



Distributional utility preference robust optimization models in multi-attribute decision making

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Abstract

Utility preference robust optimization (PRO) has recently been proposed to deal with optimal decision-making problems where the decision maker's (DM's) preference over gains and losses is ambiguous. In this paper, we take a step further to investigate the case that the DM's preference is random. We propose to use a random utility function to describe the DM's preference and develop distributional utility preference robust optimization (DUPRO) models when the distribution of the random utility function is ambiguous. We concentrate on data-driven problems where samples of the random parameters are obtainable but the sample size may be relatively small. In the case when the random utility functions are of piecewise linear structure, we propose a bootstrap method to construct the ambiguity set and demonstrate how the resulting DUPRO can be solved by a mixed-integer linear program. The piecewise linear structure is versatile in its ability to incorporate classical non-parametric utility assessment methods into the sample generation of a random utility function. Next, we expand the proposed DUPRO models and computational schemes to address general cases where the random utility

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functions are not necessarily piecewise linear. We show how the DUPRO models with piecewise linear random utility functions can serve as approximations for the DUPRO models with general random utility functions and allow us to quantify the approximation errors. Finally, we carry out some performance studies of the proposed bootstrap-based DUPRO model and report the preliminary numerical test results. This paper is the first attempt to use distributionally robust optimization methods for PRO problems.

Keywords PRO · Multi-attribute decision-making · Piecewise linear random utility function · Bootstrap ambiguity set · Mixed-integer linear program

Mathematics Subject Classification 90-10

1 Introduction

In decision-making under uncertainty, a utility representation characterizes the decision maker's (DM's) risk attitude towards the risk arising from systemic randomness [59, 66]. It is traditionally assumed that the DM makes consistent choices and hence the utility function representing the DM's preference is deterministic, i.e., the DM always makes the same decision in identical choice situations unless the DM is exactly indifferent between alternatives [9]. However, these assumptions are persistently violated in practice [2, 56, 63, 64]. Indeed, it is usually observed that the DM exhibits mutable and ambivalent preference, particularly without complete information of complex problems and uncertain environments in the real world [13, 33, 58, 70]. Ambivalent preference may also be caused by other reasons, e.g. the DM's preference does not satisfy transitivity in multi-attribute decision-making; the independence axiom in the von Neumann–Morgenstern's expected utility theory fails to hold [2]; decision-making is state dependent [3, 39] and there are inconsistencies in responses in the preference elicitation process [7]. This prompts us to adopt a random utility function to describe the DM's preference where each scenario corresponds to the DM's particular preference. This kind of random utility differs from the well-established random utility theory in discrete choice models where the latter is used to describe a group of customers' preferences to a commodity (product) and the randomness characterizes variability of customers' preferences and/or idiosyncratic product specific random shock, see e.g. [25, 41] and references therein. With the random utility function, we may consider the expected (mean) value of the DM's utility from a modeller's perspective if we know the probability distribution of the random parameter. The mean utility value captures the DM's average utility preference for the future decision-making. We refer readers to [25] and references therein for a thorough discussion of random utility representations. In the absence of complete information of the probability distribution of the random parameter, we propose a distributionally robust model where the worst-case distribution from an ambiguity set is to be considered for the calculation of the mean of the random utility. The next example explains the basic idea.

Example 1 Hannah wants to buy a jacket and hesitates to choose between A and B while considering price, style and color. Her wavering preference can be reasonably attributed to the assumption of random utility. Denote by u_1 and u_2 respectively the utility functions which characterize Hannah’s two possible inconsistent preferences over price, style and color, which are inconsistent. Suppose that $u_1(A) = 0.9, u_2(A) = 0.4$, and $u_1(B) = u_2(B) = 0.5$. Hannah’s hesitation is explained mathematically by $u_1(A) > u_1(B)$ but $u_2(A) < u_2(B)$. This vague preference can be represented as a random utility function, which is u_1 with probability p and u_2 with probability $1 - p$. Assume that, based on Hannah’s past clothing shopping choices, u_1 could be her most likely taste and thus $p \geq 0.5$. A natural way is to use the mean utility function, $\mathbb{E}_p[u] = pu_1 + (1 - p)u_2$, to reconcile the inconsistency. When the exact p is unknown, a distributionally robust model would suggest or predict Hannah to choose A since

$$\min_{p \in [0.5, 1]} \mathbb{E}_p[u(A)] = 0.65 > \min_{p \in [0.5, 1]} \mathbb{E}_p[u(B)] = 0.5.$$

Without any prior knowledge of the probability distribution, it leads to a deterministic worst-case utility approach over the entire sample space $\{u_1, u_2\}$ as follows:

$$\max_{x \in \{A, B\}} \min_{p \in [0, 1]} \mathbb{E}_p[u(x)] = \max_{x \in \{A, B\}} \min\{u_1(x), u_2(x)\} = \max\{0.4, 0.5\} = 0.5,$$

which means that Hannah should or would go with B . In our view, the mean utility approach is more reasonable, given the information on Hannah’s past shopping choices.

Research on decision-making based on the worst-case utility may be traced back to earlier work by Maccheroni [47] who considers a worst-case expected utility model for a conservative DM with an unclear evaluation of the different outcomes when facing lotteries. Hu and Mehrotra [35] (also see [36]) look into the issue from the robust optimization perspective which is later on known as preference robust optimization (PRO). They propose a moment-type framework for constructing an ambiguity set of a DM’s utility preference which covers a number of important preference elicitation approaches including certainty equivalent and pairwise comparison. To solve the resulting PRO model, they develop a step-like approximation scheme for the functions in the moment conditions and carry out some convergence analysis for the justification of the approximation.

Armbruster and Delage [4] consider an ambiguity set of utility functions which meet some criteria such as preferring certain lotteries over other lotteries and being risk averse, S -shaped, or prudent. Instead of trying to identify a single utility function satisfying the criteria, they develop a maximin PRO model where the optimal decision is based on the worst utility function from the ambiguity set and demonstrate how the maximin optimization can be reformulated as a finite-dimensional linear program. Guo et al. [28] take it further by proposing a piecewise linear approximation of the utility function and quantifying the approximation error and its propagation to the optimal value and optimal solutions of the PRO model. In the case that a DM has a nominal

utility function but lacks complete information about whether the nominal utility function is the true one, Hu and Stepanyan [37] propose to construct an ambiguity set of utility functions neighbouring the nominal one.

Over the past few years, research on the PRO models has expanded in several directions. Haskell et al. [32] propose a robust optimization model where the optimal decision is based on the worst-case utility function and the worst-case probability distribution of the underlying exogenous uncertainty amid both the true utility and the true probability distribution are ambiguous. Haskell et al. [31, 71] extend the PRO model from single-attribute decision-making to multi-attribute decision-making and from utility preference robust models to a broader class of preference robust quasi-concave choice function models. The latter gets around the Allais paradox concerning the utility-based PRO models. Liu et al. [43] present a multistage utility preference robust optimization model and demonstrate how the state-dependent ambiguity set of utility functions may be constructed and how the state dependence may affect time-consistency of the dynamic model.

The PRO models are also effectively extended to risk management problems from convex risk measures by Delage and Li [17, 19] to spectral coherent risk measures by Wang and Xu [68]. These works deal with the issue that a DM may have several risk measures at hand but lack complete information as to which one should be used for his/her decision-making. The authors propose various approaches for constructing the ambiguity set and demonstrate how to solve the resulting PRO models efficiently. More recently, Li [42] proposes an inverse optimization approach for solving PRO models when the DM's risk preference is obtained in a learning process. Most of these PRO models consider the worst-case preference. For decision-making with ambiguous utility functions, minimax regret is an alternative criterion. In the multi-attribute utility case, Boutilier et al. [11] seek decisions that achieve minimum worst-case regret, that is, regret experienced by a posterior once the true utility function is revealed. Vayanos et al. [65] consider the worst-case absolute regret to mitigate the conservatism of classical robust optimization (which maximizes worst-case utility).

Our approach fundamentally differs from the existing literature. We introduce the distributional utility PRO (DUPRO) model, which is motivated by the random utility theory. In contrast, those existing PRO models are grounded in von Neumann–Morgenstern's utility theorem [66]. This distinction in motivation has far-reaching implications for our understanding of preference robustness. Von Neumann–Morgenstern's utility theory, along with its underlying PRO models, is not suitable for the problems like Example 1 where a deterministic utility function does not exist for characterizing the DM's inconsistent preferences. By contrast, the random utility effectively addresses the inconsistency issue caused by the random preferences. In Example 1, Hannah's vague choice between jackets A and B can be described as $\text{Prob}(u(A) > u(B)) = p$ and $\text{Prob}(u(A) < u(B)) = 1 - p$. However, this inconsistent pairwise comparison, used in the PRO models, results in the emptiness of the ambiguity set of utility functions. As such, our DUPRO model complements the existing research on PRO models. The main contributions of this paper are summarized as follows.

- We investigate the case that the DM's preference is not only ambiguous but also potentially inconsistent or even displaying some kind of randomness. Differing from the classical worst-case utility maximization approaches, we propose a **DUPRO** framework where the DM's preference is represented by a random utility function and the ambiguity is described by a set of probability distributions of the random utility. An obvious advantage is that the **DUPRO** model can accommodate inconsistencies and randomness encountered in the preference elicitation process.
- In practice, a utility function is often elicited with a piecewise linear structure. In this case we show how the randomness can be represented by the increments of the each linear piece and subsequently propose two statistical approaches which suit data-driven problems for constructing an ambiguity set of the distributions of random increments. One is to construct an ellipsoidal confidence region with sample mean and sample covariance matrix which is widely used in the literature of distributionally robust optimization (DRO, see e.g. [18]); the other is to specify a nonparametric percentile- t bootstrap confidence region. While the bootstrap approach is widely used in statistics, we have not seen the approach used to construct an ambiguity set of probability distributions in DRO models. Here we adopt it because in practice it is often difficult to obtain samples of a DM's preference and hence the sample size is relatively small in our model. Moreover, compared to the ellipsoidal method, bootstrap has two potential advantages for constructing an ambiguity set. Unlike the ellipsoidal method which requires an estimation of the size of the ellipsoid for a given confidence level which could be unusually conservative (very large when sample size is small), the bootstrap-based ambiguity set is uniquely determined by the specified confidence level and the set of resamples. Although the two concepts of confidences differ slightly, they are close when the size of the original sample is sufficiently large. Moreover, we demonstrate the **DUPRO** model based on the bootstrap ambiguity set can be reformulated as an MILP which can be solved fairly efficiently.
- For the **DUPRO** model with general random utility functions, we demonstrate how to use a piecewise linear random utility function to approximate it and derive an error bound which depends only on the largest distance between two consecutive grid points of the piecewise linear random utility function for the optimal value and convergence of the optimal solutions (Proposition 5). With the piecewise linearly approximated utility maximization problem in place, we move on to discuss how to solve it with standard sample average approximation method in data-driven setting and derive exponential rate of convergence of the optimal values and almost sure convergence of the optimal solutions as the sample size increases (Proposition 7). Moreover, we consider **DUPRO** approaches for the piecewise linearly approximated model and derive exponential rate of convergence of the optimal values when the ambiguity set is constructed with ellipsoid method (Theorem 1) and almost sure convergence when the ambiguity set is constructed with the bootstrap method (Theorem 2). We have also conducted some numerical tests on the proposed **DUPRO** model and computational schemes via both an academic example and a practically oriented case study. The test results show that **DUPRO** performs

well as expected in the theoretical analysis in terms of asymptotic convergence, computational time, and out-of-sample performance.

The rest of the paper is organized as follows. Section 2 structures random multi-attribute utility functions with a piecewise linear additive structure. Section 3 proposes two DUPRO Models. The tractable reformulation and solution method of the bootstrap model are developed in Sect. 4. Section 5 extends the DUPRO models to general random utility function cases without a piecewise linear structure. Section 6 reports preliminary numerical test results. Section 7 concludes.

2 Random additive piecewise linear multi-attribute utility functions

We consider a multi-attribute decision-making problem where the DM's preference is described by an additive utility function. Additive multi-attribute utility functions are widely used in the literature of decision analysis and behavioural economics, see e.g. [26, 38, 40, 50, 62] and references therein. Here we use it primarily because the tractable reformulations and the theoretical results of the DUPRO models to be developed in the forthcoming discussions rely heavily on the additive structure. Let x_m be the performance of attribute m , for $m \in \mathfrak{M} := \{1, \dots, M\}$, and $x := (x_1, \dots, x_M)^T \in \mathbb{R}^M$ be the vector of all attribute performances. Denote by $u_m : \mathbb{R} \rightarrow \mathbb{R}$ the single-attribute utility function of attribute m . We begin by considering a piecewise linear utility function (PLUF) of each attribute and then move on to discuss general utility functions in Sect. 5. In this setup, we assume the DM's preference can be represented by a PLUF. This is not only because the PLUF will facilitate us to derive tractable DUPRO models but also because the DUPRO models with the PLUF can be conveniently applied in real-world data-driven problems. Indeed, the PLUF structure is versatile in its ability to incorporate classical non-parametric utility assessment methods [23, 67] into the random sampling processes required by the DUPRO models to generate observations of a random utility function. Let u_m be a PLUF defined over interval $[a_m, b_m]$ with breakpoints

$$a_m = t_{m,0} < \dots < t_{m,I_m} = b_m, \quad (1)$$

where $t_{m,i}$ is a certain level of attribute m and we assume that the DM's utility value at $t_{m,i}$ can be elicited. Assume without loss of generality that $u_m(a_m) = 0$ and let $v_{m,i} := u_m(t_{m,i}) - u_m(t_{m,i-1})$ be the overall increment of u_m over $[t_{m,i-1}, t_{m,i}]$ for $i \in \mathfrak{I}_m := \{1, \dots, I_m\}$ and $v_m := (v_{m,1}, \dots, v_{m,I_m})^T \in \mathbb{R}^{I_m}$. If we regard v_m as a vector of parameters, then we can obtain a class of piecewise linear functions parameterized by v_m for the m -th attribute as follows:

$$u_m(x_m; v_m) := \sum_{i \in \mathfrak{I}_m} \left[\frac{v_{m,i}}{t_{m,i} - t_{m,i-1}} (x_m - t_{m,i-1}) + \sum_{j=1}^{i-1} v_{m,j} \right] \mathbf{1}_{\{t_{m,i-1} < x_m \leq t_{m,i}\}}, \quad (2)$$

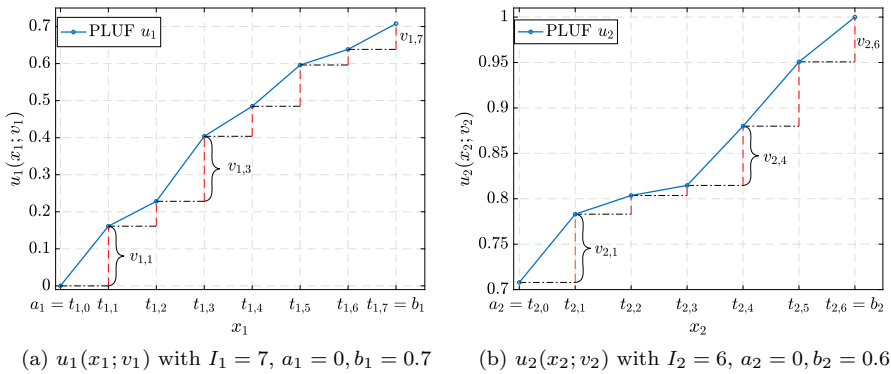


Fig. 1 PLUF $u(x; v) = u_1(x_1; v_1) + u_2(x_2; v_2)$ with $x = (x_1, x_2)^T$, $v = (v_1^T, v_2^T)^T$, $v_1 = (v_{1,1}, v_{1,2}, \dots, v_{1,7})^T$, $v_2 = (v_{2,1}, v_{2,2}, \dots, v_{2,6})^T$

where $\mathbf{1}\{\cdot\}$ is an indicator function. In other words, $u_m(x_m; v_m)$ defines a class of PLUFs parameterized by a vector of increments v_m . Let $v := (v_1^T, \dots, v_M^T)^T$ and $I := \sum_{m \in \mathfrak{M}} I_m$. We can then define the aggregate multi-attribute PLUF $u : \mathbb{R}^M \times \mathbb{R}^I \rightarrow \mathbb{R}$ with an additive form as

$$u(x; v) := \sum_{m \in \mathfrak{M}} u_m(x_m; v_m). \tag{3}$$

Without loss of generality, we assume that the utility function is nondecreasing and normalized to $[0, 1]$ (i.e., $u(a; v) = 0$ and $u(b; v) = 1$ with $a = (a_1, \dots, a_M)^T$, $b = (b_1, \dots, b_M)^T$) because the normalization does not affect its representation of the DM’s preference. Under this assumption, the utility function u_m is uniquely determined by vector v_m for $m \in \mathfrak{M}$ for a given set of breakpoints.¹ Figure 1 depicts the PLUF when $M = 2$. Moreover, the vector v lies in the simplex of \mathbb{R}^I with $v \geq 0$ and $e_I^T v = 1$, where e_I is the I -dimensional vector of all ones. Note that a traditionally defined additive multi-attribute utility function is the weighted sum of the normalized single-attribute utility functions of each attribute. The normalization of u and the definition of the utility function in (2)-(3) ensure that the criterion weight of the m -th attribute is included but hidden in u_m .

In practice, the vector v of increments is determined by elicited information on the DM’s utility scoring at certain levels of the attributes. As we discussed in the Introduction, we concentrate on the case that the DM’s preferences vary randomly and they are potentially inconsistent, which means that the values of v obtained as such cannot be used to construct a single deterministic utility function, rather they may be treated as the realizations of a random vector V . This motivates us to replace the deterministic vector of parameters v with a random vector $V : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}^I$ in (3) and subsequently consider a random PLUF mapping from \mathbb{R}^M to \mathbb{R} as

¹ In the numerical tests, these points are generated randomly.

$$u(x; V) = \sum_{m \in \mathfrak{M}} u_m(x_m; V_m), \tag{4}$$

where each realization of V may be interpreted as a ‘‘state variable’’ signifying the state of the DM (the DM’s mood, situation, etc.). This effectively randomizes the class of utility functions parameterized by random vector V with the support set

$$\mathcal{V} := \left\{ v \in \mathbb{R}^I \mid e_I^T v = 1, v \geq 0 \right\}. \tag{5}$$

In this setup, the DM’s preference is described by a random PLUF rather than a deterministic function. It means that the DM’s utility may change randomly and is uncertain. The uncertainty may arise from the DM’s inconsistent preferences or from incomplete and inaccurate observations of the DM’s preference.

In some cases, we may require that a utility function has properties on its shape. For example, the DM’s risk-averse preference should be characterized as an increasing concave utility function. The support set in the case of increasing concave utility functions is described as

$$\mathcal{V}_c := \left\{ v \in \mathbb{R}^I \mid e_I^T v = 1, Av \geq 0 \right\}, \tag{6}$$

where matrix

$$A := \begin{bmatrix} A_1 & & & \\ & \ddots & & \\ & & A_M & \\ & & & \end{bmatrix}_{(I-M) \times I},$$

in which blocks A_m for $m \in \mathfrak{M}$ are

$$A_m := \begin{bmatrix} \frac{1}{t_{m,1}-t_{m,0}} & \frac{-1}{t_{m,2}-t_{m,1}} & & & & \\ & \frac{1}{t_{m,2}-t_{m,1}} & \frac{-1}{t_{m,3}-t_{m,2}} & & & \\ & & \ddots & \ddots & & \\ & & & \frac{1}{t_{m,l_m-1}-t_{m,l_m-2}} & \frac{-1}{t_{m,l_m}-t_{m,l_m-1}} & \\ & & & & & \end{bmatrix}_{(I_m-1) \times I_m}.$$

Before concluding this section, we note that there are a number of ways to represent a piecewise linear utility function. The specific construction of $u(x; v)$ that we considered here offers significant advantages in deriving the DUPRO models discussed in the next two sections. As demonstrated in Proposition 2, the utility form $u(x; v)$ that we introduce in (3) can be expressed as a linear function of the increment vector v for a given x . This characteristic facilitates the tractable formulations of the DUPRO models. Moreover, the specific form of the utility function differs from the conventional way of representing a multi-attribute utility function as a weighted sum of normalized single-attribute utility functions. When eliciting each normalized single-attribute utility function and assessing the trade-offs between them from the same DM, we posit that the criterion weights and these single-attribute utility functions are correlated

random quantities. The random utility $u(x; V)$ described in (4) captures this correlated relationship by integrating the criterion weights into the single-attribute utility functions.

3 Data-driven distributional preference robust model

We consider a multi-attribute decision-making problem which aims to maximize the overall utility of attribute performances. Since the DM’s utility function is stochastic, we consider the expected utility $\mathbb{E}_P[u(x; V)]$ where the expectation is taken with respect to (w.r.t) the probability distribution P of V . Moreover, since the true probability distribution P is unknown in some data-driven problems, we may have to rely on partially available information such as empirical data, computational simulations, or subjective judgment to construct an ambiguity set of distributions.

3.1 The DUPRO models

Let $\mathcal{P}(\mathbb{R}^I)$ denote the set of all probability measures on the space R^I and $\mathcal{P} \subset \mathcal{P}(\mathbb{R}^I)$ represent the ambiguity set of the distributions of V . We consider the optimal decision based on the worst-case distribution from this set as follows:

$$\max_{x \in \mathcal{X}} \min_{P \in \mathcal{P}} \mathbb{E}_P[u(x; V)], \tag{DUPRO}$$

where $\mathcal{X} \subseteq \otimes_{m \in \mathcal{M}} [a_m, b_m] := [a_1, b_1] \times \dots \times [a_M, b_M]$ is the joint region of possible attribute values. There are some fundamental differences between DUPRO and the existing PRO models in the literature. First, the true utility function in DUPRO is random whereas the true utility function in the existing PRO models is deterministic. It means the former is about ambiguity of the probability distribution of the true random utility function whereas the latter is about ambiguity of the true unknown (deterministic) utility function. Second, the random utility functions at different scenarios do not have to be consistent when they are applied to a given pair of prospects. In contrast, the existing PRO models are built on von Neumann–Morgenstern’s expected utility theory [66] which means the true utility function exists and is unique up to positive linear transformation although in practice it is unknown. Thus the DM’s utility preference must be consistent. Of course, in practical applications, the DM’s answers to certain questionnaires may be inconsistent due to mistaken answers or errors in measurements and there is a way to handle this kind of inconsistency, see [4, 7]. However, the two models are different in nature. Third, in DUPRO, we assume that a random sample of V is obtainable explicitly by elicitation on the DM’s utility scoring at a specified set of levels of attributes. If we interpret $V = v$ as a scenario of V , it means that the DM’s utility $u(x; V)$ at the scenario is obtainable, there is no ambiguity about $u(\cdot; v)$. The only ambiguity lies in the true probability distribution of V . Fourth, DUPRO should be distinguished from distributionally robust expected utility optimization models where the ambiguity lies in the probability distribution of exogenous random parameters, see e.g. [49]. Fifth, the random utility model has been widely used to describe customer’s

choices of certain products, see e.g. [1, 6, 14, 61]. [48] seems to be the first to consider a distributionally robust choice model where the expected utility of a selected product is maximized under the ambiguity of correlations between customer's utility valuations of different products. There are at least three differences between our models and theirs. (a) In the discrete choice models, the random utility function represents a group of DMs' (customers') utility evaluations of certain products where each realization of the random utility represents some customers' evaluation of a product. In our models, the random utility function represents a single DM's utility evaluations in a multi-attribute decision-making problem where each realization of the random utility represents the DM's evaluation of the attributes in a particular "state". (b) We consider the expected total additive utilities rather than the expected value of the maximum utility of the M attributes. (c) The ambiguity in our model is on randomness of the single-attribute utility function of each attribute rather than correlation between these utility functions (corresponding to correlation of utility of different products).

In practice, x is often a vector-valued function of some action denoted by z , that is, $x = h(z)$. Let \mathfrak{Z} be a feasible action space. It follows that

$$\mathfrak{X} = \{x \in \mathbb{R}^M \mid x = h(z), z \in \mathfrak{Z}\}.$$

Consequently, we rewrite this variation of **DUPRO** as

$$\max_{z \in \mathfrak{Z}} \min_{P \in \mathcal{P}} \mathbb{E}_P[u(h(z); V)]. \quad (\text{DUPRO-1})$$

In the case that the relation between x and z is affected by some exogenous uncertainties denoted by a vector of random variables ξ with distribution Q , we may obtain a further variation

$$\max_{z \in \mathfrak{Z}} \min_{P \in \mathcal{P}} \mathbb{E}_P[\mathbb{E}_Q[u(h(z, \xi); V)]]. \quad (\text{DUPRO-2})$$

We next focus on discussing **DUPRO**. All the model configurations and corresponding solution methods presented in this paper can be straightforwardly extended to **DUPRO-1** and **DUPRO-2**.

3.2 Construction of the ambiguity set

A key component of **DUPRO** is the ambiguity set. In the current main stream research on DRO models, the ambiguity is concerned with the probability distribution of exogenous uncertainty, see [12, 15, 16, 27, 30, 46, 69] and the references therein. Here, the ambiguity lies in the probability distribution of endogenous uncertainty (DM's utility preference). We outline two main approaches for constructing the ambiguity set \mathcal{P} in a data-driven environment. Denote by μ and Σ the mean and covariance matrix of V . Here we consider a situation where the true μ and Σ are unknown but it is possible to obtain an approximation with sample data. Let V^1, \dots, V^N be an independent and identically distributed (i.i.d. for short) random sample of V and denote by \bar{V} the

sample mean and by S a sample covariance matrix approximating their true counterparts μ and Σ . In practice, the random samples can be generated using classical non-parametric utility assessment methods. In Sect. 6.2, we delve into the utilization of a generative logistic regression method to assemble a random dataset of consumer preference in the automobile market.

Note that traditional PRO models are not suitable in this case. This is primarily because there may not exist a deterministic utility function which fits into the random samples. For instance, let $u(\cdot; V^1), \dots, u(\cdot; V^N)$ be the N observations of Hannah’s vague preference in Example 1. It is highly likely that one observation supports the choice of jacket A , i.e., $u(A; V^i) > u(B; V^i)$, whereas another favors B with $u(A; V^j) < u(B; V^j)$. Those inconsistent observations result in emptiness of the ambiguity set of utility functions in the existing PRO models [4, 36].

In what follows, we address how to construct a set of moment uncertainty using a random sample. The support set \mathcal{V} defined as in (5) indicates that the components of V are linearly dependent, i.e., $e_I^T V = 1$ almost surely. Therefore, we can reduce by one dimension to consider the first $I - 1$ components of V in the construction of the ambiguity set. Accordingly, $V_{M,IM}$, which is the I -th component of V , is equal to 1 less the sum of the other components. Let

$$C := \begin{bmatrix} 1 & & 0 \\ & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}_{(I-1) \times I} . \tag{7}$$

Then $CV \in \mathbb{R}^{I-1}$ is the vector consisting of the first $I - 1$ components of V . Denote by μ_{I-1} and Σ_{I-1} the mean and covariance matrix of CV and by \bar{V}_{I-1} and S_{I-1} the sample counterparts of μ_{I-1} and Σ_{I-1} . Note that

$$\mu_{I-1} = C\mu, \quad \Sigma_{I-1} = C\Sigma C^T, \quad \bar{V}_{I-1} = C\bar{V}, \quad \text{and} \quad S_{I-1} = CSC^T.$$

To facilitate our analysis in the forthcoming discussions, we make a blanket assumption on Σ_{I-1} .

Assumption 1 The covariance matrix Σ_{I-1} is nonsingular.

Under this assumption, S_{I-1} is nonsingular for a sufficiently large sample size N , which is needed in the following constructions of the ambiguity set \mathcal{P} . The assumption requires that the components of CV are linearly independent. If the assumption is not satisfied, then we will repeat the procedure of dimension reduction until remaining components are linearly independent.

3.2.1 Ellipsoid approach

A popular approach in the literature of distributionally robust optimization is to specify the ambiguity set by moment conditions [8, 18, 51, 52, 73]. Here we consider an ambiguity set with ellipsoidal structure of the first moment:

$$\mathcal{P}_A(\gamma) := \left\{ P \in \mathcal{P}(\mathbb{R}^I) \mid \begin{array}{l} P(V \in \mathcal{V}) = 1 \\ \left\| S_{I-1}^{-1/2} (\mathbb{C}\mathbb{E}_P[V] - \bar{V}_{I-1}) \right\|^2 \leq \gamma \end{array} \right\}, \quad (8)$$

where $\|\cdot\|$ denotes the Euclidean norm and $S_{I-1}^{1/2}$ is a full rank matrix with $S_{I-1} = (S_{I-1}^{1/2})^T S_{I-1}^{1/2}$ and $S_{I-1}^{-1/2} = (S_{I-1}^{1/2})^{-1}$. The second condition in the set $\mathcal{P}_A(\gamma)$ specifies the range of $\mathbb{C}\mathbb{E}_P[V]$, that is, we consider only the candidate probability distributions with associated means value of V falling within the specified ellipsoid centered at the sample mean \bar{V}_{I-1} . The parameter γ determines the size of the ellipsoid and its choice depends on the DM's confidence in the sample data. The whole idea of the ambiguity set is built on the confidence region for multivariate random variables. The following proposition states conditions under which the true probability distribution of V falls in the ambiguity set $\mathcal{P}_A(\gamma)$ by a proper setting of γ for the fixed sample size N and confidence level $1 - \alpha$.

Proposition 1 For a given $\alpha \in (0, e^{-2}(2 - e^{-2}))$, let

$$\gamma_N^{(1)} = \frac{32R^2 e^2 \ln^2(1/(1 - \sqrt{1 - \alpha}))}{N}, \quad (9)$$

where $R := \left\| \Sigma_{I-1}^{-1/2} \right\|$. Then there exists a positive integer N_0 depending on α such that, for the true probability distribution P of the random vector V ,

$$\text{Prob} \left(P \in \mathcal{P}_A(\gamma_N^{(1)}) \right) \geq 1 - \alpha$$

for all $N \geq N_0$.

Proof Let $\beta := 1 - \sqrt{1 - \alpha}$. As we discussed earlier, under Assumption 1, we have

$$\left\| \Sigma_{I-1}^{-1/2} (\mathbb{C}\mathbb{E}[V] - \bar{V}_{I-1}) \right\| \leq (I - 1)R < \infty \text{ a.s.}$$

Hence, the bounded support guarantees the ‘‘Condition (G)’’ in [57], i.e., for any $p \geq 1$,

$$\mathbb{E} \left[\left\| \Sigma_{I-1}^{-1/2} (\mathbb{C}\mathbb{E}[V] - \bar{V}_{I-1}) \right\|^p \right] \leq [4R^2 p]^{p/2}.$$

It follows by Proposition 4 in [57] that

$$\text{Prob} \left(\left\| \Sigma_{I-1}^{-1/2} (\mathbb{C}\mathbb{E}[V] - \bar{V}_{I-1}) \right\| \leq \gamma_N^{(1)}/2 \right) \geq 1 - \beta.$$

On the other hand, since S_{I-1} converges to Σ_{I-1} weakly by the weak law of large numbers, and $\left\| \mathbb{C}\mathbb{E}[V] - \bar{V}_{I-1} \right\|$ is bounded by 2 a.s., then there exists N_0 depending on α such that

$$\text{Prob} \left(\left\| S_{I-1}^{-1/2} (\mathbb{C}\mathbb{E}[V] - \bar{V}_{I-1}) \right\| \leq 2 \left\| \Sigma_{I-1}^{-1/2} (\mathbb{C}\mathbb{E}[V] - \bar{V}_{I-1}) \right\| \right) \geq 1 - \beta$$

for all $N \geq N_0$. Consequently

$$\begin{aligned} & \text{Prob} \left(\left\| S_{I-1}^{-1/2} (C\mathbb{E}[V] - \bar{V}_{I-1}) \right\| \leq \gamma_N^{(1)} \right) \\ & \geq \text{Prob} \left(\left\| \Sigma_{I-1}^{-1/2} (C\mathbb{E}[V] - \bar{V}_{I-1}) \right\| \leq \gamma_N^{(1)}/2 \right) (1 - \beta) \geq (1 - \beta)^2 = 1 - \alpha, \end{aligned}$$

which completes the proof. □

The proposition states that for a fixed sample size N how large the parameter γ should be so that the true distribution falls in $\mathcal{P}_A(\gamma)$ with a specified confidence level. This means that, with probability at least $1 - \alpha$, the optimal value of the DUPRO model computed with such an ellipsoidal ambiguity set provides a lower bound for the true optimal value of the expected utility maximization problem, see Theorem 4 in [18] for a similar argument. From a decision-making perspective, we want the ambiguity set to be small (a smaller γ) so that the resulting decision based on the worst distribution from the ambiguity set is less conservative. On the other hand, we also hope the true probability distribution lies within the ambiguity set with a specified confidence level so that the model is more relevant. Nevertheless, a smaller γ would make this less likely to happen. If we are able to recover the true probability distribution with a sample of size N , then we may set $\gamma = 0$. In practice, however, this is very unlikely particularly when the sample size is small. A potential drawback of this approach is that when we use sample covariance to approximate the true covariance, R could be very large when the covariance matrix is nearly singular (the data are nearly correlated), this will result in a large $\gamma_N^{(1)}$. We refer readers to discussions in [18, 57] and our discussions in Sect. 5.3 for appropriate setting of γ . This prompts us to consider the bootstrap approach for constructing an ambiguity set.

3.2.2 The bootstrap approach

We propose to use a percentile- t bootstrap method to specify a confidence region of μ_{I-1} . Specifically, we generate K nonparametric bootstrap resamples of the first $I - 1$ components, denoted by $V_{I-1}^{(k,1)}, \dots, V_{I-1}^{(k,N)}$, $k = 1, \dots, K$, based on the empirical distribution (i.e., the uniform distribution on original sample data $V_{I-1}^1, \dots, V_{I-1}^N$). Let $\bar{V}_{I-1}^{*,k}$ and $S_{I-1}^{*,k}$ denote the sample mean and the sample covariance matrix of $V_{I-1}^{(k,1)}, \dots, V_{I-1}^{(k,N)}$. We consider the studentized statistics

$$T := \sqrt{N} S_{I-1}^{-1/2} (\bar{V}_{I-1} - \mu_{I-1})$$

and its bootstrap counterparts

$$T_k^* := \sqrt{N} \left(S_{I-1}^{*,k} \right)^{-1/2} (\bar{V}_{I-1}^{*,k} - \bar{V}_{I-1}), \quad k = 1, \dots, K. \tag{10}$$

The literature [5, 29, 72] addresses multivariate bootstrap confidence regions using data depth and likelihood. Here we state the procedure given by [72]. Recall that

Tukey’s depth of a point x under some distribution \mathcal{F} is defined as

$$\text{TD}(\mathcal{F}, x) := \inf_{\|s\|=1} \int \mathbf{1} \left\{ s^T (y - x) \geq 0 \right\} d\mathcal{F}(y).$$

Let \mathcal{F}_K^* be the empirical cumulative distribution function built using (T_1^*, \dots, T_K^*) and calculate $d_k := \text{TD}(\mathcal{F}_K^*, T_k^*)$, $k = 1, \dots, K$. Denote by $d_{(1)}, \dots, d_{(K)}$ in the decreasing order of the quantities of d_1, \dots, d_K , and let $T_{(k)}^*$ be the statistics such that $d_{(k)} = \text{TD}(\mathcal{F}_K^*, T_{(k)}^*)$. For a given $\alpha \in (0, 1)$, let $\hat{T}_{1-\alpha} := \left[T_{(1)}^*, \dots, T_{(\lceil (1-\alpha)K \rceil)}^* \right]$ be an $M \times \lceil (1-\alpha)K \rceil$ dimensional matrix. Here, $\lceil \cdot \rceil$ is the ceiling function. We construct a convex hull, denoted by $\mathfrak{W}_{1-\alpha}^*$, of $T_{(k)}^*$ for $1 \leq k \leq \lceil (1-\alpha)K \rceil$, i.e.,

$$\mathfrak{W}_{1-\alpha}^* := \left\{ \hat{T}_{1-\alpha} w \in \mathbb{R}^{I-1} \mid e_{\lceil (1-\alpha)K \rceil}^T w = 1, w \geq 0 \right\}. \tag{11}$$

Then a $100(1 - \alpha)\%$ bootstrap confidence region for μ_{I-1} is obtained as

$$\mathfrak{C}_{1-\alpha}^* := \left\{ \bar{V}_{I-1} - S_{I-1}^{1/2} \tilde{w} / \sqrt{N} \mid \tilde{w} \in \mathfrak{W}_{1-\alpha}^* \right\}. \tag{12}$$

On this basis, we define an ambiguity set of the first moment as

$$\mathcal{P}_B(\alpha) := \left\{ P \in \mathcal{P}(\mathbb{R}^I) \mid P(V \in \mathcal{V}) = 1, C\mathbb{E}_P[V] \in \mathfrak{C}_{1-\alpha}^* \right\}. \tag{13}$$

In this bootstrap approach, α is the critical value of the confidence region. A larger α means fewer sample points from the bootstrap resampling process are included. Subsequently, $\mathfrak{C}_{1-\alpha}^*$ and $\mathcal{P}_B(\alpha)$ are smaller.

Observe that for fixed K , $\mathfrak{W}_{1-\alpha}^*$ is a bounded set and by Corollary 6 in [55], $S_{I-1}^{1/2}$ converges to $\Sigma_{I-1}^{1/2}$ at an exponential rate with the increase of N . Thus $S_{I-1}^{1/2} \tilde{w} / \sqrt{N} \rightarrow 0$ as $N \rightarrow \infty$ w.p.1. which implies that the ambiguity set $\mathfrak{C}_{1-\alpha}^*$ shrinks to a singleton at an exponential rate as $N \rightarrow \infty$. That is, for a fixed α , the size of the ambiguity set is only determined by N (original sample size). In contrast, the ellipsoidal method requires one to choose an appropriate size by adjusting γ with regard to N , such as $\gamma_N^{(1)}$ in (9). When N is small, $\gamma_N^{(1)}$ could be very large to secure the true probability distribution to fall into $\mathcal{P}_A(\gamma_N^{(1)})$ by Proposition 1. However, such a large ambiguity set $\mathcal{P}_A(\gamma_N^{(1)})$ is too conservative in practice. Indeed, for a given N , the existing literature lacks a practical method for determining the size of an ellipsoid ambiguity set. Differently, the bootstrap approach offers the convenience of selecting an appropriate critical value α with a probabilistic interpretation. Here we illustrate how the confidence region constructed by the bootstrap looks like in the case that the dimension of V_{I-1} is 2, the original sample size is $N = 50$ and the number of the resamples is $K = 10,000$. Figure 2a and b depict $\mathfrak{W}_{1-\alpha}^*$ and $\mathfrak{C}_{1-\alpha}^*$ for different α values ranging from 0 to 0.25. Figure 2c depicts the minimal ellipsoid containing $\mathfrak{C}_{1-\alpha}^*$ for different α values. Since each ellipsoid may be regarded as a confidence region, we can see from the figures the sizes of the confidence regions with specified confidences of 95%, 85%

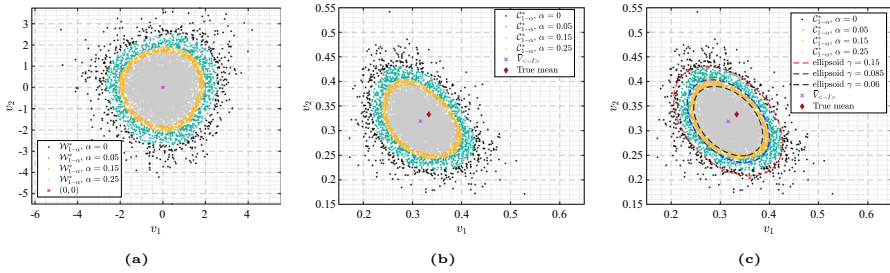


Fig. 2 $K = 1000, N = 50$. **a** convex hull $\mathfrak{W}_{1-\alpha}^*$. **b** $100(1 - \alpha)\%$ bootstrap confidence region $\mathcal{C}_{1-\alpha}^*$. **c** $\mathcal{C}_{1-\alpha}^*$ v.s. ellipsoid $\{v_{<3-1>} \in \mathbb{R}_+^2 : \|S_{<3-1>}^{-1/2}(v_{<3-1>} - \bar{V}_{<3-1>})\|^2 \leq \gamma\}$

and 75%. Moreover, we can see that sizes of these confidence regions are considerably smaller than those defined via $\mathcal{P}_A(\gamma)$. For example, when $N = 50$, and $\alpha = 0.25$, $\gamma_N^{(1)} = 70$.

In this paper, we adopt the bootstrap approach to construct the ambiguity set not only because the size of the ambiguity set is relatively smaller but also it is often difficult to obtain a large number of observations of V . Moreover, we will also see in the next section that **DUPRO** can be easily reformulated at an MILP which is relatively easier to solve.

4 Reformulation of the **DUPRO** model

In this section, we develop computational approaches for solving **DUPRO** with the ambiguity sets $\mathcal{P}_A(\gamma)$ and $\mathcal{P}_B(\alpha)$ defined as in (8) and (13), respectively. Section 4.1 addresses the case with increasing utility functions, while Sect. 4.2 discusses the case with risk-averse utility functions. We begin by presenting a reformulation of the objective function $\mathbb{E}_P[u(x; V)]$ of **DUPRO** as a linear function of the expectation of random vector V in the next proposition. To ease the presentation, we recall some of the key notations introduced in Sect. 2. $\mathbf{1}\{t_{m,i-1} < x_m \leq t_{m,i}\}$ represents an indicator function of interval $(t_{m,i-1}, t_{m,i}]$, $x = (x_1, \dots, x_M)^T \in \mathbb{R}^M$, u_m is a PLUF defined over $[a_m, b_m]$ with breakpoints $\{t_{m,i} : i \in \{0\} \cup \mathcal{J}_m\}$, for $m \in \mathfrak{M}$, $v = (v_1^T, \dots, v_M^T)^T \in \mathbb{R}^I$ with $\mathcal{J}_m = \{1, \dots, I_m\}$, $\mathfrak{M} = \{1, \dots, M\}$, $I = \sum_{m \in \mathfrak{M}} I_m$, and $u(x; v) = \sum_{m \in \mathfrak{M}} u_m(x_m; v_m)$.

Proposition 2 *Let*

$$g_{m,i}(x_m) := \mathbf{1}\{t_{m,i-1} < x_m \leq t_{m,i}\} \quad \text{and} \quad h_{m,i}(x_m) := \left(\frac{x_m - t_{m,i-1}}{t_{m,i} - t_{m,i-1}} \right) g_{m,i}(x_m),$$

for $i \in \mathfrak{I}_m, m \in \mathfrak{M}$. Let

$$f_m(x_m) := \begin{bmatrix} h_{m,1}(x_m) + \sum_{j \in \mathfrak{I}_m \setminus \{1\}} g_{m,j}(x_m) \\ h_{m,2}(x_m) + \sum_{j \in \mathfrak{I}_m \setminus \{1,2\}} g_{m,j}(x_m) \\ \vdots \\ h_{m,I_m}(x_m) \end{bmatrix}_{I_m \times 1}$$

and $f(x) := (f_1(x_1)^T, \dots, f_M(x_M)^T)^T$. Then

(i) $u_m(x_m; v_m) = v_m^T f_m(x_m)$, for $m \in \mathfrak{M}$ and

$$u(x; v) = v^T f(x). \tag{14}$$

(ii) **DUPRO** is equivalent to

$$\max_{x \in \mathfrak{X}} \min_{P \in \mathcal{P}} \mathbb{E}_P[V^T]f(x) \tag{15}$$

and

$$\max_{x \in \mathfrak{X}} \min_{v \in \mathfrak{F}(\mathcal{P})} \left\{ u(x; v) = v^T f(x) \right\}, \tag{16}$$

where $\mathfrak{F}(\mathcal{P}) := \{\mathbb{E}_P(V) \mid P \in \mathcal{P}\}$.

Formulation (16) implies that **DUPRO** can be converted into a maximin robust optimization problem where the ambiguity set $\mathfrak{F}(\mathcal{P})$ is composed of the expected values of V each of which corresponds to an expected utility function. The worst-case expected utility function $\min_{P \in \mathcal{P}} \mathbb{E}_P[V^T]f(x)$ is determined by the minimum of $\{v^T f(x) : v \in \mathfrak{F}(\mathcal{P})\}$. Thus, we may solve **DUPRO** by solving problem (16). This is indeed one of the main advantages for us to adopt the particular form of PLUF in (4). However, the structure of $f(x)$ is still too complex for us to derive a tractable formulation of **DUPRO**. To address the issue, we derive an alternative representation of $f(x)$. Let $y = (y_1^T, \dots, y_M^T)^T \in \mathbb{R}^I$ and $z = (z_1^T, \dots, z_M^T)^T \in \{0, 1\}^I$, where $y_m = (y_{m,1}, \dots, y_{m,I_m})^T \in \mathbb{R}^{I_m}$ and $z_m = (z_{m,1}, \dots, z_{m,I_m})^T \in \{0, 1\}^{I_m}$ for $m \in \mathfrak{M}$. Let

$$Y := \begin{bmatrix} Y_1 & & \\ & \ddots & \\ & & Y_M \end{bmatrix}_{I \times I}, \quad Z := \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_M \end{bmatrix}_{I \times I}, \tag{17}$$

where the blocks Y_m and Z_m for $m \in \mathfrak{M}$ are

$$Y_m := \begin{bmatrix} \frac{1}{t_{m,1} - t_{m,0}} & & \\ & \ddots & \\ & & \frac{1}{t_{m,I_m} - t_{m,I_m-1}} \end{bmatrix}_{I_m \times I_m},$$

$$Z_m := \begin{bmatrix} \frac{-t_{m,0}}{t_{m,1}-t_{m,0}} & 1 & \dots & 1 \\ \frac{-t_{m,1}}{t_{m,2}-t_{m,1}} & \dots & \dots & 1 \\ \dots & \dots & \dots & \vdots \\ \frac{-t_{m,I_m-1}}{t_{m,I_m}-t_{m,I_m-1}} & \dots & \dots & \dots \end{bmatrix}_{I_m \times I_m}.$$

Denote five other block-diagonal matrices

$$B := \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_M \end{bmatrix}_{I \times I}, \quad D := \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_M \end{bmatrix}_{I \times M}, \quad E := \begin{bmatrix} e_{I_1} & & \\ & \ddots & \\ & & e_{I_M} \end{bmatrix}_{I \times M}, \tag{18a}$$

$$H^- := \begin{bmatrix} H_1^- & & \\ & \ddots & \\ & & H_M^- \end{bmatrix}_{I \times I}, \quad H^+ := \begin{bmatrix} H_1^+ & & \\ & \ddots & \\ & & H_M^+ \end{bmatrix}_{I \times I}, \tag{18b}$$

where the blocks $B_m, D_m, H_m^-,$ and H_m^+ for $m \in \mathfrak{M}$ are

$$B_m := \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \dots & 1 \end{bmatrix}_{I_m \times I_m}, \quad D_m := \begin{bmatrix} t_{m,1} \\ \vdots \\ t_{m,I_m} \end{bmatrix}_{I_m \times 1}, \tag{19a}$$

$$H_m^- := \begin{bmatrix} t_{m,0} & & & \\ & \ddots & & \\ & & \dots & \\ & & & t_{m,I_m-1} \end{bmatrix}_{I_m \times I_m}, \quad H_m^+ := \begin{bmatrix} t_{m,1} & & & \\ & \ddots & & \\ & & \dots & \\ & & & t_{m,I_m} \end{bmatrix}_{I_m \times I_m}. \tag{19b}$$

With all these new notations, we are ready to represent $f(x)$ as follows:

$$f(x) = Yy + Zz \tag{20}$$

under the conditions below

$$E^T z = e_M, \quad E^T y = x, \quad H^- z - y \leq 0, \quad H^+ z - y \geq 0, \quad z \in \{0, 1\}^J. \tag{21}$$

It is important to note that for a fixed $x, (y, z)$ satisfying (21) is unique. Representation of $f(x)$ via (20)–(21) is a crucial step towards tractable formulation of (16) in the rest of this section. This is another main advantage that we adopt the particular form of PLUF in (4).

4.1 PLUF without concavity

We consider the case that the random utility functions are piecewise linear but are not necessarily concave, and discuss reformulation of **DUPRO** with the ambiguity set $\mathcal{P}_B(\alpha)$ (see (13))

$$\max_{x \in \mathfrak{X}} \min_{v \in \mathfrak{F}(\mathcal{P}_B(\alpha))} v^T f(x), \tag{22}$$

where

$$\mathfrak{F}(\mathcal{P}_B(\alpha)) = \left\{ v \in \mathbb{R}^I \left| \begin{array}{l} v \geq 0 \\ e_I^T v = 1 \\ Cv = \bar{V}_{I-1} - S_{I-1}^{1/2} \tilde{w} / \sqrt{N}, \tilde{w} \in \mathfrak{W}_{1-\alpha}^* \end{array} \right. \right\}. \tag{23}$$

The next proposition states that the maximin problem (22) can be reformulated as a single MILP.

Proposition 3 ***DUPRO** with the ambiguity set $\mathcal{P}_B(\alpha)$ defined as in (13) is equivalent to*

$$\max_{x, y, z, \eta, \pi, \tau} \bar{V}_{I-1}^T \eta + \pi + \tau \tag{24a}$$

$$\text{s.t. } C^T \eta + e_I \tau \leq Yy + Zz, \tag{24b}$$

$$N^{-1/2} \hat{T}_{1-\alpha}^T \left(S_{I-1}^{1/2} \right)^T \eta + e_{\lceil(1-\alpha)K\rceil} \pi \leq 0, \tag{24c}$$

$$E^T z = e_M, \tag{24d}$$

$$E^T y = x, \tag{24e}$$

$$H^- z - y \leq 0, \tag{24f}$$

$$H^+ z - y \geq 0, \tag{24g}$$

$$z \in \{0, 1\}^I, \tag{24h}$$

$$x \in \mathfrak{X}, \tag{24i}$$

where Y, Z, H^-, H^+ are defined as in (17)–(19), and $\hat{T}_{1-\alpha}$ is defined as in (11).

Proof By Proposition 2, **DUPRO** with the ambiguity set $\mathcal{P}_B(\alpha)$ is equivalent to (16) with $\mathfrak{F}(\mathcal{P}_B(\alpha))$. Also, for any $Cv \in \mathfrak{C}_{1-\alpha}^*$, there exists a $w \in \mathbb{R}^{\lceil(1-\alpha)K\rceil}$ such that $Cv = \bar{V}_{I-1} - N^{-1/2} S_{I-1}^{1/2} \hat{T}_{1-\alpha} w$, $e_{\lceil(1-\alpha)K\rceil}^T w = 1$, and $w \geq 0$. Hence, the inner minimization problem of (16) is written as

$$\begin{aligned} & \min_{v, w} v^T f(x) \\ & \text{s.t. } Cv + N^{-1/2} S_{I-1}^{1/2} \hat{T}_{1-\alpha} w = \bar{V}_{I-1}, \\ & \quad e_{\lceil(1-\alpha)K\rceil}^T w = 1, \end{aligned}$$

$$e_I^T v = 1,$$

$$v \geq 0, w \geq 0.$$

Let $\eta \in \mathbb{R}^{I-1}$, $\pi \in \mathbb{R}$, and $\tau \in \mathbb{R}$ be the dual variables regarding the first three constraints in the problem above. We obtain the dual problem as

$$\max_{\eta, \pi, \tau} \bar{V}_{I-1}^T \eta + \pi + \tau$$

$$\text{s.t. } C^T \eta + e_I \tau \leq f(x),$$

$$N^{-1/2} \hat{T}_{1-\alpha}^T \left(S_{I-1}^{1/2} \right)^T \eta + e_{\lceil(1-\alpha)K\rceil} \pi \leq 0.$$

The conclusion follows by using the unique representation of $f(x)$ via (20)–(21). □

From (24), we can see that as α increases, there will be fewer inequality constraints in (24c) and consequently the optimal value of (24) will increase. Problem (24) has I binary variables. Its scalability is contingent on the number of attributes and the number of pieces in each piecewise linear single-attribute utility function. The number of pieces depends on the complexity of the problem, empirical data, expert judgment, or various mathematical and statistical techniques. In some cases with relatively straightforward and homogeneous preferences, a PLUF may have just a few pieces or segments to capture the essential variations in utility. In other cases, analysts may choose to use more complex PLUFs with multiple segments to better capture the nonlinear nature of preferences. In Sect. 6.1, we conduct an experiment to test the computational time of problem (24). The test results exhibit that the computational time has an exponential growth rate as I increases.

4.2 Random concave PLUF case

We now turn to discuss the case that the random utility function of each single-attribute is concave. In this case, \mathcal{V}_c defined as in (6) is the support set of the random vector V of increments. In this section we substitute \mathcal{V}_c for \mathcal{V} in the ambiguity set $\mathcal{P}_B(\alpha)$ given in (13). We discuss a reformulation of DUPRO with the ambiguity set $\mathcal{P}_B(\alpha)$ in the proposition below.

Proposition 4 *DUPRO with the ambiguity set $\mathcal{P}_B(\alpha)$ is equivalent to*

$$\max_{x, \eta, \pi, \tau, \zeta, \lambda} \bar{V}_{I-1}^T \eta + \pi + \tau \tag{25a}$$

$$\text{s.t. } C^T \eta + e_I \tau + A^T \zeta - B^T \lambda \leq 0, \tag{25b}$$

$$N^{-1/2} \hat{T}_{1-\alpha}^T \left(S_{I-1}^{1/2} \right)^T \eta + e_{\lceil(1-\alpha)K\rceil} \pi \leq 0, \tag{25c}$$

$$D^T \lambda \leq x, \tag{25d}$$

$$E^T \lambda \leq e_M, \tag{25e}$$

$$\zeta \geq 0, \lambda \geq 0, \tag{25f}$$

$$x \in \mathfrak{X}, \tag{25g}$$

where A is defined as in (6), B , D , and E are defined as in (18)–(19).

Proof We first prove that the inner minimization problem of DUPRO can be represented as

$$\min_{y,z,v,w} x^T y + e_M^T z \tag{26a}$$

$$\text{s.t. } Cv + N^{-1/2} S_{I-1}^{1/2} \hat{T}_{1-\alpha} w = \bar{V}_{I-1}, \tag{26b}$$

$$e_{[(1-\alpha)K]}^T w = 1, \tag{26c}$$

$$e_I^T v = 1, \tag{26d}$$

$$Av \geq 0, \tag{26e}$$

$$Dy + Ez - Bv \geq 0, \tag{26f}$$

$$v \geq 0, w \geq 0, y \geq 0, z \geq 0. \tag{26g}$$

The \mathfrak{F} -mapping of $\mathcal{P}_B(\alpha)$ with the support \mathcal{V}_c is written as

$$\mathfrak{F}(\mathcal{P}_B(\alpha)) = \left\{ v \in \mathbb{R}_+^I \mid v \text{ satisfies conditions (26b) – (26e)} \right\}.$$

For any given $v \in \mathfrak{F}(\mathcal{P}_B(\alpha))$, $u_m(x_m; v_m)$ is a piecewise linear increasing concave function with breakpoints t_i and values $\sum_{j=1}^i v_{m,j}$, $i = 1, \dots, I_m$. It can be written as the minimization of all subgradients at points $(t_i, \sum_{j=1}^i v_{m,j})$ as

$$\begin{aligned} u_m(x_m; v_m) &= \min_{y_m, z_m, v_m} x_m y_m + z_m \\ \text{s.t. } & D_m y_m + z_m e_{I_m} - B_m v_m \geq 0, \\ & y_m \geq 0, z_m \geq 0, \end{aligned}$$

where D_m and B_m are given in (19). Therefore, we combine all u_m into the multi-attribute utility function

$$\begin{aligned} u(x; v) &= \sum_{m \in \mathfrak{M}} u_m(x_m; v_m) = \min_{y,z,v} x^T y + e_M^T z \\ \text{s.t. } & Dy + Ez - Bv \geq 0, \\ & y \geq 0, z \geq 0. \end{aligned}$$

This gives the objective (26a) and constraints (26f) and (26g).

Letting $\eta \in \mathbb{R}^{I-1}$, $\pi \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\zeta \in \mathbb{R}^{I-M}$, and $\lambda \in \mathbb{R}^I$ be the dual variables regarding the constraints, respectively, in the problem above, we can derive the Lagrange dual

$$\max_{\eta, \pi, \tau, \zeta, \lambda} \bar{V}_{I-1}^T \eta + \pi + \tau$$

$$\begin{aligned}
 \text{s.t. } & C^T \eta + e_I \tau + A^T \zeta - B^T \lambda \leq 0, \\
 & N^{-1/2} \hat{T}_{1-\alpha}^T \left(S_{I-1}^{1/2} \right)^T \eta + e_{\lceil(1-\alpha)K\rceil} \pi \leq 0, \\
 & D^T \lambda \leq x, \\
 & E^T \lambda \leq e_M, \\
 & \zeta \geq 0, \lambda \geq 0.
 \end{aligned}$$

A combination of the above dual program with the outer maximization problem yields (25). □

As in Proposition 3, the optimal value of (25) increases as α increases. Note that Proposition 3 states the MILP reformulation of DUPRO when considering general increasing piecewise linear single-attribute utility functions. Proposition 4 shows the LP reformulation of DUPRO if all the single-attribute utility functions are increasing and concave. This phenomenon is consistent with the reformulations in the literature of deterministic PRO models albeit our DUPRO framework is significantly different from deterministic PRO frameworks. This is primarily because we use the increment-based piecewise linear random utility function, which allows us to recast DUPRO as a deterministic maximin robust optimization problem (16). In the case when the piecewise linear random utility function is concave, we can use the support function method, as in the deterministic PRO reformulations, to reformulate the problem $\min_{v \in \mathfrak{F}(\mathcal{P})} u(x; v)$, where $\mathfrak{F}(\mathcal{P}) := \{v = \mathbb{E}_P(V) \mid P \in \mathcal{P}\}$, as an LP.

5 Quantification of modelling errors of the PLUF-based DUPRO

In the preceding sections, we focus on DUPRO with the DM’s true utility function of each attribute having a piecewise linear structure. In practice, the true random utility function is unknown and does not necessarily have a piecewise linear structure. What one can usually do is to use nonparametric utility assessment methods to obtain the DM’s evaluation (scoring) at different levels of attributes and then construct a PLUF as an approximation. This means that we may treat the proposed PLUF-based DUPRO as an approximation of general expected random utility maximization problems. In this section, we quantify the model (approximation) errors. Specifically, we consider the following expected utility maximization problem with deterministic attributes

$$\vartheta := \max_{x \in \mathfrak{X}} \mathbb{E}[U(x)], \tag{27}$$

where $U(x) = \sum_{m \in \mathfrak{M}} U_m(x)$, U_m is the DM’s true single-attribute utility function of attribute m . Without loss of generality, we assume that U_m is a general continuous and nondecreasing random utility function of attribute m defined on the domain $[a_m, b_m]$ for attribute $m \in \mathfrak{M}$ but it does not necessarily have a piecewise linear structure. Let $\mathfrak{X} \subseteq \otimes_{m \in \mathfrak{M}} [a_m, b_m] = [a_1, b_1] \times \dots \times [a_M, b_M]$ be a compact set. To ease the exposition, we assume that U is normalized with $U(a_1, \dots, a_M) = 0$ and $U(b_1, \dots, b_M) = 1$ almost surely.

5.1 Static piecewise linear approximation

We begin by discussing the piecewise linear approximation of $U(x)$ and its impact on the optimal value and optimal solutions. For each fixed $m \in \mathfrak{M}$, let $t_{m,i}$, for $i \in \mathcal{I}_m$, be defined as in (1). Let

$$V_{m,i} := U_m(t_{m,i}) - U_m(t_{m,i-1}), \quad i \in \mathcal{I}_m,$$

be the increment of U_m over the interval $[t_{m,i-1}, t_{m,i}]$ and $V_m = (V_{m,i}, \dots, V_{m,I_m})^T$. Throughout this section, we let the set of the breakpoints $\{t_{m,i} : i \in \mathcal{I}_m, m \in \mathfrak{M}\}$ be fixed. We construct a piecewise utility function over $[a_m, b_m]$, denoted by $u_m(\cdot; V_m)$, with breakpoints $t_{m,i}, i \in \mathcal{I}_m$, and

$$u_m(t_{m,i}; V_m) = U_m(t_{m,i}), \quad \text{for } i \in \mathcal{I}_m \tag{28}$$

and use $u_m(\cdot; V_m)$ to approximate U_m for $m \in \mathfrak{M}$. Let $V := (V_1^T, \dots, V_M^T)^T$ and

$$u(x; V) := \sum_{m \in \mathfrak{M}} u_m(x_m; V_m).$$

Note that the dimension of vector V is $I = \sum_{m=1}^M I_m$. We propose to obtain an approximated optimal value and optimal solution of problem (27) by solving the following piecewise linear approximated expected utility maximization problem:

$$\vartheta_I := \max_{x \in \mathfrak{X}} \mathbb{E}[u(x; V)]. \tag{29}$$

Let X^* and X_I be the respective sets of optimal solutions of problems (27) and (29). The subscript I indicates the optimal value depends on the total number of breakpoints I in the piecewise linear approximation. Of course, it also depends on the location of these points. Let

$$\Delta := \max_{m \in \mathfrak{M}} \max_{i \in \mathcal{I}_m} (t_{m,i} - t_{m,i-1}). \tag{30}$$

Obviously in order to secure a good approximation of ϑ by ϑ_I , we need Δ to be sufficiently small. In the case when the breakpoints are evenly spread, this is equivalent to setting I to a large value. To see this more clearly, let us consider the simplest case that $I_1 = I_2 = \dots = I_M$ where I_m is the number of breakpoints in piecewise approximation of the random utility function of attribute m . In that case $I_m = \frac{I}{M}$, where I denotes the total number of breakpoints of piecewise linear utility functions of all attributes. The requirement on $\Delta \leq \epsilon$ means that

$$\max_{m \in \mathfrak{M}} \frac{b_m - a_m}{I_m} = \max_{m \in \mathfrak{M}} \frac{b_m - a_m}{I/M} \leq \epsilon.$$

The latter is equivalent to $I \geq \max_{m \in \mathfrak{M}} \frac{M(b_m - a_m)}{\epsilon}$, which means $I = O(1/\epsilon)$. Unless specified otherwise, we assume in the rest of discussions that $I \rightarrow \infty$ ensures $\Delta \rightarrow 0$.

The following proposition addresses the approximation of problem (27) by problem (29) in terms of the optimal value and optimal solutions.

Proposition 5 *Suppose that for $m \in \mathfrak{M}$, U_m is Lipschitz continuous over $[a_m, b_m]$ with random Lipschitz modulus L_m almost surely, where $\mathbb{E}[L_m] < \infty$ and the expectation is taken w.r.t. the probability distribution of the random factors underlying $U(x)$. Then the following assertions hold.*

(i) For any $\epsilon > 0$,

$$|\vartheta - \vartheta_I| \leq \mathbb{E}[L]\Delta, \tag{31}$$

where $L := \sum_{m \in \mathfrak{M}} L_m$.

(ii) Let $\{x_I\}$ be a sequence of optimal solutions obtained from solving problem (29). Then every cluster point of the sequence is an optimal solution of problem (27), that is,

$$\lim_{I \rightarrow \infty} \mathbb{D}(X_I, X^*) = 0, \tag{32}$$

where $\mathbb{D}(A, B)$ denotes the access distance of set A over set B .

Proof Part (ii) follows directly from Part (i) and the well-known stability results in parametric programming, see e.g. [45, Lemma 3.8]. We only prove Part (i). Observe first that

$$|\vartheta - \vartheta_I| \leq \sup_{x \in \mathfrak{X}} |\mathbb{E}[u(x; V)] - \mathbb{E}[U(x)]|.$$

By the monotonicity, Lipschitz continuity of the utility function, and equality (28),

$$|u_m(x_m; V_m) - U_m(x_m)| \leq |U_m(t_{m,i}) - U_m(t_{m,i-1})| \leq L_m |t_{m,i} - t_{m,i-1}| \leq L_m \Delta, \text{ a.s.}$$

for any $x_m \in [t_{m,i-1}, t_{m,i}]$, which ensures that

$$\sup_{x_m \in [a_m, b_m]} |\mathbb{E}[u_m(x_m; V_m)] - \mathbb{E}[U_m(x_m)]| \leq \mathbb{E}[L_m]\Delta,$$

for $m \in \mathfrak{M}$ and hence,

$$\begin{aligned} \sup_{x \in \mathfrak{X}} |\mathbb{E}[u(x; V)] - \mathbb{E}[U(x)]| &\leq \sup_{x \in \otimes_{m \in \mathfrak{M}} [a_m, b_m]} |\mathbb{E}[u(x; V)] - \mathbb{E}[U(x)]| \\ &\leq \sum_{m \in \mathfrak{M}} \mathbb{E}[L_m]\Delta = \mathbb{E}[L]\Delta. \end{aligned}$$

The proof is complete. □

The result provides a theoretical guarantee for using a piecewise linear utility function model parameterized by a vector of random increments to approximate the true random utility function in absence of complete information on the latter. In other words, one may use a random PLUF as we discussed in Sect. 2 to represent a DM’s

true random utility function provided that it is Lipschitz continuous over the specified compact set and the breakpoints of the PLUF are sufficiently dense in the set. Lipschitz continuity over a compact set is fulfilled by most utility functions in the literature.

5.2 Sample average approximation

In practice, the true probability distribution of V is often unknown, but it is possible to obtain some observations from empirical data such as scoring. This means that instead of solving problem (29), we often solve the sample average approximation of the problem. Research on SAA is well-documented, see e.g. [34, 53, 54]. Let V^1, \dots, V^N denote an iid random sample of V and \bar{V} the sample mean. By Proposition 2, we propose to approximate $\mathbb{E}[U(\cdot)]$ using the sample average

$$\frac{1}{N} \sum_{n=1}^N u(\cdot; V^n) = \frac{1}{N} \sum_{n=1}^N f(\cdot)^T V^n = f(\cdot)^T \bar{V} = u(\cdot; \bar{V}).$$

The following proposition gives a qualitative description of such approximation.

Proposition 6 *Under the settings and conditions of Proposition 5, for any $\epsilon > 0$ and $\delta > 0$, there exists $N_0 > 0$ such that*

$$\text{Prob} \left(\sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[U(x)]| \geq \epsilon \right) \leq \delta, \tag{33}$$

for all $N \geq N_0(\epsilon, \delta) = O(\ln \delta / \epsilon^2)$ and $\Delta \leq \frac{\epsilon}{2\mathbb{E}[L]}$, where Δ is defined as in Proposition 5.

Proof We write $\bar{V} = (\bar{V}_1^T, \dots, \bar{V}_M^T)^T$, where \bar{V}_m is the associated sample average for the vector of increments regarding the single-attribute utility function of attribute m . By the triangle inequality, we have

$$\begin{aligned} |u_m(x; \bar{V}_m) - \mathbb{E}[U_m(x)]| &\leq |u_m(x; \bar{V}_m) - \mathbb{E}[u_m(x; V_m)]| \\ &\quad + |\mathbb{E}[u_m(x; V_m)] - \mathbb{E}[U_m(x)]|, \text{ for } m = 1, \dots, M. \end{aligned} \tag{34}$$

By the monotonicity, Lipschitz continuity of the utility function, and equality (28),

$$|u_m(x_m; V_m) - U_m(x_m)| \leq |U_m(t_{m,i}) - U_m(t_{m,i-1})| \leq L_m |t_{m,i} - t_{m,i-1}| \leq L_m \Delta \text{ a.s.},$$

for all $x_m \in [t_{m,i-1}, t_{m,i}]$, which implies that

$$\sup_{x_m \in [a_m, b_m]} |\mathbb{E}[u_m(x; V_m)] - \mathbb{E}[U_m(x)]| \leq \sup_{x_m \in [a_m, b_m]} \mathbb{E}|u_m(x; V_m) - U_m(x)| \leq \mathbb{E}[L_m]\Delta.$$

Thus

$$\begin{aligned} \sup_{x \in \mathfrak{X}} |\mathbb{E}[u(x; V)] - \mathbb{E}[U(x)]| &\leq \mathbb{E} \left[\sum_{m \in \mathfrak{M}} \sup_{x_m \in [a_m, b_m]} |u_m(x; V_m) - U_m(x)| \right] \\ &\leq \mathbb{E}[L]\Delta. \end{aligned} \tag{35}$$

Let Δ be sufficiently small such that $\mathbb{E}[L]\Delta \leq \epsilon/2$. By (34) and (35), we have

$$\begin{aligned} \sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[U(x)]| &\leq \sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[u(x; V)]| + \mathbb{E}[L]\Delta \\ &\leq \sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[u(x; V)]| + \frac{\epsilon}{2}. \end{aligned}$$

Together with Proposition 2, we obtain

$$\begin{aligned} \text{Prob} \left(\sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[U(x)]| \geq \epsilon \right) &\leq \text{Prob} \left(\sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[u(x; V)]| \geq \frac{\epsilon}{2} \right) \\ &\leq \text{Prob} \left(\|\bar{V} - \mathbb{E}[V]\| \sup_{x \in \mathfrak{X}} \|f(x)\| \geq \frac{\epsilon}{2} \right). \end{aligned}$$

By the definition of f in Proposition 2, we know that $\sup_{x \in \mathfrak{X}} \|f(x)\| < \infty$. Thus by Cramér’s large deviation theorem, there exists a positive integer N_0 and positive constant $\Upsilon(\epsilon)$ (depending on ϵ with $\Upsilon(0) = 0$) such that for all $N \geq N_0$

$$\text{Prob} \left(\|\bar{V} - \mathbb{E}[V]\| \sup_{x \in \mathfrak{X}} \|f(x)\| \geq \frac{\epsilon}{2} \right) \leq e^{-\Upsilon(\epsilon)N}. \tag{36}$$

Specifically, it follows by Hoeffding’s inequality (see e.g. [55, Theorem 2]), for fixed $\delta \in (0, 1)$, we can set $N_0(\epsilon, \delta) := -\frac{\ln \delta}{\epsilon^2}$ and subsequently obtain

$$\text{Prob} \left(\|\bar{V} - \mathbb{E}[V]\| \sup_{x \in \mathfrak{X}} \|f(x)\| \geq \frac{\epsilon}{2} \right) \leq \delta \tag{37}$$

for all $N \geq N_0(\epsilon, \delta)$. The proof is complete. □

With the theoretical justification of approximation of $\mathbb{E}[U(x)]$ with $u(x; \bar{V})$, we consider the sample data-based utility maximization problem

$$\max_{x \in \mathfrak{X}} u(x; \bar{V}). \tag{38}$$

Differing from the standard sample average approximation scheme in stochastic programming, problem (38) consists of two layers of approximation: piecewise linear approximation of utility function $U(x)$ by $u(x; V)$ and sample average approximation of $\mathbb{E}[u(x; V)]$. Let $\vartheta_{N,I}$ denote the optimal value of problem (38) and $X_{N,I}$ the set of the optimal solutions. Here the subscripts N and I are used to indicate that these quantities depend on both the sample size N and the number of breakpoints I . By combining Propositions 5 and 6, we can establish convergence of $u(x; \bar{V})$ to $\mathbb{E}[U(x)]$ and associated optimal values and optimal solutions as both N and I go to infinity. The following proposition addresses this.

Proposition 7 *Assume the settings and conditions of Propositions 5 and 6. Then the following assertions hold.*

- (i) *Let I (the set of breakpoints) be fixed. For any $\epsilon > 0$ and $\delta > 0$, there exists $N_0(\epsilon, \delta) = O(\ln \delta/\epsilon^2)$ such that*

$$\text{Prob}(|\vartheta_{N,I} - \vartheta| \geq \epsilon) \geq \delta, \tag{39}$$

for all $N \geq N_0(\epsilon, \delta)$ and $\Delta \leq \frac{\epsilon}{2\mathbb{E}[L]}$.

- (ii) *For any positive number ϵ ,*

$$\lim_{N,I \rightarrow \infty} \text{Prob}(|\vartheta_{N,I} - \vartheta| \geq \epsilon) = 0 \tag{40}$$

and

$$\lim_{N,I \rightarrow \infty} \text{Prob}(\mathbb{D}(X_{N,I}, X^*) \geq \epsilon) = 0. \tag{41}$$

Proof Observe that

$$|\vartheta_{N,I} - \vartheta| \leq \sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[U(x)]|. \tag{42}$$

Part (i) follows directly from (33) and (42). Part (ii). Equality (40) follows from (31), (39) and (42) whereas (41) follows from (40) and classical stability results (see e.g. [10]). □

The proposition provides a theoretical guarantee that one can solve problem (38) to obtain an approximation of the optimal value and optimal solution with specified confidence provided that the sample size is sufficiently large.

5.3 DUPRO approximation and convergence

The convergence results established in Proposition 7 are based on the assumption that the sample size N can be arbitrarily large. In some practical data-driven problems, this assumption may not be fulfilled. This motivates us to use DUPRO with a piecewise linear random utility function to approximate problem (27). In Sect. 3.2, we propose two approaches to construct an ambiguity set of DUPRO: an ellipsoid moment region \mathcal{P}_A in (8) and a bootstrap confidence region \mathcal{P}_B in (13).

We begin with \mathcal{P}_A based **DUPRO** and move on to the one based on \mathcal{P}_B . To facilitate reading, we repeat the definition of $\mathcal{P}_A(\gamma_N)$

$$\mathcal{P}_A(\gamma_N) := \left\{ P \in \mathcal{P}(\mathbb{R}^I) \mid \begin{array}{l} P(V \in \mathcal{V}) = 1 \\ \left\| S_{I-1}^{-1/2} (C\mathbb{E}_P[V] - \bar{V}_{I-1}) \right\|^2 \leq \gamma_N \end{array} \right\}. \tag{43}$$

By Proposition 2, we can reformulate **DUPRO** with $\mathcal{P}_A(\gamma_N)$ as

$$\vartheta_{\text{elp}} := \max_{x \in \mathfrak{X}} \min_{v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} u(x; v), \tag{44}$$

where

$$\mathfrak{F}(\mathcal{P}_A(\gamma_N)) := \left\{ v \in \mathcal{V} \mid (Cv - \bar{V}_{I-1})^T S_{I-1}^{-1} (Cv - \bar{V}_{I-1}) \leq \gamma_N \right\}. \tag{45}$$

The reformulation is sensible in that the objective function only depends on $\mathbb{E}[V]$ and so does the moment constraint in the ambiguity set. Analogous to $\gamma_N^{(1)}$ defined as in (9), we require that γ_N decreases to 0 as N goes to ∞ . Then the ambiguity set $\mathfrak{F}(\mathcal{P}_A(\gamma_N))$ converges to the singleton $\{\mu = \mathbb{E}[V]\}$. We investigate convergence of ϑ_{elp} to ϑ (the optimal value of problem (27)) as N and I go to ∞ and γ_N is driven to 0. The next theorem states the convergence.

Theorem 1 (**DUPRO** with ellipsoidal ambiguity (45)) *Let γ_N in (43) monotonically decrease to 0 as $N \rightarrow \infty$. Then for any $\alpha \in (0, 1)$ and small $\epsilon > 0$, there exist positive numbers $I_0(\epsilon), N_0(\epsilon, \alpha)$ such that*

$$\text{Prob}(|\vartheta_{\text{elp}} - \vartheta| \geq \epsilon) \leq \alpha \tag{46}$$

for all $N \geq N_0(\epsilon, \alpha) = O((\ln \alpha)/\epsilon^2)$, $I \geq I_0(\epsilon) = O(1/\epsilon)$.

Proof For fixed $v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))$, it follows by (14) that

$$\begin{aligned} |u(x; v) - \mathbb{E}[U(x)]| &\leq |u(x; v) - u(x; \bar{V})| + |u(x; \bar{V}) - \mathbb{E}[U(x)]| \\ &= |f(x)^T (v - \bar{V})| + |u(x; \bar{V}) - \mathbb{E}[U(x)]|. \end{aligned} \tag{47}$$

Let E denote the event that $S_{I-1}^{-1} \succ (2\Sigma_{I-1})^{-1}$ and F denote the event that $\sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[U(x)]| \leq \epsilon/2$. Since $2\Sigma_{I-1} \succ \Sigma_{I-1}$, by Corollary 6 in [55], there exists $\hat{N}_0(\alpha) = O(-\ln \alpha)$ such that

$$\text{Prob}(E) = \text{Prob}\left(S_{I-1}^{-1} \succ (2\Sigma_{I-1})^{-1}\right) \geq 1 - \alpha \tag{48}$$

for all $N \geq \hat{N}_0(\alpha)$. Moreover, it follows by Proposition 6 and its proof, that there exists $\tilde{N}_0(\epsilon/2, \alpha) = O((\ln \alpha)/\epsilon^2)$ such that

$$\text{Prob}(F) = \left(\sup_{x \in \mathfrak{X}} |u(x; \bar{V}) - \mathbb{E}[U(x)]| \leq \epsilon/2 \right) \geq \alpha \tag{49}$$

for all $N \geq \tilde{N}_0(\epsilon/2, \alpha)$. Thus

$$\text{Prob}(E \cap F) \geq 1 - \alpha \tag{50}$$

for all $N \geq N_0(\epsilon, \alpha) := \max\{\hat{N}_0(\alpha), \tilde{N}_0(\epsilon/2, \alpha)\} = O((\ln \alpha)/\epsilon^2)$. By (47),

$$\begin{aligned} & \text{Prob}(|\vartheta_{\text{elp}} - \vartheta| \geq \epsilon) \\ & \leq \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} |u(x; v) - \mathbb{E}[U(x)]| \geq \epsilon\right) \\ & = \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} |u(x; v) - \mathbb{E}[U(x)]| \geq \epsilon \cap (E \cap F)\right) \\ & \quad + \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} |u(x; v) - \mathbb{E}[U(x)]| \geq \epsilon \cap \overline{E \cap F}\right) \\ & \leq \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} |u(x; v) - u(x; \bar{V})| + |u(x; \bar{V}) - \mathbb{E}[U(x)]| \geq \epsilon \mid E \cap F\right) \\ & \quad \text{Prob}(E \cap F) + \text{Prob}(\overline{E \cap F}) \\ & \leq \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} |u(x; v) - u(x; \bar{V})| \geq \frac{\epsilon}{2}\right) \text{Prob}(E \cap F) \\ & \quad + \text{Prob}(\overline{E \cap F}) \quad (\text{By (49)}) \\ & \leq \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} |f(x)^T(v - \bar{V})| \geq \frac{\epsilon}{2}\right) + \alpha. \quad (\text{By (50)}) \end{aligned} \tag{51}$$

Let $f_{I-1}(x)$, v_{I-1} and \bar{V}_{I-1} denote respectively the vectors which consist of the first $I - 1$ components of $f(x)$, v and \bar{V} , let $f_I(x)$, v_I , \bar{V}_I denote the I -th component of $f(x)$, v and \bar{V} , and e_{I-1} denotes an $(I - 1)$ -dimensional vector with all ones. Then

$$v_I - \bar{V}_I = 1 - e_{I-1}^T v_{I-1} - (1 - e_{I-1}^T \bar{V}_{I-1}) = e_{I-1}^T (\bar{V}_{I-1} - v_{I-1})$$

and subsequently

$$f(x)^T(v - \bar{V}) = f_{I-1}(x)^T(v_{I-1} - \bar{V}_{I-1}) + f_I(x)e_{I-1}^T(\bar{V}_{I-1} - v_{I-1}).$$

By the Hölder inequality

$$\begin{aligned} |f(x)^T(v - \bar{V})| & \leq \|(2\Sigma)_{I-1}^{1/2} f_{I-1}(x)\| \|(2\Sigma)_{I-1}^{-1/2} (v_{I-1} - \bar{V}_{I-1})\| \\ & \quad + |f_I(x)| \|(2\Sigma)_{I-1}^{1/2} e_{I-1}\| \|(2\Sigma)_{I-1}^{-1/2} (v_{I-1} - \bar{V}_{I-1})\| \\ & \leq (I - 1) \|(2\Sigma)_{I-1}^{1/2}\| \|f(x)\| \|(2\Sigma)_{I-1}^{-1/2} (v_{I-1} - \bar{V}_{I-1})\|. \end{aligned} \tag{52}$$

Let $N \geq N_0(\epsilon, \alpha)$ be sufficiently large such that

$$\frac{\epsilon}{2(I - 1)\|(2\Sigma)_{I-1}^{1/2}\| \sup_{x \in \mathcal{X}} \|f(x)\|} > \sqrt{\gamma_N} \tag{53}$$

for all $N \geq N_0(\epsilon, \alpha)$. Combining (51)–(53), we have

$$\begin{aligned} & \text{Prob}(|\vartheta_{\text{elp}} - \vartheta| \geq \epsilon) \\ & \leq \text{Prob}\left(\sup_{x \in \mathcal{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} |f(x)^T(v - \bar{V})| \geq \frac{\epsilon}{2}\right) + \alpha \\ & \leq \text{Prob}\left(\sup_{x \in \mathcal{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} (I - 1)\|(2\Sigma)_{I-1}^{1/2}\| \|f(x)\| \|(2\Sigma)_{I-1}^{-1/2}(v_{I-1} - \bar{V}_{I-1})\| \geq \frac{\epsilon}{2}\right) + \alpha \\ & \leq \text{Prob}\left(\sup_{v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} \|(2\Sigma)_{I-1}^{-1/2}(v_{I-1} - \bar{V}_{I-1})\| \geq \frac{\epsilon}{2 \sup_{x \in \mathcal{X}} (I - 1)\|(2\Sigma)_{I-1}^{1/2}\| \|f(x)\|}\right) + \alpha \\ & \leq \text{Prob}\left(\sup_{v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} \|(2\Sigma)_{I-1}^{-1/2}(v_{I-1} - \bar{V}_{I-1})\| > \sqrt{\gamma_N}\right) + \alpha \\ & \leq \text{Prob}\left(\sup_{v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N))} (Cv - \bar{V}_{I-1})^T S_{I-1}^{-1}(Cv - \bar{V}_{I-1}) > \gamma_N\right) + \alpha \\ & = 0 + \alpha = \alpha, \end{aligned}$$

where the last inequality is due to (48) and the last equality is because of (45). □

The theorem provides a theoretical guarantee that the optimal value obtained from solving problem (44) approximates the true one with a specified confidence when the sample size goes to infinity and $\gamma_N \rightarrow 0$. This kind of convergence result should be distinguished from those in the literature of DRO (see e.g. [60]) in that here problem (44) is a robust optimization model instead of a DRO model, and the ambiguity set $\mathfrak{F}(\mathcal{P}_A(\gamma_N))$ in terms of v in (45) shrinks to a singleton whereas the ambiguity set in terms of P in (43) does not. One of the main challenges that we have to tackle in the proof is that the ellipsoid in (45) is defined for v_{I-1} rather than v whereas the objective function $u(x; v)$ depends on v . The mismatch requires us to derive an upper bound for $|f(x)^T(v - \bar{V})|$ in terms of $S_{I-1}^{-1/2}(v_{I-1} - \bar{V}_{I-1})$.

We now move on to investigate convergence of the optimal value of DUPRO to ϑ when the ambiguity set is constructed by the bootstrap samples. Recall that $\mathfrak{W}_{1-\alpha}^*$ defined as in (11) is the convex hull of points $T_{(1)}^*, \dots, T_{(\lceil(1-\alpha)K\rceil)}^*$. Here we consider the case that $K = \infty$ and use $\mathfrak{W}_{1-\alpha}$ to denote the 100(1 - α)% interior points of the convex hull of $T_{(1)}^*, \dots, T_{(\infty)}^*$ based on Tukey’s depth. Consider DUPRO with

$$\mathcal{P}_{\hat{B}}(\alpha) = \left\{ P \in \mathcal{P}(\mathbb{R}^J) \mid P(V \in \mathcal{V}) = 1, C\mathbb{E}_P[V] = \bar{V}_{I-1} - S_{I-1}^{1/2}\tilde{w}/\sqrt{N}, \tilde{w} \in \mathfrak{W}_{1-\alpha} \right\}. \tag{54}$$

The difference between $\mathcal{P}_{\hat{B}}(\alpha)$ and $\mathcal{P}_B(\alpha)$ in (13) is that $\mathcal{P}_B(\alpha)$ is defined with $\mathfrak{W}_{1-\alpha}^*$ while $\mathcal{P}_{\hat{B}}(\alpha)$ is based on $\mathfrak{W}_{1-\alpha}$. The next lemma refers to Theorem 1 in [72] and the comments following the theorem.

Lemma 1 *If P , the true probability measure of V , is absolutely continuous with respect to Lebesgue measure in \mathbb{R}^I , then there exists $\delta_{1-\alpha} > 0$ depending on α such that*

$$\lim_{N \rightarrow \infty} \mathfrak{W}_{1-\alpha} \subseteq \mathfrak{B}(\delta_{1-\alpha}), \text{ a.s.}, \tag{55}$$

where $\mathfrak{B}(\delta_{1-\alpha})$ is an I -dimensional ball centered at 0 with radius $\delta_{1-\alpha}$.

The lemma indicates that $\mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha))$ is contained in an ellipsoid centered at \bar{V}_{I-1} when N goes to infinity. We will use this fact and Theorem 1 to establish the convergence property of DUPRO with $\mathcal{P}_{\hat{B}}(\alpha)$ in the next theorem.

Theorem 2 (DUPRO with bootstrap ambiguity (54)) *Let*

$$\vartheta_{\text{bts}} := \max_{x \in \mathcal{X}} \min_{v \in \mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha))} u(x; v) \tag{56}$$

and $\tau \in (0, 1)$, where

$$\mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha)) = \left\{ v \in \mathbb{R}^I \left| \begin{array}{l} v \geq 0 \\ e_I^T v = 1 \\ Cv = \bar{V}_{I-1} - S_{I-1}^{1/2} \tilde{w} / \sqrt{N}, \tilde{w} \in \mathfrak{W}_{1-\alpha} \end{array} \right. \right\}. \tag{57}$$

Suppose that P , the true probability measure induced by V , is absolutely continuous in \mathbb{R}^I . Then for any small positive number ϵ , there exist positive numbers I_0, \hat{N}_0 such that

$$\text{Prob}(|\vartheta_{\text{bts}} - \vartheta| \geq \epsilon) \leq \tau \tag{58}$$

for all $N \geq \hat{N}_0, I \geq I_0$.

It is important to highlight that (56) is a maximin robust optimization problem rather than a DRO problem. Moreover, by Lemma 1, $\mathfrak{W}_{1-\alpha}$ is a bounded set a.s. as N goes to infinity and by Corollary 6 in [55], $S_{I-1}^{1/2}$ converges to $\Sigma_{I-1}^{1/2}$ at an exponential rate with the increase of N , then $S_{I-1}^{1/2} \tilde{w} / \sqrt{N} \rightarrow 0$ as $N \rightarrow \infty$ w.p.1. This means the ambiguity set $\mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha))$ shrinks to a singleton w.p.1 as $N \rightarrow \infty$.

Proof The thrust of the proof is to show

$$\mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha)) \subseteq \mathfrak{F}(\mathcal{P}_A(\gamma_N^{(2)})) \tag{59}$$

for some $\gamma_N^{(2)}$ when N is sufficiently large and then the conclusion follows from a similar proof to that of Theorem 1. To this end, we choose $\eta \in (0, \tau)$ and let

$\beta = 1 - (\tau - \eta)$. It follows by Lemma 1 that there exist $\delta_{1-\alpha} > 0$ and $N_1 > 0$ such that

$$\text{Prob}(\mathfrak{M}_{1-\alpha} \subseteq \mathfrak{B}(\delta_{1-\alpha})) \geq \beta, \tag{60}$$

for all $N > N_1$. Let $\gamma_N^{(2)} := \delta_{1-\alpha}^2/N$. Note that

$$\mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha)) = \{v \in \mathcal{V} \mid S_{I-1}^{-1/2}(Cv - \bar{V}_{I-1}) = -\tilde{w}/\sqrt{N}, \tilde{w} \in \mathfrak{M}_{1-\alpha}\}$$

and

$$\mathfrak{F}(\mathcal{P}_A(\gamma_N^{(2)})) = \{v \in \mathcal{V} \mid S_{I-1}^{-1/2}(Cv - \bar{V}_{I-1}) = -\tilde{w}/\sqrt{N}, \tilde{w} \in \mathfrak{B}(\delta_{1-\alpha})\}.$$

Let G denote the event that (59) holds. We then obtain

$$\text{Prob}\left(\mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha)) \subseteq \mathfrak{F}(\mathcal{P}_A(\gamma_N^{(2)}))\right) \geq \beta \tag{61}$$

for $N > N_1$. Consequently, for $N > N_1$, we have

$$\begin{aligned} \text{Prob}(|\vartheta_{\text{bts}} - \vartheta| \geq \epsilon) &\leq \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha))} |u(x; v) - \mathbb{E}[U(x)]| \geq \epsilon\right) \\ &\leq \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha))} |u(x; v) - \mathbb{E}[U(x)]| \geq \epsilon \mid G\right) \text{Prob}(G) \\ &\quad + \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_{\hat{B}}(\alpha))} |u(x; v) - \mathbb{E}[U(x)]| \geq \epsilon \mid \bar{G}\right) \text{Prob}(\bar{G}) \\ &\leq \text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N^{(2)}))} |u(x; v) - \mathbb{E}[U(x)]| \geq \epsilon\right) + \text{Prob}(\bar{G}), \end{aligned} \tag{62}$$

where \bar{G} is the complement. It follows from a similar analysis to the proof of Theorem 1 that, there exist N_2 and I_0 such that

$$\text{Prob}\left(\sup_{x \in \mathfrak{X}, v \in \mathfrak{F}(\mathcal{P}_A(\gamma_N^{(2)}))} |u(x; v) - \mathbb{E}[U(x)]| \geq \epsilon\right) \leq \eta, \tag{63}$$

for $N > N_2$ and $I > I_0$. Let $\hat{N}_0 = \max\{N_1, N_2\}$. Then for $N \geq \hat{N}_0$, we have from (61)–(63) that

$$\text{Prob}(|\vartheta_{\text{bts}} - \vartheta| \geq \epsilon) \leq \eta + (1 - \beta) = \tau.$$

The proof is complete. □

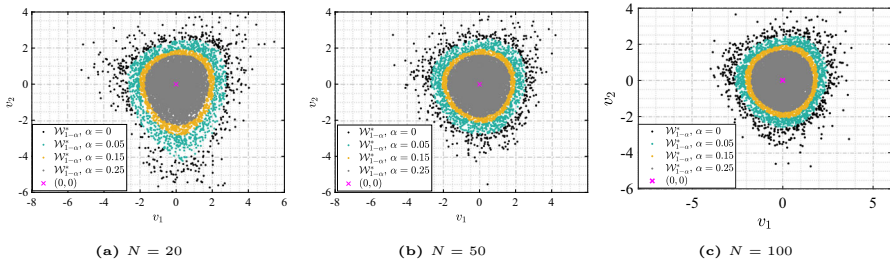


Fig. 3 $N = 20, N = 50, N = 100, K = 10,000$. The sets of coloured points represent the set $\mathfrak{W}_{1-\alpha}^*$ with different values of α

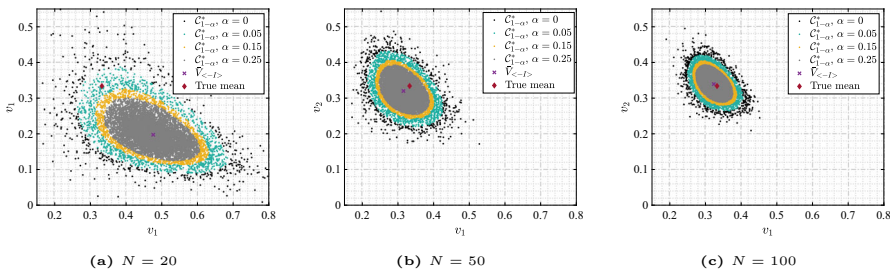


Fig. 4 $N = 20, N = 50, N = 100, K = 10,000$. The sets of coloured points represent the set $\mathfrak{C}_{1-\alpha}^*$ with different values of α

The theorem guarantees that the optimal value obtained from solving problem (56) converges to the true one with a specified confidence as the sample size N of the original sample and the number K of bootstrap resamples go to infinity. The constant \hat{N}_0 depends heavily on N_1 to ensure (61), and this is essentially down to the rate of convergence in (55). In the literature of statistics, it is shown that the empirical cumulative distribution function of statistical estimator based on bootstrap resamples converges to the one based on the original sample at a rate of $o(N^{-1/2})$ in single variate case, see Theorem 3.10 and follow-up discussions in [20]. The situation here is much more complex since here v_{I-1} is multivariate and the ambiguity set is constructed via Tukey’s depth which trims down some of the resamples. It remains an open challenging question as to whether the classical result may be generalized to this case, we leave this for future research. However, we have done some numerical studies on the convergence of $\mathfrak{W}_{1-\alpha}^*$ as N increases. We further set $K = 10,000$ and such a large K guarantees that $\mathfrak{W}_{1-\alpha}^*$ can closely approximate $\mathfrak{W}_{1-\alpha}$. On this basis, we examine the convergence of $\mathfrak{W}_{1-\alpha}^*$ as N increases from 20 to 50 and 100 for different α values. Figure 3 displays the tendency of the convergence. Figure 4 depicts the respective ambiguity sets $\mathfrak{C}_{1-\alpha}^*$ in (12). We can see that when original sample size N reaches 50, the true mean lies within the ambiguity set $\mathfrak{C}_{1-\alpha}^*$ with $\alpha = 0.25$.

Table 1 The breakpoints of the tree attributes

Attributes	$t_{m,1}$	$t_{m,2}$	$t_{m,3}$	$t_{m,4}$	$t_{m,5}$	$t_{m,6}$
Attribute 1	0.0667	0.4	0.6667	1	–	–
Attribute 2	0.05	0.3	0.4	0.5	0.75	1
Attribute 3	0.0667	0.2	0.5333	0.6667	1	–

6 Numerical tests

We have carried out some numerical studies of the proposed DUPRO model. In this section, we report the test results. We begin by examining the performance of DUPRO with bootstrap ambiguity set where we look into the effect of the sample size of the random parameter V , the number of the bootstrap resamples, and the critical value determining the size of \mathcal{P}_B (see (13)) and then move on to comparative analysis of DUPRO relative to the sample average approximation model.

6.1 Performance studies of the bootstrap-based DUPRO model

The first set of tests are aimed to analyse the performance of DUPRO with the ambiguity set $\mathcal{P}_B(\alpha)$ defined as in (13):

- (a) Report the optimal values, worst-case expected utility functions, and computational time of DUPRO based on different settings of the critical value α .
- (b) Investigate the impact of increase of the original sample size N in relation to the convergence result in Theorem 2.
- (c) Validate the reliability and performance of DUPRO using out-of-sample tests.

The test problem has three attributes with a feasible set $\mathcal{X} := \{x \in \mathbb{R}_+^3 \mid e_3^T x = 1\}$. The set of breakpoints of the piecewise utility functions can be set arbitrarily and we fix them here, see Table 1. Thus $I_1 = 4$, $I_2 = 6$ and $I_3 = 5$ and $I = 15$. We assume that the true probability distribution of vector V (of dimension $I = 15$) follows Dirichlet distribution with order 15 with parameters $(0.5, 0.5, \dots, 0.5) \in \mathbb{R}^{15}$. Thus, $\mathbb{E}[V] = (1/15, \dots, 1/15) \in \mathbb{R}^{15}$. In this subsection, we report our findings. All of the tests are carried out by Matlab 2022a installed on a PC (16GB RAM, CPU 2.3 GHz) with Intel Core i7 processor. We use Gurobi solver to solve the mixed integer problem (24).

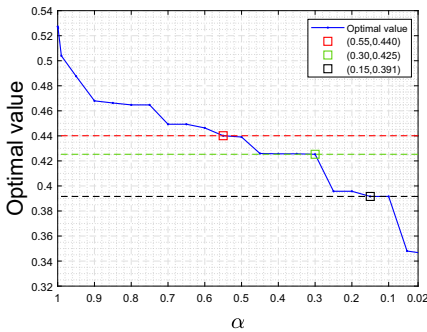
6.1.1 Test case (a): Optimal value, worst-case expected utility function and computational time

We generate a random sample, V^1, \dots, V^{50} , with Dirichlet distribution and then generate $K = 10,000$ bootstrap resamples $V^{k,1}, \dots, V^{k,50}$, $k = 1, \dots, 10,000$. Table 2 displays the sample mean of V .

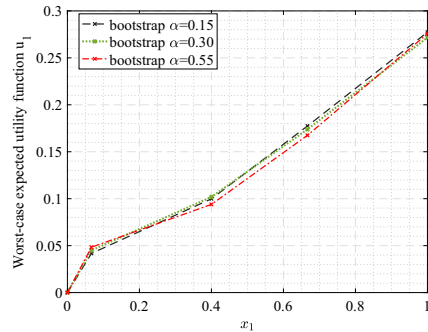
With the sample data, we are able to calculate Tukey’s depth $d_{(1)}, \dots, d_{(K)}$ and then construct a $100(1 - \alpha)\%$ bootstrap confidence region $\mathcal{C}_{1-\alpha}^*$, based on which, we can obtain the ambiguity set $\mathcal{P}_B(\alpha)$ in (13) (or $\mathfrak{F}(\mathcal{P}_B(\alpha))$ in (23)). Next, we solve

Table 2 The sample mean $\bar{V}_{m,k}$ at the breakpoints

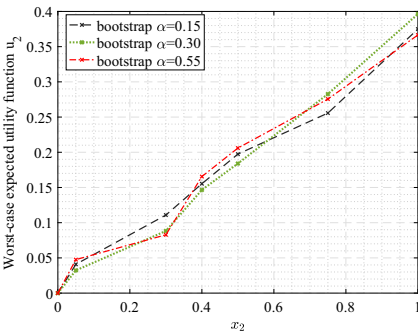
Attributes	$\bar{V}_{m,1}$	$\bar{V}_{m,2}$	$\bar{V}_{m,3}$	$\bar{V}_{m,4}$	$\bar{V}_{m,5}$	$\bar{V}_{m,6}$
Attribute 1	0.0591	0.0537	0.0642	0.0719	–	–
Attribute 2	0.0514	0.0592	0.0756	0.0551	0.1052	0.0686
Attribute 3	0.0706	0.0675	0.0594	0.0830	0.0555	–



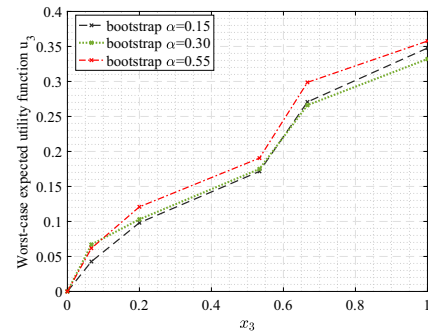
(a) optimal values



(b) utility function of attribute 1.



(c) utility function of attribute 2.



(d) utility function of attribute 3.

Fig. 5 **a** The optimal value of DUPRO with ambiguity set by bootstrap; **b–d** The worst expected utility functions when $\alpha = 0.15, \alpha = 0.30$ and $\alpha = 0.55$

problem (22), equivalent to DUPRO, via solving program (24) and obtain an optimal solution x^* . By substituting x^* into the inner problem $\min_{v \in \mathcal{F}(\mathcal{P}_B(\alpha))} v^T f(x^*)$, we obtain the worst v^* and subsequently the worst expected utility function. Figure 5a depicts convergence of the optimal value of problem (24) as α reduces from 99.99% to 2%. For example, we present the worst-case expected utility functions u_1, u_2 and u_3 when $\alpha = 0.15, 0.30$, and 0.55 , respectively, in Fig. 5b–d.

Next, we test the scalability of DUPRO when I , the total number of the linear pieces of the utility functions, increases. Concomitantly, the complexity of problem (24) increases with more binary variables. We equally partition each of the intervals between the breakpoints given in Table 1 into τ subintervals with $I = 15\tau$. In the study,

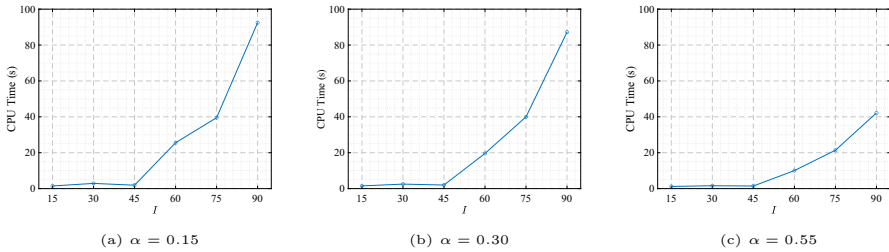


Fig. 6 Running time of DUPRO (problem (24))

we vary τ from 1 to 6. Figure 6 depicts the running CPU time for solving problem (24) with $N = 50$, $K = 10,000$, and α varying from 0.15 to 0.55. It shows that the increase in I yields an exponential increase of the running time.

6.1.2 Test case (b): convergence analysis

We investigate the convergence result in Theorem 2. In the test problem, the true random utility function is step-like which means

$$\mathbb{E}[u(x; V)] = \mathbb{E}[V]^T f(x) = \frac{e^{15^T f(x)}}{15},$$

where $f(x)$ is defined as in (14) and all of its breakpoints are given in Table 1. Let $\vartheta = \max_{x \in \mathcal{X}} e^{15^T f(x)}/15$. We use the bootstrap approach to construct the ambiguity set $\mathcal{P}_B(\alpha)$ in (13). Let $\tilde{\vartheta}_{\text{bts}}$ be the optimal value of DUPRO (problem (24)). The set $\mathcal{P}_B(\alpha)$ is an approximation to $\mathcal{P}_{\tilde{B}}(\alpha)$ defined as in (57), and thus $\tilde{\vartheta}_{\text{bts}}$ approximates ϑ_{bts} defined as in (56). We choose $K = 10,000$ replications for bootstrapping in this test. Such a large K guarantees the quality of the approximation. Hence, we can obtain

$$\text{Prob}(|\vartheta_{\text{bts}} - \vartheta| \geq \epsilon) \simeq \text{Prob}\left(|\tilde{\vartheta}_{\text{bts}} - \vartheta| \geq \epsilon\right), \tag{64}$$

where the right-hand side is the complementary cumulative distribution function (ccdf) of $|\tilde{\vartheta}_{\text{bts}} - \vartheta|$ with respect to ϵ .

We run a Monte Carlo simulation to generate 1000 observations of $|\tilde{\vartheta}_{\text{bts}} - \vartheta|$. On this basis, we depict the empirical distribution to approximate the probability distribution function (pdf) of $|\tilde{\vartheta}_{\text{bts}} - \vartheta|$. Note that the ccdf is the area under the pdf on the right of a given critical value ϵ . To ease the exposition, we omit “empirical” in the statement below. Figure 7 depicts the pdfs when α is 0.05 or 0.15, and N (the size of the original random sample) varies from 30 to 200. In comparison, we also plot the pdf’s of $|\vartheta_{\text{elp}} - \vartheta|$ in the case of using ellipsoidal ambiguity set in (46). For a fixed α , an increase of N results in a narrower and more concentrated pdf curve, which shifts to the left and its right tail gradually diminishing or becoming lighter. This shows the convergence of $\tilde{\vartheta}_{\text{bts}}$ to ϑ in probability as Theorem 2 claims. That is, the right-hand-side probability in (64) dwindles for any given ϵ . For example, when $\alpha = 0.15$ and $\epsilon = 0.05$, the probability is 0.999, 0.994, 0.882, 0.758, 0.558 and 0.390,

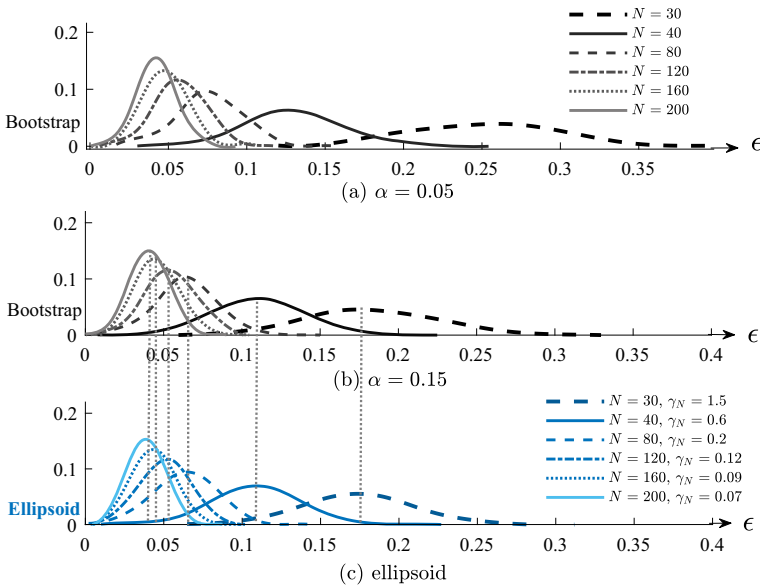


Fig. 7 p.d.f. of $|\vartheta_{\text{bts}} - \vartheta_I|$ and p.d.f. of $|\vartheta_{\text{elp}} - \vartheta_I|$

respectively, as N increases from 30 to 200. Moreover, we observe that the pdf differs with respect to varying α value when N is small ($N = 30, 40$ in this case), becomes indifferent N is large. Theorem 1 indicates that the probability of convergence of ϑ_{elp} relies on a sequence of γ_N in infinite descent. We choose γ_N regarding N to obtain the results similar to the case with $\alpha = 0.15$. It is worth noting that Proposition 1 recommends too large γ_N , under which we would have $\vartheta_{\text{elp}} = 0$ in our tests.

6.1.3 Test case (c): out-of-sample test

Analogous to [22], we analyze the out-of-sample performance of the bootstrap-based DUPRO. In the out-of-sample performance tests, we change the true probability distribution of vector V (of dimension $I = 15$) to an asymmetric Dirichlet distribution with order 15 with parameters $(0.25e_5^T, 0.75e_5^T, 0.50e_5^T) \in \mathbb{R}^{15}$. For a given random sample, V^1, \dots, V^N , we solve DUPRO (problem (24)) with a fixed α to obtain an optimal solution $\hat{x}_N(\alpha)$ and then examine its performance via the conditional expectation of the utility value

$$J_N(\alpha) = \mathbb{E}[u(\hat{x}_N(\alpha); V) \mid V^1, \dots, V^N]. \tag{65}$$

This study runs a Monte Carlo simulation to generate 300 observations of $J_N(\alpha)$. Figure 8a–c, depicts the 20–80% sample quantiles of $J_N(\alpha)$ (shaded area) and the sample means (solid curve in the area) when $1 - \alpha$ varies from 10^{-3} to 1, for $N = 20, 100$, and 200, respectively. Also, the dotted curves visualize the reliability that

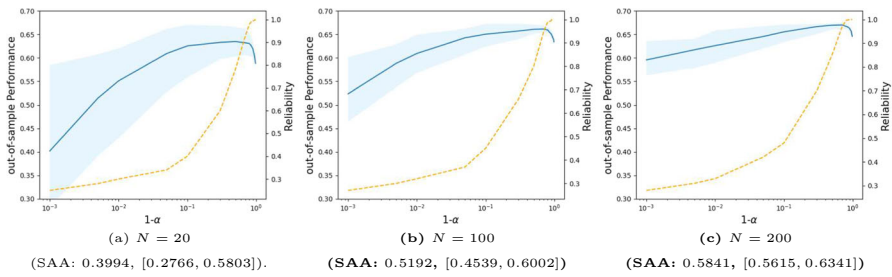


Fig. 8 The out-of-sample performance of **DUPRO** ((i) the solid line for the sample mean of $J_N(\alpha)$, the shaded area for its 20–80% sample quantiles, the dotted line for its reliability; (ii) the out-of-sample performance of the SAA solution given in the caption of each subfigure in the form of (sample mean, [20% sample percentile, 80% sample percentile]))

represents the empirical probability of the event $J_N(\alpha) \geq \tilde{\vartheta}_{\text{bts}}$ via the simulation. Recall that $\tilde{\vartheta}_{\text{bts}}$ is the optimal value of **DUPRO** (problem (24)).

The out-of-sample performance improves until $1 - \alpha$ reaches a large critical value (close to 0.8 in this test) and then deteriorates. Note that Fig. 8 uses a log-scale axis to draw the effectiveness of **DUPRO** with a small-sized ambiguity set. The concave curves of sample mean indicate that the out-of-sample performance has a higher rate of improvement when $1 - \alpha$ is small, while **DUPRO** becomes too conservative when $1 - \alpha$ approaches 1. The robustness of **DUPRO** is also reflected by the range between the 20–80% sample quantiles shrinking as $1 - \alpha$ increases. We also investigate reliability of $J_N(\alpha)$. In Fig. 8, the reliability attains the maximum when $1 - \alpha = 0.8$.

On the other hand, we can see that the increase of original sample size N improves the out-of-sample performance entirely, not only lifting the curve of the sample mean but also narrowing down the range of the 20–80% sample quantiles. The results are consistent with the convergence analysis in Sect. 6.1.2. Moreover, Fig. 8 shows that **DUPRO**, as a data-driven approach, is affected by the level of distributional uncertainty. As N increases, we deduce from (12)–(13) that the gap between the ambiguity sets $\mathcal{P}_B(0)$ and $\mathcal{P}_B(\alpha)$ diminishes for any $\alpha \in (0, 1]$, thereby compromising the superior performance of **DUPRO** in comparison to the sample average approximation (SAA) approach (a trivial **DUPRO** case with the singleton ambiguity set $\mathfrak{F}(\mathcal{P}_B(0))$ including the sample mean only). Given that $1 - \alpha$ changes in $[10^{-3}, 1]$, the largest relative improvement of **DUPRO** over the SAA approach is $(0.6322 - 0.3994)/0.6322 = 36.82\%$ when $N = 20$ in Fig. 8a and is $(0.6656 - 0.5841)/0.6656 = 7.23\%$ when $N = 200$ in Fig. 8c.

6.2 Project investment

We now illustrate how to implement **DUPRO** into a project investment problem. In this problem, an automotive manufacturer needs to learn consumer preference towards passenger vehicles, and on this basis, to pursue the best portfolio investment with a fixed budget among the 10 projects listed in Table 4, see Appendix A. These candidate projects include

- *safety promotion*: design a new structure to alleviate the impacts of the collision;
- *new car model development*: create a concept car with the aim of a more fashion style and higher market acceptance before actually producing it;
- *engine upgrade*: upgrade the current engine and its vibration sensor system to achieve more horsepower and less engine noise;
- *e-platform development*: improve the human-vehicle interaction experience and visualize the performances of the car;
- *computational fluid dynamics (CFD) testing system development*: implement related fluid mechanics and numerical analysis into a testing platform to analyze the performance of concept cars;
- *common modular platform (CMP) development*: develop a platform used for subcompact and compact car models with internal combustion engine and battery-electric cars;
- *checking fixture promotion*: enhance the performance of checking fixture to control the dimensions of auto parts (such as trim edge, surface profile, flatness, etc.) in a more convenient way for mass production of parts detection;
- *noise, vibration, harshness (NVH) digitalization*: incorporate the characteristics of the noise, vibration and harshness of vehicles to a digital platform with the target of more efficient computation in evaluating the driver satisfactions;
- *driving assistance system development*: incorporate the latest interface standards and running multiple vision-based algorithms to support real-time multimedia, vision co-processing, and sensor fusion subsystems;
- *digitalization of marketing network*: build a digital platform used for all the dealers across different regions to share the customer resources, the service standards and the marketing strategies.

These projects may lead to enhancement of eight attributes related to vehicle performance, economy, after-sales service, etc, among which the attributes including *sale price*, *fuel consumption*, *depreciation rate (yearly)*, and *estimated maintain and repair (M&R) fee (in five years)* are related to consumers' major economic considerations when purchasing new cars, *wheel base*, *acceleration*, *comfort rating* are related to consumers' major consideration of vehicle performance, and *dealership*, represented by the number of dealers able to deal with certain services, directly influences consumers' satisfactions on after-sale services. Table 4 displays the estimated consequences and costs of the projects. The column of "Base model" gives the baseline attribute values before the investment, of which the vector is denoted by x^0 , while the other columns show the vectors of attribute increment values, denoted by y^d for $d = 1, \dots, 10$, after the investment on the projects.

Application of DUPRO: Denote by $z \in \{0, 1\}^{10}$ the decision vector of which a component $z_d = 1$ means the project d is selected in the investment and otherwise $z_d = 0$. Assume that the improvement on the attributes yielded by each project is independent of the one by any other project. As a result, the attribute values achieved in the investment are obtained as

$$\hat{x}(z) = x^0 + \sum_{d=1}^{10} z_d y^d. \quad (66)$$

Let Φ be the budget limit of the investment and b_d the cost of project d given in Table 4. A budget constraint is thus described as

$$\sum_{d=1}^{10} b_d z_d \leq \Phi. \tag{67}$$

Combining (66) and (67), we formulate the feasible region of the attribute value achieved by the investment as

$$\mathfrak{X} = \{\hat{x}(z) \mid z \in \{0, 1\}^{10} \text{ satisfies (67)}\}. \tag{68}$$

Consequently, DUPRO in this case is specified as

$$\max_{x \in \mathfrak{X}} \min_{P \in \mathcal{P}_B(\alpha)} \mathbb{E}_P[u(x; V)], \tag{69}$$

where $u(\cdot; V)$ represents the uncertain consumer preference and $\mathcal{P}_B(\alpha)$ is the ambiguity set of the distribution specified by means of bootstrap given in (13).

Sample Generation: We next generate a random sample of V using a logistic regression method and construct $\mathcal{P}_B(\alpha)$. The logistic regression method is developed in [24] to assess the consumer preference toward passenger vehicles in the US market. The dependent variable of the logistic regression model is the market share of the car industry in years 2013 and 2014, which is regarded as the probability distribution of consumers' choices among major vehicle brands. The independent variables are the performance of those brands regarding the attributes listed in Table 4. Following the approach in [24], we select a set of representative car brands and survey the monthly market shares and attribute values for each brand during two years. Let S_j^ℓ be the market share of brand j at the ℓ -th month, x^j be the vector of the attribute values of brand j , and $u(\cdot; v^\ell)$, the utility function with the parameter vector v^ℓ , represent the consumer preference at the ℓ -th month. The probability that consumers purchase brand j in month ℓ is

$$S_j^\ell = \frac{\exp(\eta u(x^j; v^\ell))}{\sum_{i=1}^J \exp(\eta u(x^i; v^\ell))}, \tag{70}$$

where η is the parameter ensuring the normalization of u . Estimating the parameters η and v^ℓ in the logistic model (70) using the survey data, we generate a random sample of V including 24 month-wise observations, i.e., the estimate of v^ℓ for $\ell = 1, \dots, 24$. The case of "Sample mean" in Fig. 9 draws each single-attribute component of $u(\cdot; \bar{V})$ parameterized by the sample average \bar{V} . Note that, with attributes such as *Price*, *Fuel Consumption*, *Acceleration*, *Depreciation Rate*, and *M&R fee*, a lower utility value corresponds to a higher attribute value. To enhance readability, we present their disutility functions in Fig. 9. The utility functions, employed in DUPRO, are the reflections of these disutility functions over the y-axis. In addition, we interpret a population of customer's preferences as a single idiosyncratic customer's random

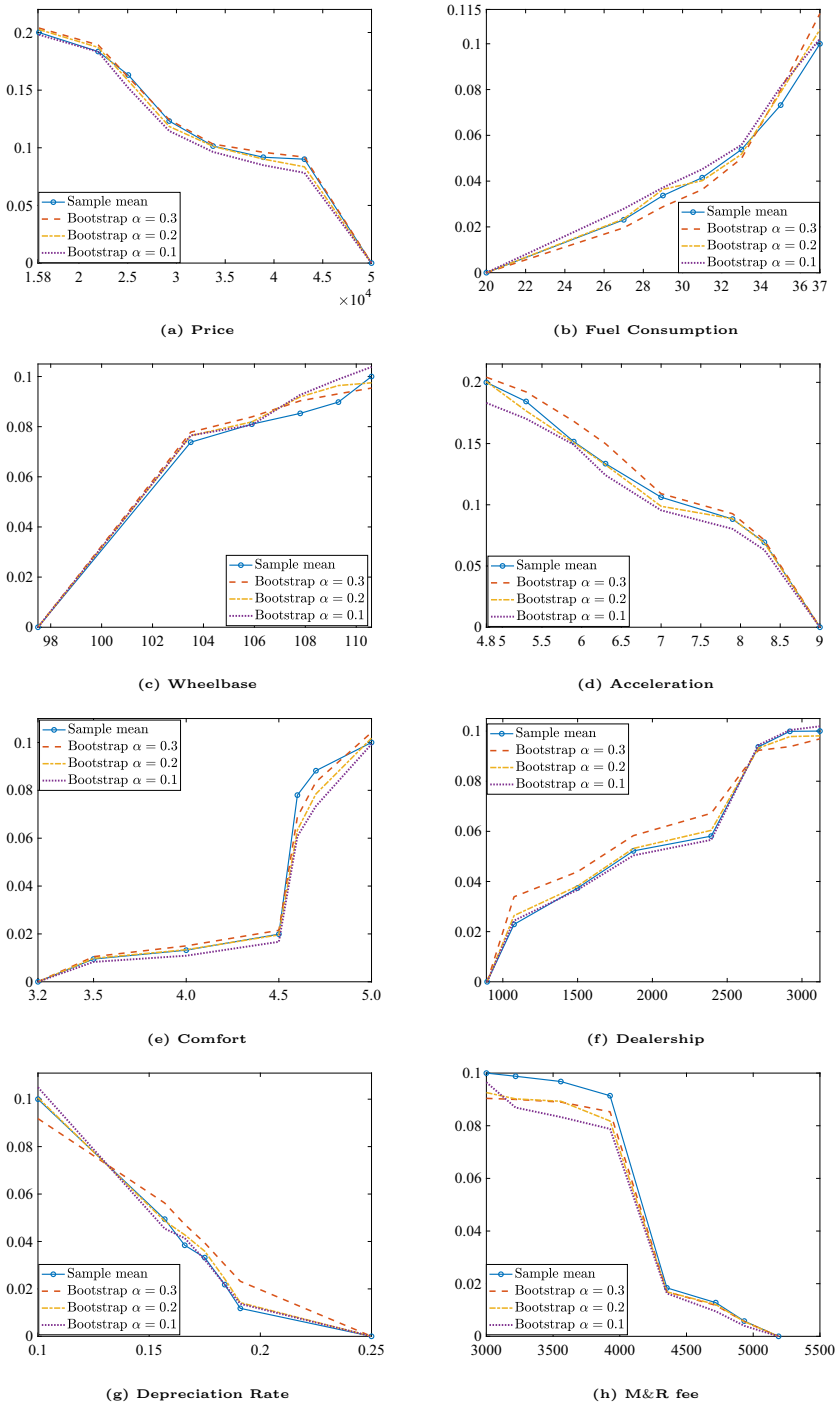


Fig. 9 The single-attribute components of average consumer preference. These components include the disutility functions for *Price (a)*, *Acceleration (d)*, *Depreciation Rate (g)*, and *M&R fee (h)*, and the utility functions for *Fuel Consumption (b)*, *Wheelbase (c)*, *Comfort (e)*, and *Dealership (f)*

preference at different “states” whose average utility captures the one of the population. This kind of approach is used by [44] in their recent work on Stackelberg risk preference design.

Computational Results: In this study, we choose the size of bootstrap samples $K = 1000$ and budget limit $\Phi = \$200$ million. The value of α is varied from 0.10 to 0.30 in the setting of $\mathcal{P}_B(\alpha)$. Note that the size of $\mathcal{P}_B(\alpha)$ shrinks with the increase of α and subsequently the level of robustness dwindles. Table 3 presents the results of model (69) showing the best investment portfolio and the optimal value of consumer preference, and Fig. 9 displays the worst-case utility functions. For comparative purposes, the last row in Table 3 gives the baseline attribute values x^0 before the investment. When α is small, the ambiguity set is large. The robust approach with a high level of confidence requires the DM to take a more conservative action than the bottom line of the majority of customers would prefer. Note that the SAA case represents the maximization of the observed sample mean of the random utility function, a method that does not incorporate considerations for distributional ambiguity. In the SAA case, the proposed strategic recommendations include the implementation of “CFD testing system development”, “CMP development”, “NVH digitalization”, and “Digitalization of marketing network”. These initiatives are aimed at enhancing attributes such as “Acceleration”, “Comfort”, “Dealership”, “Depreciation rate”, and “M&R fee”. It should be acknowledged, however, that these recommendations are made with the observation that there could be a potential trade-off, notably a sacrifice in the “Wheelbase” attribute. In practical terms, the “Wheelbase” attribute often receives less attention during the vehicle purchasing process by customers. When the value of α is set to 0.30, there is a noticeable shift towards investments that aim to improve “Price” competitiveness and “Fuel consumption” through the “New car model development” project. Historical data shows that retail gasoline prices peaked in both current and constant dollars during the years 2013 and 2014, making fuel efficiency a priority at that time [21]. This trend also supports the observed investments in enhancing acceleration and dealership services, which contribute to the overall optimal value of the expected car utility. As α is further decreased to 0.20, a more conservative strategy emerges, suggesting the replacement of “New car model development” with “Engine update”. Although this approach yields a smaller improvement in “Fuel consumption”, it simultaneously reduces the risks and costs associated with the development of a new car model. An even more conservative stance is taken when $\alpha = 0.10$, where the “Comfort rating” attribute gains prominence. This attribute enjoys stable demand from customers and is less susceptible to random economic fluctuations.

7 Conclusions

The past decade has witnessed a surge of research interest in PRO which handles decision-making problems under ambiguity of the DM’s utility preferences. The traditional utility theory, which existing PRO models rely on, assumes preference representations to be deterministic and consistent. Here we propose a novel PRO model which, combining the stochastic utility theory and distributionally robust optimization

Table 3 The optimal solutions and values of model (69) with different α

α	Safety pro- motion	New car model develop- ment	Engine upgrade	E-platform develop- ment	CFD test- ing system develop- ment	CMP develop- ment	Checking fixture promotion	NVH digi- talization	Driving assistance system develop- ment	Digitalization of mar- keting network
0.10 (Optimal solution z)	$z_1 = 0$	$z_2 = 0$	$z_3 = 0$	$z_4 = 1$	$z_5 = 1$	$z_6 = 1$	$z_7 = 1$	$z_8 = 1$	$z_9 = 0$	$z_{10} = 0$
0.10 (Attribute values $\hat{x}(z)$)	Price \$ 37.7K, Fuel Consumption 30MPG, Dealership 1250, M&R fee \$ 4.9K									
0.10 (Optimal value)	0.5162									
0.20 (Optimal solution z)	$z_1 = 0$	$z_2 = 0$	$z_3 = 1$	$z_4 = 0$	$z_5 = 1$	$z_6 = 1$	$z_7 = 0$	$z_8 = 1$	$z_9 = 0$	$z_{10} = 0$
0.20 (Attribute values $\hat{x}(z)$)	Price \$ 37.7K, Fuel Consumption 32MPG, Dealership 1200, M&R fee \$ 4.8K									
0.20 (Optimal value)	0.5381									
0.30 (Optimal solution z)	$z_1 = 0$	$z_2 = 1$	$z_3 = 0$	$z_4 = 1$	$z_5 = 1$	$z_6 = 1$	$z_7 = 0$	$z_8 = 0$	$z_9 = 0$	$z_{10} = 0$
0.30 (Attribute values $\hat{x}(z)$)	Price \$ 35.2K, Fuel Consumption 36MPG, Dealership 1400, M&R fee \$ 5.5K									
0.30 (Optimal value)	0.5507									
SAA (Optimal solution z)	$z_1 = 0$	$z_2 = 0$	$z_3 = 0$	$z_4 = 0$	$z_5 = 1$	$z_6 = 1$	$z_7 = 0$	$z_8 = 1$	$z_9 = 0$	$z_{10} = 1$
SAA (Attribute values $\hat{x}(z)$)	Price \$ 37.7K, Fuel Consumption 30MPG, Dealership 1200, M&R fee \$ 4.5K									
SAA (Optimal value)	0.5669									
x^0	Price \$ 38.0K, Fuel Consumption 30MPG, Dealership 110in, M&R fee \$ 5.5K									

techniques, is capable of dealing with decision-making problems with inconsistent and mutable utility preferences.

We concentrate on the case that the samples of the random utility function are difficult to obtain and use the well-known bootstrap method for constructing the ambiguity set with relatively small sample size. We reformulate the resulting DUPRO model as an MILP when the random utility function is piecewise linear and represented in the specific increment-based form, and LP when it is concave. We conduct some numerical tests to analyze the effects of the crucial parameters in the bootstrap method such as the size of original sample data, the number of the bootstrap resamples, the critical value of the confidence region, and the variance of the underlying true distribution. The preliminary results show that the proposed model and computational scheme work very well. Of course, there is a gap between bootstrap confidence region (based on resamples) and the confidence region based on the original samples. In order for the two confidence regions to be close, the size of the original samples should be sufficiently large, see [72, Theorem 1].

To extend the scope of applicability of the proposed model and computational schemes, we consider the PRO model with general random utility functions and discuss approximation of general random utility functions by piecewise linear random utility functions. Specifically, we quantify propagation of the error of the approximation to the optimal value and optimal solutions. In the case when the data of the piecewise linear random utility functions are obtained from samples, we demonstrate convergence of the optimal value obtained from solving the sample-based piecewise linear approximated PRO model to its true counterpart as the sample size increases.

While this work effectively extends the existing research of PRO from deterministic utility to random utility, it also advances the literature of random utility theory by providing new modelling paradigms, computational schemes and underlying theory if we interpret a single DM's random preferences as a population of customer preferences. We leave it for future research as to how we may use them to study discrete-choice demand models.

Appendix A Data of project investment problem—Sect. 6.2.

See Table 4.

Table 4 The list of candidate projects

	Base model	Safety pro- motion	New model develop- ment	Engine upgrade	E-platform develop- ment	CFD test- ing system develop- ment	CMP develop- ment	Checking fixture promotion	Noise, vibration, harshness (NVH) digitaliza- tion	Driving assistance system develop- ment	Digitalization of mar- keting network
	(x ⁰)	(y ¹)	(y ²)	(y ³)	(y ⁴)	(y ⁵)	(y ⁶)	(y ⁷)	(y ⁸)	(y ⁹)	(y ¹⁰)
Price (\$k)	38	+7	-1	0	0	0	-1.8	0	+1.5	+1.5	0
Fuel consumption (MPG)	30	0	+6	+2	0	0	0	0	0	+2	0
Wheelbase (in)	110	0	-2	0	0	0	-3	0	0	0	0
Acceleration (0-60 miles, sec)	8	0	-1	-1	0	-1.5	0	0	0	0	0
Comfort rating (0-5)	3.8	+0.02	+0.04	0	+0.1	+0.04	0	+0.05	+0.08	+0.02	+0.03
Dealership (# of dealers)	1050	0	+150	+150	+200	0	0	0	0	0	+150
Depreciation rate (0-1)	0.25	-0.08	-0.03	-0.05	0	0	-0.03	0	-0.03	-0.02	-0.05
M&R fee (\$K)	5.5	-0.5	-0.5	+0.3	+0.5	-0.5	0	-0.1	-0.5	-0.2	0
Project costs (\$ million)	-	50	100	70	50	20	30	20	80	50	70

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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