary 3.1. *Model (NM) always yields an exact estimation of the maximum possible loss for any balanced options strategy.*

Although we are not able to prove the tractability of the above integer programming model at this stage, our model of size six only involves $2d$ constraints and $11d - 17$ integer variables with d being the number of strike prices, which is significantly less than the counterparts in the model by Matsypura and Timkovsky (2013), $n_0^2 + n_0$ integer variables and $2dn_0 + n_0$ constraints, where, according to Matsypura and Timkovsky (2013),

$$n_0 = \sum_{w=1}^{d-1} [6(d-w) + 8(d-2w)^+ + 8(d-3w)^+]. \tag{9}$$

For instance, when we consider 5 different strike prices, there are only 10 constraints and 38 integer variables involved in our size-six model, while the integer programming model in Matsypura and Timkovsky (2013) introduces 1,188 constraints and 11,772 integer variables. In fact, we find the computational time of our netting model of size six is less than one second for all the cases where the number of different strike prices does not exceed twenty in our numerical experiments using ILOG CPLEX 12.6.

Although we cannot theoretically present the computational complexity for the netting model (NM), we can prove that this model of a smaller size is solvable in polynomial time. If we limit the coefficient matrix A of model (NM) to $\bar{A}$, whose columns are base offsets with up to four legs, and the vector m to the corresponding maximum possible loss vector $\bar{m}$, we have the following netting model of size four,

$$\min_a \{\bar{m}^T a : \ \bar{A}a = \bar{A}q, \ a \in \mathbb{Z}_+^{\bar{n}}\}. \tag{NM$_1$}$$

We can prove that the netting model (NM$_1$) of size four is equivalent to its continuous relaxation, which is of polynomial time complexity.

Proposition 3.3. *Our netting model* (NM$_1$) *is equivalent to its continuous relaxation,* $\min_a \{\bar{m}^T a : \ \bar{A}a = \bar{A}q, \ a \in \mathbb{R}_+^{\bar{n}}\}$.

Having a size smaller than that of our netting model (NM), the restricted model (NM$_1$) cannot guarantee an exact MPL estimation for all balanced strategies. However, we can still figure out the set of balanced strategies for which model (NM$_1$) can provide an exact estimation of the MPL.

Proposition 3.4. *If a balanced options strategy v satisfying one of the following three conditions: i) v is a call spread; ii) v is a put spread; or iii) v is a strategy with up to four legs, then the restricted netting model* (NM$_1$) *provides an exact estimation of the MPL for the strategy v.*

[10] Recall that the size of a strategy-based model is the maximum number of legs of A_i, $i = 1, \ldots, n$, but not the number of legs of v. In other words, we can calculate MPLs by model (NM) for any balanced strategy v with legs possibly up to $2d$. More importantly, the model complexity does not depend on the legs of v.

3.2. Unbalanced Portfolio

As we discussed above, unbalanced option portfolios could have unbounded MPLs, defined in (2), leading to ill-posedness of our optimization problem. Thereupon, one can not directly apply our netting model to unbalanced option portfolios. In this subsection, we show how to apply our model to unbalanced ones under the assumption that the underlying price is bounded.

Note that we consider the maximum possible loss in terms of all the possible underlying price $S_T \in [0, \infty)$, which results in infinite possible loss of unbalanced portfolios. Thus, to deal with unbalanced ones, we have to adopt the assumption that the underlying price is bounded. With such an assumption, we set virtual strike prices K_u and K_l so that $S_T \in [K_l, K_u]$. The net payoff at maturity of a naked long call option with strike price $K \in (K_l, K_u)$ is then also bounded by $(K_u - K)$. As a result, the naked long call option has the same payoff curve as a bull call spread (See Figure 1 for an illustration with $K_{i_1} = K$ and $K_{i_2} = K_u$). Similarly, a naked short call option would have the same payoff curve as a bear call spread.

For example, let's consider an unbalanced portfolio where there is one naked short call option, but no naked put option, i.e.,

$$\sum_{i=1}^{d} v_i = -1, \quad \sum_{i=d+1}^{2d} v_i = 0.$$

Without the assumption on underlying price S_T, the naked position might incur infinite possible loss. With an assumption that the price is bounded by $S_T < K_u$, the naked call option's loss is bounded. Our model is then ready to be applied. For an illustrative purpose, we assume in this example that the upper bound $K_u = K_d + (K_d - K_{d-1})$ so that we only need to add one virtual strike price $K_{d+1} = K_u$ and express the options as $\bar{v} = (\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_{d+1}, \bar{v}_{d+2}, \cdots, \bar{v}_{2(d+1)})^T$. A naked short call option at strike price K_i, $i = 1, \ldots, d$, has the same terminal payoff as the balanced portfolio $\bar{v} = -e_i + e_{d+1}$, where e_i is a vector with $i-$th element being one and all other elements being zero.

In general, with the assumption that $S_T \in [K_l, K_u]$[11] and that $K_l \leq K_1, K_d \leq K_u$, to keep a common difference among the strike prices, besides the original strike prices $K_1 < K_2 < \cdots < K_d$, we add $(\underline{d} + \bar{d})$ $(\underline{d}, \bar{d} \in \mathbb{Z}_+)$ virtual strike prices as

$$K_l = K_{1-\underline{d}} < K_{2-\underline{d}} < \cdots < K_1 < K_2 < \cdots < K_d < K_{d+1} < K_{d+\bar{d}} = K_u, \tag{10}$$

where

$$K_{1-i} = K_1 - i(K_2 - K_1), \ \ i = 1, \cdots, \underline{d}, \ \ \underline{d} = \frac{K_1 - K_l}{K_2 - K_1},$$
$$K_{d+i} = K_d + i(K_2 - K_1), \ \ i = 1, \cdots, \bar{d}, \ \ \bar{d} = \frac{K_u - K_d}{K_2 - K_1}.$$

The option portfolios are then expressed as a vector of dimension $2(\bar{d} + \underline{d} + d)$,

$$\bar{v} = (\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_{\bar{d}+\underline{d}+d}, \bar{v}_{\bar{d}+\underline{d}+d+1}, \cdots, \bar{v}_{2(\bar{d}+\underline{d}+d)})^T, \tag{11}$$

where $\bar{v}_{i+\underline{d}}$ and $\bar{v}_{d+\bar{d}+2\underline{d}+i}$ are the numbers of call and put positions on the option at strike price K_i, $i = 1, \cdots, d$.

Proposition 3.5. *Assuming that the underlying price at maturity is bounded by $S_T \in [K_{1-\underline{d}}, K_{d+\bar{d}}]$, with the virtual strikes defined in (10) and the options expression (11),*

1) a naked long or short call option at strike price K_i, $i = 1, \ldots, d$, has the same payoff function as a balanced option portfolio $\bar{v} = e_{i+\underline{d}} - e_{d+\bar{d}+\underline{d}}$ or $\bar{v} = -e_{i+\underline{d}} + e_{d+\bar{d}+\underline{d}}$.

2) a naked long or short put option at strike price K_i, $i = 1, \ldots, d$, has the same payoff function as a balanced option portfolio $\bar{v} = e_{d+\bar{d}+2\underline{d}+i} - e_{d+\bar{d}+\underline{d}+1}$ or $\bar{v} = -e_{d+\bar{d}+2\underline{d}+i} + e_{d+\bar{d}+\underline{d}+1}$.

With such an expression and price bounds, the naked options are equivalent to balanced portfolios in terms of the maximum possible loss at maturity. As a result, the unbalanced option portfolios are turned into balanced ones, in terms of the MPL. We can apply the models proposed above to calculate the MPLs for these unbalanced option portfolios. We will illustrate the application in Example 4.7.

With the same spirit, we can also take the underlying asset into consideration by the following proposition.

Proposition 3.6. *Assuming that the underlying price at maturity is bounded by $S_T \in [K_{1-\underline{d}}, K_{d+\bar{d}}]$, with the virtual strikes defined in (10) and the options expression (11),*

1) a long position of one unit of the underlying asset, up to the option multiplier, has the same payoff function as that of a balanced option portfolio $v = e_1 - e_{d+\bar{d}+\underline{d}}$ plus $K_{1-\underline{d}}$;

2) a short position of one unit of the underlying asset, up to the option multiplier, has the same payoff function as that of a balanced option portfolio $v = -e_{d+\bar{d}+\underline{d}+1} - e_{2(d+\bar{d}+\underline{d})}$ minus $K_{d+\bar{d}}$;

[11] Note that we still assume that the difference between upper/lower bounds and the original strike price K_1 being integer multiple of a common difference among the strike prices in this part. However, this assumption would not be critical to our model, since one can set the common difference small enough to establish it.

As we show above, the unbalanced portfolios can be turned into balanced ones, in terms of the MPL, with the assumption of price bounds. A related question would be how would the price bounds' assumption affect our main result on the balanced portfolios? In the next proposition, we will show that, with the price bounds, the balanced portfolios are still dominated by the base offsets, as shown in Theorem 3.1. As a result, the main results and our NM models still hold.

Proposition 3.7. *Assuming that the underlying price at maturity is bounded by $S_T \in [K_{1-\underline{d}}, K_{d+\bar{d}}]$, with the virtual strikes defined in (10) and the options expression (11), an arbitrary balanced options strategy can be expressed into a no-hedging combination of base offsets.*

4. Numerical Experiments and Applications

This section compares the maximum possible loss estimated by our netting models with three prevailing portfolio margin models, the SPAN, the historical VaR and the SIMM, to demonstrate the netting efficiency and the procyclicality issue. In particular, with the historical data of both exchange-based and over-the-counter European options, we apply our netting model and the three different risk-sensitive margin models to illustrate how the increase of our netting model's size from two to six improves the netting efficiency and why the netting model can be invoked into the prevailing risk-based models to tackle the procyclicality issue. In addition, we propose mixed models for incorporating our model (NM) with the risk-sensitive margin models to counter cyclicality.

Dataset. For the comparison between our netting model and the existing methodologies for portfolio margin, we consider two types of European option portfolios: the options on S&P 500 index (SPX) and the USD swaptions. Our dataset includes the underlying price and values of options on SPX from Sep. 2, 2008 to Aug. 31, 2018; and the prices of 3M×10Y USD forward swap and swaptions from July 1, 2011 to Aug. 20, 2019.

4.1. Comparison with the Standard Portfolio Analysis of Risk (SPAN)

The SPAN approach, practiced by the Chicago Mercantile Exchange, is a scenario-based methodology used by most exchanges to calculate margins for option portfolios. The SPAN provides both "gross margin", which provides the margin for portfolios by summarizing all the positions' estimated maximum possible losses among prescribed potential scenarios, and the "net margin", which identifies the margin for a portfolio as the estimated maximum possible loss of this portfolio as a total among these prescribed potential scenarios. As mentioned before, the netting model of size one corresponds to the "gross margin", while the netting model of a larger size reduces the estimated market risk corresponding to an inefficient netting and thus corresponds to the "net margin". In this subsection, we compare the MPLs by our netting models with the SPAN for net margin calculation.

For each portfolio, as documented in CME (2019), we simulate sixteen different SPAN scenarios by shocking the underlying price and its volatility up or down. For instance, if the underlying asset's price follows geometric Brownian motion, we set the scan range of the underlying price to be[12]

$$(S_t \times exp(-2.326 \times \sqrt{14/365}\sigma_t),\ S_t \times exp(2.326 \times \sqrt{14/365}\sigma_t)),$$

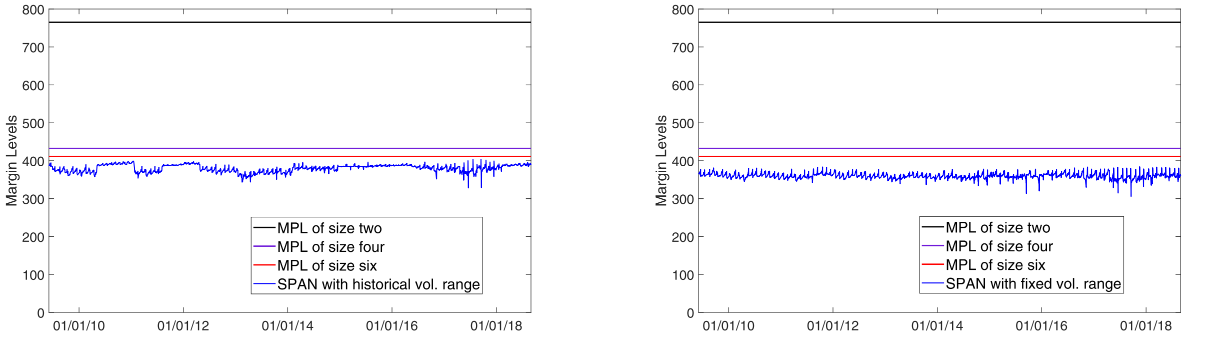
where σ_t is the at-the-money implied volatility at day t. The scan range of the volatility is either a fixed one $0.2\sigma_t$, or a historical range to approximate a 10-day 99% VaR shock. Two extreme scenarios are generated by considering three times the underlying price scan range and one time of the volatility scan range. Afterward, we identify the margin by SPAN as the maximum possible loss among the fourteen normal scenarios and two extreme scenarios, with the weight of the extreme scenarios is set to be 0.33.

Example 4.1 (Netting Efficiency Compared with SPAN)**.** We randomly generate balanced portfolios on SPX and USD swap, separately, and calculate the maximum possible losses yielded by our netting model of different size s[13] to demonstrate the importance of netting efficiency. In particular, we first randomly generate ten balanced portfolios on 21 different strike prices with the number of positions at each strike price uniformly distributed in [-9,9]. For each trading day t, we consider the ten portfolios of one-month constant maturity options starting from day t with strike prices $S_t \pm 5i$, $i = 0, 1, \cdots, 10$, where S_t is the price of SPX. We then calculate the margin levels for such ten sets of option portfolios by SPAN with historical and fixed volatility ranges, respectively, and the MPLs by the netting model of sizes two, four, and six. The averaged margin levels and MPLs at each trading day are documented in Figure 3.

Similarly, we investigate the applications of SPAN and our netting models on ten sets of randomly generated balanced portfolios of 3M×10Y USD swaptions with unit notional and strike prices $F_t \pm 5i/10000$, $i = 0, 1, \cdots, 10$, where F_t is the 3M×10Y USD forward swap price at day t. The averaged margin levels and MPLs are documented in Figure 4.

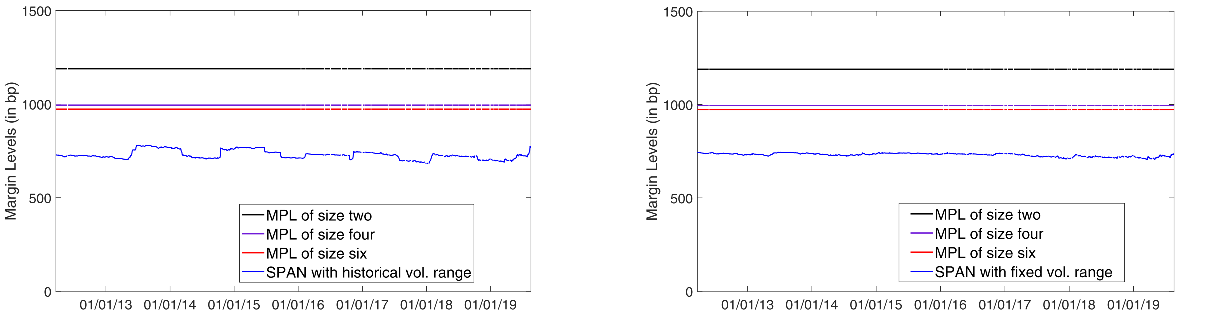
[12] We set the scan range to approximate a 10-day 99% VaR shock. In particular, 2.326 is the multiplier for 99% confidence using a normal distribution, and $\sqrt{14/365}$ is the scaling factor for a two-week period of risk.

[13] As we discussed in the last section, one can easily construct our model of size $s (s \le 6)$ via limiting the coefficient matrix A to a sub-matrix consisting of base offsets with up to s legs.



(a) Margin levels by SPAN with historical volatility range. (b) Margin levels by SPAN with fixed volatility range.

Fig. 3. The MPLs by netting model of different sizes and the margin levels by SPAN on ten randomly generated one-month maturity SPX option portfolios from Jun. 4, 2009 to Aug. 31, 2018. We apply SPAN with the historical volatility range in the left figure, based on the volatility change during the last 180 days. In the right figure, we apply SPAN with a fixed volatility range $0.2\sigma_t$. With the increase of size, the MPL by netting models is significantly reduced.



(a) Margin levels by SPAN with historical volatility range. (b) Margin levels by SPAN with fixed volatility range.

Fig. 4. The MPLs by netting models of different sizes and the margin levels by SPAN on ten randomly generated 3M×10Y USD swaption portfolios from Mar. 22, 2012 to Aug. 20, 2019. In the left figure, we apply SPAN with the historical volatility range, based on the volatility change during the last 180 days. In the right figure, we apply SPAN with a fixed volatility range $0.2\sigma_t$. With the increase of size, the MPL by netting models is reduced.

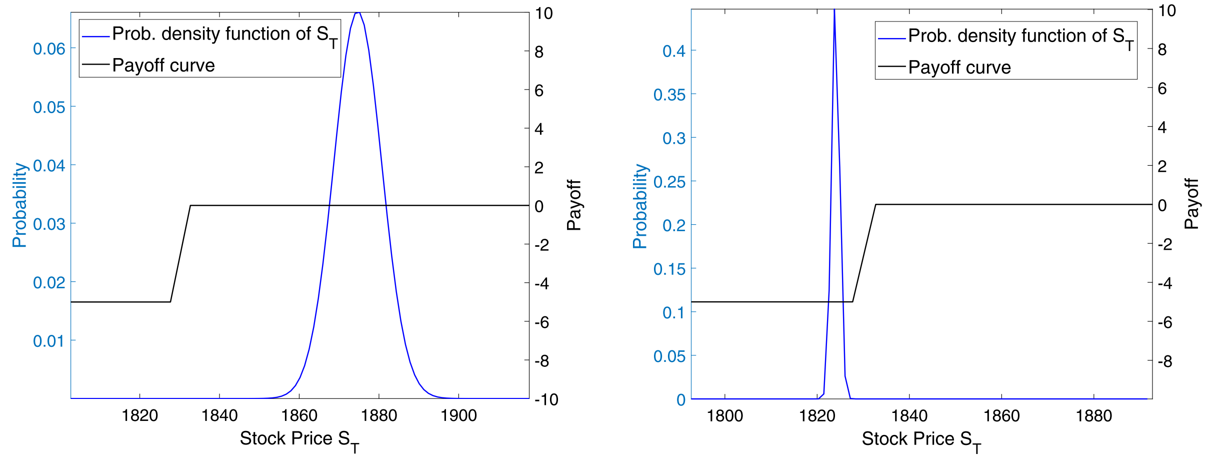
We have proved in Section 3 that the netting model (NM) yields an exact estimation of the maximum possible loss. In other words, we offer the tightest upper bound for the market risk for the CCPs. However, there is often a relatively small chance for the realized loss of an option portfolio to attain the maximum possible loss. The SPAN provides the maximum possible loss among sixteen scenarios, which are generated by the possible shocks on the underlying price and the volatility. As a result, the MPL calculated by the netting model could be much higher than the margins by the SPAN, as shown in Figures 3-4.

However, we observe from Figures 3-4 that the estimated MPLs tighten drastically when the model size increases from two to six. Furthermore, we calculate the average ratios of MPLs by the netting models of sizes two, four, and six over the margin levels by SPAN for each trading day and document the mean, the 75$^{\text{th}}$ percentile, the 95$^{\text{th}}$ percentile, and the maximum of the average ratios in Table 2. We can observe that the ratio decreases sharply when we increase the size from two to six, which implies improved netting efficiency. The reduction is more significant for SPX options than USD swaptions. Meanwhile, SPAN's margin levels on SPX options are closer to the MPLs of size six than these on USD swaptions. Recall that our netting model aims to reduce the requirement induced by an inefficient netting, but not that by small probability events. That is to say, by eliminating the part of inefficient netting, the maximum possible loss by our netting model (NM) can provide a reasonable level for the margin requirement of fluctuated assets, such as the SPX options. As for the USD swaptions, the fluctuations of both the underlying assets' price and the volatility are less significant. The probability that the realized loss of this portfolio attains the MPL is relatively small. Therefore, the difference between the MPL and the margin requirement by risk-sensitive model comes from the small probability events with significant losses for these less fluctuated assets. It is worth mentioning that the probability depends on the scenario setting, which is subjective.

We illustrate the difference between the maximum possible loss and SPAN's margin levels with a simple one-month maturity bull put spread on SPX starting from Oct. 14, 2014. In particular, the spread includes a long put at the strike price $S_t - 50$ and a short put at the strike $S_t - 45$, where S_t is the price of SPX on Oct. 14, 2014. In Figure 5, we plot the payoff curve of this put spread and the probability density functions in terms of the terminal price S_T of SPX conditional on the stock price and volatility settings in the generated scenarios. In particular, Figure 5a) corresponds to the scenario generated

Table 2
Statistics of the ratios of MPLs by netting models over the margins by SPAN. The ratio is significantly reduced with the increase of size. The reduction is more significant for SPX options than USD swaptions.

MPLs v.s. margins by SPAN with fixed volatility range on SPX options.				
	mean	75th percentile	95th percentile	maximum
netting model of size two	2.1278	2.1510	2.1905	2.5001
netting model of size four	1.2030	1.2161	1.2384	1.4135
netting model of size six	1.1432	1.1556	1.1769	1.3432
MPLs v.s. margins by SPAN with historical volatility range on SPX options.				
	mean	75th percentile	95th percentile	maximum
netting model of size two	2.0157	2.0533	2.1048	2.3312
netting model of size four	1.1396	1.1609	1.1900	1.3179
netting model of size six	1.0829	1.1032	1.1308	1.2524
MPLs v.s. margins by SPAN with fixed volatility range on USD swaptions.				
	mean	75th percentile	95th percentile	maximum
netting model of size two	1.6278	1.6436	1.6664	1.6882
netting model of size four	1.3616	1.3747	1.3938	1.4120
netting model of size six	1.3321	1.3450	1.3636	1.3815
MPLs v.s. margins by SPAN with historical volatility range on USD swaptions.				
	mean	75th percentile	95th percentile	maximum
netting model of size two	1.6332	1.6683	1.7084	1.7421
netting model of size four	1.3660	1.3954	1.4289	1.4572
netting model of size six	1.3365	1.3653	1.3980	1.4257



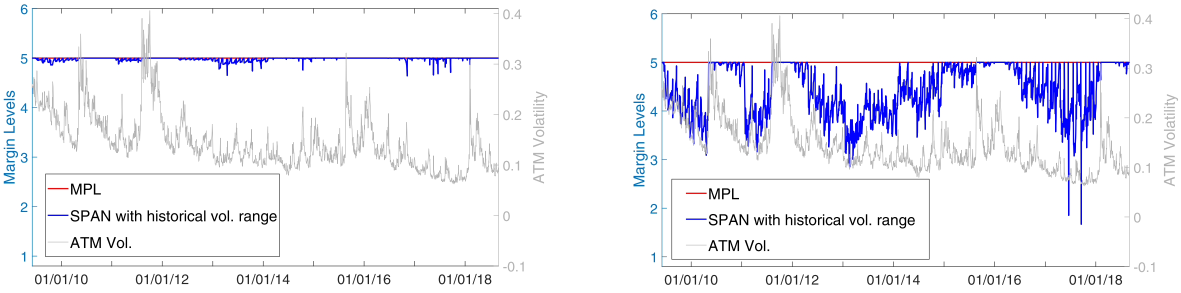
(a) Scenario I: no shock on the underlying price or volatility.

(b) Scenario II: shocks on both the underlying price and the volatility.

Fig. 5. Illustration of scenarios generated in SPAN to margin a one-month maturity bull put spread on SPX. The left figure presents the terminal payoff of the spread and the probability density function of the terminal price S_T in a scenario without any price or volatility shock; the right figure corresponds to the scenario where the underlying price decreases by one price scan range and the volatility decreases by a third of the historical volatility scan range. The estimated margin in the left scenario approaches zero, while the one in the right scenario approaches MPL$= -5$.

without any price or volatility shock. In such a scenario, the probability of the terminal payoff of the bull put spread being negative is extremely small. Thus, the corresponding potential loss is almost zero.

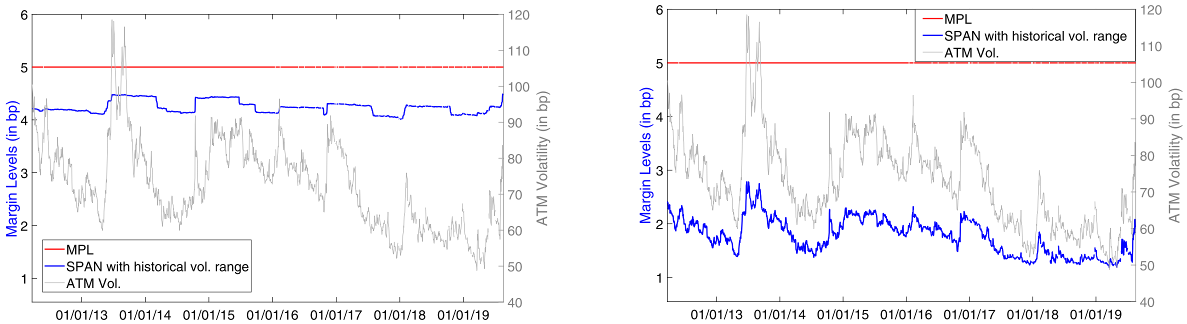
On the other hand, by moving the underlying price down by one price scan range and the volatility down by a third of the historical volatility scan range, another scenario can be generated for the SPAN. The corresponding probability density function, as shown in Figure 5b), states that the probability of the terminal payoff of the bull put spread being -5 approaches one. Thus, the corresponding potential loss approaches the maximum possible loss identified by our netting model (NM).



(a) Margin levels for bull put spreads close to the underlying price.

(b) Margin levels for out-the-money bull put spreads.

Fig. 6. Procyclicality of SPAN's margin levels with historical volatility range for SPX option portfolios from Jun. 4, 2009 to Aug. 31, 2018. We apply SPAN with historical volatility range to two sets of one-month maturity bull put spread on SPX: the spread corresponding to the left figure includes a long put at the strike price $S_t - 5$ and a short put at the strike price S_t; the one corresponding to the right figure includes a long put at the strike price $S_t - 50$ and a short put at the strike price $S_t - 45$. S_t is the price of SPX at day t. The trend of margin levels by SPAN is similar to that of the at-the-money option implied volatility. As shown in the right figure, the margin levels for out-the-money portfolios are much lower but have a higher degree of procyclicality than those in the left figure.



(a) Margin levels for swaptions close to the underlying price.

(b) Margin levels for out-the-money swaptions.

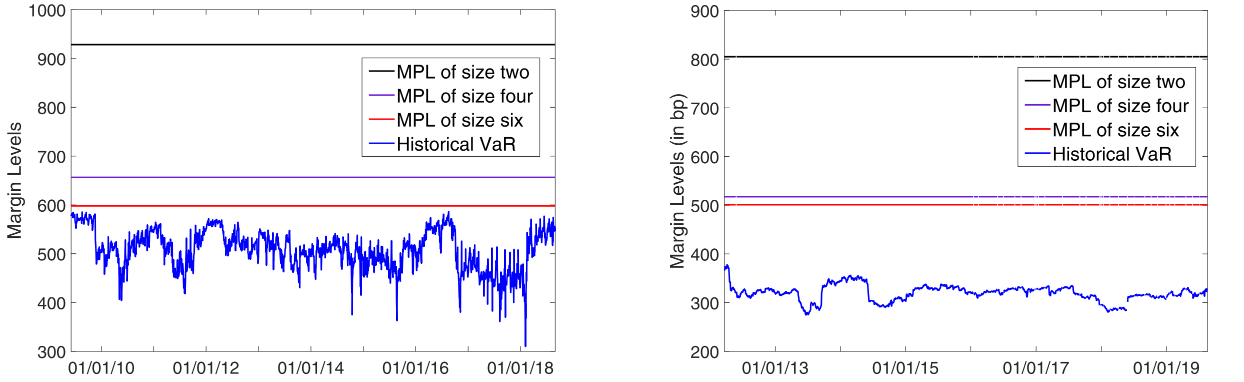
Fig. 7. Procyclicality of SPAN's margin levels with historical volatility range for USD swaption portfolios from Mar. 22, 2012 to Aug. 20, 2019. We apply SPAN to two sets of 3M×10Y USD swaption portfolios: the one corresponding to the left figure includes a long receiver swaption at the strike price $F_t - 5/10000$ and a short receiver at the strike price F_t; the one corresponding to the right figure includes a long receiver at the strike price $F_t - 50/10000$ and a short receiver at the strike price $F_t - 45/10000$. F_t is the 3M×10Y forward swap rate at day t. The trend of margin levels by SPAN is similar to that of the at-the-money option implied volatility. As shown in the right figure, the margin levels for out-the-money portfolios are much lower but have a higher degree of procyclicality than those in the left figure.

Example 4.2 (Procyclicality of SPAN)**.** We consider a simple type of balanced portfolio, a bull put spread, to present the procyclicality of risk-sensitive margin levels for an illustrative purpose. In particular, for each trading day t, we consider two one-month maturity bull put spreads on SPX: one includes a long put at the strike price $S_t - 5$ and a short put at the strike S_t; while the other includes a long put at the strike price $S_t - 50$ and a short put at the strike $S_t - 45$. S_t is the price of SPX at day t. In Figure 6, we present the comparison of margin levels by SPAN with historical volatility range[14] and at-the-money option implied volatility, with the MPLs calculated by our netting model of size six as a benchmark level, for these two sets of bull put spreads on SPX.

Similarly, we also consider two sets of portfolios of the 3M×10Y USD swaptions: one includes a long receiver swaption at strike price $F_t - 5/10000$ and a short receiver swaption at F_t; while the other includes a long receiver swaption at strike price $F_t - 50/10000$ and a short receiver swaption at $F_t - 45/10000$. The results on USD swaption portfolios are shown in Figure 7.

With such a setting, Figures 6(a) and 7(a) are corresponding to a case where the strike prices of the option portfolios are closer to the underlying price, comparing with the case in Figure 6(b) and Figure 7(b). We can observe from Figures 6(a) and 7(a) that the margin requirement by the risk-sensitive model SPAN is quite close to the MPL by our netting model (NM) for these option portfolios whose strikes are close to the current underlying price. Meanwhile, the margin requirement by SPAN could be much lower than the MPL if the option portfolios' strike prices are far away from the underlying price,

[14] The results corresponding to SPAN with fixed volatility range can be found in our online supplement.



(a) Application on balanced SPX option portfolios. (b) Application on balanced USD swaption portfolios.

Fig. 8. The MPLs by netting models of different sizes and the margin levels by historical VaR on ten randomly generated one-month maturity SPX option portfolios from Jun. 4, 2009 to Aug. 31, 2018 or 3M×10Y USD swaption portfolios from Mar. 22, 2012 to Aug. 20, 2019. The left figure presents the results on SPX option portfolios, while the right figure shows those on USD swaption portfolios. With the increase of size, the MPL by netting models is reduced.

Table 3
Statistics of the ratios of MPLs by our netting model (NM) over the margins by Historical VaR.

	mean	75th percentile	95th percentile	maximum
Applications on SPX options	1.1853	1.2430	1.3546	1.9289
Applications on USD swaptions	1.5721	1.6041	1.7414	1.8225

as shown in Figures 6(b) and 7(b). This is consistent to the essence of the scenario-based model SPAN: it identifies the maximum possible loss among potential scenarios generated according to the current market state, and thus may overlook the potential loss of these deeply out-the-money options. Meanwhile, our netting model (NM) corresponds to the worst case, which is independent of the current market state.

As a result, we can observe from Figures 6(b) and 7(b) that the risk-sensitive margin levels are much lower than the MPL provided by model (NM) when the at-the-money implied volatility is small and the market sentiment is optimistic. Because people believe that the worst case is a small probability event when the market sentiment is optimistic. When markets become volatile, investors face the possibility of the worst-case and thus increase the margin requirement. Therefore, the risk-sensitive margin requirement would be high at bad times and low at good times, and thus is pro-cyclical, especially for these out-of-the-money option portfolios.

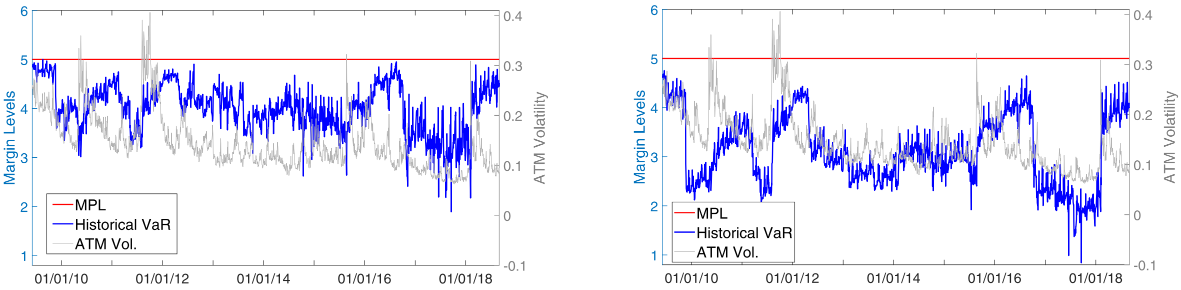
4.2. Comparison with the Historical Value-at-Risk (HVAR)

The Historical VaR approach is a historical simulation methodology, which evaluates the value at risk of a portfolio by simulations based on the past performance. Unlike parametric VaR models, historical VaR does not assume a particular distribution of the asset returns. In particular, we simulate the portfolio performance, based on the 10-day price and volatility change in last 180 days, and identify the 99% quantile of the portfolio loss as the margin requirement.

Example 4.3 (Netting Efficiency Compared with the Historical VaR)**.** With the same data set and experiment process as Example 4.1, we compare the MPLs by our netting model of sizes two, four and six for twenty sets of option portfolios with the margin levels by historical VaR. The averaged margin levels and MPLs at each trading day are documented in Figure 8. In addition, we calculate the average ratios of MPLs (of size six) by our netting model (NM) over the margin levels by the historical VaR for each trading day, and document the mean, the 75th percentile, the 95th percentile, and the maximum of the average ratios in Table 3.

Similar to Example 4.1, we can observe from Figure 8 that the margin levels are significantly reduced by increasing the model size from two to six. In addition, as discussed in Example 4.1, the MPL is closer to the risk-sensitive margin levels for fluctuated assets, such as the SPX options, than for less fluctuated ones, such as the USD swaptions. The reason is that based on the historical performance, these less fluctuated assets are less likely to achieve the maximum possible loss.

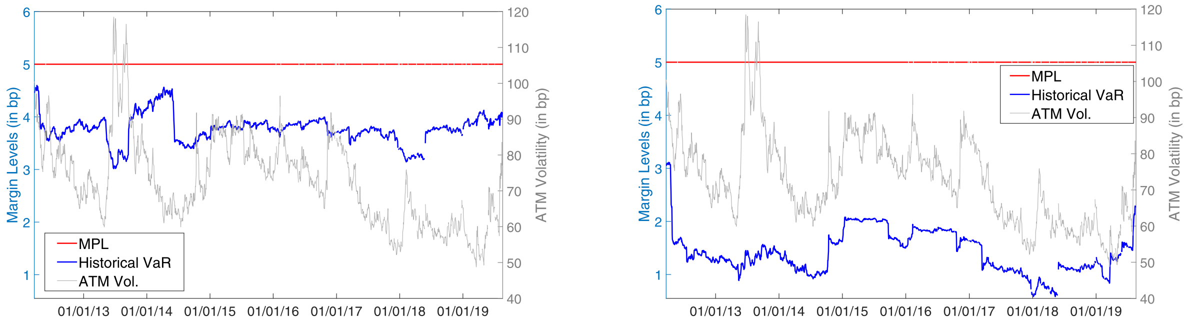
Example 4.4 (Procyclicality of Historical VaR)**.** We consider the same data and portfolio sets as Example 4.2. In Figures 9 - 10, we present the comparison of the margin level by historical VaR and at-the-money option implied volatility for these two sets of bull put spreads, with the MPL by our netting model (NM) as a benchmark.



(a) Margin levels for bull put spreads close to the underlying price.

(b) Margin levels for out-the-money bull put spreads.

Fig. 9. Procyclicality of the margin levels by historical VaR for SPX option portfolios from Jun. 4, 2009 to Aug. 31, 2018. We apply historical VaR to two sets of one-month maturity bull put spread on SPX: the spread corresponding to the left figure includes a long put at the strike price $S_t - 5$ and a short put at the strike price S_t; the one corresponding to the right figure includes a long put at the strike price $S_t - 50$ and a short put at the strike price $S_t - 45$. S_t is the price of SPX at day t. The trend of margin levels by historical VaR is similar to that of the at-the-money option implied volatility. As shown in the right figure, the margin levels for out-the-money portfolios are much lower but have a higher degree of procyclicality than those in the left figure.



(a) Margin levels for swaptions close to the underlying price.

(b) Margin levels for out-the-money swaptions.

Fig. 10. Procyclicality of the margin levels by historical VAR for USD swaption portfolios from Mar. 22, 2012 to Aug. 20, 2019. We apply historical VAR to two sets of 3M×10Y USD swaption portfolios: the one corresponding to the left figure includes a long receiver swaption at the strike price $F_t - 5/10000$ and a short receiver at the strike price F_t; the one corresponding to the right figure includes a long receiver at the strike price $F_t - 50/10000$ and a short receiver at the strike price $F_t - 45/10000$. F_t is the 3M×10Y forward swap rate at day t. As shown in the right figure, the margin levels for out-the-money portfolios are much lower but have a higher degree of procyclicality than those in the left figure.

Compared with the results in Figures 6-7, we can find that the margin levels by the historical VaR is lower and less procyclical than those by the SPAN. Because we do not consider stressed VaR in our historical VaR, the margin levels are much less than the maximum possible loss, especially for the out-of-money spread and less fluctuated asset.

4.3. Comparison with the Standard Initial Margin Model (SIMM)

The ISDA's SIMM approach is a sensitivity-based methodology widely used for the over-the-counter derivatives' margin calculation. The idea is to approximate the potential value change by expanding the price change through local sensitivities with respect to risk factors (F, σ) as follows,

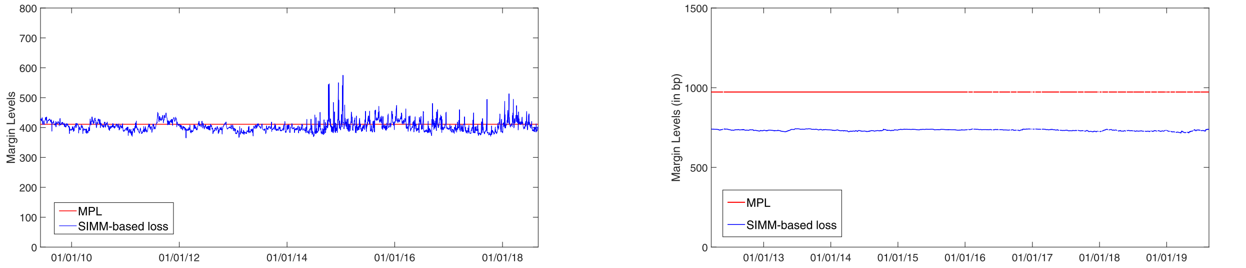
$$\frac{\partial P}{\partial F}(t)\Delta F + \frac{1}{2}\frac{\partial^2 P}{\partial F^2}(t)(\Delta F)^2 + \frac{\partial P}{\partial \sigma}(t)\Delta\sigma,$$

where P is the option's market price, F is the forward price of the underlying asset and σ is the option's implied volatility. Take the swaption for an example, according to Andersen and Pykhtin (2018), we approximate the potential value change of a swaption by

$$\Delta(F_t, \sigma_t) \times \Delta F + 0.5\Gamma(F_t, \sigma_t) \times (\Delta F)^2 + V(F_t, \sigma_t) \times VRW\sigma_t,$$

where $\Delta(\cdot,\cdot)$, $\Gamma(\cdot,\cdot)$ and $V(\cdot,\cdot)$ are Delta, Gamma and Vega of the option. $\Delta F = 2.326 \times \sqrt{14/365}\sigma_t$, $VRW = 0.16$ and σ_t is the implied volatility of the option at day t.

Different from both our netting model (NM) and the SPAN, which provide an estimation for the possible loss, SIMM approximates the negative of possible value change. To compare them, we provide the SIMM-based loss as $(SIMM - P_t)^+$,



(a) Application on balanced SPX option portfolios. (b) Application on balanced USD swaption portfolios.

Fig. 11. The MPLs by our size-six netting model and SIMM's margin levels on ten randomly generated one-month maturity SPX option portfolios from Jun. 4, 2009 to Aug. 31, 2018 or 3M×10Y USD swaption portfolios from Mar. 22, 2012 to Aug. 20, 2019. The left figure presents the results on SPX option portfolios, while the right figure shows those on USD swaption portfolios. With the MPL as a benchmark, SIMM's margins could overestimate the market risk for SPX option portfolios.

Table 4
Statistics of the ratios of MPLs by our netting model (NM) over the SIMM-based losses.

Applications on SPX options.				
	mean	75^{th} percentile	95^{th} percentile	maximum
MPLs v.s. SIMM-based losses	1.0179	1.0476	1.0761	1.1252
Applications on USD swaptions.				
	mean	75^{th} percentile	95^{th} percentile	maximum
MPLs v.s. SIMM-based losses	1.3269	1.3332	1.3426	1.3597

where P_t is the portfolio value at time t. As we discussed above, SIMM estimates the negative of possible value change $\Delta P_t = -(P_{t+\Delta t} - P_t)$, while our netting model provides MPL as $(\max(-P_T))^+$. Thus, it is easy to derive that

$$(\Delta P_t - P_t)^+ = (-P_{t+\Delta t})^+ \le (\max(-P_T))^+ = MPL.$$

Meanwhile, we also provide the MPL-based possible value change as $(MPL + P_t)^+$. With the fact that $(x+y)^+ \le (x^+ + y)^+$ holds for arbitrary x and y, we can derive that

$$(\Delta P_t)^+ \le (\max(-P_T) + P_t)^+ \le (MPL + P_t)^+. \tag{12}$$

With the further relaxation introduced by the non-negativity operator, the MPL-based possible value change, $(MPL + P_t)^+$, is not the tightest upper bound for the possible value change. However, it does serve as an upper bound for the possible value change, as shown in (12). We present the comparison between MPLs and the SIMM-based losses in the following part of this section and defer the comparison between MPL-based value change and the margin levels by SIMM to the online supplement.

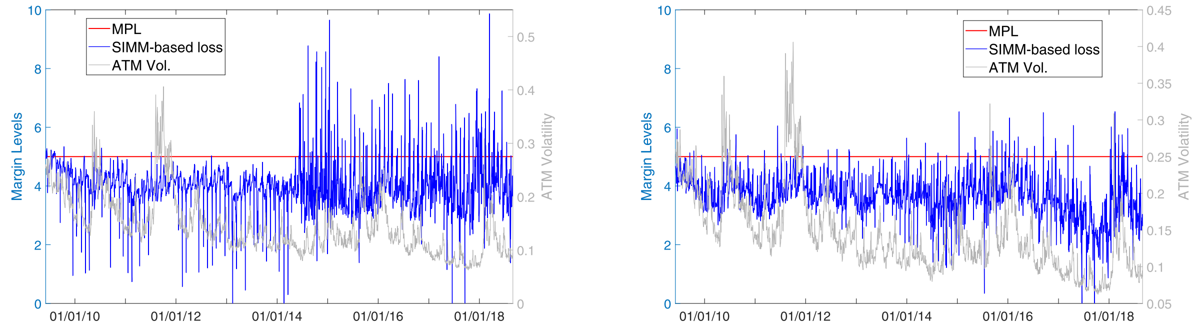
Example 4.5 (Comparison between our netting model and SIMM)**.** With the same data set and generated portfolios as Example 4.1, we compare the MPLs by our netting model of size six for such twenty sets of option portfolios with the SIMM-based losses in this example. The SIMM-based losses and MPLs at each trading day are documented in Figure 11.

As discussed previously, the MPLs calculated by our netting model could be much higher than the margins by the risk-sensitive model. However, note that the SIMM-based loss could be higher than the MPL, for SPX option portfolios, in Figure 11. The reason is that SIMM approximates the negative of possible value change by Greeks. This approximation is based on Taylor's formula and thus could possibly over-estimate the possible value change, especially for an underlying with large fluctuations, e.g., SPX.

Furthermore, we calculate the average ratios of MPLs by our netting model (NM) over the SIMM-based losses for each trading day, and document the mean, the 75^{th} percentile, the 95^{th} percentile, and the maximum of the average ratios in Table 4. Similar to Example 4.1, we can observe that the difference between the MPLs and the SIMM-based losses is more significant for USD swaptions than SPX options.

Example 4.6 (Procyclicality of SIMM)**.** We consider the same data and portfolio sets as Example 4.2. In Figures 12 - 13, we present the comparison of the SIMM-based loss and at-the-money option implied volatility for these two sets of bull put spreads, with the MPLs by our netting model (NM) as a benchmark.

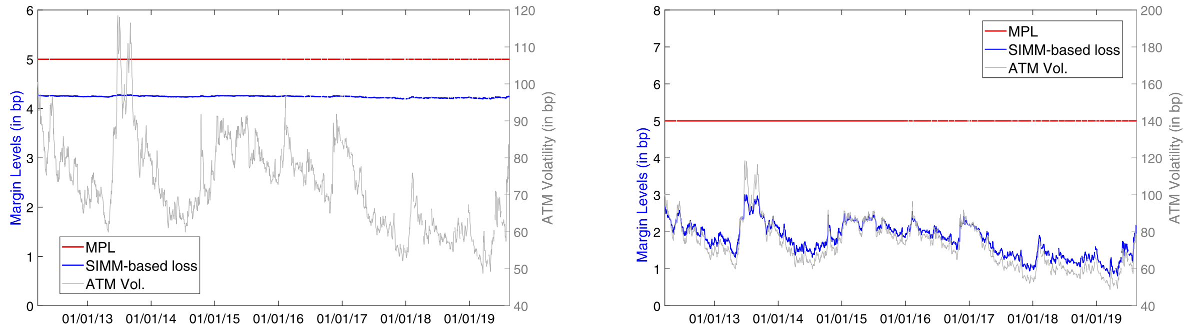
As shown in Figures 12(a) and 13(a), the magnitude of SIMM-based losses is quite close to the MPLs for these option portfolios whose strikes are close to the underlying price. In fact, the SIMM-based losses could even possibly be higher than



(a) Margin levels for options whose strikes are close to the underlying price.

(b) Margin levels for out-the-money option portfolios.

Fig. 12. Pro-cyclicality of the SIMM-based losses for SPX option portfolios from Jun. 4, 2009 to Aug. 31, 2018. We apply SIMM to two sets of one-month maturity bull put spread on SPX: the one in the left figure includes a long put at the strike price $S_t - 5$ and a short put at the strike price S_t; the one in the right figure includes a long put at the strike price $S_t - 50$ and a short put at the strike price $S_t - 45$. S_t is the price of SPX at day t. The SIMM-based losses in the left figure could overestimate the market risk, with the MPL as a benchmark. The SIMM-based losses in the right figure are lower, but of a higher degree of pro-cyclicality.



(a) Margin levels for swaptions whose strikes are close to the underlying price.

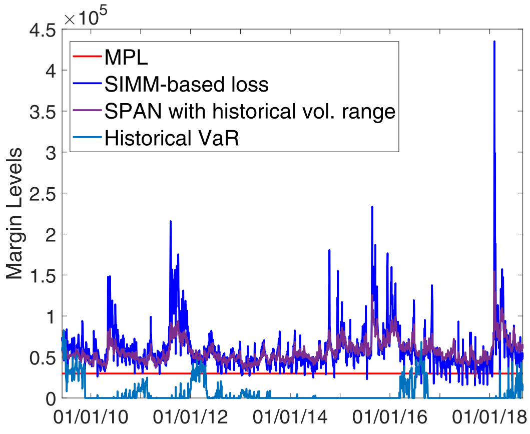
(b) Margin levels for out-the-money swaption portfolios.

Fig. 13. Pro-cyclicality of the SIMM-based losses for USD swaption portfolios from Mar. 22, 2012 to Aug. 20, 2019. We apply SIMM to two sets of 3M×10Y USD swaption portfolios: the one in the left figure includes a long receiver swaption at the strike price $F_t - 5$ and a short receiver at the strike price F_t; the one in the right figure includes a long receiver at the strike price $F_t - 50$ and a short put at the strike price $F_t - 45$. F_t is the 3M×10Y forward swap rate at day t. The trend of SIMM-based losses is similar to that of the at-the-money option implied volatility. The SIMM-based losses for out-the-money portfolios, as shown in the right figure, are much lower, but of a higher degree of pro-cyclicality, than the ones in the left figure.

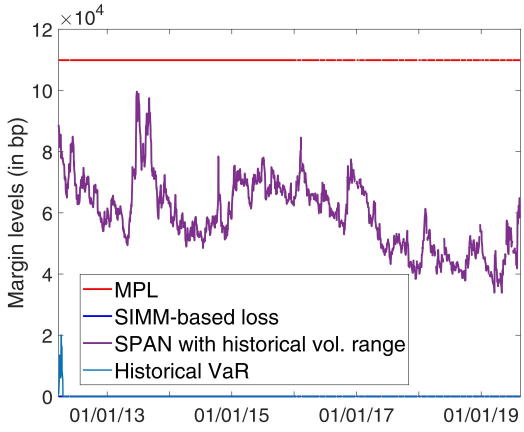
the MPLs by our netting model for this set of portfolios. Meanwhile, the SIMM-based loss could be much lower than the MPL if the option portfolios' strike prices are far away from the underlying price, as shown in Figure 13(b). This is consistent with the essence of SIMM, as we discussed in Example 4.5: the sensitivity-based model SIMM approximates the potential value change by Greeks. The parameters are generated according to the current market state, and thus may overlook the potential loss of these deeply out-the-money options, with the belief that the worst case is a small probability event. Moreover, the approximation could overestimate the potential loss for the at-the-money options on fluctuated assets, such as SPX options.

Example 4.7 (Unbalanced portfolio)**.** Inspired by the statistics on options market by SEC[15], we randomly generate ten sets unbalanced SPX option portfolios and ten sets unbalanced portfolios of 3M×10Y USD swaptions with unit notional and strike prices $S_t \pm 5i$ or $F_t \pm 5i/10000$, with 80,000 long call positions and 20,000 positions at short call or long/short put. The bounds of underlying price at maturity, S_T or F_T, are set as $[S_t - 50,\ S_t + 50]$ or $[F_t - 50/10000, F_t + 50/10000]$, respectively, where S_t or F_t is the price of SPX or the 3M×10Y forward swap rate at day t. We apply our netting model and the SPAN, the historical VaR and SIMM on these unbalanced portfolios and document the results in Figure 14. As shown in Figure 14, with the bounds' assumption, the MPLs are closer to these risk-sensitive margin levels, and could even be lower than those by the SPAN, the SIMM-based losses and the historical VaR, especially for the option portfolios on fluctuated assets.

[15] https://www.sec.gov/files/staff-report-equity-options-market-struction-conditions-early-2021.pdf



(a) Margin levels by different methodologies on SPX unbalanced option portfolios.



(b) Margin levels by different methodologies on unbalanced USD swaption portfolios.

Fig. 14. The MPLs by our netting model of size six and the margin levels by SPAN with historical volatility range, SIMM and historical VaR on ten randomly generated one-month maturity SPX unbalanced option portfolios from Jun. 4, 2009 to Aug. 31, 2018 or one-month maturity unbalanced USD swaption portfolios from Mar. 22, 2012 to Aug. 20, 2019. The left figure presents the results on SPX unbalanced option portfolios, while the right figure is on the swaption portfolios. To apply our netting model to unbalanced portfolios, we set the bounds of SPX price or the 3M×10Y forward swap rate to be $[S_t - 50, S_t + 50]$ or $[F_t - 50/10000, F_t + 50/10000]$, where S_t or F_t is the SPX price or the swap rate at day t.

4.4. The Mixed Models

In this section, we take the MPL by our netting model (NM) as a benchmark in the margin calculation process and incorporate it into a risk-sensitive model to balance the margin level and counter-cyclicality by proposing two simple mixed models.

Denote the margin calculated by a risk-sensitive model as Riskmargin and the optimal value of the netting model (NM) as MPL. Recall the three measures in European Union rules (European Commission, 2013) referred in Section 1. Because the second measure is vague, we consider two simple modifications corresponding to the first and third measures as the current models:

Mod. 1 (floor tool): max (Riskmargin, Riskmargin with 10-year volatility);

Mod. 2 (buffer tool): 1.25*Riskmargin.

Next, we present two similar mixed models by incorporating a risk-sensitive model with our netting model. Regulators may set a multiplier to combine both a risk-sensitive model and our netting model to provide reliable risk protection against the price fluctuation as follows:

Mmod. 1 (floor tool): min (max (Riskmargin, β_1*MPL), MPL);

Mmod. 2 (buffer tool): $(1 - \beta_2)*$ Riskmargin + β_2*MPL.

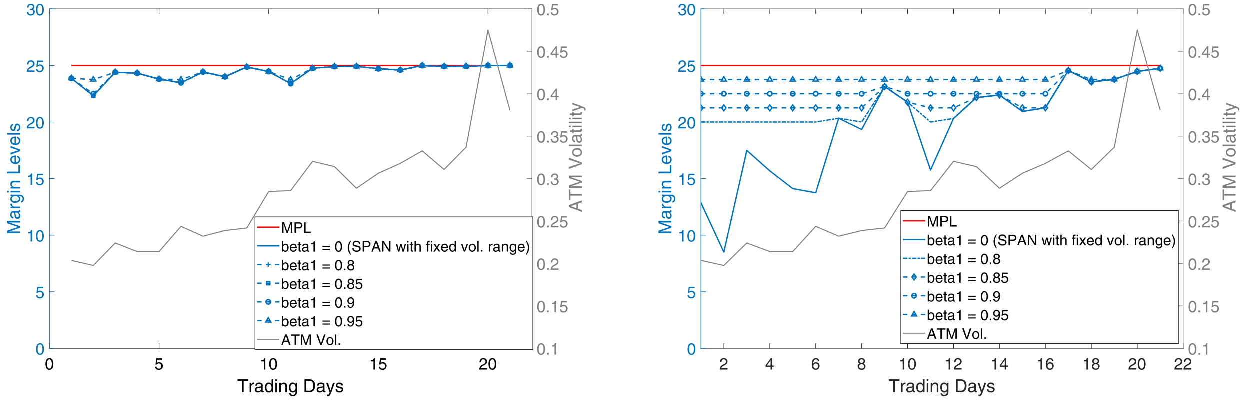
Similar to the weights in Murphy et al. (2016), both multipliers β_1 and β_2 are adjustable parameters determined by regulators. A large β_1 or β_2 corresponds to a heavy emphasis on the counter-cyclicality rather than the risk-sensitivity. For an illustrative purpose, we present the application of Mmod.1 to SPAN with fixed volatility range on two simple bull put spread on SPX in the following example. We would vary β_1 to present how a large β_1 can counter procyclicality by modifying the optimistic sentiment with an objective estimation of the potential loss. The application of Mmod.2 and the applications to other risk-based models are similar.

Example 4.8 (Mixed Model's Application)**.** We consider two bull put spreads on SPX, starting from September 5, 2008 and expiring on October 3, 2008. The index price was \$1242.31 on Sept. 5, 2008.

1) Bull put spread I includes a long put at the strike price \$1217.31 and a short put at the strike price \$1242.31;
2) Bull put spread II includes a long put at the strike \$1125 and a short put at the strike \$1150.

By buying such two bull put spreads, the investor expected to make a profit as long as the index price on the expiration date, Oct. 3, 2008, was not too low. With the historical prices observed on Sept. 5, 2008, it is highly possible to find these two bull put spreads profitable. However, the realized price was \$1099.23 and thus induced a realized loss of \$25 for each bull put spread.

In Figure 15, we present the margin levels provided by SPAN, Mmod. 1 with different β_1, and the MPLs by the netting model, for these two spreads from day 1 (Sept. 5, 2008) to day 21 (Oct. 3, 2008). We can observe from Figure 15(a) that the mixed model Mmod.1 provides almost the same margin levels as SPAN for bull put spread I, where the strike prices are close to the underlying price. In other words, when the margin requirement by SPAN has already covered a large part



(a) Margin levels for bull put spread I.

(b) Margin levels for bull put spread II.

Fig. 15. Margin Levels provided by mixed models with different multipliers β_1 for two bull put spreads on SPX in 2008. Both spreads started from September 5, 2008 and expired on October 3, 2008. The bull put spread I includes a long put at the strike $S_1 - 25$ and a short put at the strike price S_1, where S_1 =\$1242.31 is the price of SPX at day 1 (September 5, 2008). The bull put spread II includes a long put at the strike $S_1 - 117.31$ and a short put at the strike $S_1 - 92.31$, and thus are deeply out-the-money. The x-axis is the trading day t from September 5, 2008 to October 3, 2008. The realized price of SPX on October 3, 2008 was \$1099.23. The investors face the possibility of the worst-case and the run-ups of the risk-sensitive margin levels during the trading period. The difference between the margins by SPAN and the MPL indicates the market sentiment.

of the potential loss, our mixed model would not affect the margin by SPAN a lot further. However, for the cases where the risk-sensitive model underestimates the potential loss, our mixed model would modify the margin result, as shown in Figure 15(b). Because the current market condition predicts that the price is highly unlikely to drop below \$1125, the risk-sensitive model, based on the historical and current market data, determines a relatively low margin requirement for bull put spread II, as shown in Figure 15(b).

The difference between the margin levels and the MPLs is the deviation of our subjective estimation with the optimistic sentiment from the objective estimation. In other words, the difference can provide an indicator of the market sentiment: when the difference is large, the sentiment is optimistic and investors would prefer to underestimate possible risks and invest in risky opportunities; meanwhile, when the difference is small, the sentiment is pessimistic, and investors are rational, forced or not. With a larger β_1, our mixed model Mmod. 1 would provide a margin level closer to the MPL, which is an objective level. A larger β_1 thus corresponds to a higher margin requirement, but smaller margin run-ups when the market realizes that winter is coming. As a result, regulators can set a large β_1 when the market sentiment is so optimistic that the irrational tradings may be encouraged, and a small β_1 when the investors are relatively rational.

5. Conclusion

This paper develops a novel strategy-based margin calculation model for balanced options strategies. As an alternative to the risk-sensitive approach, the strategy-based approach offers an objective estimation for market risks of strategies against all possible price movements of the underlying asset. Although the strategy-based approach enjoys a distribution-free estimation and is fully counter-cyclical, it has suffered from two crucial issues of possible overestimation and high computational complexity and has become not too attractive in recent years. Fortunately, our proposed novel strategy-based netting model has satisfactorily addressed both issues for the balanced options strategies.

With our introduction of base offsets, we have proved that our size-six model always provides an exact estimation of the market risk measured by the maximum possible loss, for all balanced options strategies. Although we cannot prove the tractability for our netting model of size six, according to our numerical experiments, the computational time of our netting model of size six is less than one second for all the cases where the number of strike prices is less than twenty. Note that the number of different strike prices on one option is unlikely to be larger than twenty in the options markets. Therefore, our model of size six, which is proved to provide an exact estimation of the market risk for every balanced options strategy, does serve the purpose as a good netting model in practice. To conclude, by proposing the base offsets, we 1) show the essential hedging properties inside all the balanced strategies which can be easily documented into the margin manual book for traders' reference, and 2) solve the strategy-based model of size six in less than one second for multi-leg strategies.

We also incorporate the margin level calculated by our strategy-based model with the prevalent risk-sensitive margin model to counter procyclicality. With data of SPX options and USD swaptions, we demonstrate the netting efficiency of our model and the procyclicality of the risk-sensitive margin levels. We find that incorporating the maximum possible loss can counter procyclicality, but also increase the margin level. We only apply simple formulations for the incorporation to provide a possible counter-cyclical way. Regulators can further adjust the formula and the multipliers to identify the balance between margin level and the degree of procyclicality.

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Appendix A. Proofs of Properties of the Balanced Strategy and Dominance

Proof of Proposition 3.1

We first prove that an arbitrary given balanced option strategy can always be represented by an integer combination of basic spreads. If v is a balanced option strategy, i.e., $v_1 + \cdots + v_d = v_{d+1} + \cdots + v_{2d} = 0$, then we can rewrite vector v as

$$v = v_1(e_1 - e_d) + v_2(e_2 - e_d) + \cdots + v_{d-1}(e_{d-1} - e_d) +$$
$$v_{d+1}(e_{d+1} - e_{2d}) + v_{d+2}(e_{d+2} - e_{2d}) + \cdots + v_{2d-1}(e_{2d-1} - e_{2d}),$$

where e_i is the vector with i-th element being one and all other elements being zero. Because all of the vectors $e_i - e_d$ and $e_{d+i} - e_{2d}$, $i = 1, \cdots, d-1$, are basic spreads, v is an integer combination of basic spreads.

Meanwhile, any given basic spread can be easily represented by an integer combination of basic bull spreads of width one. For example, $b = e_1 - e_3$ is a basic spread of width two and can be represented as $b = B_1 + B_2$, where B_1 and B_2 are defined in Equation (4). Therefore, there exist integers $\bar{k}_1, \ldots, \bar{k}_{2(d-1)}$ such that $\sum_{i=1}^{2(d-1)} \bar{k}_i B_i = v$. Note that some of $\bar{k}_i$, $i = 1, \ldots, 2(d-1)$, could be negative. We further define non-negative coefficients $k_i = (\bar{k}_i)_+$ and $k_{2(d-1)+i} = (-\bar{k}_i)_+$, for all $i = 1, \ldots, 2(d-1)$. Also notice that $B_{i+2(d-1)} = -B_i$ hold for all $i = 1, \ldots, 2(d-1)$. Thus, we can express any balanced option strategy v as

$$v = \sum_{i=1}^{2(d-1)} \bar{k}_i B_i = \sum_{i=1}^{2(d-1)} (k_i - k_{i+2(d-1)}) B_i = \sum_{i=1}^{2(d-1)} (k_i B_i + k_{i+2(d-1)} B_{i+2(d-1)}) = \sum_{i=1}^{4(d-1)} k_i B_i.$$

On the other hand, according to the definition, basic spreads have the same number of long and short positions in puts or calls, and thus belong to the set of balanced option strategies. Therefore, an arbitrary integer combination of basic spreads, $B_1, \ldots, B_{4(d-1)}$, is also balanced. To conclude, an option strategy can be represented by a non-negative integer combination of basic spreads, $B_1, \ldots, B_{4(d-1)}$, if and only if it is a balanced option strategy.□

Proof of Proposition 3.2

From the relationship among v, v^i and b^j stated in (8), we can easily derive that

$$v = \sum_{i=1,\ i\neq u}^{t} z_i v^i + \sum_{j=1}^{n} z_u k_j b^j, \ \ m(v) \geq \sum_{i=1,\ i\neq u}^{t} z_i m(v^i) + \sum_{j=1}^{n} z_u k_j m(b^j),$$

where $z_i,\ z_u k_j \geq 0$, $i = 1, \ldots, t$, and $j = 1, \ldots, n$. Thus, according to Definition 2, v is dominated by $\{v^i, i = 1, \cdots, u-1, u+1, \cdots, t\} \cup \{b^1, \cdots, b^n\}$. □

Proof of Lemma 3.1

According to Definition 3, there exist $k_1, \ldots, k_n \in \mathbb{Z}_+$ such that $v = \sum_{i=1}^{n} k_i b_i$ is a combination satisfying either condition i) or condition ii) in Definition 3. i) If condition i) holds, denote f_{b_i} as the payoff function of b_i, $i = 1, \ldots, n$. From condition i) stated in Definition 3, we have $f_{b_i}(S_T) \geq 0$, for all $S_T \in [0, \infty)$. Thus, according to the definition of maximum possible loss(MPL) in (2), all the MPLs of these strategies b_i, $i = 1, \ldots, n$, are zeros. Meanwhile, the MPL of any strategy is non-negative. The resulting MPL, $\sum_{i=1}^{n} k_i m(b_i)$, equals to zero and thus is no higher than the MPL of v. ii) If condition ii) holds, which is to say, all the payoff functions f_{b_i}, $i = 1, \ldots, n$, achieve their non-positive minimal values at S_T^0, then the payoff function $f_v(S_T) = \sum_{i=1}^{n} k_i f_{b_i}(S_T)$, also achieves its non-positive minimal value at S_T^0. Thus, the resulting MPL equals to the MPL of v, since

$$m(v) = (-\min_{S_T} \sum_{i=1}^{n} k_i f_{b_i}(S_T))^+ = -\min_{S_T} \sum_{i=1}^{n} k_i f_{b_i}(S_T) = -\sum_{i=1}^{n} k_i f_{b_i}(S_T^0)$$

$$= \sum_{i=1}^{n} k_i(-\min_{S_T} f_{b_i}(S_T))^+ = \sum_{i=1}^{n} k_i m(b_i). \qquad \square$$

Appendix B. Proof of Theorem 3.1

Our proof is based on Proposition 3.2 and Lemma 3.1 and will be carried out in a few steps. The main idea is to factorize any balanced option strategy into a combination of option strategies whose payoff functions either are all non-negative or all achieve their non-positive minimal values at one same point. Then according to Lemma 3.1, the balanced option strategy is dominated by these option strategies. We keep on factorizing these option strategies and prove that these option strategies are either base offsets or dominated by base offsets. Thus, we can conclude that any balanced option strategy is dominated by base offsets with Proposition 3.2 and can be expressed into a no-hedging combination of base offsets.

To proceed, we firstly introduce Form 2'-7' option strategies and prove that they are dominated by base offsets in Lemma Appendix B.1. We devote Lemma Appendix B.2 to prove that box spreads are dominated by base offsets. Both the newly introduced Form 2'-7' option strategies and the box spread will be used in the proof of the theorem during the factorizing process. Then we present Lemma Appendix B.3 to describe the case where some of Form 2'-7' option strategies achieve their non-positive minimal values at one same point. Lemma Appendix B.4 states that if an option strategy is dominated by base offsets, it can be expressed into a no-hedging combination of base offsets. Finally, we prove our main theorem.

B1. Preparatory Lemmas

Lemma B.1. *Define a series of Form 2'-7' option strategies, which are generalized base offsets, as follows:*
Form 2': $B_{i+2(d-1)} + B_j$, where $i,\ j \in \{1, ..., d-1\},\ i > j$;
Form 3': $B_{i+2(d-1)} + B_j + B_{j+3(d-1)}$, where $i,\ j \in \{1, ..., d-1\},\ i < j$;
Form 4': $B_{i+d-1} + B_{j+3(d-1)}$, where $i,\ j \in \{1, ..., d-1\},\ i < j$;
Form 5': $B_{i+d-1} + B_j + B_{j+3(d-1)}$, where $i,\ j \in \{1, ..., d-1\},\ i > j$;
Form 6': $B_{i+2(d-1)} + B_{j+d-1}$, where $i,\ j \in \{1, ..., d-1\},\ i \geq j$;
Form 7': $B_{i+2(d-1)} + B_{u+d-1} + B_j + B_{j+3(d-1)}$, where $i,\ u,\ j \in \{1, ..., d-1\}$, $u \leq i$ and $j \in [1, u) \cup (i, d-1]$.
Form 2'-7' option strategies are all dominated by base offsets.

Proof of Lemma Appendix B1.. We first reveal the relationship between Form 2'-7' option strategies and Form 2-7 base offsets by examining the payoff functions of Form 2'-7' option strategies in Figure B.16. Recall the payoff functions of base offsets presented in Figure 3 of the paper. We can observe that Form 2'-7' option strategies are generalized Form 2-7 base offsets. We next show that an arbitrary Form 2' option strategy $B_{i+2(d-1)} + B_j$, where $i,\ j \in \{1, ..., d-1\},\ i > j$, is dominated by base offsets. Because $B_{i+2(d-1)} + B_i = 0$ holds for all $i = 1, \cdots, 2(d-1)$, we can rewrite the Form 2' option strategy as

$$B_{i+2(d-1)} + B_j = \sum_{s=j+1}^{i} (B_{s+2(d-1)} + B_{s-1}).$$

All these strategies $B_{s+2(d-1)} + B_{s-1}$, $s = j+1, \cdots, i$, are Form 2 base offsets. According to the payoff functions of base offsets plotted in Figure 3, Form 2 base offsets have zero maximum possible losses. Thus, the combination of them cannot furthermore reduce the MPL. Form 2' option strategies are dominated by base offsets. Similar proof can be applied to all the Form 3'- Form 7' option strategies. □ □

Lemma B.2. *For an arbitrary box spread $\bar{b} = B_i + B_{i+3(d-1)}$, $i \in \{1, \ldots, d-1\}$, the following statements are true:*
i) It is dominated by base offsets;
ii) The payoff function of $\bar{b} + v$ achieves its minimal value at the same points as that of v, where v is an arbitrary option strategy;
iii) For any basic spread of width one or Form 2'-7' option strategy b, $\bar{b} + b$ has a non-negative payoff function and is dominated by base offsets.

Proof of Lemma Appendix B2.. i) For any $i \in \{1, \ldots, d-1\}$, basic spreads B_i and $B_{i+3(d-1)}$ are bull call and bear put spreads, respectively. Recall that both basic bull call and bear put spreads have non-negative payoff functions. According to Lemma 3.1, $\bar{b}$ is dominated by them and thus by base offsets. ii) Box spread $\bar{b}$ has the following constant payoff function, $f_{\bar{b}}(S_T) = 1$, for all $S_T \in [0, \infty)$. As a result, for any option strategy v, the payoff function of $\bar{b} + v$,

$$f_{\bar{b}+v}(S_T) = 1 + f_v(S_T),$$

achieves its minimal value at the same points as that of v. iii) Firstly, Figure B.16 shows that the payoff function of any Form 2'-7' option strategy b satisfies $f_b(S_T) \geq -1$. Because the payoff function of $\bar{b}$ is always one, the strategy $b + \bar{b}$ has a non-negative payoff function.

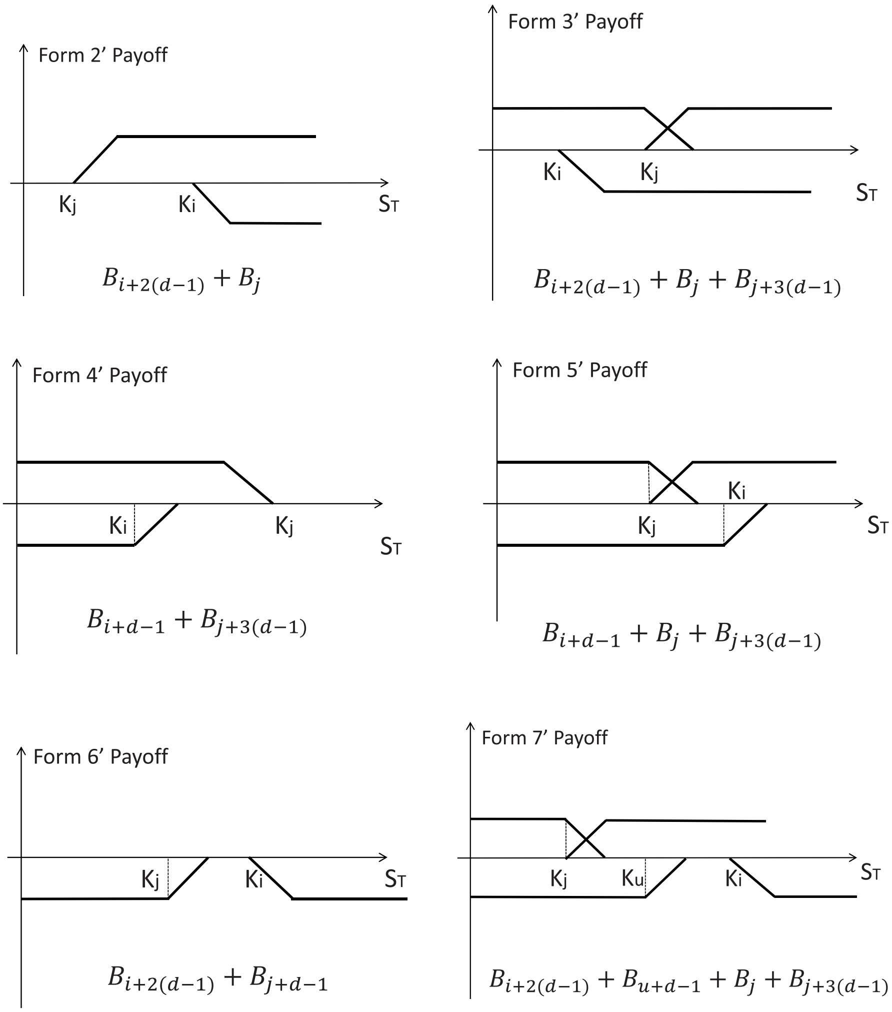

Fig. B1. Payoff functions of Forms 2' to 7' option strategies.

We continue to prove the dominance part. According to Lemma 3.1, the proof is trivial if b has a non-negative payoff function. Otherwise, according to the payoffs of basic spreads and Form 2'-7' option strategies presented in Figures 2 and B.16, strategy b is a basic bull put spread, a basic bear call spread or a Form 6' option strategy.

If b is a basic bull put spread, $b = B_j$, with $j \in \{d, \cdots, 2(d-1)\}$. From the definitions of Form 2'-7' option strategies in Lemma Appendix B.1, we know that $\bar{b} + b$ is either a Form 5' option strategy when $j - d + 1 > i$ or dominated by $B_j + B_{i+3(d-1)}$ and B_i otherwise, because

$$\bar{b} + b = (B_j + B_{i+3(d-1)}) + B_i, \ \ m(\bar{b} + b) \geq 0 = m(B_j + B_{i+3(d-1)}) + m(B_i).$$

Moreover, $B_j + B_{i+3(d-1)}$ is either a zero vector when $j - d + 1 = i$ or a Form 4' option strategy otherwise. According to the transitivity property in Proposition 3.2 and Lemma Appendix B.1, b is dominated by the base offsets.

If b is a basic bear call spread or a Form 6' option strategy, the proof is similar. □ □

Lemma B.3. *We consider the following five types of option strategies:*

i) Basic bull call spreads B_i, $i \in I \subseteq \{1, 2, ..., d-1\}$;

ii) Basic bear call spreads, $B_{t_1}, \cdots, B_{t_n}$;

iii) Form 2' option strategy, $\hat{b}_{n+1} = B_{t_{n+1}} + B_{j_{n+1}}, \cdots, \hat{b}_c = B_{t_c} + B_{j_c}$, *with* B_{t_i} *and* B_{j_i}, $i = n+1, \cdots, c$, *being basic bear and bull call spreads, respectively;*

iv) Form 6' option strategy, $\tilde{b}_1 = B_{g_1} + B_{u_1}, \cdots, \tilde{b}_h = B_{g_h} + B_{u_h}$, *with* B_{g_i} *and* B_{u_i}, $i = 1, \cdots, h$, *being basic bull put and bear call spreads, respectively;*

v) Basic bull put spreads, $B_{g_{h+1}}, \cdots, B_{g_p}$.

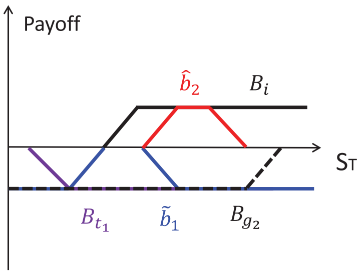

Fig. B2. Illustration of a Common Worst-case Scenario in Lemma Appendix B.3. The figure plots the payoff functions of five option strategies: the black solid and dot lines stand for a basic bull call and put spread, respectively. The purple line plots the payoff function of a basic bear call spread, while the red and blue curves are corresponding to the payoff functions of a Form 2' and a Form 6' option strategies, respectively.

If the indices of these five types of option strategies satisfy all the following five conditions:

$$\begin{cases} C1: \ t_1 \le t_2 \le \cdots \le t_n \le t_{n+1} \le \cdots \le t_c < g_{h+1} + d - 1; \\ C2: \ g_1 \le g_2 \le \cdots \le g_h \le g_{h+1} \le \cdots \le g_p; \\ C3: \ \text{Either } \ t_r < g_s + d - 1 \text{ or } t_r \ge u_s \text{ holds for all } r = 1, \cdots, c \text{ and } s = 1, \cdots, h; \\ C4: \ \text{For all } i \in I, \ i > t_n - 2(d-1); \\ C5: \text{Either } \ i > t_r - 2(d-1) \text{ or } i \le j_r \text{ holds for all } i \in I \text{ and } r = n+1, \cdots, c, \end{cases} \tag{B.1}$$

then all the payoff functions of these five types of option strategies have (non-positive) minimal values at one common point.

Proof of Lemma Appendix B3.. According to Figure B.16, all the payoff functions of Form 2'-7' option strategies have non-positive minimal values. Denote the smallest element of I as $\hat{i}$. We will carry out our proof for the following two different situations: $\hat{i} \le t_c - 2(d-1)$ and $\hat{i} > t_c - 2(d-1)$.

Case 1. If $\hat{i} \le t_c - 2(d-1)$, conditions C1 and C4 reveal that $\hat{i} \in (t_n - 2(d-1), t_c - 2(d-1)]$. Thus, there must exist $l \in \{n, ..., c-1\}$ such that $\hat{i} \in (t_l - 2(d-1), t_{l+1} - 2(d-1)]$. Denote S_T^0 as $1 + K_{t_l - 2(d-1)}$. We will prove in the following that all these five types of option strategies satisfying Conditions C1 - C5 in (B.1) achieve their minimal values at S_T^0.

We firstly illustrate the basic idea of this proof with Figure B.17. To make the figure clear, we consider a simple case in Figure B.17 where I contains one unique element i, and indices are set as $n = h = 1$ and $c = p = 2$. In other words, Figure B.17 plots the payoff curves of one basic bull call spread, B_i, one basic bear call spread, B_{t_1}, one Form 2' option strategy, $\hat{b}_2 = B_{t_2} + B_{j_2}$, one Form 6' option strategy, $\tilde{b}_1 = B_{g_1} + B_{u_1}$, and one basic bull put spread, B_{g_2}, whose indices satisfy conditions C1-C5. It is easy to figure out that $l = 1$, since $\hat{i} = i \in (t_1 - 2(d-1), t_2 - 2(d-1))$, and that all the payoff functions of these five strategies achieve their non-positive minimal values at point $S_T^0 = 1 + K_{t_1 - 2(d-1)}$.

The mathematical proof is then presented as follows. We invoke Figure 2 to find where the payoff functions of basic spreads achieve their minimal values.

Firstly, for all $i \in I$, according to Figure 2, the payoff functions of basic bull call spreads, f_{B_i}, achieve their minimums at S_T^0, as the definitions of $\hat{i}$ and l give rise to $S_T^0 \le K_{\hat{i}} \le K_i$.

Secondly, the fact that $t_1 \le t_2 \le \ldots \le t_n \le t_l$ leads to a conclusion that all the payoff functions of the basic bear call spreads $B_{t_1}, \cdots, B_{t_n}$ achieve their minimal values at S_T^0.

Thirdly, according to the property of Form 2' option strategy stated in Figure B.16, the payoff functions of $\hat{b}_i$, $i = n+1, \ldots, c$, achieve their minimums at $[0, K_{j_i}] \cup [1 + K_{t_i - 2(d-1)}, \infty)$. Thus, all the payoff functions of $\hat{b}_{n+1}, \ldots, \hat{b}_l$ achieve their minimal values at S_T^0, since $t_l \ge t_i$ holds for all $i = n+1, \ldots, l$.

As for $i = l+1, \ldots, c$, we have $\hat{i} \le t_i - 2(d-1)$, according to the definition of l. Thus, condition C5 dictates that $\hat{i} \le j_i$ holds for any $i = l+1, \ldots, c$. Therefore, for all $i = l+1, \ldots, c$, the payoff functions of Form 2' option strategies $\hat{b}_i$ also achieve their minimums at S_T^0, since $S_T^0 \le K_{\hat{i}} \le K_{j_i}$.

Fourthly, condition C3 states that either $t_l < g_i + d - 1$ or $t_l \ge u_i$ holds for all $i = 1, \ldots, h$. As a result, $S_T^0 \in [0, K_{g_i - d + 1}] \cup [1 + K_{u_i - 2(d-1)}, \infty)$, $\forall i = 1, \ldots, h$. According to the property of Form 6' option strategy presented in Figure B.16, all the payoff functions of $\tilde{b}_i$, $i = 1, \ldots, h$, achieve their minimal values at S_T^0.

Fifth, all the payoff functions of the basic bull put spreads $B_{g_{h+1}}, \ldots, B_{g_p}$ achieve their minimums at S_T^0, because conditions C1 and C2 require that $t_l - 2(d-1) \le t_c - 2(d-1) < g_{h+1} - d + 1 \le \cdots \le g_p - d + 1$.

To conclude, all the payoff functions of these five kinds of option strategies achieve their non-positive minimal values at $S_T^0 = 1 + K_{t_l - 2(d-1)}$.

Case 2. If $\hat{i} > t_c - 2(d-1)$, we can prove that all the payoff functions of these option strategies achieve their non-positive minimal values at $S_T^0 = 1 + K_{t_c - 2(d-1)}$ in a way similar to Case 1. □ □

Lemma B.4. *If an option strategy v is dominated by base offsets $b_1, \ldots, b_n$, then v can be expressed into a no-hedging combination of $b_1, \ldots, b_n$.*

Proof of Lemma Appendix B4.. From the definitions of dominance and margin requirement in Definition 2 and Equation (2), we know that there exist $k_1, \dots, k_n \in \mathbb{Z}_+$ such that

$$v = \sum_{i=1}^{n} k_i b_i,\ \ m(v) = (-\min_{S_T} \sum_{i=1}^{n} k_i f_{b_i}(S_T))^+ \geq \sum_{i=1}^{n} k_i m(b_i) = \sum_{i=1}^{n} k_i (-\min_{S_T} f_{b_i}(S_T))^+, \tag{B.2}$$

where $f_{b_i}(\cdot)$ are the payoff functions of base offsets b_i, $i = 1, \dots, n$. Without loss of generality, we assume that $k_i > 0$ for all $i = 1, \dots, n$. According to the payoff functions of base offsets in Figure 3, all of these payoff functions have non-positive minimal values. If the minimal values of the payoff functions, $f_{b_i}(\cdot)$, $i = 1, \dots, n$, are all zeros, all of these payoff functions are non-negative. The combination is no-hedging, since condition i) of the no-hedging definition is satisfied.

If at least one of the minimal values of the payoff functions is negative, i.e. there exists some $j \in \{1, \dots, n\}$ such that $m(b_j) > 0$, then Equation (B.2) reveals that

$$m(v) \geq \sum_{i=1}^{n} k_i m(b_i) > 0 \ \Rightarrow\ m(v) = -\min_{S_T} \sum_{i=1}^{n} k_i f_{b_i}(S_T) \geq \sum_{i=1}^{n} k_i (-\min_{S_T} f_{b_i}(S_T))^+ = -\sum_{i=1}^{n} k_i \min_{S_T} f_{b_i}(S_T).$$

The last equality follows the fact that all the payoff functions of base offsets have non-positive minimal values. Meanwhile, the minimum of a summation of functions is no smaller than the summation of the minimums of these functions, i.e.

$$\min_{S_T} \sum_{i=1}^{n} k_i f_{b_i}(S_T) \geq \sum_{i=1}^{n} \min_{S_T} k_i f_{b_i}(S_T).$$

Therefore, we have $\min_{S_T} \sum_{i=1}^{n} k_i f_{b_i}(S_T) = \sum_{i=1}^{n} \min_{S_T} k_i f_{b_i}(S_T)$, and thus that there exists one point $S_T^0 \in [0, \infty)$ such that all of these payoff functions, f_{b_i}, achieve their minimal values at S_T^0. As a result, this combination is no-hedging with condition ii) in the no-hedging definition being satisfied.□ □

B2. Proof of Theorem 3.1

Step 1. Express the balanced option strategy into a combination of Form 1 base offsets. From Lemma Appendix B.4, to prove Theorem 1, it suffices to prove that any balanced strategy expressed in the following form,

$$v = \sum_{i=1}^{4(d-1)} k_i B_i,\ \ k_i \in \mathbb{Z}_+,\ \ i = 1, \dots, 4(d-1), \tag{B.3}$$

is dominated by the set of base offsets, where B_i and k_i, $i = 1, \dots, 4(d-1)$, are basic spreads of width one and the number of these basic spreads involved, respectively. Note that the expression is not unique, because of the fact that $B_i + B_{i+2(d-1)} = 0$ holds for all $i = 1, \dots, 2(d-1)$. We thus require the expression (B.3) to satisfy that for all $i = 1, \dots, 2(d-1)$, either k_i or $k_{i+2(d-1)}$ equals to zero, i.e., we avoid the redundant cases where both the bull spread B_i and the corresponding bear spread $B_{i+2(d-1)}$ exist in the expression (B.3).

Define $L_1 := \{l \in \{2d-1, ..., 3(d-1)\} \mid k_l > 0\}$ as the set of indices of basic bear call spreads involved in v, and $L_2 := \{l \in \{d, ..., 2(d-1)\} \mid k_l > 0\}$ as the set of indices of basic bull put spreads involved in v.

If $L_1 = L_2 = \emptyset$, no basic bear call or bull put spread is involved in v. Thus, v only contains basic bull call and bear put spreads. Because all the basic bull call and bear put spreads have non-negative payoff functions, according to Lemma 3.1, v is dominated by them and thus dominated by base offsets. Otherwise, go to Step 2.

Step 2. Construct box spreads. For all the basic bear put spread B_i, $i = 3d-2, \dots, 4(d-1)$, if any, construct box spreads by adding $0 = B_{i-3(d-1)} + B_{i-d+1}$ to v. Combine B_i with $B_{i-3(d-1)}$ to get box spreads whose payoffs are always ones: Because $B_j = -B_{j+2(d-1)}$ holds for all $j = 1, ..., 2(d-1)$, we can reexpress v in (B.3) as a linear combination of basic spreads and box spreads,

$$v = \sum_{i=1}^{3(d-1)} k_i^0 B_i + \sum_{i=3d-2}^{4(d-1)} k_i (B_i + B_{i-3(d-1)}), \tag{B.4}$$

where $k_i^0 = k_i + k_{i+d-1}$, for all $i = 2d-1, ..., 3(d-1)$ and $k_i^0 = k_i$, for all $i = 1, ..., 2(d-1)$. Denote all these box spreads $B_i + B_{i-3(d-1)}$ as $\bar{b}_1, ..., \bar{b}_{\bar{n}}$ with $\bar{n} = \sum_{i=3d-2}^{4(d-1)} k_i$. Because $k_i \geq 0$ holds in (B.3) for all $i = 1, \cdots, 4(d-1)$, we have $k_l^0 \geq 0$ for all $l = 1, \cdots, 3(d-1)$. Go to Step 3.

Step 3. Combine the bear call and bull put spreads. Order all the remaining bull put spreads, if any, in a sequence of non-decreasing strike prices of the first leg. From the first to the last in the sequence, combine the bull put spread with the bear call spread whose first-leg strike price is the smallest among all the unpaired bear call spreads which can generate a new offset with this bull put spread. In particular, order the bull put spreads with indices in L_2, in terms of g_i, as $B_{g_1}, ..., B_{g_p}$, $d \leq g_1 \leq \cdots \leq g_p \leq 2(d-1)$.

For example, in the case where $L_2 = \{5, 6\}$, $k_5^0 = 2$ and $k_6^0 = 1$, we have $p = 3$, $g_1 = g_2 = 5$ and $g_3 = 6$.

Case 1. If there exists some bear call spread which can offset bull put spread B_{g_1}, i.e.,

$$U_1 := \{u \in \{2d-1, \dots, 3(d-1)\} \mid k_u^0 > 0, u \geq g_1 + d - 1\} \neq \emptyset, \tag{B.5}$$

denote the smallest element of U_1 as u_1. We can combine B_{u_1} and B_{g_1} as an offset $\tilde{b}_1$ and have

$$\sum_{i=d}^{3(d-1)} k_i^0 B_i = \tilde{b}_1 + B_{g_2} + \dots + B_{g_p} + \sum_{u=2d-1}^{3(d-1)} k_u^1 B_u,$$

where $k_{u_1}^1 = k_{u_1}^0 - 1,\ k_i^1 = k_i^0,\ \forall i \in \{2d-1, \dots, 3(d-1)\}/\{u_1\},\ \tilde{b}_1 = B_{u_1} + B_{g_1}$. Note that $\tilde{b}_1$ is a Form 6' option strategy, as shown in Figure B.16, whose payoff function achieves its minimum -1 at any $S_T \in [0, K_{g_1-d+1}] \cup [1 + K_{u_1-2(d-1)}, \infty)$. Similarly we can recursively define the set $(U_l)_{l=1}^p$, the index $(u_l)_{l=1}^p$ and the coefficient $(k_i^l)_{l=1}^p$, $i = 2d-1, \dots, 3(d-1)$, via

$$\begin{cases} U_l := \{u \in \{2d-1, \dots, 3(d-1)\} \mid k_u^{l-1} > 0, u \geq g_l + d - 1\}, \\ u_l = \min\{j | j \in \{3d-2\} \cup U_l\}, \\ k_{u_l}^l = k_{u_l}^{l-1} - 1,\ k_i^l = k_i^{l-1},\ \forall i \in \{2d-1, \dots, 3(d-1)\}/\{u_l\}, \\ \tilde{b}_l = B_{g_l} + B_{u_l},\ \text{if } u_l \leq 3(d-1). \end{cases} \tag{B.6}$$

In fact, U_l is the set of indices of unpaired basic bear call spreads which can offset the bull put spread B_{g_l}. To facilitate the proof, we define $U_{p+1} = \emptyset$, $g_{p+1} = \infty$ and $K_\infty = \infty$. There exists $h \in \{1, \dots, p\}$ such that $U_{h+1} = \emptyset,\ U_1, \dots, U_h \neq \emptyset$. At the end of the iteration, we can express the bear call and bull put spreads as

$$\sum_{i=d}^{3(d-1)} k_i^0 B_i = \tilde{b}_1 + \dots + \tilde{b}_h + B_{g_{h+1}} + \dots + B_{g_p} + \sum_{s=2d-1}^{3(d-1)} k_s^h B_s,$$

with $\tilde{b}_l = B_{g_l} + B_{u_l}$, $l = 1, \dots, h$. Denote $c = \sum_{s=2d-1}^{3(d-1)} k_s^h \geq 0$ as the number of remaining unpaired basic bear call spreads and order these bear call spreads, in terms of t_i, as

$$\sum_{s=2d-1}^{3(d-1)} k_s^h B_s = B_{t_1} + \dots + B_{t_c},\ 2d-1 \leq t_1 \leq \dots \leq t_c \leq 3(d-1).$$

Notice that there does not exist any B_{t_i} in the above representation if $c = 0$. Go to Step 4 for the remaining bear call spreads.

We next show that these indices $g_1, \cdots, g_p, u_1, \cdots, u_h$, and $t_1, \cdots, t_c$ satisfy conditions C1-C3 in Lemma Appendix B.3 to facilitate the final proof in Step 5. Condition C2 is naturally satisfied with the order of these bull put spreads. Because $U_{h+1} = \emptyset$, from the definition of U_{h+1} in (B.6), we have

$$\text{(C1)}:\ g_{h+1} + d - 1 > t_c \geq \dots \geq t_1. \tag{B.7}$$

In addition, for all $r = 1, \dots, c$, if $t_r \geq g_s + d - 1$ holds for any $s \in \{1, \dots, h\}$, the definition of U_s in (B.6) reveals that $t_r \in U_s$. Because B_{t_r} remains unpaired after h iterations, i.e., $k_r^{s-1} \geq k_r^h > 0$, the definition of u_s in (B.6) claims that u_s is the smallest element in U_s, i.e., $t_r \geq u_s$. That is to say,

$$\text{(C3)}:\ \ \text{Either } t_r < g_s + d - 1 \text{ or } t_r \geq u_s \text{ holds for all } \ r = 1, \dots, c,\ \text{ and } s = 1, \dots, h. \tag{B.8}$$

Case 2. Otherwise, there does not exist any bear call spread which can offset the bull put spread B_{g_1}, i.e., $U_1 = \emptyset$. It can be considered as a special case of Case 1 with $h = 0$. Go to Step 4.

Step 4. Combine the bull and bear call spreads. From the bear call spread with the largest strike price to that with the smallest one, combine the bear call spread with the bull call spread whose first-leg strike price is the largest one among all the unpaired bull call spreads which can offset this bear call spread. In particular, order c basic bear spreads, in terms of t_i, as $B_{t_1}, \dots, B_{t_c}$, with $2d-1 \leq t_1 \leq \dots \leq t_c \leq 3(d-1)$.

Case 1. If there exists a bull call spread which can offset the basic bear spread B_{t_c}, i.e.,

$$J_c = \{j \in \{1, \dots, d-1\} \mid k_j^0 > 0,\ j \leq t_c - 2(d-1)\} \neq \emptyset, \tag{B.9}$$

denote the largest element of J_c as j_c. We can combine B_{t_c} and B_{j_c} as an offset $\hat{b}_c$ and have

$$B_{t_1} + \dots + B_{t_{c-1}} + B_{t_c} + \sum_{i=1}^{d-1} k_i^0 B_i = B_{t_1} + \dots + B_{t_{c-1}} + \sum_{i=1}^{d-1} k_i^1 B_i + \hat{b}_c,$$

where $k_i^1 = k_i^0,\ \forall i \in \{1, \dots, d-1\}/\{j_c\},\ k_{j_c}^1 = k_{j_c}^0 - 1$. The strategy $\hat{b}_c = B_{t_c} + B_{j_c}$ is either a zero vector when $j_c = t_c - 2(d-1)$ or a Form 2' option strategy defined in Lemma Appendix B.1 otherwise. Similarly, we define the following backward recursion of the sets J_l, the indices j_l and the coefficients k_i^{c-l+1}, $i = 1, ..., d-1$, in l from $c-1$ to 1:

$$\begin{cases} J_l = \{j \in \{1, \dots, d-1\} \mid k_j^{c-l} > 0,\ j \leq t_l - 2(d-1)\}, \\ j_l = \max\{j |\ j \in \{0\} \cup J_l\}, \\ k_i^{c-l+1} = k_i^{c-l},\ \forall i \in \{1, \dots, d-1\}/\{j_l\}, \\ k_{j_l}^{c-l+1} = k_{j_l}^{c-l} - 1,\ \hat{b}_l := B_{t_l} + B_{j_l},\ \text{if } j_l > 0. \end{cases} \tag{B.10}$$

If the iteration can be carried out for c times, i.e., all the basic bear call spreads, $B_{t_1},\dots,B_{t_c}$, are offset by bull spreads, define J_0 as empty set. Otherwise, the recursion must stop at some $n\in\{1,2,\dots,c\}$ with $J_n=\emptyset$ and $J_{n+1}\neq\emptyset$. To conclude, there must exist some $n\in\{0,\dots,c\}$, such that $J_n=\emptyset$, $J_i\neq\emptyset$, for all $i\in\mathbb{Z}\cap[n+1,c]$. The strategy v is expressed as

$$v=\tilde{b}_1+\dots+\tilde{b}_h+B_{g_{h+1}}+\dots+B_{g_p}+B_{t_1}+\dots+B_{t_n}+\hat{b}_{n+1}+\dots+\hat{b}_c+\sum_{i=1}^{d-1}k_i^{c-n}B_i+\sum_{i=1}^{\bar{n}}\bar{b}_i.$$

Denote a set of indices of basic bull call spreads as $I_n:=\{i\in\{1,2,\cdots,d-1\}\mid k_i^{c-n}>0\}$. Similar to Step 3, we can prove that these indices satisfy conditions C4-C5 in Lemma Appendix B.3 with the definitions of J_r and j_r, $r=n+1,\dots,c$, in (B.10). Go to Step 5.

Case 2. Otherwise, there does not exist any bull call spread which can completely offset the basic bear spread B_{t_c}, i.e., $J_c=\emptyset$. This case can be considered as a special case of Case 1 with $n=c$. Go to Step 5.

Step 5. Apply the box spreads to offset combinations or base offsets with positive MPLs. After the above four steps, we can express v as

$$v=\tilde{b}_1+\dots+\tilde{b}_h+B_{g_{h+1}}+\dots+B_{g_p}+B_{t_1}+\dots+B_{t_n}+\hat{b}_{n+1}+\dots+\hat{b}_c+\sum_{i=1}^{d-1}k_i^{c-n}B_i+\sum_{i=1}^{\bar{n}}\bar{b}_i, \tag{B.11}$$

where the involved strategies are i) Form 6' option strategies, $\tilde{b}_1,\dots,\tilde{b}_h$, ii) basic bull put spreads, $B_{g_{h+1}},\dots,B_{g_p}$, iii) basic bear call spreads, $B_{t_1},\dots,B_{t_n}$, iv) Form 2' option strategies, $\hat{b}_{n+1},\dots,\hat{b}_c$, v) basic bull call spreads, B_i, $i\in\{i=1,\dots,d-1: k_i^{c-n}>0\}$ and vi) box spreads, $\bar{b}_1,\dots,\bar{b}_{\bar{n}}$. Note that one or several of the six kinds may not be involved in v. For example, there is no Form 6' option strategy if $h=0$. And strictly speaking, the strategy $\hat{b}_l$, $l=n+1,\dots,c$, is either a zero vector when $j_l=t_l-2(d-1)$ or a Form 2' option strategy otherwise. We can ignore the zero vectors. The MPLs of the first three kinds of strategies are ones, while those of the last three kinds of strategies are zeros.

If $\bar{n}\geq p+n$, combine the box spreads with Form 6' option strategies, basic bull put and bear call spreads. Because the payoffs of box spreads are always one, all these combinations have non-negative payoff functions. As a result, the strategy v is dominated by these strategies with non-negative payoff functions. According to Lemma 3.1 and Lemma Appendix B.2, v is dominated by base offsets.

Otherwise, to simplify the notation, we denote

$$\bar{v}_1=\tilde{b}_1,\dots,\bar{v}_h=\tilde{b}_h,\ \bar{v}_{h+1}=B_{g_{h+1}},\dots,\bar{v}_p=B_{g_p},\ \bar{v}_{p+1}=B_{t_1},\dots,\bar{v}_{p+n}=B_{t_n}.$$

we verify that i) Form 6' option strategies, $\tilde{b}_1,\dots,\tilde{b}_h$, ii) basic bull put spreads, $B_{g_{h+1}},\dots,B_{g_p}$, iii) basic bear call spreads, $B_{t_1},\dots,B_{t_n}$, iv) Form 2' option strategies, $\hat{b}_{n+1},\dots,\hat{b}_c$ and v) basic bull call spreads, B_i, $i\in I:=\{i=1,\dots,d-1:k_i^{c-n}>0\}$ all satisfy conditions C1 - C5 in (B.1) of Lemma Appendix B.3. According to Lemma Appendix B.3 and the properties of box spreads proved in Lemma Appendix B.2, all the payoff functions of option strategies,

$$\bar{b}_1+\bar{v}_1,\dots,\bar{b}_{\bar{n}}+\bar{v}_{\bar{n}},\ \bar{v}_{\bar{n}+1},\dots,\ \hat{b}_{n+1},\dots,\ \hat{b}_c,\ B_i,\ i\in I,$$

achieve their non-positive minimal values at one same point. From Lemma 3.1 and the transitivity property, we know that v in (B.11) is dominated by these strategies and thus by base offsets.□

Appendix C. Proofs of Properties of Margin Calculation Model

Proof of Proposition 3.3

To prove this proposition, we need to investigate the special structure of the matrix $\bar{A}$. In particular, we document the structure of $\bar{A}$ and a supplementary matrix D in Algorithm 1.

We then construct a null space matrix of $\bar{A}$ as

$$N:=\left(\begin{array}{c|c} I_{2(d-1)} & D \\ I_{2(d-1)} & \\ \hline \mathbf{0} & I_{n_1}\end{array}\right),\ \text{with } n_1=\bar{n}-4(d-1). \tag{C.1}$$

We claim two properties of this matrix N in the following two lemmas, whose proofs are deferred immediately after the proof of Proposition 3.3.

Lemma C.1. *The matrix N defined in (C.1) is a null space matrix of $\bar{A}$.*

Lemma C.2. *The matrix N defined in (C.1) is totally unimodular.*

With these two lemmas, we can prove Proposition 3.3 as follows. Because N is a null space matrix of $\bar{A}$, as stated in Lemma Appendix C.1, and q is a special solution to $\bar{A}a=\bar{A}q$,

$$\bar{A}a=\bar{A}q,\ a\in\mathbb{Z}_+^{\bar{n}}\ \Leftrightarrow\ a=q-Ny\geq\mathbf{0},\ Ny\in\mathbb{Z}^{\bar{n}}. \tag{C.2}$$

Moreover, the transpose of matrix N, defined in (C.1),

$$N^T = \left(\begin{array}{c|c|c} I_{2(d-1)} & I_{2(d-1)} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{D^T} & I_{n_1} \end{array}\right),$$

is of full row rank. It is easy to verify that there exists a unimodular matrix U such that $(I_{n_1+2(d-1)} \quad \mathbf{0}) = UN^T$. That is to say, the Hermite Normal Form of N^T is $(I, 0)$. Denote the lattice generated by N as $L(N) = \{z \in \mathbb{R}^{\bar{n}} \; |z = Ny, \; y \in \mathbb{Z}^{\bar{n}-2(d-1)}\}$. According to Observation 1 in Aardal and Wolsey (2010), $L(N)$ is a pure lattice, i.e.,

$$L(N) = \{z \in \mathbb{R}^{\bar{n}} \; |z = Ny, \; y \in \mathbb{R}^{\bar{n}-2(d-1)}\} \cap \mathbb{Z}^{\bar{n}}. \tag{C.3}$$

Therefore,

$$\begin{aligned} &\min\{\bar{m}^T a : \bar{A}a = \bar{A}q, \; a \in \mathbb{Z}_+^{\bar{n}}\} = \bar{m}^T q - \max\{\bar{m}^T Ny : Ny \le q, \; Ny \in \mathbb{Z}^{\bar{n}}\} \\ &= \bar{m}^T q - \max\{\bar{m}^T Ny : Ny \le q, \; y \in \mathbb{Z}^{\bar{n}-2(d-1)}\}, \end{aligned}$$

where the first and second equalities come from (C.2) and (C.3), respectively. Because N is totally unimodular, as shown in Lemma Appendix C.2, we have the equivalence of the above model and its continuous relaxation,

$$\max\{\bar{m}^T Ny : Ny \le q, \; y \in \mathbb{Z}^{\bar{n}-2(d-1)}\} = \max\{\bar{m}^T Ny : Ny \le q, \; y \in \mathbb{R}^{\bar{n}-2(d-1)}\}.$$

As a result,

$$\begin{aligned} &\min\{\bar{m}^T a : \bar{A}a = \bar{A}q, \; a \in \mathbb{Z}_+^{\bar{n}}\} = \bar{m}^T q - \; \max\{\bar{m}^T Ny : Ny \le q, \; y \in \mathbb{Z}^{\bar{n}-2(d-1)}\} \\ &= \bar{m}^T q - \; \max\{\bar{m}^T Ny : Ny \le q, \; y \in \mathbb{R}^{\bar{n}-2(d-1)}\} \\ &= \min\{\bar{m}^T a : \bar{A}a = \bar{A}q, \; a \in \mathbb{R}_+^{\bar{n}}\}. \qquad \square \end{aligned}$$

Proof of Lemma Appendix C1.. It suffices to prove that

i) $\mathrm{rank}(N) = n - \mathrm{rank}(\bar{A})$ and ii) $M := \bar{A} \cdot N = \mathbf{0}_{4(d-1)\times(\bar{n}-2(d-1))}$.

Recall that the first $2(d-1)$ columns of the matrix $\bar{A}$ generated in Algorithm 1 are basic bull spreads $B_1, \cdots, B_{2(d-1)}$, and

Algorithm 1 Generate matrices $\bar{A}$ and D.

Set $D = \mathbf{0}_{4(d-1)\times n_1}$, $n_1 = 3d-5$;
For $i = 1 : 4(d-1)$ **do**
Set the i-th column of matrix $\bar{A}$ as $\bar{A}_i = B_i$;
endfor
For $i = 1 : d-2$ **do**
$\bar{A}_{i+4(d-1)} = B_{i+2d-1} + B_i$; $D_{i+1,i} = 1$; $D_{i+2(d-1),i} = 1$;
$\bar{A}_{i+5d-6} = B_{i+d-1} + B_{i+3d-2}$; $D_{i+3(d-1),i+d-2} = 1$; $D_{i+d,i+d-2} = 1$;
endfor
For $i = 1 : d-1$ **do**
$\bar{A}_{i+6d-8} = B_{i+2(d-1)} + B_{i+d-1}$; $D_{i,i+2(d-2)} = 1$; $D_{i+3(d-1),i+2(d-2)} = 1$;
endfor

that all the columns of matrix $\bar{A}$ are base offsets, and thus belong to the space $V = \{v \in \mathbb{Z}^{2d} | v_1 + \cdots + v_d = v_{d+1} + \cdots + v_{2d} = 0\}$. Because $B_1, \cdots, B_{2(d-1)}$ defined in (4) and (5) are linearly independent and form a basis for V, $\mathrm{rank}(\bar{A}) = 2(d-1)$. Meanwhile, according to the definition of matrix N in (C.1), we can derive that

$$\mathrm{rank}\ (N) = \mathrm{rank} \left(\begin{array}{c|c} \mathbf{0}_{2(d-1)} & \mathbf{0}_{4(d-1)\times n_1} \\ I_{2(d-1)} & \\ \hline 0 & I_{n_1} \end{array}\right) = \bar{n} - 2(d-1).$$

Thus, $\mathrm{rank}\ (N) = \bar{n} - \ \mathrm{rank}\ (\bar{A})$.

We next prove that for any $l = 1, \cdots, \bar{n} - 2(d-1)$, the l-th column of M, $M_l = \bar{A} \cdot N_l$, is a zero vector. From Algorithm 1, we know that for all $i = 1, \cdots, 2(d-1)$,

$$B_i + B_{i+2(d-1)} = \mathbf{0} \;\Rightarrow\; \bar{A}_i + \bar{A}_{i+2(d-1)} = \mathbf{0} \;\Rightarrow\; M_i = \bar{A} \cdot (e_i + e_{i+2(d-1)}) = \mathbf{0}.$$

Moreover, for all $l = 4d-3, \; 4d-2, \cdots, \bar{n}$, according to Algorithm 1, there exist indices $i \in \{1, ..., 2(d-1)\}$ and $j \in \{2d-1, ..., 4(d-1)\}$ such that

$$\bar{A}_l = B_i + B_j, \;\; D_{i+2(d-1), \; l} = D_{j-2(d-1), \; l} = 1.$$

Therefore, $M_l = B_{i+2(d-1)} + B_{j-2(d-1)} + \bar{A}_l = \mathbf{0}$, for all $l = 4d-3, \; 4d-2, \cdots, \bar{n}$. □ □

Proof of Lemma Appendix C2.. Note that matrix N in (C.1) is a matrix with entries 0 and 1. From Theorem 19.3 in Schrijver (1998), N is totally unimodular if and only if:

Each collection of columns can be split into two parts, M_1 and M_2, such that $\sum\limits_{i\in M_1} N_{.i} - \sum\limits_{j\in M_2} N_{.j}$ is a vector with only 0 and ±1.

Because the total unimodularity is preserved when taking the transpose, it suffices to prove the existence of such a split for each collection of rows of N. Denote M as a collection of arbitrary rows of N. We can split it into M_1 and M_2 as follows:

Step 1: Place $M\cap\{1,2,\cdots,2(d-1)\}$ into M_1 and $M\cap\{2d-1,2d,\cdots,4(d-1)\}$ into M_2;

Step 2: For any row $l\in\{4d-3,\cdots,\bar{n}\}$, there exist indices $i=1,\cdots,2(d-1)$ and $j=2d-1,\cdots,4(d-1)$ such that

$$\bar{A}_l = B_i + B_j,\ \ D_{i+2(d-1),\ l} = D_{j-2(d-1),\ l} = 1.$$

If $i\in M_1, j\notin M_2$, put l into M_2; if $i\notin M_1, j\in M_2$, put l into M_1. Otherwise, you can arbitrarily put l into M_1 or M_2.

After these two steps, $\sum\limits_{i\in M_1} N_{i.} - \sum\limits_{j\in M_2} N_{j.}$ is a vector with only 0 and ±1. □ □

Proof of Proposition 3.4

To prove this proposition, we introduce the following lemma first. The proof of Lemma Appendix C.3 is deferred immediately after the proof of Proposition 3.4.

Lemma C.3. *If option strategy $v=(v_1,...,v_d,v_{d+1},...,v_{2d})^T$ satisfies one of the three conditions in Proposition 3.4, then v is dominated by the set of the base offsets with up to four legs.*

With this lemma, we can prove Proposition 3.4 as follows. Denote V_P as the optimal value of model (NM_1). Firstly, we prove that $V_P\geq m(v)$. Denote the optimal solution to model (NM_1) as a^*. We know that $\bar{A}a^*=v$ and that $V_P=\sum\limits_{i=1}^{\bar{n}} a_i^* m(\bar{A}_i)$. If $m(v)=0$, it is obvious that $V_P\geq m(v)=0$. Otherwise,

$$m(v)>0\Rightarrow m(v)=(-\min_{S_T} f_v(S_T))^+ = -\min_{S_T} f_v(S_T) = -\min_{S_T}\sum_{i=1}^{\bar{n}} a_i^* f_{\bar{A}_i}(S_T).$$

The last equality in the above expression owes to the fact that $\bar{A}a^*=v$. Note that $\bar{A}_i$ is a base offset whose payoff function has a non-positive minimal value, for any $i=1,\cdots,\bar{n}$ and that a^* is a non-negative integer vector. As a result,

$$\begin{aligned} m(v) &= -\min_{S_T}\sum_{i=1}^{\bar{n}} a_i^* f_{\bar{A}_i}(S_T) \\ &\leq \sum_{i=1}^{\bar{n}} a_i^*(-\min_{S_T} f_{\bar{A}_i}(S_T)) = \sum_{i=1}^{\bar{n}} a_i^*(-\min_{S_T} f_{\bar{A}_i}(S_T))^+ = \sum_{i=1}^{\bar{n}} a_i m(\bar{A}_i) = V_P. \end{aligned}$$

On the other hand, according to Lemma Appendix C.3, the fact that the strategy v satisfies one of the three conditions in this proposition results in that v is dominated by base offsets with up to four legs. Therefore, according to the definition of dominance, there exists a feasible solution to model (NM_1), $a=(a_1,\cdots,a_{\bar{n}})^T\in\mathbb{Z}_+^{\bar{n}}$, such that

$$v=\sum_{i=1}^{\bar{n}} a_i\bar{A}_i = \bar{A}a,\ \ m(v)\geq\sum_{i=1}^{\bar{n}} a_i m(\bar{A}_i) = \bar{m}^T a \geq V_P.$$

To conclude, $V_P=m(v)$ holds for v satisfying one of the three conditions in this proposition. □

Proof of Lemma Appendix C.3

The proof of Lemma Appendix C.3 follows a similar idea as the proof of Theorem 3.1: factorizing the strategy v into a no-hedging combination of base offsets and Form 2'-7' option strategies with up to four legs. We thus introduce a useful lemma similarly to Lemma Appendix B.3, before giving the proof of Lemma Appendix C.3.

Lemma C.4. *For all the following three kinds of option strategies:*

i) Basic bear put spreads, B_i, $i\in I\subseteq\{3d-2,3d-1,\ldots,4(d-1)\}$;

ii) Form 4' option strategies, $\hat{b}_1'=B_{g_1}+B_{w_1},\ldots,\hat{b}_h'=B_{g_h}+B_{w_h}$, with B_{g_i} and B_{w_i}, $i=1,\ldots,h$, being basic bull and bear put spreads, respectively;

iii) Basic bull put spreads, $B_{g_{h+1}},\ldots,B_{g_p}$,

if their indices satisfy the following three conditions,

$$\begin{cases} g_1\leq\ldots\leq g_h\leq g_{h+1}\leq\ldots\leq g_p, \\ i<g_{h+1}+2(d-1),\ \forall i\in I, \\ \text{ either } i<g_l+2(d-1) \text{ or } i\geq w_l \text{ holds for all } i\in I \text{ and } l=1,\ldots,h, \end{cases} \tag{C.4}$$

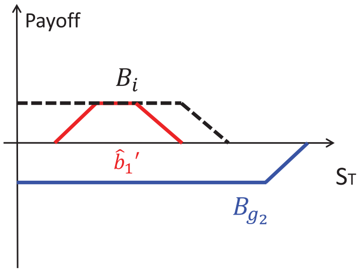

Fig. C1. Illustration of a Common Worst-case Scenario in Lemma Appendix C.4. We plot the payoff functions of three option strategies in this figure. The black and blue curves are the payoff functions of basic bear and bull put spreads, respectively, while the red one stands for a Form 4' option strategy.

then all the payoff functions of these strategies achieve their non-negative minimal values at a common point.

Proof. We first illustrate the basic idea of this proof in Figure C.18. To present a clear picture, we display in Figure C.18 a simple case where only a single strategy is involved for each kind of the three option strategies involved in this lemma. More specifically, the set I contains one single element i and indices are $h = 1$ and $p = 2$. Figure C.18 shows that the payoff functions of these three strategies whose indices satisfy Conditions (C.4) achieve their non-negative minimal values at $1 + K_{i-3(d-1)}$.

Denote $\hat{i}$ as the largest element of I, which is the set of indices of basic bear put spreads stated in this lemma. Similar to the proof of Lemma Appendix B.3, We can prove that the payoff functions of all these three kinds of strategies achieve their minimums at $S_T^0 = 1 + K_{\hat{i}-3(d-1)}$ by looking into the payoff functions of these strategies and the conditions (C.4). □ □

Proof of Lemma Appendix C.3

Denote the set of the indices corresponding to the nonzero components of call options involved in the strategy v as $I_1 = \{i = 1, \ldots, d,\ s.t.\ v_i \neq 0\}$, and the set of indices corresponding to nonzero components of put options in the strategy v as $I_2 = \{i = d+1, \ldots, 2d,\ s.t.\ v_i \neq 0\}$. If $I_1 = I_2 = \emptyset$, v is a zero vector and the case is trivial. We consider the cases where at least one of I_1 and I_2 is non-empty as follows.

Case 1. If condition i) is satisfied, i.e. $v_1 + \cdots + v_d = 0, v_{d+1} = \cdots = v_{2d} = 0$, then $I_1 \neq \emptyset$ and $I_2 = \emptyset$. There are only calls but not any put involved in the strategy v. According to the proof of Theorem 3.1, we can order the involved basic bear call spreads, combine bear and bull call spreads as Form 2' option strategies with (B.10) and prove that v is dominated by a set of basic bull and bear call spreads and Form 2' option strategies and thus is dominated by base offsets with up to four legs.

Case 2. If condition ii) is satisfied, $I_1 = \emptyset$ and $I_2 \neq \emptyset$. We can express v as

$$v = \Big(\sum_{i=d}^{2(d-1)} + \sum_{i=3d-2}^{4(d-1)}\Big)\ k_i^0 B_i,\ \ k_i^0 \in \mathbb{Z}_+, \forall i = 1, \ldots, 4(d-1). \tag{C.5}$$

There might exist some offsets among these basic bull and bear put spreads. Similarly to Case 1, we order all these bull put spreads in terms of the strike prices as $B_{g_1}, \ldots, B_{g_p},\ d \le g_1 \le g_2 \le \ldots \le g_p \le 2(d-1)$, and combine the bull and bear put spreads. In particular, for $l = 1, \ldots, p$, we recursively define that

$$\begin{cases} W_l = \{j = 3d-2, \ldots, 4(d-1),\ s.t.\ k_j^{l-1} > 0,\ j \ge g_l + 2(d-1)\},\ W(p+1) = \emptyset, \\ w_l = \min\{j | j \in W_l \cup \{4d-3\}\}, \\ k_{w_l}^l = k_{w_l}^{l-1} - 1,\ k_i^l = k_i^{l-1},\ \forall i \in \{3d-2, \ldots, 4(d-1)\}/\{w_l\},\ \hat{b}_l' = B_{g_l} + B_{w_l}. \end{cases} \tag{C.6}$$

The set W_l, $l = 1, \ldots, h$, is the set of remaining basic bear put spreads that can offset the l-th basic bull put spread B_{g_l}. The basic idea of this recursion is similar to the recursion (B.10) in the proof of Theorem 3.1: from the bull puts with the smallest strike price to puts with the highest one, we combine the bull puts, B_{g_l}, with the basic bear puts, B_{w_l}, which have the smallest strike prices among all the unpaired basic bear puts which can offset the bull puts. If $W_1 \neq \emptyset$, there exists $h = 1, \ldots, p$, such that $W_{h+1} = \emptyset$ and $W_1, \ldots, W_h \neq \emptyset$, i.e. there does not exist any basic bear put spreads that can offset the put $B_{g_{h+1}}$. The iteration ends at h. Otherwise, we define $h = 0$.

As $\hat{b}_i' = B_{g_i} + B_{w_i}$ is either a zero vector when $w_l = g_l + 2(d-1)$ or a Form 4' option strategy defined in Lemma Appendix B.1 otherwise, we can express v in (C.5) as a linear combination of Form 4' option strategies, basic bull and bear put spreads:

$$v = \hat{b}_1' + \ldots + \hat{b}_h' + B_{g_{h+1}} + \ldots + B_{g_p} + \sum_{i=3d-2}^{4(d-1)} k_i^h B_i.$$

Notice that there does not exist any $\hat{b}_i'$ in the above representation if $h = 0$. Similar to the proof in Theorem 3.1, one can verify that the indices of these three strategies satisfy the three conditions in Lemma Appendix C.4 with the recursion form

(C.6). Therefore, according to Lemma Appendix C.4, all of the payoff functions of these basic bear and bull put spreads and Form 4' option strategies achieve their non-negative minimal values at a common point. According to Lemma 3.1 and Lemma Appendix B.1, v is dominated by them. In addition, basic bear and bull put spreads are base offsets with two legs. Form 4' option strategies are generalized Form 4 base offsets and dominated by Form 4 base offsets, which have three legs. According to the transitive property of dominance, the strategy v is dominated by base offsets with up to four legs.

Case 3. As for condition iii), a similar proof can be applied for the case where only one of I_1 and I_2 is non-empty. If both I_1 and I_2 are non-empty, i.e., the strategy v contains both calls and puts, because strategy v is balanced, i.e., $v_1 + \ldots + v_d = v_{d+1} + \ldots + v_{2d} = 0$, both sets I_1 and I_2 have at least two non-zero elements, respectively. That is to say, $leg(v) \geq 4$. Furthermore, condition iii) of Proposition 3.4 states that $leg(v) \leq 4$. Therefore, we have $leg(v) = 4$. Without loss of generality, we denote

$$I_1 = \{j_1, j_2\},\ \ I_2 = \{j_3, j_4\}, \quad \text{with } v_{j_1} + v_{j_2} = v_{j_3} + v_{j_4} = 0 \text{ and } v_{j_i} \neq 0, i = 1, \cdots, 4.$$

That is to say, $v = v_{j_1}(e_{j_1} - e_{j_2}) + v_{j_3}(e_{j_3} - e_{j_4})$, where $e_i \in \mathbb{Z}^{2d}$ is a vector with i-th element being one and the other elements being zero. Denote $b^1 = e_{j_1} - e_{j_2}$ and $b^2 = e_{j_3} - e_{j_4}$. Then we know that

$$v = v_{j_1} b^1 + v_{j_3} b^2.$$

According to the payoff functions of basic spreads in Figure 2, the payoff function of the basic bull call spread b^1 achieves its minimum, zero, at any $S_T \in [0, K_{j_2}]$, and b^2 is a basic bull put spread whose payoff function achieves its minimum, -1, at any $S_T \in [0, K_{j_4}]$.

Case 3.1. If $v_{j_1} > 0$ and $v_{j_3} > 0$, applying Lemma 3.1 concludes that v is dominated by b^1 and b^2, since both payoff functions of b^1 and b^2 achieve their minimums at the same point $S_T = 0$. It is easy to verify that basic spreads, b^1 and b^2, are dominated by basic spreads of width one. Thus, v is dominated by base offsets with up to two legs.

Case 3.2. If $v_{j_1} < 0$ and $v_{j_3} < 0$, we can rewrite v as $v = (-v_{j_1})(-b^1) + (-v_{j_3})(-b^2)$. Both the basic bear call spread $-b^1$ and the bear put spread $-b^2$ achieve their minimums at the same point $S_T = 1 + \max\{K_{j_2}, K_{j_4}\}$. The strategy v is dominated by basic spreads and thus base offsets with up to two legs.

Case 3.3. If $v_{j_1} > 0$ and $v_{j_3} < 0$, we can rewrite v as $v = v_{j_1} b^1 + (-v_{j_3})(-b^2)$. Then $-b^2$ is a basic bear put spread whose payoff function is non-negative. Thus, v is dominated by b^1 and $-b^2$ and thus dominated by base offsets with up to two legs.

Case 3.4. If $v_{j_1} < 0$ and $v_{j_3} > 0$, we can rewrite v as $v = \sum_{i=d}^{3(d-1)} k_i B_i$, since there is no basic bull call or bear put spreads involved. We thus do not have any box spreads in such a case. Similar to the proof in Step 3 of Theorem 3.1, v is dominated by the set of basic bear call spreads, bull put spreads and Form 6' option strategies. In addition, we can verify from the definition of Form 6' option strategies in Lemma Appendix B.1 that Form 6' option strategies are dominated by the set of Form 4 and Form 6 base offsets. According to the transitivity in Proposition 3.2, v is dominated by the set of basic bear call spreads, bull put spreads, Form 4 and Form 6 base offsets, which are base offsets with up to four legs. □

Proof of Proposition 3.5

The terminal payoffs of a naked long call option at strike K_i and a balanced portfolio $v = e_{i+\underline{d}} - e_{d+\bar{d}+\underline{d}}$ are

$$f_{lc}(S_T) = \begin{cases} 0, & \text{if } S_T < K_i, \\ S_T - K_i, & \text{if } S_T \in [K_i, \infty], \end{cases} \quad f_v(S_T) = \begin{cases} 0, & \text{if } S_T < K_i, \\ S_T - K_i, & \text{if } S_T \in [K_i, K_u], \\ K_u - K_i, & \text{if } S_T > K_u. \end{cases}$$

As a result, when we consider the bounds of future underlying price $S_T \in [K_l, K_u]$, both of the payoffs are the same. Similar analysis can be applied to a naked long/short call/put option.

Proof of Proposition 3.6

Similar to the proof of Proposition 3.5, with the price bound $S_T \in [K_l, K_u]$, the terminal payoffs of a long position of one unit of the underlying asset, up to the option multiplier, is the same as

$$f_s(S_T) = \begin{cases} K_l, & \text{if } S_T < K_l, \\ S_T, & \text{if } S_T \in [K_l, K_u], \\ K_u, & \text{if } S_T > K_u. \end{cases}$$

Meanwhile, the balanced portfolio $e_1 - e_{d+\bar{d}+\underline{d}}$ has the terminal payoff

$$f_v(S_T) = \begin{cases} 0, & \text{if } S_T < K_l, \\ S_T - K_l, & \text{if } S_T \in [K_l, K_u], \\ K_u - K_l, & \text{if } S_T > K_u. \end{cases}$$

As a result, we have $f_s(S_T) = K_l + f_v(S_T)$ and conclude part 1). The proof of part 2) is similar.

Proof of Proposition 3.7

According to Theorem 3.1, an arbitrary balanced portfolio ν can be expressed into a no-hedging combination of base offsets A_i, $i \in I$. That is to say, there exists one point $S_T^0 \in [0, \infty)$ such that the payoff of all these base offsets A_i, $i \in I$, all achieve their non-positive minimal values at S_T^0. Next, we will prove that there exists one point $\bar{S}_T^0 \in [K_l, K_u]$ such that all the non-positive minimal values are achieved at $\bar{S}_T^0$. As a result, according to the definition of the no-hedging combination in Definition 3.3, with such a price bound, theorem 3.1 still holds.

If there does not exist such a point within the interval $[K_l, K_u]$, there must exist one base offset A_k, $k \in I$, whose minimal values all lie outside the interval. However, as shown in Figure 2, all the base offsets achieve their non-positive minimal values at one of the strike prices K_i, $i = 1, \cdots, d$. That is to say, there exists at least one of the strike prices K_i, $i = 1, \cdots, d$ lying outside the interval $[K_l, K_u]$, which contradicts the definition of virtual strikes (10) in the main body of our paper that $K_l < K_1 < \ldots < K_d < K_u$.

Appendix D. Comparison with the Model by Matsypura and Timkovsky (2013)

The salient properties of our models are mainly ascribed to the introduction of base offsets. In this section, we furthermore compare our netting model of size four with the state-of-the-art strategy-based model of Matsypura and Timkovsky (2013) using their proposed main spreads to demonstrate the significant advantages of our formulation.

We replace the columns of matrix $\bar{A}$ in our model (NM_1), which is a collection of base offsets with up to four legs, by matrix O, whose columns, $O_1, \cdots, O_{n_0}$, are a collection of main spreads defined in Matsypura and Timkovsky (2013). Essentially, the main spreads are bull/bear call/put spreads, long/short box spreads, call/put condor and butterfly strategies of different widths. Denote the margin requirement $m_i = m(O_i)$, $i = 1, \ldots, n_0$, we present the model in Matsypura and Timkovsky (2013) as

$$\min_a \{m^T a : \ Oa = Oq, \ a \in \mathbb{Z}_+^n\}. \tag{P_1}$$

We prove in the following that, in comparison with our model (P_1), the original model by Matsypura and Timkovsky (2013),

$$\begin{aligned} (P_0) \qquad & \min \ m^T a \\ & \text{s.t. } Ox_i = a_i O_i, \ i = 1, \ldots, n_0, \\ & \qquad \sum_{i=1}^{n_0} x_i = q, \ x_1, \ldots, x_{n_0}, \ a \in \mathbb{Z}_+^{n_0}. \end{aligned}$$

tends to overestimate the market risk. More precisely, our model (P_1) dominates model (P_0).

Proposition D.1. *The optimal value of model (P_0), V_{P_0}, is no less than that of model (P_1), V_{P_1}, i.e., $V_{P_1} \le V_{P_0}$.*

Proof. Model (P_1) is a surrogate relaxation of the model (P_0), as evidenced from the following.

Let $\mu = (I_{2d}, \cdots, I_{2d}, -O) \in \mathbb{Z}^{(2dn+n)\times 2d}$. We have

$$\mu \cdot \begin{pmatrix} Ox^1 - a_1 O_1 \\ \vdots \\ Ox^n - a_n O_n \\ \sum_{i=1}^{n} x^i - q \end{pmatrix} = 0 \Leftrightarrow Oa - Oq = 0.$$

Therefore, $V_{P_1} \le V_{P_0}$ because of the property of a surrogate relaxation. □

The financial implication of this finding is that model (P_0) by Matsypura and Timkovsky (2013) only considers the margin reduction of centipedes, which is defined as a portfolio of main spreads $\{x_i : \ \exists \ a_i \in Z_+ \ s.t. \ Ox_i = a_i O_i\}$ (Definition 4 in Matsypura and Timkovsky (2013)), while our model (P_1) carries out the margin reduction by exploring all possible combinations. There would exist some main spread portfolio which cannot be replicated by centipedes. Therefore, it is not surprised to find that model (P_1) provides a margin requirement no higher than (P_0). More precisely, because all of the main spreads are balanced strategies with up to four legs, according to Proposition 3.4, they are all dominated by the base offsets with up to four legs[16]. We can thus conclude Proposition Appendix D.2 that model (P_1) is dominated by model (P).

Proposition D.2. *The optimal value of model (P_1) is no less than that of model (P).*

Therefore, model (NM_1) yields a margin requirement no higher than (P_1) and thus also no higher than (P_0) by Matsypura and Timkovsky (2013). Recall that our netting model (NM_1) is tractable, while the integer programming model

[16] In fact, one can prove that the main spreads are equivalent to our base offsets with up to four legs in terms of the margin calculation, i.e., these two sets dominate each other.

(P_0) has $n_0^2 + n_0$ integer variables and $2dn_0 + n_0$ constraints. As a result, our netting model (NM_1) provides a margin requirement no higher than model (P_0) by Matsypura and Timkovsky (2013), and, at the same time, achieves a significant shorter computational time with the dominance increasing in d. We demonstrate this difference in the following example.

Example D.1. We randomly generate 1000 main spread portfolios[17], $v = (v_1, \cdots, v_{2d})'$, with v_i uniformly distributed in [0,9] with strike prices $\{45, 50, 55, 60, 65\}$. Although the margins calculated by model (P) and model (P_0) are the same to each other for these 1000 portfolios, the average computational times using ILOG CPLEX 12.6 are significantly different, 0.0022 seconds for our model (P) and 26.5178 seconds for model (P_0).

The difference between the computational time of model (P_0) and that of model (P) increases in the number of strike prices, d. Our model (P) is equivalent to a linear programming and thus has a polynomial computational time, while the integer programming model (P_0) by Matsypura and Timkovsky (2013) may take more than one hour for the cases when $d \geq 7$.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.jedc.2022.104572

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[17] We conduct the experiment on main spread portfolios, which is a subset of the set of balanced strategies, to apply the model by Matsypura and Timkovsky (2013).