FEWER BATTLES, MORE GAIN: AN INFORMATION-EFFICIENT FRAMEWORK FOR ARENA-BASED LLM EVALUATION

Anonymous authorsPaper under double-blind review

000

001

002

004 005 006

008 009 010

011 012 013

014

015

016

017

018

019

021

023

025

026

027

028

029

030

031

033

035

037

040

041

042

043

044

045

046 047

048

051

052

ABSTRACT

Arena-based evaluation has become a key method for assessing large language models (LLMs) through head-to-head model comparisons, closely reflecting human preferences. However, current arena rating systems (e.g., ELO rating system) often suffer from inefficiencies due to exhaustive or random model pair annotations, leading to redundant evaluations, longer evaluation times, and lower overall efficiency. To address these challenges, we propose a novel adaptive modelpair selection algorithm. By leveraging the asymptotic normality of LLM ability estimation under sparse conditions, our approach strategically selects highvalue model pairs, focusing on confrontations with the lowest variance. Specifically, we introduce Fisher information as a metric to guide model pair selection, optimizing the evaluation process through A-optimality and D-optimality. A-optimality minimizes estimation variance, ensuring balanced reliability across models, while D-optimality reduces uncertainty by maximizing the determinant of the Fisher Information Matrix. Extensive experiments on both simulated and real-world datasets demonstrate that our method outperforms existing approaches in terms of information efficiency and result reliability. Notably, our method offers a flexible, general toolkit that can be easily integrated into existing arena-based platforms, greatly improving scalability and efficiency for large-scale LLM evaluations. Our code is publicly available to promote reproducibility at https://anonymous.4open.science/r/Adaptive-Arena-870D.

1 Introduction

I INTRODUCTION

The rapid advancements in large language models (LLMs) have led to their widespread use in various fields, including content generation (Wang et al., 2024a), customer service, and intelligent tutoring (Kasneci et al., 2023). As LLM architectures (Liu et al., 2024a), fine-tuned variants (Hu et al., 2023), and domain-specific models (Patil & Gudivada, 2024) continue to evolve, thus the demand for efficient and scalable evaluation frameworks is growing (Mizrahi et al., 2024; Zhou et al., 2023; Li et al., 2024b). Among existing evaluation methods, arena-based evaluation is notable (Chiang et al., 2024): unlike traditional benchmark-based approaches (Wang et al., 2024b; Mirzadeh et al., 2024) that rely on predefined prompts and fixed answers, arena-based frameworks assess models through head-to-head confrontations. This approach captures real-world performance dynamics and produces rankings that closely align with human judgments (Zheng et al., 2023; Luo et al., 2024), making it a valuable tool for assessing LLMs' practical utility.

However, current arena-based evaluation methods face significant issues. First, is the **efficiency** (Luo et al., 2024; Spangher et al., 2025). As shown in Figure 1, these frameworks often rely on exhaustive or random pairwise annotations to generate reliable rankings, which becomes increasingly resource-intensive. Second, is the **scalability**. As the number of newly proposed models grows, platforms like Chatbot Arena (Chiang et al., 2024), hosting over 190 models, require tens of thousands of confrontations to produce reliable rankings (Patil & Gudivada, 2024). This not only consumes considerable resources but also results in substantial evaluation cycles, making it difficult to keep up with the rapid development of LLMs (Min et al., 2025).

Figure 1: The diagram of Arena-based Evaluation. Models engage in pairwise battles, annotators label the outcomes, and the ELO ratings are then estimated based on these annotations.

To address these challenges, we propose a novel adaptive arena-based evaluation method. Leveraging the asymptotic normality of ability estimation under sparse conditions, we can strategically select high-value model pairs for evaluation and improve efficiency by minimizing the variance of the asymptotic distribution. Specifically, we introduce two key methods: A-optimality, which minimizes the variance in model estimates, and D-optimality, which reduces uncertainty by maximizing the determinant of the Fisher Information Matrix (Rissanen, 1996; Ly et al., 2017). Through these techniques, we achieve *more gain* in each annotation, leading to *fewer battles*.

Our contributions are as follows:

- We first introduce the concept of efficiency in arena-based LLM evaluation, using statistical uncertainty to minimize redundant evaluations, significantly improving both speed and resource usage in ranking.
- We apply A-optimality and D-optimality to model-pair selection, backed by rigorous theoretical analysis, to reduce annotation costs while preserving ranking reliability.
- Through extensive experiments, we show that our method outperforms existing approaches in efficiency and reliability, and can seamlessly integrate with current arena-based platforms. This makes our method not just a theoretical innovation, but a practical and scalable solution for large-scale LLM evaluations.

2 Related Work

Arena-based Evaluation Arena-based evaluation is a key approach in LLM evaluation, distinguished from traditional benchmark methods by its focus on direct head-to-head model competitions (Wang et al., 2025). This design provides a more realistic measure of models' relative abilities (Zellers et al., 2019; Hendrycks et al., 2020; Cobbe et al., 2021). Unlike traditional approaches that use predefined datasets (Liang et al., 2022; Liu et al., 2024b), arena settings emphasize dynamic model confrontations that better reflect real-world usage scenarios. Ranking systems are central to Arena-based evaluation (Busa-Fekete et al., 2014; Szörényi et al., 2015; Chernoff, 1992), with most methods relying on the ELO Rating System to estimate model proficiency (Coulom, 2007; Pelánek, 2016). However, ELO's sensitivity to sample order can lead to instability in performance estimates (Boubdir et al., 2024). To address this, recent studies have proposed adaptations, such as statistical refinements (Ameli et al., 2024; Yin et al., 2024; Liu et al., 2025) and bias-mitigation methods like UDA (Zhang et al., 2025). Notably, all existing Arena-based methods aim to improve ranking accuracy but overlook the critical issue of annotation efficiency. Whether using standard ELO, adapted variants, or frameworks like UDA, these approaches often involve unnecessary confrontations, leading to low annotation efficiency and high evaluation overhead.

Efficiency of LLM Evaluation Efficiency has become a key focus in LLM evaluation due to the high costs of annotating large datasets and conducting extensive tests (Marion et al., 2023). Existing methods (Xie et al., 2023; Chung et al., 2023; Saranathan et al., 2024; Li et al., 2024a) share the goal of reducing the number of evaluation examples while maintaining accuracy, but they are designed primarily for traditional static question-answering (QA) tasks, where models are assessed against predefined prompts and answers. For example, Polo et al. (2024a) introduced TinyBenchmarks, which evaluates LLMs with fewer examples than traditional benchmarks by prioritizing information-dense samples, thus reducing annotation and computational costs while maintaining

reliable performance estimation. Similarly, Kipnis et al. (2024) proposed MetaBench, which selects diverse, compact prompts based on statistical analysis, avoiding redundant evaluations while capturing core model abilities. Based on this, Polo et al. (2024b) developed an efficient multi-prompt evaluation method, which optimizes prompt selection across rounds to reduce redundant data collection and computation without sacrificing accuracy. However, these efficiency-driven methods are not applicable to dynamic Arena-based evaluation for two key reasons: First, Arena scenarios involve pairwise head-to-head model confrontations rather than static QA tasks; second, rankings in Arena settings rely on iterative human judgments (Kruglanski & Ajzen, 1983), breaking the assumption of fixed criteria inherent in traditional QA methods. Therefore, the dynamic and interactive nature of Arena evaluation makes current efficiency methods unsuitable.

Given these gaps—existing efficiency methods being incompatible with dynamic Arena scenarios, and Arena-based methods neglecting annotation efficiency—our study aims to address these issues. We propose optimizing Arena-based evaluation through strategic sample selection techniques to both accelerate system convergence and improve overall evaluation efficiency.

3 Adaptive Arena Method

3.1 Modeling ELO Ratings

 Arena-based evaluation has emerged as a key approach for LLM evaluation, where models compete head-to-head on benchmarks or datasets, with outcomes annotated by evaluators. Let $\mathcal{S} = \{(i,j,w_{ij})|i< j\in [N]\}$ represent the comparative records, where N is the total number of models being evaluated. Each element $(i,j,w_{ij})\in\mathcal{S}$ contains important information: it indicates that model i and model j have engaged in a competition, and w_{ij} is the annotation result. Specifically, if $w_{ij}=1$, model i wins; if $w_{ij}=0$, model j wins; and if $w_{ij}=0.5$, it indicates a tie. The goal of arena-based evaluation is to estimate the ranking scores (r_1,\ldots,r_N) . Since the ranking is relative, we assume that the ability of the last model is 0, i.e., $r_N=0$, and in the following, we focus on the ability estimation of the first N-1 models, denoted as $\mathbf{r}=(r_1,\ldots,r_{N-1})$.

To model the win probabilities, we use the Bradley-Terry model (Rao & Kupper, 1967), which forms the statistical foundation for modern ELO-style rating systems. It posits that the probability of model i winning against model j, denoted P_{ij} , is a logistic function of their ability difference:

$$P_{ij} = \frac{1}{1 + e^{-C(r_i - r_j)}},\tag{1}$$

where C is a scaling constant.

With the obtained winning probability, many methods can be adopted to get the ranking scores. Among these, Maximum Likelihood Estimation (MLE)-based **ELO** finds the ratings that best explain all observed match results simultaneously. It is a more statistically robust batch approach to estimate the abilities r by MLE. It is achieved by minimizing the negative log-likelihood function:

$$\hat{r} = \arg\min_{r} \mathcal{L}_{\mathcal{S}}(r), \text{ where } \mathcal{L}_{\mathcal{S}}(r) = -\sum_{(i,j,w_{ij})\in\mathcal{S}} [w_{ij} \ln P_{ij} + w_{ji} \ln P_{ji}].$$
 (2)

Adopting the MLE framework is crucial as it allows us to rigorously analyze the statistical properties of the estimated ratings, which is essential for developing our adaptive selection strategy. The detailed properties are as follows.

Statistical Properties of ELO. Consider modeling the arena as an Erdős-Rényi (E-R) Graph $G(N, q_N)$ (Kumar et al., 2000; Guimera et al., 2004), where N is variable and the sample size $|\mathcal{S}|$ increases with N. Here, q_N represents the probability that a confrontation occurs between models i and j. Under this framework, the estimation results of ELO exhibit both uniqueness and asymptotic normality (Simons & Yao, 1999; Han et al., 2020):

Lemma 1. Let $G(N, q_N)$ be the Erdős-Rényi Graph, and \hat{r}_n, r_n^* be the estimated and true abilities of the first n models, respectively. We have:

(1) (Uniqueness) If $q_N = \omega\left(\frac{\log N}{N}\right)$, then ELO almost surely has a unique solution \hat{r}_{N-1} .

(2) (Asymptotic Normality) If $q_N \cdot \frac{N^{1/10}}{(\log N)^{1/5}} \to \infty$ as $N \to \infty$, then for any n < N, $\sqrt{|\mathcal{S}|}(\hat{r}_n - r_n^*) \stackrel{d}{\longrightarrow} \mathcal{N}(0, \mathcal{I}_{\mathcal{S}}(r_n^*)^{-1})$, where $\mathcal{I}_{\mathcal{S}}(r^*)$ is the Fisher information matrix.

In the ELO method, Fisher information (Frieden, 2004) is calculated by solving the Hessian matrix:

$$\mathcal{I}_{\mathcal{S}}(\boldsymbol{r}) = -\mathbb{E}[\mathcal{H}(\mathcal{L}_{\mathcal{S}}(\boldsymbol{r}))] = C^2 \sum_{(i,j,w_{ij})\in\mathcal{S}} P_{ij} P_{ji} (\boldsymbol{e}_i - \boldsymbol{e}_j) (\boldsymbol{e}_i - \boldsymbol{e}_j)^\top, \tag{3}$$

where e_i is the *i*-th standard basis vector, and $e_N = 0$. To explain the Fisher information matrix intuitively, consider a toy example with three models, X, Y, and Z, and two battle records $S = \{(X > Y), (X = Z)\}$. Assuming all models have an initial ability of 0 and that model Z is fixed during the estimation process, the calculation of the information quantity is as follows:

$$C^{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{X > Y} C^{2} \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix} \xrightarrow{X = Z} C^{2} \begin{bmatrix} 0.5 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}, \tag{4}$$

and we also provide a case study of the information matrix based on real dataset in Appendix E.

Considering Lemma 1 with Fisher information matrix like Equation 4, it can be confirmed that ELO has a unique estimated solution and this estimation is asymptotically unbiased and normally distributed, even in sparse scenarios. Therefore, from a statistical perspective, the efficiency problem in arena evaluation essentially revolves around how to reduce variance more quickly during the evaluation process. Specifically, the key lies in reducing the inverse of the Fisher information matrix $\mathcal{I}_{\mathcal{S}}(r^*)$ under ELO context.

3.2 SELECTION STRATEGY FOR ARENA-BASED EVALUATION

As mentioned earlier, our direct goal is to reduce the inverse of the information matrix. However, the true ability values of candidate models, r^* , are unknown, making it impossible to directly select the model pairs. Therefore, we dynamically select each subsequent model pair based on the accumulated records S_{t-1} . Specifically, by approximating the true abilities r^* with the estimated abilities \hat{r}^{t-1} derived from S_{t-1} , we can compute an approximate information matrix. Using the Fisher information of the current abilities, we propose the following A-optimality method (Chan, 1982; Wanyonyi et al., 2021) for Arena-based Evaluation:

The A-optimality method. This method aims to select the model pair that minimizes the sum of the sampling variances of the ability estimators, which is equivalent to minimizing the trace of the inverse of the information matrix:

$$(i_t, j_t) = \arg\min_{1 \le i < j \le N} \operatorname{tr} \left[\left(\mathcal{I}_{\mathcal{S}_{t-1}}(\hat{\boldsymbol{r}}^{t-1}) + \mathcal{I}_{\{(i,j)\}}(\hat{\boldsymbol{r}}^{t-1}) \right)^{-1} \right]. \tag{5}$$

For this method to be feasible, the Fisher information matrix must be invertible. We have the following theorem:

Theorem 1. Consider the Erdős-Rényi Graph $G(N, q_N)$ for ELO. If $q_N = \omega\left(\frac{\log N}{N}\right)$, then the information matrix $\mathcal{I}_{\mathcal{S}}(\boldsymbol{r})$ is almost surely positive definite.

From Theorem 1, it follows that as long as there are sufficient battle samples, the information matrix is guaranteed to be positive definite (Johnson, 1970). This not only ensures that the information matrix is invertible but also guarantees that the determinant is positive.

However, A-optimality method still requires calculating the inverse of a matrix, which increases computational complexity. To address this, we consider maximizing the determinant of information matrix and propose the D-optimality method (John & Draper, 1975) for Arena-based Evaluation:

The D-optimality method. The D-optimality method aims to select the model pair that maximizes the determinant of the information matrix in the current state, expressed as:

$$(i_t, j_t) = \arg \max_{1 \le i \le j \le N} \left| \mathcal{I}_{\mathcal{S}_{t-1}}(\hat{r}^{t-1}) + \mathcal{I}_{\{(i,j)\}}(\hat{r}^{t-1}) \right|. \tag{6}$$

Table 1: This table summarizes the characteristics of each strategy. Using \hat{r} indicates whether the strategy utilizes currently estimated abilities; Global Info indicates whether global information is used, and min q_N represents the minimum edge probability that ensures effective selection.

Method	Random	Nearest	Ada-Pair	A-optimality	D-optimality
Using \hat{r}	X	√	√	✓	√
Global Info	X	X	X	✓	\checkmark
$\min q_N$	0	$w(\frac{\log N}{N})$	1	$w(\frac{\log N}{N})$ $O(N^{\beta+3})$	$w(\frac{\log N}{N}) O(N^{\beta+2})$
Complexity	O(1)	O(N)	$O(N^2)$	$O(N^{\beta+3})$	$O(N^{\beta+2})$

3.3 THEORETICAL ANALYSIS

In this subsection, we analyze our proposed method from three perspectives: algorithmic prerequisites, numerical computation, and time complexity.

Algorithmic Prerequisites. Different selection strategies depend on distinct prerequisites. Table 1 summarizes the conditions for each method.

For example, the Random selection method requires no additional variables, so it is often used to initialize other strategies. In contrast, methods that consider model abilities \hat{r} must meet at least the conditions outlined in Lemma 1(1), which represents the theoretical lower bound of q_N for these methods. More stringent conditions apply to methods like Ada-Pair, which considers the confidence interval of each model pair (i,j). To ensure the existence of confidence intervals, Ada-Pair requires that each pair has competed at least once, implying that the density of the E-R graph must satisfy $q_N=1$. This requirement becomes restrictive as the number of models N increases. This highlights a key advantage of our proposed method: it only requires the minimal condition, $\omega(\frac{\log N}{N})$, to ensure the positive definiteness of the information matrix (Theorem 1) without requiring additional samples for initialization. This feature demonstrates the broad applicability of our approach.

Numerical Computation. To avoid numerical instability and ensure computational validity, we need to consider the relationship between Fisher information and the number of samples. We present the following theorem to address this:

Theorem 2. Let r be the ability parameter vector, and let $\mathcal{I}_{\mathcal{S}_t}(r)$ denote the Fisher information matrix based on the subset \mathcal{S}_t with size t. Define:

- $A(S_t) = \frac{1}{tr[\mathcal{I}_{S_t}(r)^{-1}]}$ (average information efficiency),
- $\mathcal{D}(\mathcal{S}_t) = |\mathcal{I}_{\mathcal{S}_t}(r)|^{\frac{1}{N-1}}$ (normalized geometric information density).

Assume that each element in S_t contributes bounded positive information and that $I_{S_t}(r)$ is positive definite for sufficiently large t. Then as $t \to \infty$:

- 1. $A(S_t) = O(t)$ (there exist constants C_1 and t_1 such that $A(S_t) \leq C_1 \cdot t$ for all $t \geq t_1$),
- 2. $\mathcal{D}(\mathcal{S}_t) = O(t)$ (there exist constants C_2 and t_2 such that $\mathcal{D}(\mathcal{S}_t) \leq C_2 \cdot t$ for all $t \geq t_2$).

Theorem 2 provides the upper bound for the variation of Fisher information with respect to the number of competitions. Specifically, the inverse of the A-optimality measure grows linearly with the increase in the number of samples, while the (N-1)-th root of the D-optimality measure grows linearly. Therefore, when practically implementing the D-optimality method, it is advisable to transform the calculation into the form defined in Theorem 2 to prevent numerical explosion.

Time Complexity. Algorithm 1 presents the pseudo-code for model selection in the Arena. The main computational cost in the model pair selection phase comes from calculating the determinant or the inverse of the matrix. Both operations have a complexity of $O(N^{\beta})$ (Kaltofen & Villard, 2005), where N is the matrix size or the number of models in the arena. Therefore, for the D-optimality method, which traverses all model pairs and computes the determinant of the information matrix at each step, the complexity is $O(N^{\beta+2})$. For the A-optimality method, whether calculating the trace of the inverse matrix or computing N submatrices, it requires N additional calculations per iteration, resulting in a complexity of $O(N \times N^2 \times N^\beta) = O(N^{\beta+3})$. The complexity for calculating the

Algorithm 1: The Adaptive Arena Framework

Initialize: Step $t \leftarrow 1$, Model's ability $\hat{\boldsymbol{r}}^0$, Battle records $\mathcal{S}_0 \leftarrow \emptyset$.

1 while True do

2 | Select model pair: $(i_t, j_t) \leftarrow \arg\max_{1 \leq i < j \leq N} \left| \mathcal{I}_{\mathcal{S}_{t-1}}(\hat{\boldsymbol{r}}^{t-1}) + \mathcal{I}_{\{(i,j)\}}(\hat{\boldsymbol{r}}^{t-1}) \right|$.

3 | Get Annotation for model pair: $w_{i_tj_t} \leftarrow Anno(i_t, j_t)$.

4 | Add record in \mathcal{S} : $\mathcal{S}_t \leftarrow \mathcal{S}_{t-1} \cup \{(i_t, j_t, w_{i_tj_t})\}$.

Update model's ability \hat{r} : $\hat{r}^t \leftarrow \arg \max_{r} \mathcal{L}_{\mathcal{S}_t}(r), t \leftarrow t + 1$.

Output: The models' estimated abilities \hat{r}^t .

information matrix involves summing over t iterations, making the overall time complexity for the A-optimality method $O(|S|^2 + |S| \cdot N^{\beta+3})$, and for the D-optimality method, $O(|S|^2 + |S| \cdot N^{\beta+2})$.

Although our method has a higher time complexity than previous approaches, its primary goal is to reduce human-annotated samples by leveraging computational time, ultimately minimizing the overall annotation effort. Therefore, we only require our method to meet the operational needs of the Arena system—providing model pairs for evaluation within a limited time for each user request. To achieve this, we have implemented the following improvements: (1) The information matrix is updated directly by adding the current model pair's information to the matrix from the previous step. (2) To address system concurrency, the top-K model pairs with the highest information gain are selected for each request.

We discuss the time cost and the efficiency of these improvements in section 4.2. Although our method does not directly reduce the complexity related to the annotation volume, its value becomes evident within the iterative, ongoing Arena framework. For example, if our method improves efficiency by 50%, it means that the annotation results from our approach can achieve the same outcomes as twice the original annotation volume, given the Arena's continuous operation. This offers a significant long-term advantage, enabling comparable or even superior results with fewer resources over extended periods.

4 EXPERIMENTS

In this section, we evaluate the performance of the proposed A-optimality and D-optimality methods through experiments on both simulated and real-world datasets, analyzing the abilities of each method and discussing the time cost for proposed method.

4.1 EXPERIMENTAL SETUP

Evaluation Metric. Since the ELO score is relative, we use the pairwise index (Hastie & Tibshirani, 1997) to measure the consistency between the estimated abilities and the true abilities:

$$Pairwise = \frac{2}{N(N-1)} \sum_{1 \le i < j \le N} \mathbb{I}\left[(r_i^* - r_j^*)(\hat{r}_i - \hat{r}_j) > 0 \right], \tag{7}$$

where $\mathbb{I}[\cdot]$ is the indicator function, r^* is true abilities of the models, and \hat{r} is the estimated abilities. Since the true abilities are not available in real-world scenarios, we use *the results estimated from the full dataset by ELO* (Equation 2) as the ground truth (Zhuang et al., 2023; Liu et al., 2024c). Lemma 1 ensures that r^* derived from this process is unique and unbiased, making this ground truth feasible.

Dataset. We conducted experiments on two real-world datasets: **Chatbot** (Chiang et al., 2024; Zheng et al., 2023) and **PPE** (Frick et al., 2024). The Chatbot dataset¹ was collected from 13,000 distinct IP addresses on the Chatbot Arena platform, while the PPE dataset² contains human preference evaluations for Preference Proxy Evaluations. Detailed statistics and experimental settings are provided in Appendix C.1.

https://huggingface.co/datasets/lmsys/chatbot_arena_conversations

²https://huggingface.co/datasets/lmarena-ai/PPE-Human-Preference-V1

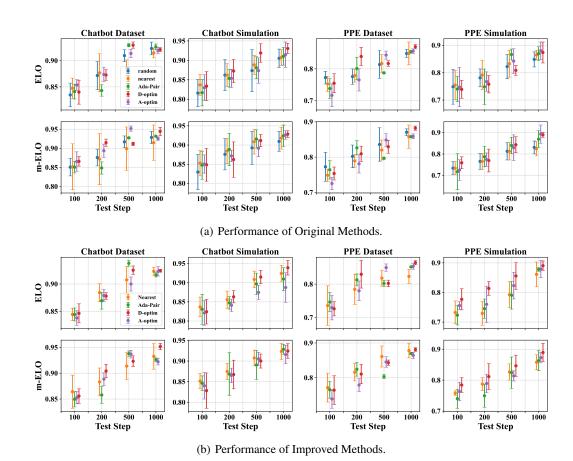


Figure 2: This Error bar illustrates pairwise metrics for each dataset and ELO method. "Dataset" indicates randomly sampled competition results, while "Simulation" denotes results simulated based on real model abilities. Error bars represent standard deviation.

Compared Approaches. This paper compares two model ability evaluation methods: **ELO** (Albers & Vries, 2001) and **m-ELO** (Liu et al., 2025). ELO refers to the traditional ELO, it updates each model's rating iteratively by adding the difference between the annotation w_{ij} and the prediction P_{ij} , while m-ELO uses a MLE process like Equation 2. For model pair selection, we compare several common strategies used in both traditional competitions and modern LLM arenas. These strategies include: **Random**, which randomly selects a model pair for annotation at each step and serves as a baseline; **Nearest**, which selects the two models with the closest abilities at each step, a typical strategy in competitive games; and **Ada-Pair** (Chiang et al., 2024), an official selection strategy from Chatbot Arena, which takes into account the confidence interval of each model pair and selects the one that maximizes the reduction of the interval. Additionally, for each method, we consider both the original version and the time-improved version discussed in Section 3.3, referred to as **Original** and **Improved**, respectively.

4.2 EXPERIMENT RESULTS AND ANALYSIS

To evaluate the effectiveness of our proposed selection method, we conducted experiments on real-world datasets. The following conclusions were drawn:

Performance on Pairwise Metrics. Figure 2(a) shows the performance of different selection strategies at each step, and Table 2 presents the average pairwise for each strategy. Our proposed method, D-optim, outperforms all others across all scenarios and is the only method that significantly surpasses the Random baseline. Additionally, as shown in Figure 2(a), D-optim effectively reduces variance in most cases, indicating that it lowers evaluation uncertainty and enhances efficiency.

Table 2: Performance of Pairwise Metrics for Different Strategies. Results are averaged over five experiments and four steps. Bold values indicate significant improvements over the baseline.

Method		Dataset		Is Simu		ELO Method		Total
		Chatbot	PPE	Real	Simu	ELO	m-ELO	101111
	Random	0.8797	0.8021	0.8501	0.8316	0.8379	0.8439	0.8409
Original	Nearest	0.8825	0.7995	0.8422	0.8398	0.8412	0.8407	0.8410
	Ada-Pair	0.8835	0.8040	0.8452	0.8423	0.8379	0.8496	0.8437
	A-optim	0.8880	0.8033	0.8494	0.8418	0.8406	0.8506	0.8456
	D-optim	0.8941	0.8122	0.8596	0.8466	0.8486	0.8577	0.8531
Improved	Nearest	0.8897	0.8020	0.8525	0.8392	0.8351	0.8566	0.8459
	Ada-Pair	0.8849	0.7993	0.8509	0.8332	0.8376	0.8465	0.8421
	A-optim	0.8807	0.8061	0.8488	0.8380	0.8372	0.8495	0.8434
	D-optim	0.8918	0.8243	0.8581	0.8581	0.8546	0.8616	0.8581

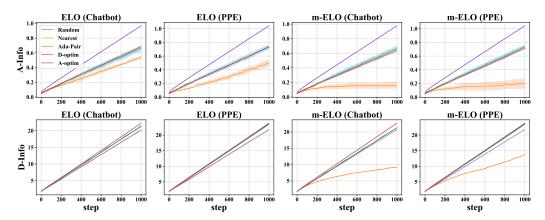


Figure 3: This line chart shows A-Info and D-Info for each dataset and ELO method.

The Nearest method, on the other hand, performs suboptimally in the LLM arena. This is due to the nature of the arena: rather than actual battles, models answer a single question, making ties more likely than in traditional competitions. This diminishes the effectiveness of pairing models with similar abilities. We also observe that while m-ELO incurs additional time costs, its impact on selection strategies that rely on current model abilities is notable. In practical applications, engineers need to balance computational efficiency and accuracy, such as using m-ELO for high-precision tasks and ELO in resource-constrained settings.

Performance of Time-Improved Method on Pairwise Metrics. To assess the impact of our improvements on proposed methods, we compared the Nearest, Ada-Pair, A-optimality, and D-optimality methods, where the Top-10 samples were selected in each sampling round (i.e., K=10). The experimental results are presented in Figure 2(b) and Table 2.

We can observe that our D-optimality method not only maintains its original performance but also shows improvements, where a total 0.5% gain is observed compared to the time-improved version. This suggests that model pairs with higher D-Info are indeed prioritized for selection, supporting the effectiveness of our D-Info design. This result confirms the effectiveness of our improved D-optim. In the appendix C.3, we also present its performance on larger-scale datasets.

Performance on Fisher Information Gain. Figure 3 shows the variation of $\mathcal{A}(\mathcal{S}_t) = \frac{1}{\operatorname{tr}[\mathcal{I}_{\mathcal{S}_t}(r)^{-1}]}$

(A-Info) and $\mathcal{D}(\mathcal{S}_t) = |\mathcal{I}_{\mathcal{S}_t}(r)|^{\frac{1}{N-1}}$ (D-Info) as the number of samples increases. As shown in Theorem 2, both A-Info and D-Info exhibit linear growth. The experimental results confirm that under most strategies, their growth is indeed linear. Furthermore, our method achieves the fastest improvement rate (steepest slope) in both A-Info and D-Info. In contrast, Ada-Pair shows similar information gain to the Random method. This suggests that our method, despite using estimated abilities, **enhances information gain** from samples, improving evaluation efficiency.

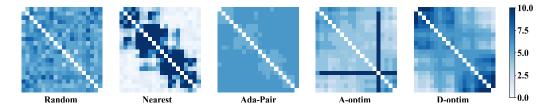


Figure 4: Heatmap of model pair sampling frequency for different selection methods. Each cell represents the number of battles between Model i (ranked by ability) and Model j (ranked by ability).

Visualization of Sampling Distribution. To visualize the sampling distribution of model pairs selected by each strategy, we sorted the models by true abilities and used heatmaps to represent the sampling frequency of each model pair. Figure 4 shows that D-optim selects model pairs with similar true abilities more often (darker colors near the diagonal), while ensuring diverse coverage. In contrast, Ada-Pair samples more uniformly across all pairs, and Nearest focuses too much on similar models.

Interestingly, the A-optimality method results in many samplings involving a specific model, since this model's ability is fixed in the ELO system. This creates a unique role for the model pair selection: The A-optimality method, while ensuring sparsity (i.e., most pairs have battled), prefer to use this model as an "anchor" to estimate others' abilities. As shown in Figure 3, this method actually leads to high A-Info, but may introduce bias and impact the Arena's sustainability. An improvement could be to fix each model sequentially, calculate A-Info for each, and average the results, but this would increase complexity. In contrast, D-optimality is minimally impacted by this constraint by ELO, making D-Info a more suitable metric for Arena-based evaluation.

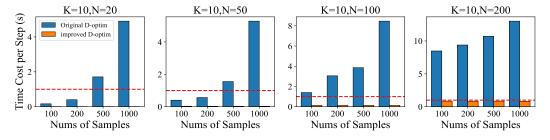


Figure 5: Time costs for Original D-optimality and Improved D-optimality Methods as the number of samples and models increases.

Performance on Time Cost. Figure 5 shows the time required for each sample selection as the number of samples increases. The red line indicates a time cost of 1 second. While the D-optimality method does not achieve the lowest time cost, its time cost is significantly lower than the time consumed by LLM API calls and human annotations in practical scenarios. Moreover, increasing the value of K can further reduce the time cost. These observations confirm that our method is well-suited for real-world arena-based evaluation scenarios and can be easily integrated into existing arena-based platforms.

5 CONCLUSION

In this work, we addressed the efficiency bottleneck in arena-based LLM evaluation. By leveraging the asymptotic normality of ELO-based ability estimation and introducing Fisher information as a guiding principle, we proposed adaptive model-pair selection strategies grounded in A-optimality and D-optimality. These methods systematically reduce redundant battles while preserving ranking reliability, thereby accelerating evaluation and lowering annotation costs. Extensive experiments on both simulated and real-world datasets confirm that our framework consistently outperforms existing approaches in efficiency and robustness. Beyond improving scalability for current arenas, our approach can be regarded as a general and flexible toolkit that can be readily integrated into future evaluation platforms. We believe this work opens a promising direction toward information-efficient, scalable, and reliable assessment of large language models.

ETHICS STATEMENT

This research does not involve any human subjects, animals, or sensitive data. The proposed methods and experiments focus solely on the evaluation of LLMs using publicly available datasets. We ensure that all data used in our experiments are publicly accessible, adhering to ethical standards and relevant data usage policies. Furthermore, our work does not raise any concerns related to bias, fairness, or social impact, as it is focused on improving the efficiency of model evaluations in a controlled, academic setting. In summary, our research does not present any ethical issues, and we are committed to conducting future work in line with ethical best practices and standards.

7 REPRODUCIBILITY STATEMENT

To ensure the reproducibility of our results, we provide the complete implementation of our methods, along with all necessary code and data, on an anonymous GitHub repository. The repository includes scripts to replicate the experiments presented in the paper, including data preprocessing, model training, and evaluation procedures. All hyperparameters, random seeds, and environment configurations are specified to ensure consistent results across different platforms. The repository can be accessed at the following anonymous link: https://anonymous.4open.science/r/Adaptive-Arena-870D.

REFERENCES

- Paul CH Albers and Han de Vries. Elo-rating as a tool in the sequential estimation of dominance strengths. *Animal behaviour*, pp. 489–495, 2001.
- Siavash Ameli, Siyuan Zhuang, Ion Stoica, and Michael W Mahoney. A statistical framework for ranking llm-based chatbots. *arXiv preprint arXiv:2412.18407*, 2024.
- Meriem Boubdir, Edward Kim, Beyza Ermis, Sara Hooker, and Marzieh Fadaee. Elo uncovered: Robustness and best practices in language model evaluation. *Advances in Neural Information Processing Systems*, 37:106135–106161, 2024.
- Róbert Busa-Fekete, Eyke Hüllermeier, and Balázs Szörényi. Preference-based rank elicitation using statistical models: The case of mallows. In *International conference on machine learning*, pp. 1071–1079. PMLR, 2014.
- NN Chan. A-optimality for regression designs. *Journal of Mathematical Analysis and Applications*, 87(1):45–50, 1982.
- Sourav Chatterjee and SR Srinivasa Varadhan. The large deviation principle for the erdős-rényi random graph. *European Journal of Combinatorics*, 32(7):1000–1017, 2011.
- Herman Chernoff. Sequential design of experiments. Springer, 1992.
- Wei-Lin Chiang, Lianmin Zheng, Ying Sheng, Anastasios Nikolas Angelopoulos, Tianle Li, Dacheng Li, Banghua Zhu, Hao Zhang, Michael I. Jordan, Joseph E. Gonzalez, and Ion Stoica. Chatbot arena: An open platform for evaluating llms by human preference. In *Forty-first International Conference on Machine Learning, ICML 2024, Vienna, Austria, July 21-27, 2024*. OpenReview.net, 2024. URL https://openreview.net/forum?id=3MW8GKNyzI.
- Hyung Won Chung, Noah Constant, Xavier Garcia, Adam Roberts, Yi Tay, Sharan Narang, and Orhan Firat. Unimax: Fairer and more effective language sampling for large-scale multilingual pretraining. *arXiv preprint arXiv:2304.09151*, 2023.
- Karl Cobbe, Vineet Kosaraju, Mohammad Bavarian, Mark Chen, Heewoo Jun, Lukasz Kaiser, Matthias Plappert, Jerry Tworek, Jacob Hilton, Reiichiro Nakano, et al. Training verifiers to solve math word problems. *arXiv preprint arXiv:2110.14168*, 2021.
- Rémi Coulom. Computing "elo ratings" of move patterns in the game of go. *ICGA journal*, 30(4): 198–208, 2007.

Joel N Franklin. *Matrix theory*. Courier Corporation, 2000.

- Evan Frick, Tianle Li, Connor Chen, Wei-Lin Chiang, Anastasios N. Angelopoulos, Jiantao Jiao, Banghua Zhu, Joseph E. Gonzalez, and Ion Stoica. How to evaluate reward models for rlhf, 2024. URL https://arxiv.org/abs/2410.14872.
 - B Roy Frieden. Science from Fisher information, volume 974. Citeseer, 2004.
 - Roger Guimera, Marta Sales-Pardo, and Luís A Nunes Amaral. Modularity from fluctuations in random graphs and complex networks. *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics*, 70(2):025101, 2004.
 - Ruijian Han, Rougang Ye, Chunxi Tan, and Kani Chen. Asymptotic theory of sparse bradley-terry model. 2020.
 - Trevor Hastie and Robert Tibshirani. Classification by pairwise coupling. *Advances in neural information processing systems*, 10, 1997.
 - Dan Hendrycks, Collin Burns, Steven Basart, Andy Zou, Mantas Mazeika, Dawn Song, and Jacob Steinhardt. Measuring massive multitask language understanding. *arXiv preprint arXiv:2009.03300*, 2020.
 - Zhiqiang Hu, Lei Wang, Yihuai Lan, Wanyu Xu, Ee-Peng Lim, Lidong Bing, Xing Xu, Soujanya Poria, and Roy Ka-Wei Lee. Llm-adapters: An adapter family for parameter-efficient fine-tuning of large language models. *arXiv preprint arXiv:2304.01933*, 2023.
 - RC St John and Norman R Draper. D-optimality for regression designs: a review. *Technometrics*, 17(1):15–23, 1975.
 - Charles Royal Johnson. Positive definite matrices. *The American Mathematical Monthly*, 77(3): 259–264, 1970.
 - Erich Kaltofen and Gilles Villard. On the complexity of computing determinants. *Computational complexity*, 13(3):91–130, 2005.
 - Enkelejda Kasneci, Kathrin Seßler, Stefan Küchemann, Maria Bannert, Daryna Dementieva, Frank Fischer, Urs Gasser, Georg Groh, Stephan Günnemann, Eyke Hüllermeier, et al. Chatgpt for good? on opportunities and challenges of large language models for education. *Learning and individual differences*, 103:102274, 2023.
 - Alex Kipnis, Konstantinos Voudouris, Luca M Schulze Buschoff, and Eric Schulz. metabench—a sparse benchmark of reasoning and knowledge in large language models. *arXiv preprint* arXiv:2407.12844, 2024.
 - Arie W Kruglanski and Icek Ajzen. Bias and error in human judgment. *European Journal of Social Psychology*, 13(1):1–44, 1983.
 - Ravi Kumar, Prabhakar Raghavan, Sridhar Rajagopalan, D Sivakumar, Andrew Tomkins, and Eli Upfal. Stochastic models for the web graph. In *Proceedings 41st Annual Symposium on Foundations of Computer Science*, pp. 57–65. IEEE, 2000.
 - Conglong Li, Zhewei Yao, Xiaoxia Wu, Minjia Zhang, Connor Holmes, Cheng Li, and Yuxiong He. Deepspeed data efficiency: Improving deep learning model quality and training efficiency via efficient data sampling and routing. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 38, pp. 18490–18498, 2024a.
 - Jiatong Li, Renjun Hu, Kunzhe Huang, Yan Zhuang, Qi Liu, Mengxiao Zhu, Xing Shi, and Wei Lin. Perteval: Unveiling real knowledge capacity of llms with knowledge-invariant perturbations. *arXiv preprint arXiv:2405.19740*, 2024b.
 - Percy Liang, Rishi Bommasani, Tony Lee, Dimitris Tsipras, Dilara Soylu, Michihiro Yasunaga, Yian Zhang, Deepak Narayanan, Yuhuai Wu, Ananya Kumar, et al. Holistic evaluation of language models. *arXiv preprint arXiv:2211.09110*, 2022.

- Aixin Liu, Bei Feng, Bing Xue, Bingxuan Wang, Bochao Wu, Chengda Lu, Chenggang Zhao, Chengqi Deng, Chenyu Zhang, Chong Ruan, et al. Deepseek-v3 technical report. *arXiv preprint* arXiv:2412.19437, 2024a.
 - Yuan Liu, Haodong Duan, Yuanhan Zhang, Bo Li, Songyang Zhang, Wangbo Zhao, Yike Yuan, Jiaqi Wang, Conghui He, Ziwei Liu, et al. Mmbench: Is your multi-modal model an all-around player? In *European conference on computer vision*, pp. 216–233. Springer, 2024b.
 - Zirui Liu, Zhuang Yan, Qi Liu, Jiatong Li, Yuren Zhang, Zhenya Huang, Jinze Wu, and Shijin Wang. Computerized adaptive testing via collaborative ranking. In *Neural Information Processing Systems*, 2024c. URL https://api.semanticscholar.org/CorpusID:276259892.
 - Zirui Liu, Jiatong Li, Yan Zhuang, Qi Liu, Shuanghong Shen, Jie Ouyang, Mingyue Cheng, and Shijin Wang. am-elo: A stable framework for arena-based llm evaluation, 2025. URL https://arxiv.org/abs/2505.03475.
 - Haipeng Luo, Qingfeng Sun, Can Xu, Pu Zhao, Qingwei Lin, Jianguang Lou, Shifeng Chen, Yansong Tang, and Weizhu Chen. Arena learning: Build data flywheel for llms post-training via simulated chatbot arena. *arXiv preprint arXiv:2407.10627*, 2024.
 - Alexander Ly, Maarten Marsman, Josine Verhagen, Raoul PPP Grasman, and Eric-Jan Wagenmakers. A tutorial on fisher information. *Journal of Mathematical Psychology*, 80:40–55, 2017.
 - Max Marion, Ahmet Üstün, Luiza Pozzobon, Alex Wang, Marzieh Fadaee, and Sara Hooker. When less is more: Investigating data pruning for pretraining llms at scale. *arXiv preprint arXiv:2309.04564*, 2023.
 - Rui Min, Tianyu Pang, Chao Du, Qian Liu, Minhao Cheng, and Min Lin. Improving your model ranking on chatbot arena by vote rigging. In *ICLR 2025 Workshop on Foundation Models in the Wild*, 2025.
 - Iman Mirzadeh, Keivan Alizadeh, Hooman Shahrokhi, Oncel Tuzel, Samy Bengio, and Mehrdad Farajtabar. Gsm-symbolic: Understanding the limitations of mathematical reasoning in large language models. *arXiv preprint arXiv:2410.05229*, 2024.
 - Moran Mizrahi, Guy Kaplan, Dan Malkin, Rotem Dror, Dafna Shahaf, and Gabriel Stanovsky. State of what art? a call for multi-prompt llm evaluation. *Transactions of the Association for Computational Linguistics*, 12:933–949, 2024.
 - Onorato Timothy O'Meara. *Introduction to quadratic forms*, volume 117. Springer, 2013.
 - Rajvardhan Patil and Venkat Gudivada. A review of current trends, techniques, and challenges in large language models (llms). *Applied Sciences*, 14(5):2074, 2024.
 - Radek Pelánek. Applications of the elo rating system in adaptive educational systems. *Computers & Education*, 98:169–179, 2016.
 - Felipe Maia Polo, Lucas Weber, Leshem Choshen, Yuekai Sun, Gongjun Xu, and Mikhail Yurochkin. tinybenchmarks: evaluating llms with fewer examples. *arXiv* preprint *arXiv*:2402.14992, 2024a.
 - Felipe Maia Polo, Ronald Xu, Lucas Weber, Mírian Silva, Onkar Bhardwaj, Leshem Choshen, Allysson Flavio Melo de Oliveira, Yuekai Sun, and Mikhail Yurochkin. Efficient multi-prompt evaluation of llms. *arXiv preprint arXiv:2405.17202*, 2024b.
 - PV Rao and Lawrence L Kupper. Ties in paired-comparison experiments: A generalization of the bradley-terry model. *Journal of the American Statistical Association*, 62(317):194–204, 1967.
 - Jorma J Rissanen. Fisher information and stochastic complexity. *IEEE transactions on information theory*, 42(1):40–47, 1996.
 - Gayathri Saranathan, Mahammad Parwez Alam, James Lim, Suparna Bhattacharya, Soon Yee Wong, Martin Foltin, and Cong Xu. Dele: Data efficient llm evaluation. In *ICLR 2024 Workshop on Navigating and Addressing Data Problems for Foundation Models*, 2024.

- Gordon Simons and Yi-Ching Yao. Asymptotics when the number of parameters tends to infinity in the bradley-terry model for paired comparisons. *The Annals of Statistics*, 27(3):1041–1060, 1999.
 - Lucas Spangher, Tianle Li, William F Arnold, Nick Masiewicki, Xerxes Dotiwalla, Rama Kumar Pasumarthi, Peter Grabowski, Eugene Ie, and Daniel Gruhl. Chatbot arena estimate: towards a generalized performance benchmark for llm capabilities. In *Proceedings of the 2025 Conference of the Nations of the Americas Chapter of the Association for Computational Linguistics: Human Language Technologies (Volume 3: Industry Track)*, pp. 1016–1025, 2025.
 - Balázs Szörényi, Róbert Busa-Fekete, Adil Paul, and Eyke Hüllermeier. Online rank elicitation for plackett-luce: A dueling bandits approach. *Advances in neural information processing systems*, 28, 2015.
 - Kelly Tang, Wei-Lin Chiang, and Anastasios N. Angelopoulos. Arena explorer: A topic modeling pipeline for llm evals & analytics, 2025.
 - Kangyu Wang, Hongliang He, Lin Liu, Ruiqi Liang, Zhenzhong Lan, and Jianguo Li. Inclusion arena: An open platform for evaluating large foundation models with real-world apps. *arXiv* preprint arXiv:2508.11452, 2025.
 - Xianquan Wang, Likang Wu, Shukang Yin, Zhi Li, Yanjiang Chen, Hufeng Hufeng, Yu Su, and Qi Liu. I-am-g: Interest augmented multimodal generator for item personalization. In *Proceedings of the 2024 Conference on Empirical Methods in Natural Language Processing*, pp. 21303–21317, 2024a.
 - Yubo Wang, Xueguang Ma, Ge Zhang, Yuansheng Ni, Abhranil Chandra, Shiguang Guo, Weiming Ren, Aaran Arulraj, Xuan He, Ziyan Jiang, et al. Mmlu-pro: A more robust and challenging multitask language understanding benchmark. *Advances in Neural Information Processing Systems*, 37:95266–95290, 2024b.
 - Samson W Wanyonyi, Ayubu A Okango, and Julius K Koech. Exploration of d-, a-, i-and g-optimality criteria in mixture modeling. 2021.
 - Hermann Weyl. Inequalities between the two kinds of eigenvalues of a linear transformation. *Proceedings of the national academy of sciences*, 35(7):408–411, 1949.
 - Sang Michael Xie, Shibani Santurkar, Tengyu Ma, and Percy S Liang. Data selection for language models via importance resampling. *Advances in Neural Information Processing Systems*, 36: 34201–34227, 2023.
 - Yanbin Yin, Zhen Wang, Kun Zhou, Xiangdong Zhang, Shibo Hao, Yi Gu, Jieyuan Liu, Somanshu Singla, Tianyang Liu, Eric P. Xing, Zhengzhong Liu, Haojian Jin, and Zhiting Hu. Decentralized arena via collective llm intelligence: Building automated, robust, and transparent llm evaluation for numerous dimensions, 10 2024. URL https://de-arena.maitrix.org/.
 - Rowan Zellers, Ari Holtzman, Yonatan Bisk, Ali Farhadi, and Yejin Choi. Hellaswag: Can a machine really finish your sentence? *arXiv preprint arXiv:1905.07830*, 2019.
 - Yang Zhang, Cunxiang Wang, Lindong Wu, Wenbo Yu, Yidong Wang, Guangsheng Bao, and Jie Tang. Uda: Unsupervised debiasing alignment for pair-wise llm-as-a-judge. 2025. URL https://api.semanticscholar.org/CorpusID:280642317.
 - Lianmin Zheng, Wei-Lin Chiang, Ying Sheng, Siyuan Zhuang, Zhanghao Wu, Yonghao Zhuang, Zi Lin, Zhuohan Li, Dacheng Li, Eric. P Xing, Hao Zhang, Joseph E. Gonzalez, and Ion Stoica. Judging llm-as-a-judge with mt-bench and chatbot arena, 2023.
 - Kun Zhou, Yutao Zhu, Zhipeng Chen, Wentong Chen, Wayne Xin Zhao, Xu Chen, Yankai Lin, Ji-Rong Wen, and Jiawei Han. Don't make your llm an evaluation benchmark cheater. *arXiv* preprint arXiv:2311.01964, 2023.
 - Yan Zhuang, Qi Liu, GuanHao Zhao, Zhenya Huang, Weizhe Huang, Zachary Pardos, Enhong Chen, Jinze Wu, and Xin Li. A bounded ability estimation for computerized adaptive testing. *Advances in Neural Information Processing Systems*, 36, 2023.

PROOFS OF THEOREM 1

 Theorem 1. Consider the Erdős-Rényi Graph $G(N,q_N)$, if $q_N = \omega(\frac{logN}{N})$, then the infomation matrix $\mathcal{I}_{\mathcal{S}}(\mathbf{r})$ is almost surely positive definite.

Proof. Given that $q_N = \omega(\frac{logN}{N})$ implies that the graph $G(N,q_N)$ is almost surely connected (Guimera et al., 2004; Chatterjee & Varadhan, 2011), we next prove the following: if there exists a path between any two models i and j, then the information matrix is positive definite. Assume $r_N = 0$ and consider the remaining variables $r = (r_1, \dots, r_{N-1})$. Consider the log-likelihood function for records S:

$$\mathcal{I}_{\mathcal{S}}(\mathbf{r}) = C^2 \sum_{(i,j,w_{ij})\in\mathcal{S}} P_{ij} P_{ji} (\mathbf{e}_i - \mathbf{e}_j) (\mathbf{e}_i - \mathbf{e}_j)^{\top}$$
(8)

where e_i is the *i*-st standard basis vector and let $e_N = 0$. Let the number of matches between model i and model j be δ_{ij} and define $a_{ij} = \delta_{ij}C^2P_{ij}P_{ji}$, then the quadratic form (O'Meara, 2013) for $\mathcal{I}_{\mathcal{S}}(r)$ can be expressed as:

$$x\mathcal{I}_{\mathcal{S}}(r)x^{\top} = \sum_{1 \le i < j \le N-1} a_{ij}(x_i - x_j)^2 + \sum_{i=1}^{N-1} a_{iN}x_i^2 + \sum_{j=1}^{N-1} a_{jN}x_j^2.$$

Note that $a_{ij} \geq 0$, therefore:

$$\boldsymbol{x}\mathcal{I}_{\mathcal{S}}(\boldsymbol{r})\boldsymbol{x}^{\top} \geq 0.$$

The following proves that the condition for the equality of this inequality can only be achieved at 0.

Suppose there exists a non-zero vector $\hat{x} = (\hat{x}_1, \dots, \hat{x}_{N-1})$ such that $\hat{x}\mathcal{I}_{\mathcal{S}}(r)\hat{x}^{\top} = 0$. Without loss of generality, assume that $\hat{x}_1 \neq 0$. Then, since the graph formed by the pairwise battles among the models is connected, it means that there must exist a path $(c_0 = 1, c_1, c_2, \cdots, c_{k-1}, c_k = N)$ that connects 1 and N. Then $a_{c_i c_{i+1}} > 0$, from which we can deduce that $\hat{x}_1 = \hat{x}_{c_1} = \cdots = \hat{x}_{c_{k-1}}$. Since c_{k-1} and N are connected, it indicated that $a_{c_{k-1}N} > 0$. Therefore, $\hat{x}_{c_{k-1}} = 0$, and thus $\hat{x}_1 = 0$, which is a contradiction! So $x\mathcal{I}_{\mathcal{S}}(r)x^{\top} = 0$ if and only if x is the zero vector. Therefore, $\mathcal{I}_{\mathcal{S}}(r)$ is a positive definite matrix (Johnson, 1970).

PROOFS OF THEOREM 2

Lemma 2. Let $0 < \lambda_1^t \le \lambda_2^t \le \cdots \le \lambda_N^t$ denote the eigenvalues of the information matrix $\mathcal{I}_{\mathcal{S}_t}(r)$, then $\lambda_N^t = O(t)$.

Proof. Without loss of generality, we can assume that model i and j compete against each other at time t+1. It is not hard to see that the information increment $C^2P_{ij}P_{ji}(e_i-e_j)(e_i-e_j)^{\top}$ is a rank-1 matrix and its only positive eigenvalue a satisfies that $0 < a \le 2C^2 P_{ij} P_{ji} \le C^2$. Therefore, by Weyl's Inequality for matrix perturbations (Weyl, 1949; Franklin, 2000), the eigenvalues of the information matrix at time t+1 satisfy $\lambda_N^t \leq \lambda_N^{t+1} \leq \lambda_N^t + a \leq \lambda_N^t + C^2$. By recursively expanding this inequality, we obtain that $\lambda_N^t \leq \lambda_N^{t-1} + C^2 \leq \cdots \leq N \cdot C^2, i.e. \lambda_N^t = O(t)$.

Theorem 2. Let $\mathcal{A}(\mathcal{S}_t) = \frac{1}{tr[\mathcal{I}_{\mathcal{S}_t}(\boldsymbol{r})^{-1}]}$ and $\mathcal{D}(\mathcal{S}_t) = |\mathcal{I}_{\mathcal{S}_t}(\boldsymbol{r})|^{\frac{1}{N-1}}$, then it can be proved that: $(1)\mathcal{A}(\mathcal{S}_t) = O(t)$ $(2)\mathcal{D}(\mathcal{S}_t) = O(t).$

Proof. Since $tr(\mathcal{I}_{\mathcal{S}_t}(\boldsymbol{r})^{-1}) = \frac{1}{\lambda_1^t} + \frac{1}{\lambda_2^t} + \dots + \frac{1}{\lambda_N^t} \geq \frac{N}{\lambda_N^t}$ and $|\mathcal{I}_{\mathcal{S}_t}(\boldsymbol{r})| = \lambda_1^t \lambda_2^t \dots \lambda_N^t \leq (\lambda_N^t)^N$, we can derive the following bounds for the optimality criteria:

For the A-optimality criterion $\mathcal{A}(\mathcal{S}_t) = \frac{1}{tr[\mathcal{I}_{\mathcal{S}_{\star}}(\mathbf{r})^{-1}]} \leq \frac{\lambda_N^t}{N}$.

For the D-optimality criterion $\mathcal{D}(\mathcal{S}_t) = |\mathcal{I}_{\mathcal{S}_t}(\boldsymbol{r})|^{\frac{1}{N-1}} \leq \lambda_N^t$. By Lemma 2, we know $\lambda_N^t = O(t)$, substituting this into the above inequalities yields $\mathcal{A}(\mathcal{S}_t) = O(t)$ and $\mathcal{D}(\mathcal{S}_t) = O(t)$.

C DETAILS OF EXPERIMENTS

C.1 STATISTICS OF THE DATASET

In addition to the two datasets (**Chatbot** and **PPE**) mentioned in the main text, we have also included a larger dataset **Arena-100k** (Chiang et al., 2024; Tang et al., 2025) in the appendix. The Arena-100k¹ dataset comprises conversation data collected from June 2024 to August 2024, as well as English human preference evaluations, which were used to develop Arena Explorer. Each record in datasets includes a question ID, the names of the two competing models, their full conversation transcripts and the annotator's vote. The statistics of the datasets can be seen in Table 3.

Table 3: Statistics of the dataset				
Dataset	Chatbot	PPE	Arena-100k	
#Models	20	20	55	
#Response logs	33000	16038	106134	
#Response logs per model	1650.0	801.9	1929.7	
#Response logs per model pair	173.68	89.10	71.47	

C.2 DETAILS OF EXPERIMENTAL SETTING.

In this experiment, 100 sample pairs will be randomly selected during the model initialization phase to initialize ELO, and the performance of different selection strategies will be recorded at steps 100, 200, 500, and 1000. After a selection strategy chooses a model pair, we consider two methods to simulate the competition result: (1) Conducting without-replacement random sampling on the real competition results in the dataset; (2) Simulating based on the real model abilities. If the abilities of the two models are r_i^* , r_j^* and $p = \sigma(C(r_i^* - r_j^*))$, then each battle will return 1 with a probability of p^2 , 0 with a probability of p^2 , 0 with a probability of p^2 , and 0.5 with a probability of p^2 , 0 with a probability of p^2 , and 0.5 with a probability of p^2 , 0 with a

C.3 Performance on Large Dataset

To further validate the effectiveness of the D-optimality method, we also conducted experiments on a larger dataset, namely Arena-100k. The key difference from the previous experiments lies in the following: as the number of models increased to 55, the number of initial samples we used was also increased to 1000. All other experimental settings remained unchanged. The experimental results are presented in Figure 6 and Table 4.

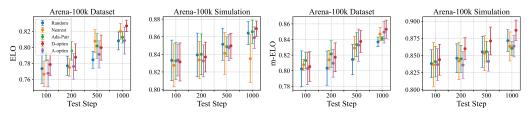


Figure 6: This line chart illustrates the pairwise metrics for Arena-100k dataset and each ELO method. Here, "Dataset" indicates that the competition results are randomly sampled from the dataset, while "Simulation" denotes that the results are simulated based on the real model abilities. The error bars represent the standard deviation.

It can be observed that with an increase in the number of models, the errors introduced by the traditional ELO method into the information matrix become amplified, leading to inaccurate information estimation. This explains why the D-optim method exhibits superior performance specifically in terms of m-ELO. However, at step 1000, the D-optim method achieves the best performance across all experimental settings. Therefore, overall speaking, as long as the current abilities of the models can be effectively estimated, the D-optimality method remains fully effective even in scenarios with a relatively large number of models.

https://huggingface.co/datasets/lmarena-ai/arena-human-preference-100k

Table 4: The Performance of pairwise for Different Selection Algorithms. Each experimental result represents the average of five experiments and four selection steps, the bold font represents a significant improvement in statistics compared to the baseline.

Method	Is Simu		ELO I	Total	
	Real	Simu	ELO	m-ELO	
Random	0.8000	0.8499	0.8164	0.8335	0.8249
Nearest	0.8083	0.8415	0.8133	0.8365	0.8249
Ada-Pair	0.8138	0.8486	0.8236	0.8388	0.8312
A-optim	0.8047	0.8438	0.8145	0.8340	0.8243
D-optim	0.8132	0.8561	0.8225	0.8468	0.8346

D USAGE OF LLM

810

811

823 824

825

826

827

828

829

830 831

832 833

834

835

836

837

838

839

840

841

844

845

846

847

848

849

850 851

852

853

854

855

856

858

859

861

862

According to the rules about LLM use set by the ICLR 2026 conference, when we worked on this paper, we only used LLMs to polish the text. This includes making the flow of language smoother and improving how well sentences connect logically. We did not use LLMs for key steps like coming up with main points, making research content, or handling data. Also, we made sure that the polishing process does not change the main ideas or weaken the academic strictness of the original text, and we fully follow the standards for proper LLM use set by ICLR 2026.

E CASE STUDY OF FISHER INFOMATION MATRIX

To facilitate a better understanding of our method, we directly visualize the changes in the information matrix throughout the experimental process. Specifically, we adopt the D-optimality method: in each iteration, a model pair is selected, and the model ability estimates are updated in real time. Below, we present the information matrix of 5 iterations under the condition of 100 initial samples:

```
0.0
                                             0.0
                                                  -0.5 -0.2 -0.2 -0.5 0.0
                                                                                               -0.2
       -0.4
             -0.2
                   -0.2
                                0.0
                                       0.0
                                                                                  -0.5 -0.2
                                                                                                      0.0
                                                   \begin{array}{ccc} 0.0 & -0.5 \\ 0.0 & -0.2 \end{array}
-0.4
      2.3
             0.0
                   0.0
                                0.0
                                      0.0
                                                                0.0
                                                                      -0.4
                                                                            -0.2 \quad 0.0 \quad -0.2
                                                                                                      0.0
                                                                                                             0.0
                                                                                                                  -0.2 \\ -0.2
                                             0.0
-0.2
      0.0
             2.0
                   -0.2
                          0.0
                                0.0
                                      -0.5
                                             0.0
                                                                0.0
                                                                      0.0
                                                                             0.0
                                                                                  -0.2 -0.2
                                                                                                0.0
                                                                                                      0.0
                                                                                                             0.0
-0.2
             -0.2 2.9
                                      0.0
                                                  -0.2
                                                         -0.2
                                                                                                            -0.2
                                -0.2
                                             -0.2
                                                                -0.4 \quad 0.0
                                                                             0.0
                                                                                                                  0.0
      0.0
                          0.0
                                                                                  -0.5 -0.2
                                                                                               0.0
                                                                                                      0.0
                   0.0
                                0.0
                                      0.0
                                                   0.0
                                                          0.0
                                                                0.0
                                                                                                      -0.2
                                                                                               -0.2
0.0
      0.0
             0.0
                   -0.2
                         0.0
                                1.6
                                      -0.2
                                             0.0
                                                   0.0
                                                          0.0
                                                                0.0
                                                                      0.0
                                                                            0.0
                                                                                   0.0
                                                                                        -0.2
                                                                                               -0.4
                                                                                                      0.0
                                                                                                             0.0
                                                                                                                   -0.2
      0.0
             -0.5 0.0
                                -0.2 2.7
                                             -0.5
                                                  0.0
                                                         0.0
                                                                     -0.2
                                                                            -0.4
                                                                                         -0.2
                                                                                                      0.0
0.0
                          0.0
                                                                -0.4
                                                                                   0.0
                                                                                               0.0
                                                                                                             0.0
                                                                                                                   0.0
      0.0
                   -0.2
                                0.0
                                      -0.5
                                             2.7
                                                    -0.2
                                                         -0.2
                                                                0.0
                                                                      0.0
                                                                            0.0
0.0
                                                                                                       -0.4
             0.0 - 0.2
                                            -0.2
-0.5
      0.0
                          0.0
                                0.0
                                       0.0
                                                   1.6
                                                          0.0
                                                                0.0
                                                                      0.0
                                                                            -0.4
                                                                                   0.0
                                                                                          0.0
                                                                                                0.0
                                                                                                      0.0
                                                                                                             0.0
                                                                                                                   0.0
             -0.2 -0.2
                                            -0.2
                                                                                                             -0.2
-0.2
       -0.5
                                      0.0
                                                                -0.2 -0.6
                          0.0
                                0.0
                                                   0.0
                                                         2.7
                                                                            0.0
                                                                                   0.0
                                                                                         0.0
                                                                                                0.0
                                                                                                      0.0
                                                                                                                   0.0
-0.2
      0.0
             0.0
                   -0.4
                         0.0
                                0.0
                                      -0.4
                                             0.0
                                                   0.0
                                                         -0.2
                                                                2.8
                                                                      -0.4
                                                                            0.0
                                                                                                      0.0
                                                                                    -0.2
                                                                                         -0.7
                                                                                                                   0.0

\begin{array}{ccc}
-0.4 & 3.1 \\
0.0 & -0.2
\end{array}

-0.5
      -0.4
             0.0
                   0.0
                         -0.2
                                0.0
                                     -0.2
                                             0.0
                                                   0.0
                                                         -0.6
                                                                            -0.2
                                                                                   0.0
                                                                                         0.0
                                                                                               -0.2
                                                                                                     -0.2
                                                                                                             0.0
                                                                                                                   0.0
0.0
      -0.2
             0.0
                   0.0
                         0.0
                                0.0
                                      -0.4
                                             0.0
                                                   -0.4
                                                         0.0
                                                                0.0
                                                                            1.9
                                                                                         -0.5
                                                                                   -0.2
                                                                                                0.0
                                                                                                      0.0
-0.5
      0.0
             -0.2
                   -0.5
                         0.0
                                0.0
                                      0.0
                                             0.0
                                                   0.0
                                                          0.0
                                                                -0.2
                                                                      0.0
                                                                            -0.2
                                                                                   1.6
                                                                                          0.0
                                                                                                0.0
                                                                                                      0.0
                                                                                                                   0.0
-0.2
      -0.2
            -0.2
                  -0.2
                         -0.2
                               -0.2
                                      -0.2
                                             0.0
                                                   0.0
                                                          0.0
                                                               -0.7
                                                                      0.0
                                                                            -0.5
                                                                                   0.0
                                                                                          3.1
                                                                                                0.0
                                                                                                     -0.2
                                                                                                            0.0
                                                                                                                   0.0
-0.2
                                                          0.0
                                                                     -0.2
      0.0
             0.0
                   0.0
                         -0.2
                               -0.4
                                       0.0
                                            -0.4
                                                   0.0
                                                                0.0
                                                                             0.0
                                                                                         0.0
                                                                                                     -0.2
                                                                                                             -0.2
                                                                                                                  0.0
                                0.0 \\ 0.0
      0.0
             0.0
                   0.0
                         -0.2
                                       0.0
                                             -0.4
                                                   0.0
                                                         0.0
                                                                0.0
                                                                      -0.2
                                                                            0.0
                                                                                   0.0
                                                                                         -0.2
                                                                                               -0.2
                                                                                                                  -0.2
0.0
                                                                                                     0.0
                   -0.2
                        0.0
                                                         -0.2
                                                                                               -0.2
      0.0
             0.0
                                       0.0
                                             0.0
                                                   0.0
                                                                0.0
                                                                      0.0
                                                                                   0.0
                                                                                         0.0
                                                                                                                  -0.2
                                                                             0.0
                                      0.0
                                                                0.0
            -0.2 \quad 0.0
                               -0.2
                                            -0.2
                                                   0.0
                                                         0.0
```

 \downarrow Step 1:(17,4,1)

```
-0.5 -0.2 -0.2 -0.5 0.0
                                     0.0
                                           0.0
                                                                              -0.5 -0.2 -0.2
                                                                                                  0.0
                                                 0.0
                                                       -0.5
                                                             0.0
                                                                   -0.4
                                                                          -0.2 \quad 0.0
                                                                                                   0.0 + 0.0 + -0.2
                  -0.2 \pm 0.0 \pm 0.0
-0.2
      0.0
            2.0
                                     -0.5
                                           0.0
                                                 0.0
                                                      -0.2 \quad 0.0
                                                                    0.0
                                                                          0.0
                                                                               -0.2 -0.2
                                                                                                   0.0 \pm 0.0 \pm -0.2
                                                 -0.2
            -0.2 2.9 | 0.0 | -0.2
                                     0.0
                                           -0.2
                                                             -0.4 \quad 0.0
                                                                          0.0
                                                                               -0.5 - 0.2
                                                                                                   0.0 1-0.2 0.0
           0.0
                                                                  0.0
                               \overline{0.0}
                                     0.0
                                           0.0
                                                              \overline{0.0}
                                                                                0.0
                                                                                       -0.\bar{2}
                                                                                                   -0.2
                            -\frac{5.0}{1.6}
                                                                                    -\frac{0}{0.2}
                                           0.0
      0.0
                                     -0.2^{-}
                                                 0.0
                                                       0.0
                                                             \overline{0.0}
                                                                               0.0^{-}
                                                                                                  0.0
0.0
                               -0.2
                                    2.7
                                           -0.5
                                                       0.0
                                                             -0.4 - 0.2
                                                                                     -0.2
0.0
      0.0
                                                0.0
                                                                         -0.4
                                                                                0.0
                                                                                            0.0
                  -0.2^{||}0.0^{||}0.0
                                                                    0.0
                                     -0.5
                                                 -0.2
                                                       -0.2
                                                             0.0
                                                                         0.0
                                                       0.0 \\ 2.7
                                                                                0.0
-0.5
      0.0
             0.0
                  -0.2 \mid 0.0 \mid 0.0
                                     0.0
                                           -0.2
                                                 1.6
                                                              0.0
                                                                    0.0
                                                                         -0.4
                                                                                      0.0
                                                                                                   0.0 \pm 0.0 \pm 0.0
-0.2
      -0.5
            -0.2 \quad -0.2 \quad 0.0 \quad +0.0
                                     0.0
                                          -0.2
                                                 0.0
                                                                   -0.6
                                                                         0.0
                                                                                0.0
                                                                                                   0.0 \mid -0.2 \mid 0.0
                                                             -0.2
                                                                                      0.0
                                                                                            0.0
      0.0
                   -0.4 | 0.0 | 0.0
                                     -0.4
                                                       -0.2
                                                             2.8
                                                                    -0.4
                                                                         0.0
                                                                                                  0.0 \\ -0.2
                         -0.2
                                                                                                        0.0
-0.5
     -0.4
             0.0
                   0.0
                               0.0
                                    -0.2
                                           0.0
                                                 0.0
                                                       -0.6
                                                             -0.4 3.0
                                                                          -0.2 \quad 0.0
                                                                                      0.0
                                                                                            -0.2
                   0.0 \begin{array}{c} -0.2 \\ 0.0 \end{array}
                                                                   -0.2
0.0
      -0.2
            0.0
                               0.0
                                    -0.4
                                           0.0
                                                 -0.4
                                                       0.0
                                                             0.0
                                                                         1.9
                                                                               -0.2
                                                                                      -0.5
                                                                                            0.0
                                                                                                   0.0
                                                                                                               0.0
                  -0.5 | 0.0 | 0.0
                                                                                                   0.0 | 0.0 |
      0.0
                                     0.0
                                                 0.0
                                                              -0.2
                                                                   0.0
                                                                                1.6
                                                                                      0.0
                  -0.2 \mid -0.2 \mid -0.2
                                                                                                  -0.2 \pm 0.0 \pm 0.0
-0.2
     -0.2 -0.2
                                    -0.2
                                           0.0
                                                 0.0
                                                        0.0
                                                             -0.7
                                                                    0.0
                                                                         -0.5
                                                                                0.0
                                                                                      3.1
                                                                                            0.0
-0.2 \quad 0.0
                  0.0 \mid -0.2 \mid -0.4
                                     0.0
                                                                   -0.2
                                                       0.0
                                                              0.0
                                                                         0.0
                                                                                0.0
                                                                                                  -0.2 \mid -0.2 \mid 0.0
            0.0
                                          -0.4
                                                 0.0
                                                                                      0.0
     -0.4
                                                 0.0
                                                       0.0
                                                              0.0
                                                                   -0.2
                                                                         0.0
                                                                                      -0.2
                                                       -\frac{0.0}{-0.2}
                                                 0 0
                                                                  -0.0 - 0.0 - 0.0
                                                                                -0.0
                                                                                      0.0
                                                      -0.0
                                                                                    -\frac{0.0}{0.0}
                                                                               0.0
                                                 0.0
```

 \downarrow Step 2:(17,13,0)

```
864
865
                                                                                                                                             3.7
                                                 -0.4 -0.2 -0.2 0.0
                                                                                                               0.0
                                                                                                                              0.0
                                                                                                                                                                                                                                                                                       -0.4
                                                2.3
                                                                 0.0 0.0
                                                                                                0.0
                                                                                                               0.0
                                                                                                                              0.0
                                                                                                                                                            0.0 - 0.2
                                                                                                                                                                                                                          0.01 - 0.21 - 0.2
                                 -0.2
                                                 0.0
                                                                 2.0
                                                                                -0.2
                                                                                               0.0
                                                                                                               0.0
                                                                                                                             -0.5
                                                                                                                                             0.0
                                                                                                                                                                                         0.0 0.0
                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                       0.0 \pm 0.0 \pm -0.2
                                    -0.2
                                                                                                                                            -0.2 \ -0.2 \ -0.2 \ -0.4 \ 0.0
                                                                 -0.2 2.9
                                                                                                0.0
                                                                                                              -0.2 \quad 0.0
                                                                                                                                                                                                                          0.0 \mid -0.5 \mid -0.2
                                                  0.0
                                                                                                                                                                                                                                                                                       0.0 \ | -0.2 \ | \ 0.0
                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                         0.0 + 0.0 + -0.2
                                                                                0.0
                                                                                                                              0.0
                                                                                                                                                            0.0
                                                                                                                                                                                            0.0
                                                                                                                                                                                                      -0.2
                                   0.0
                                                  0.0
                                                                 0.0
                                                                                                               0.0
                                                                                                                                             0.0
                                                                                                                                                                            0.0
                                                                                                                                                                                                                                                                      -0.2
                                                                                                                                                                                                                                                                                     -0.2 | -0.2 | 0.0
868
                                                                                                                                            \begin{array}{ccc} 0.0 & 0.0 \\ -0.5 & 0.0 \end{array}
                                                                                                                                                                                          0.0
                                                  0.0
                                                                 0.0
                                                                               -0.2
                                                                                              0.0
                                                                                                              1.6
                                                                                                                             -0.2
                                                                                                                                                                            0.0
                                                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                                                      -0.2
                                                                                                                                                                                                                                                                                       0.0 | 0.0 | -0.2
                                                                                                              -0.2 2.7
                                                                                                                                                                         0.0
                                   0.0
                                                  0.0
                                                                 -0.5 \quad 0.0
                                                                                                0.0
869
                                                                                                                             \begin{array}{ccc} 0.0 & -0.2 \\ 0.0 & -0.2 \end{array}
                                                                                                                                                                                                          0.0
                                                                                                                                                                                                                        0.0 \pm 0.0 \pm 0.0
                                                                                                                                                                                                                                                                                      -0.4 | 0.0 | -0.2
                                   0.0
                                                  0.0
                                                                                               0.0
                                                                                                               0.0
                                                                                                                                                                          -0.2 	ext{ } 0.0
                                                                                                                                                                                                                                                                        -0.4
                                                                                                                                                                         0.0 \\ 2.7
                                  -0.5
                                                 0.0
                                                                                               0.0
                                                                                                               0.0
                                                                                                                                                                                           0.0
                                                                                                                                                                                                          0.0
                                                                                                                                                                                                                        -0.4 \mid 0.0 \mid 0.0
                                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                       0.0 + 0.0 + 0.0
870
                                                                                                               0.0
                                 -0.2
                                                 -0.5
                                                                -0.2 -0.2
                                                                                               0.0
                                                                                                                                                                                           -0.2 -0.6 0.0 | 0.0 | 0.0
                                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                       0.0 \mid -0.2 \mid 0.0
                                                                                                                                                           \begin{array}{cccc} 0.0 & -0.4 & 0.0 \\ 0.0 & -0.2 & 0.0 \end{array}
                                                                                                                                                                                                                                                                                     0.0 \\ -0.2
                                   -0.2 	0.0
                                                                 0.0
                                                                               -0.4
                                                                                              0.0
871
                                                -0.4
                                                                 0.0
                                                                                0.0
                                                                                              -0.2
                                   -0.5
                                                                                                                                                                                                                                                                       -0.2
                                                                                                                                                                                                                                                                                                       0.0
872
                                                  -0.2
                                                                                 0.0
                                                                                             -0.0 \\ -0.0
                                                                                                               0.0
                                                                                                                              -0.4 \quad 0.0
                                                                                                                                                                                                                                                                                   \begin{array}{c} -0.0 \\ -0.0 \\ -0.2 \end{array} - \begin{array}{c} 0.0 \\ \overline{0.0} \\ -0.2 \end{array}
                                                                                                                           -0.5
                                                              0.0
873
                                  -0.2
                                                 -0.2
                                  -0.2 \quad 0.0
                                                                0.0
                                                                                0.0 \quad -0.2 \quad -0.4 \quad 0.0 \quad -0.4 \quad 0.0 \quad 0.0 \quad 0.0 \quad -0.2 \quad 0.0 \quad \mid 0.0 \quad \mid 0.0 \quad \mid 0.2 \quad -0.2 \mid -0.2 \mid -0.2 \mid 0.0 \mid \mid 0.0 \quad \mid
874
                                                                -\frac{0.0}{0.0} - \frac{0.0}{0.0}
                                                                                           0.0
875
                                    0.0
                                                  0 0
                                                                                 -0.2
                              \begin{bmatrix} -\frac{0.0}{-0.2} & \frac{0.0}{-0.2} & -\frac{0.0}{-0.2} & 0.0 \end{bmatrix}
876
                                                                                                                                                                        Step 3:(8,4,0)
877
878
                                                 0.0 \\ 0.0
                                                                                                                                             0.0 \\ 0.0
                                                                                                                                                                                                                                                                                                                   -0.2 \\ -0.2
879
                                  -0.4
                                                                                -0.2 \mid 0.0 \mid 0.0
                                                                                                                                                                                                                                     -0.2 -0.2
                                  -0.2
                                                                                                                            -0.5
                                                                                                                                             0.0 + 0.0 + -0.2 + 0.0 + 0.0
                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                                                                     -0.2
880
                                                            -0.2
                                                 0.0
                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                                      -0.2
                                                                                                                                                                                                                                                                                                                     0.0
                                                                                                                                                                                                                                                                                     -0.0
-0.2
-0.0
                                   0.0_{-}
                                                                                                                                                                                                                                                                                                       -0.2
                                                                                                                                                                                                                                                                                                                    0.0
881
                                                                                                                                                                                                                                                                      -0.2
-0.4
                                                 0.0
                                                 0.0
                                                                                                                                                                                                                                                                                                                     -0.2
                                   0.0
                                                                                                                                                                                         -0.4 \ -0.2 \ -0.4 \ 0.0 \ -0.2 \ 0.0 \ 0.0
                                                                                                                                                                                                                                                                                                                     0.0
                                                                                                                                                                                                                                                                                     -0.4
                                   0.0
                                                                                                                                                                                                                                                                        -0.4
                                                                                                                                                                                                                                                                                                                      -0.2
883
                                                                                                                                                                                                                                                                                                                    -0.2
                                    -0.5
                                                0.0
                                -0.2
884
                                                -0.5
                                                                                                                                                                                                                                                                                                                     -0.0
                                                              -0.2 -0.7 0.0
                                 -0.2 \quad 0.0
                                                                                                                                                                                                                                                                                      0.0
885
                                  -0.5
                                                -0.4
                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                     -0.2
                                                                                                                                                                                                                                                                       -0.2
                                                                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                       -0.5 \quad 0.0
                                                -0.2 \quad 0.0
886
                                   -0.5 \quad 0.0
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                                      -0.2
                                                                                                                                                                                                                                                                                                                     0.0
                                 -0.2
                                                -0.2
                                                                                                                                                                                                                                                                                     -0.2 	 0.0
887
                                                                                                                                                                                                                                                         3.1
                                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                                                      0.0
                                  -0.2
                                                                                 0.0 \;\; | \, -0.2 \, | \, -0.4
                                                                                                                                             -0.4 \mid 0.0 \mid 0.0
                                                                                                                                                                                            0.0 - 0.2
                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                                     -0.2
                                                                                                                                                                                                                                                                                                                     0.0
                                                 0.0
                                                                                                                              0.0
                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                                                      -0.2
888
                                 0.0 - 0.2 0.0
                                                                                                                                                                                                                                                      -0.2 -0.2
                                                                                                                                            -0.4 | 0.0 | 0.0
                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                                                     1.8
                                                                                                                                                                                                                                                                                                      0.0 - 0.2
                                                                                                                                           0.0
                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                        -0.2 \quad 0.0 \quad -0.2 \quad 0.0
                                                                                                                                                                                                                                                                                                                     -0.2
                                                                                                                                                                                                          0.0 0.0
                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                                                     -0.2 -0.2
890
                                                                                                                                                                    Step 4:(17,5,1)
891
                                              892
                                                                                                                                             0.0 \quad -0.5 \quad -0.2 \quad -0.2 \quad -0.5 \quad 0.0 \quad -0.5 \quad -0.2 \quad -0.2 \quad 0.0 \quad 0.0 \quad -0.2 \quad 0.0 \quad 0.
                                   3.7
                                                                                                                                                                                                                                                                                       0.0 + 0.0 + -0.2
                                 -0.4
                                                                                                                                             0.0
                                                                                                                                                            -0.2
                                                                              -0.2 \quad 0.0 \quad | \quad 0.0 \quad | \quad -0.5 \quad 0.0
                                                0.0
                                                                 2.0
                                                                                                                                                                                                                                                                                       0.0 \pm 0.0 \pm -0.2
                                   -0.2
                                                 0.0
                                                                 -0.2 2.8
                                                                                               0.0\ |-0.2\ |\ 0.0
                                                                                                                                            -0.2 -0.2 -0.2
                                                                                                                                                                                          -0.4 \quad 0.0
                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                       -0.5 -0.2
                                                                                                                                                                                                                                                                                        0.0\ |-0.2\ |\ 0.0
894
                                                             0.0
                                                 0.0
                                                                                                                                                                                                                                                                      \underline{-0.2}\ \underline{-0.2}\rfloor \underline{-0.2}\rfloor \underline{-0.2}\rfloor \underline{-0.2}
                                                                                                                                                                                                                                                                       -0.4
                                                                                                                                                                                                                                                                                   -\frac{0.0}{0.0} + \frac{-0.2}{0.0} - \frac{-0.2}{0.0}
                                0.0
                                                 0.0
                                                 0.0
896
                                                                                                                                                                                                                                                                      0.0
                                                  0.0
                                   -0.5
897
                                                0.0
                                                 -0.5
                                                                -0.2 -0.2 -0.0 + 0.0 + 0.0
                                                                                                                                           -0.2
                                                                                                                                                          0.0
                                                                                                                                                                            2.6
                                                                                                                                                                                          -0.2 -0.6
                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                                       0.0 \mid -0.2 \mid 0.0
                                                                                                                                                                                                                                                                        0.0
                                                                898
                                  -0.2
                                                0.0
                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                        -0.2
                                                                                                                                                                                                                                                        -0.7
                                                                                                                                                                                                                                                                                       0.0\ |\ 0.0\ |\ 0.0
                                                                                                                                                                                                                                                                        0.0
                                   -0.5 -0.4
                                                                                                                                                                                                                        -0.2 \quad 0.0
                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                       -0.2 \quad -0.2 \quad 0.0
                                                                                                                                                                                                                                                                                                                      0.0
                                                 -0.2
                                                                                                                                                                                                                                        -0.2
                                                                                                                                                                                                                                                        -0.5
                                                                                                                                                                                                                                                                                      0.0 \ | \ 0.0 \ | \ 0.0 \ | \ 0.2 \ |
                                                                                                                                                                                         -0.2 \quad 0.0 \quad -0.2
                                  -0.5
                                                0.0
                                                                                                                                                                                                                                        1.9
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                                                     0.0
900
                                                                                                                                                                                         -0.7 \quad 0.0 \quad -0.5 \quad 0.0
                                                -0.2 \ -0.2 \ -0.2 \ -0.2 \ -0.2 \ -0.2 \ -0.2 \ 0.0
                                                                                                                                                                                                                                                                        0.0 \quad -0.2 \mid 0.0 \mid 0.0
                                                                                                                                                             0.0
                                 -0.2
                                                                                                                                                                            0.0
                                                                                                                                                                                                                                                         3.1
                              \begin{array}{c} 0.0 & 0.2 & 0.02 & 0.02 & 0.0 \\ 0.0 & 2.2 & -0.2 & -0.2 & 0.0 \\ -0.2 & -0.2 & 1.8 & 0.0 & -0.2 \\ -0.0 & -0.2 & -0.0 & 1.6 & -0.2 \\ -0.0 & 0.0 & -0.2 & -0.2 & 2.1 \end{array}
901
902
903
                                            904
905
                                -0.4
                                                                                                                                                                                                                                                                                                                   \frac{-0.2}{-0.2}
906
                                                                                                                                                                                                                                                                                                      0.0
                                   -0.\overline{2}
                                                                                                                                                                                                                                                                                                       \overline{0.0}
                                                                                                                                                                                                                                                                                    -\frac{0.0}{0.0} - \frac{0.0}{-0.2}
907
                                                                                                                                                                                                                                                                                                                    0.0
                                                                                                                                                                                                                                                                                                      -0.\overline{2}
                                                                                                                                                                                                                                                                                                                     0.0
908
                                                                                                                                                                                                                                                                                                      -0.\overline{2}
                                   0.0
                                                                                                                                                                                                                                                                                                                    -0.2
909
                                   0.0
                                                                                                                                                                                                                                                                                                                     0.0
910
                                   -0.5
                                                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                                                                      -0.2
                                 -0.2
                                                                                                                                                                                                                                                                                                                      0.0
911
                                                 0.0 + 0.0 + -0.4 + 0.0 + 0.0
                                                                                                                            -0.4 \quad 0.0
                                                                                                                                                             0.0
                                                                                                                                                                           -0.2
                                                                                                                                                                                         2.6 -0.4 0.0
                                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                         -0.2
                                                                                                                                                                                                                                                        -0.7
                                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                                                      0.0
                                               \begin{array}{c} -0.4 \mid 0.0 \mid 0.0 \mid -0.2 \mid 0.0 \quad -0.2 \quad 0.0 \\ -0.2 \mid 0.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad -0.4 \quad 0.0 \\ 0.0 \mid -0.2 \mid -0.5 \mid 0.0 \mid 0.0 \quad 0.0 \quad 0.0 \\ -0.2 \mid -0.2 \mid -0.2 \mid -0.2 \mid -0.2 \quad -0.2 \quad 0.0 \end{array}

\begin{array}{ccc}
-0.2 & 0.0 \\
1.8 & -0.2
\end{array}

912
                                   -0.5
                                                                                                                                                            0.0
                                                                                                                                                                          -0.6 -0.4 2.9
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                        -0.2 -0.2
                                                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                            -0.4
                                                                                                                                                                            0.0
                                                                                                                                                                                         0.0
                                                                                                                                                                                                         -0.2
                                                                                                                                                                                                                                        -0.2
                                                                                                                                                                                                                                                        -0.5
                                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                                                                                      0.0
913
                                                                                                                                                                                         1.9
                                                                                                                                                             0.0
                                                                                                                                                                             0.0
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                                       -0.2
                                                                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                                                    -0.2 \quad 0.0
                                 -0.2
                                                                                                                                                             0.0
                                                                                                                                                                            0.0
                                                                                                                                                                                                                        -0.5
                                                                                                                                                                                                                                                         3.0
                                                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                                                                                      0.0
914
                                                                                                                                                                                           0.0
                                   -0.2
                                                 0.0 + 0.0 + 0.0 + -0.2 + -0.4
                                                                                                                             0.0
                                                                                                                                            -0.4
                                                                                                                                                             0.0
                                                                                                                                                                             0.0
                                                                                                                                                                                                      -0.2
                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                                      -0.2
                                                                                                                                                                                                                                                                                                      -0.2
                                                                                                                                                                                                                                                                                                                     0.0
                                                                                                                                                                                                                                                                        2.2
                                 -0.4
                                                                                                                                                            0.0
                                                                                                                                                                            0.0
                                                                                                                                                                                           0.0 - 0.2
                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                        -0.2
                                                                                                                                                                                                                                                                     -0.2
                                                                                                                                                                                                                                                                                     1.8
                                                                                                                                                                                                                                                                                                       0.0
                                                                                                                                                                                                                                                                                                                     -0.2
915
                                                                                                                                                                            -0.2 \quad 0.0
                                                                                                                                                                                                                                         -0.2 \quad 0.0
                                                                                                                                                                                                                                                                       -0.2 \quad 0.0
                                                                                                                                           0.0
                                                                                                                                                                                                          0.0
                                                                                                                                                             0.0
                                                                                                                                                                                                                          0.0
                                                                                                                                                                                                                                                                                                      1.6
                                                                                                                                                                                                                                                                                                                     -0.2
916
                                                                                                                                           -0.2
                                                                                                                                                           0.0
                                                                                                                                                                            0.0
                                                                                                                                                                                           0.0
                                                                                                                                                                                                          0.0
                                                                                                                                                                                                                          0.0
                                                                                                                                                                                                                                        0.0
                                                                                                                                                                                                                                                         0.0
                                                                                                                                                                                                                                                                      0.0
                                                                                                                                                                                                                                                                                    -0.2 -0.2
```