Sampling via Generating Functions

Stephen D. Casey
Department of Mathematics and Statistics,
American University, Washington, DC 20016 USA
Email: *scasey@american.edu*

***Abstract*—We develop connections between some of the most powerful theories in analysis, tying the Shannon sampling formula to Cauchy's integral and residue formulae, Jacobi interpolation, and Levin's sine-type functions. The techniques use tools from complex analysis, and in particular, the Cauchy theory and the theory of entire functions, to realize sampling sets Λ as zero sets of well-chosen entire functions (sampling set *generating functions*). We then reconstruct the signal from the set of samples using the Cauchy-Jacobi machinery. These methods give us powerful tools for creating a variety of general sampling formulae, e.g., allowing us to derive Shannon sampling and Papoulis generalized sampling via Cauchy theory and sampling in radial domains.**

I. Introduction

We develop connections between some of the most powerful theories in analysis, tying the Shannon sampling formula to Cauchy's integral and residue formulae, Jacobi interpolation, and Levin's sine-type functions. The main techniques in this paper use tools from complex analysis, and in particular, the Cauchy theory and the theory of entire functions, to realize sampling sets Λ as zero sets of well-chosen entire functions (sampling set *generating functions*). We then reconstruct the signal from the set of samples using the Cauchy-Jacobi machinery. These methods give us powerful tools for creating a variety of general sampling formulae, e.g., allowing us to derive Shannon sampling and Papoulis generalized sampling via Cauchy theory. The techniques developed are also manifest in solutions to the analytic Bezout equation associated with certain multi-channel deconvolution problems, and we show how these lead to multi-rate sampling. We give specific examples of non-commensurate lattices associated with multi-channel deconvolution, and use a generalization of B. Ya. Levin's sine-type functions to develop interpolating formulae on these sets[1].

The *Jacobi interpolation formula* works with the Cauchy integral formula to extract information from an analytic function by integrating it against an interpolating function which places the "right poles" at the "right spots" to extract that information. *This makes it an excellent tool for sampling.* The key to make this work is to choose the correct Jacobi interpolator G, which is a sampling set *generating function*. Given the analytic function f and Jacobi interpolator G, for our purposes it has the general form

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta - z)} \cdot \frac{G(\zeta)-G(z)}{G(\zeta)} d\zeta \qquad (1)$$

$$+ \frac{1}{2\pi i}\oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta - z)} \cdot \frac{G(z)}{G(\zeta)} d\zeta\,, \qquad (2)$$

where Γ_m is a sequence of Jordan curves chosen to avoid zeroes of G. We extract the information from (1) using the residue calculus, while (2) plays the role of the remainder R_m, with $|(2)| \longrightarrow 0$ as $m \longrightarrow \infty$. The evaluation of (1) gives the sampling formula, while $|(2)|$ gives us the convergence rate of the sampling formula. We will see (1) throughout the paper, with $G(z)$ equal to $\sin(\pi z)$ for Shannon sampling, $\sin^2\left(\frac{\pi z}{2}\right)$ for Papoulis sampling, $\sin(2\pi z)\cdot\sin(2\pi\varphi z)$ for multi-rate sampling, and the appropriate Bessel functions for radial domains.

II. Shannon and Papoulis Sampling

Papoulis gave a generalization of WKS Sampling in the paper "Generalized sampling expansion," *IEEE Trans. Circuits and Systems*, **24** (11), 652–654 (1977). His technique was to write sampling down in terms of linear systems and then solve the resulting system of equations. This gave us formulae for derivative and bunched samples.

We will use Jacobi interpolation to derive the Papoulis theorem for derivative sampling. In particular, we derive the "double point formula," where the sampling rate is half the rate of WKS sampling, but for which twice the information, namely the values of f and f', is required at each of the sample points.

Theorem 1: Let $f \in \mathbb{PW}_\Omega$ and let $T > 0$ be a fixed sampling rate. Let $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. If $T \le 1/2\Omega$, then for all $t \in \mathbb{R}$,

$$f(t) = \sum_{n\in\mathbb{Z}} \Big[f(2nT) + (t-2nT)f'(2nT)\Big]\left[\operatorname{sinc}\left(\frac{(t-2nT)}{2T}\right)\right]^2 . \qquad (3)$$

If $T \le 1/2\Omega$ and $f(2nT) = 0$, $f'(2nT) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

Proof of Papoulis via Cauchy and Jacobi: Let $T = 1$. Then $T \le \frac{1}{2\Omega}$. Also let

$$G(z) = \sin^2\left(\frac{\pi z}{2}\right). \qquad (4)$$

An Ω band-limited function $f(t)$ is real analytic and has an analytic continuation $f(z)$ to $\mathbb{C}$. Moreover, $f(z)$ satisfies the

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Paley-Wiener growth bound for Ω, i.e., there exists $C = C(n)$ such that for all $n \in \mathbb{N}$

$$|f(z)| \le C(n)(1+|z|)^{-n} e^{2\pi\Omega|\Im z|} .$$

The function $G(z)$ is the Jacobi interpolating function, and has zeros $\mathcal{Z} = 2k$, $k \in \mathbb{Z}$. To avoid these zeros, let Γ_m be a circle centered at the origin with radius $(2m+1)$, for $m \in \mathbb{N}$. We apply the Jacobi interpolation formula, getting for $m < N$,

$$\begin{aligned}
f(z) &\overset{(1.)}{=} \frac{G(z)}{2\pi i}\oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta-z)}\cdot\left[\frac{1}{G(z)} - \frac{1}{G(\zeta)}\right] d\zeta \;+\; R_m \\
&\overset{(2.)}{=} \lim_{N\to\infty} \frac{g_N(z)}{2\pi i} \\
&\qquad \oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta-z)}\cdot\left[\frac{1}{g_N(z)} - \frac{1}{g_N(\zeta)}\right] d\zeta \;+\; R_m \\
&\overset{(3.)}{=} \lim_{N\to\infty} \frac{g_N(z)}{2\pi i} \\
&\qquad \oint_{\Gamma_m} \sum_{|n|\le N}\left[\frac{1}{(2n-z)^2(\zeta-2n)g_N''(n)}\right. \\
&\qquad\qquad \left. + \frac{(-1)}{(2n-z)(\zeta-2n)^2 g_N''(n)}\right] d\zeta \;+\; R_m \\
&\overset{(4.)}{=} \lim_{N\to\infty} g_N(z) \\
&\qquad \sum_{|n|\le N} \frac{1}{2\pi i}\oint_{\Gamma_m}\left[\frac{f(\zeta)}{(2n-z)^2(\zeta-2n)g_N''(n)}\right. \\
&\qquad\qquad \left. + \frac{(-1)f(\zeta)}{(2n-z)(\zeta-2n)^2 g_N''(n)}\right] d\zeta \;+\; R_m \\
&\overset{(5.)}{=} \lim_{N\to\infty} g_N(z) \\
&\qquad \sum_{|n|\le N}\left[\frac{f(2n)}{(z-2n)^2 g_N''(n)}\right. \\
&\qquad\qquad \left. + \frac{f'(2n)}{(z-2n)g_N''(n)}\right] \;+\; R_m \\
&\overset{(6.)}{=} \frac{4}{\pi^2}\sin^2\left(\frac{\pi z}{2}\right) \\
&\qquad \sum_{|n|\le m}\left[\frac{f(2n)}{(z-2n)^2} + \frac{f'(2n)}{(z-2n)}\right] \;+\; R_m \\
&\overset{(7.)}{=} \sum_{|n|\le m}\Big[f(2n) + (z-2n)f'(2n)\Big] \\
&\qquad \cdot\left[\operatorname{sinc}\left(\frac{(z-2n)}{2}\right)\right]^2 \;+\; R_m
\end{aligned} \tag{5}$$

where (1.) is Jacobi interpolation, (2.) is the Weierstrass product, (3.) is the Mittag-Leffler decomposition, (4.) is the switch between integration and a finite sum, (5.) is the Cauchy residue calculus, (6.) is the Weierstrass product (and trigonometric evaluation), and (7.) is the definition of the sinc function. By showing that $|R_m| \longrightarrow 0$ as $m \longrightarrow \infty$, we get the sampling formula. We have used the Weierstrass product representation of a sine function, getting $G(z) =$

$$\sin^2\left(\frac{\pi\zeta}{2}\right) = \lim_{N\to\infty} g_N(\zeta) = \lim_{N\to\infty} (\pi\zeta)^2 \prod_{j=1}^{N}\left(1 - \frac{\zeta^2}{2j^2}\right)^2 .$$

The terms of the Mittag-Leffler partial fraction are repeating, and therefore telescope. The formula generalizes to the expansion

$$\begin{aligned}
\frac{1}{(\zeta-z)\cdot g_N(\zeta)} &= \frac{1}{(\zeta-z)g_N(z)} \\
&+ \sum_{|n|\le N}\left[\frac{(-1)}{(2n-z)^2(\zeta-2n)g_N''(n)}\right. \\
&\left. + \frac{1}{(2n-z)(\zeta-2n)^2 g_N''(n)}\right].
\end{aligned} \tag{6}$$

Finally, for f analytic in a neighborhood of z_0, the residues needed are

$$\operatorname{Res}\frac{f(z)}{(z-z_0)} = f(z_0)\,,\ \operatorname{Res}\frac{f(z)}{(z-z_0)^2} = f'(z_0)\,.$$

To finish, we show that the sampling set $2\mathbb{Z}$ is a set of uniqueness for this sampling scheme.

Remark: The result generalizes. Using the Jacobi interpolating function

$$G(z) = \left[\sin\left(\frac{\pi z}{K}\right)\right]^K ,$$

we can sample at $1/K$ the rate, namely sample points at $K\mathbb{Z}$. However, at each point, we now require a "K-tuple" of information, namely the values of f, f', f'', $\dots f^{(K-1)}$ at the sample points. The sampling formulae are as follows.

$$f(t) = \sum_{n\in\mathbb{Z}} f(nT)\left[\operatorname{sinc}\left(\frac{(t-nT)}{T}\right)\right]$$

$$f(t) = \sum_{n\in\mathbb{Z}}\Big[f(2nT) + (t-2nT)f'(2nT)\Big]\left[\operatorname{sinc}\left(\frac{(t-2nT)}{2T}\right)\right]^2 ,$$

and, for general $K \in \mathbb{N}$,

$$\begin{aligned}
f(t) = \sum_{n\in\mathbb{Z}}\Big[& f(KnT) + (t-KnT)f'(KnT) + \dots \\
&+ \frac{(t-KnT)^{(K-1)}}{(K-1)!} f^{(K-1)}(KnT)\Big]\left[\operatorname{sinc}\left(\frac{(t-KnT)}{KT}\right)\right]^K .
\end{aligned}$$

This last formula naturally leads to a discussion of the coding of information for functions $f \in \mathbb{PW}_\Omega$. We are exchanging a slower rate of gathering information with a requirement of an exactly corresponding increase in the amount of information gathered at each sample point. As $K \longrightarrow \infty$, we are approaching encoding all of the information of the function at a single point. However, we are requiring an infinite amount of information about the function f at that point, namely the values of $f, f' f'' \dots$ – the information in the Taylor series.

III. Multi-Channel Deconvolution

We consider, in this section, an overview of the problem of recovering information from linear translation-invariant systems (deconvolution). The details of this work are presented in two papers of Casey and Walnut [4], [5]. A key step in our solutions of deconvolution problems is the interpolation from discrete data, using the Cauchy residue calculus and Jacobi interpolation. This key step essentially boiled down to a sampling problem.

Multi-channel deconvolution utilizes information recovery from a given signal by taking several "looks" at the signal, each of which recovers information possibly missed by one of the other "looks." The "looks" are sensors, and can be modeled as a collection of compactly supported distributions $\{\mu_i\}_{i=1}^m \subseteq \mathcal{E}'(\mathbb{R}^d)$. We discuss the first two steps in this recovery process. The first is how one chooses $\{\mu_i\}_{i=1}^m$. The framework of how this is done is given in a theorem of Hörmander [10]. This first step gives conditions on the sensors $\{\mu_i\}_{i=1}^m$ which allow for this reconstruction. This step gives us the discrete sets which will act as our sampling sets.

The second step in multi-channel deconvolution is to recover an arbitrary signal, a function $f \in C^\infty(\mathbb{R}^d)$ from the data $\{s_i\}_{i=1}^m = \{f * \mu_i\}_{i=1}^m$ This second step is a sampling problem, an interpolation from discrete data. This step involves the construction of *deconvolvers*, which come in a variety of types but which are essentially a collection of distributions which (1) depend only on the convolvers $\{\mu_i\}_{i=1}^m$ and (2) allow for the solution with only simple linear operations on the data $\{s_i\}_{i=1}^m$. The deconvolvers are constructed via interpolation from the discrete sets, using the Cauchy residue calculus and Jacobi interpolation.

We construct a set of distributions $\{\nu_i\}_{i=1}^m \subseteq \mathcal{D}'(\mathbb{R}^d)$ which satisfy

$$\sum_{i=1}^m \mu_i * \nu_i = \delta \tag{7}$$

or equivalently,

$$\sum_{i=1}^m \widehat{\mu_i}(\gamma)\widehat{\nu_i}(\gamma) = 1\,. \tag{8}$$

The collection $\{\nu_i\}_{i=1}^m$ is a set of deconvolvers. In this case, f may be recovered by

$$\begin{aligned}\sum_{i=1}^m s_i * \nu_i &= \sum_{i=1}^m (f * \mu_i) * \nu_i = \sum_{i=1}^m f * (\mu_i * \nu_i) \\ &= f * \sum_{i=1}^m \mu_i * \nu_i = f * \delta = f\,, \end{aligned}\tag{9}$$

provided that the associative law holds.

Equation (8) is a type of *Bezout equation*. Many theorems in elementary number theory, such as Euclid's lemma or Chinese remainder theorem, are derived from the basic Bezout equation, which holds in principal ideal domains. Equation (8) is an *analytic Bezout equation*, in which we are dealing with transcendental entire functions, rather than finite number theoretic problems. Bezout problems involving transcendental entire functions have been extensively studied in a variety of contexts, including the study of division problems, interpolation, analytic continuation, complexity theory, number theory, and solution to systems of PDE's. For the purposes of this paper, we require the following result of Hörmander which gives necessary and sufficient conditions under which compactly supported solutions of (7) exist. Hörmander's result gives a framework for solving the first step of the problem.

Theorem 2: [10] There exist compactly supported distributions

$$\{\nu_i\}_{i=1}^m \subseteq \mathcal{E}'(\mathbb{R}^d)$$

solving (7) if and only if there exist constants A, B, $N > 0$ such that

$$\sum_{i=1}^m |\widehat{\mu_i}(z)| \geq A(1+|z|)^{-N}\, e^{-B|\Im z|} \quad \text{for all } z \in \mathbb{C}^d. \tag{10}$$

A collection $\{\mu_i\}_{i=1}^m \subseteq \mathcal{E}'(\mathbb{R}^d)$ which satisfies (10) is said to be *strongly coprime*[2]. Several different classes of strongly coprime convolving systems are discussed in [4], [5]. The most relevant systems for our discussion on sampling are due to Petersen and Meisters.

Definition 1: A real number α is poorly approximated by rationals provided that there exist constants C, $N > 0$ such that for all integers p, q,

$$|\alpha - p/q| \geq C|q|^{-N}. \tag{11}$$

For example, quadratic irrationals of the form $\sqrt{n}$, where $n \in \mathbb{N}$ is not a perfect square, are poorly approximated by rationals. The Golden Mean $\varphi = (1+\sqrt{5})/2$ is the most poorly approximated, as discussed in Hardy and Wright (see [4]).

Theorem 3: [4] Let $0 < r_1 < \cdots < r_m$, $m \geq d+1$ satisfy r_i/r_j is poorly approximated by rationals whenever $i \neq j$. Then the collection $\{\chi_{[-r_i,r_i]^d}\}_{i=1}^m$ is a strongly coprime set.

The next step of the problem involves solving an interpolation problem, reconstructing functions (the deconvolvers) in a space of restricted growth ($\widehat{\mathcal{E}'}$) from discrete data (their values on the zero sets of the convolvers). This gives solutions to the Bezout equation. *Note, this step is essentially a sampling problem.* See [4], [5] for details.

IV. Multi-Rate Sampling

Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the Golden Mean, and let f be a $(1+\varphi)$-band-limited function. We use the functions $G_1(z) = \sin(2\pi z)$ and $G_2(z) = \sin(2\pi\varphi z)$, which each generate sampling sets. The functions can be multiplied, which gives the generating function $G(z) = G_1 \cdot G_2(z)$ for a *multi-rate* sampling. Let

$$\Lambda_1 = \left\{\frac{\pm k}{2}\right\}\,,\, \Lambda_2 = \left\{\frac{\pm k}{2\varphi}\right\},$$

for $k \in \mathbb{N}$, and let $\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2 \cup \{0\}$. We have that $G(z)$ is an entire function, which is almost periodic on

[2]It is of interest to compare the envelope condition above with the Paley-Wiener-Schwartz growth condition. Note that Hörmander's envelope condition is essentially the inversion of the Paley-Wiener-Schwartz growth condition.

$\mathbb{R}$, has simple zeros on $\Lambda \setminus \{0\}$, and a double zero at $\{0\}$. Following Levin ([13]), we *conditionally* reconstructed f from $\{f(\lambda_k)\} \cup \{f(0), f'(0)\}$.

The sampling set Λ has infinitely many pairs of "clustering points" of the form $\{\frac{j}{2}, \frac{k}{2\varphi}\}$ – Λ is not separated. This interpolation problem requires tools beyond the "standard toolbox," which can be found in Levin ([13]), and which are described in Rom and Walnut ([17]).

By the Cauchy-Jacobi machinery,

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta - z)} \cdot \frac{G(\zeta) - G(z)}{G(\zeta)} d\zeta \quad (12)$$

$$+ \frac{1}{2\pi i}\oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta - z)} \cdot \frac{G(z)}{G(\zeta)} d\zeta, \quad (13)$$

where Γ_m is a sequence of circles with increasing radii chosen to avoid zeroes of G. Let ρ_m be the radius of Γ_m. We choose the circles so that $\rho_m \longrightarrow \infty$ as $m \longrightarrow \infty$. We also let the second integral be denoted by R_m (the remainder), which will $\longrightarrow 0$ as $m \longrightarrow \infty$.

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma_n} \frac{f(\zeta)}{(\zeta - z)} d\zeta$$
$$= \frac{1}{2\pi i}\oint_{\Gamma_m} \frac{f(z)[G_1(\zeta)G_2(\zeta) - G_1(z)G_2(z)]}{(\zeta - z)(G_1(\zeta)G_2(\zeta))} d\zeta + R_m.$$

Now, for $z \in \Lambda_1$ or $z \in \Lambda_2$, $(G_1(z)G_2(z)) = 0$, but $\frac{d}{d\zeta}(G_1(z)G_2(z)) \neq 0$. Thus,

$$\frac{f(z)[G_1(\zeta)G_2(\zeta) - G_1(z)G_2(z)]}{(\zeta - z)(G_1(\zeta)G_2(\zeta))}$$

has simple poles in $\Lambda_1 \cup \Lambda_2$, and so by the Cauchy Residue Theorem,

$$\oint_{\Gamma_m} \frac{f(\zeta)[G_1(\zeta)G_2(\zeta) - G_1(z)G_2(z)]}{(\zeta - z)(G_1(\zeta)G_2(\zeta))} d\zeta$$
$$= \sum_{\substack{z\in\Lambda_2 \\ |z|<\rho_m}} \frac{f(z)}{G_1(z)\frac{d}{d\zeta}G_2(z)} \left(\frac{G_1(\zeta)G_2(\zeta)}{(\zeta - z)}\right)$$
$$+ \sum_{\substack{z\in\Lambda_1 \\ |z|<\rho_n}} \frac{f(z)}{G_2(z)\frac{d}{d\zeta}G_1(z)} \left(\frac{G_1(\zeta)G_2(\zeta)}{(\zeta - z)}\right).$$

Because the sampling set Λ is not separated, we have to proceed in several steps.

We first establish that Λ is a set of uniqueness for $\mathbb{PW}_{(1+\varphi)}$. We denote elements in Λ as $\lambda_{m,n}$, containing elements that are multiples of both $\frac{1}{2}$ and $\frac{1}{2\varphi}$.

Lemma 1: Let f be a $(1+\varphi)$-bandlimited function. Then f is uniquely determined by $\{f(\lambda_{m,n})\} \cup \{f(0), f'(0)\}$. In other words, Λ is a set of uniqueness for $\mathbb{PW}_{(1+\varphi)}$. Therefore, $\{e^{2\pi i\lambda_{m,n}t}\} \cup \{t, t^2\}$ is complete.

The lemma is equivalent to the statement that $\{e^{2\pi i\lambda_{m.n}t}\} \cup \{t, t^2\}$ is complete. It is not, however, minimal, for it is not a Riesz basis because sample points are not separated. The sampling set has to be split up, into parts that are separated and clusters of points where the set is not. Convergence is conditional and works because, given any collection of clustering sample points, the cluster contains only two elements. Moreover, given any $\epsilon > 0$, there exist infinitely many pairs of sample points of the form $\{n/2, m/(2\varphi)\}$ in an interval of length ϵ centered at either point. We reconstruct Λ as follows. Let η be given, $0 < \eta < \frac{1}{4(1+\varphi)}$. Let

$$\Lambda_\eta = \{\lambda \in \Lambda : \text{dist}(\lambda, \Lambda \setminus \{\lambda\}) < \eta\}. \quad (14)$$

Elements in Λ_η occur in pairs, with each pair containing one element from $\Lambda_1 = \left\{\frac{\pm k}{2}\right\}$ and the other from $\Lambda_2 = \left\{\frac{\pm j}{2\varphi}\right\}$ for $k, j \in \mathbb{N}$. Let $\Lambda_\sigma = \Lambda \setminus \Lambda_\eta$, and so $\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2 = \Lambda_\eta \cup \Lambda_\sigma$. The set Λ_σ is separated.

Lemma 2: Let $\lambda \in \Lambda_\sigma$. The sequence

$$\left\{\frac{G(z)}{G'(\lambda)(z-\lambda)}\right\}_{\lambda\in\Lambda_\sigma}$$

is a Bessel sequence in $\mathbb{PW}_{(1+\varphi)}$.

Following Levin, we treat the sample points in Λ_η, letting them form their own "sampling blocks."

Definition 2: Let $\mathbb{H}$ be a Hilbert space, and $\mathcal{H} = \{H_i\}$ be a collection of subspaces of $\mathbb{H}$. Then $\mathcal{H}$ is a *generalized basis* if there exist projection operators $\mathcal{P}_n$ such that $\mathcal{P}_n$ restricted to H_k equals $\delta_{n,k}$, and given $x \in \mathbb{H}$, $x = \sum \mathcal{P}_n x$ unconditionally.

Lemma 3: Let $\lambda_k, \lambda_{k'} \in \Lambda_\eta$. The sequence

$$\left\{\frac{G(z)}{G'(\lambda_k)(z-\lambda_k)}, \frac{G(z)}{G'(\lambda_{k'})(z-\lambda_{k'})}\right\}$$

is a Bessel sequence in $\mathbb{PW}_{(1+\varphi)}$.

Theorem 4: Let f be a $(1+\varphi)$-bandlimited function. Then f can be reconstructed conditionally (in the sense of a generalized basis) by $\{f(\lambda_{m,n})\} \cup \{f(0), f'(0)\}$. In other words, Λ is a set of conditional reconstruction for $\mathbb{PW}_{(1+\varphi)}$, and so $\{e^{2\pi i\lambda_{m,n}t}\} \cup \{t, t^2\}$ is minimal.

Let $G(z) = \sin(2\pi z) \cdot \sin(2\pi\varphi z)$. The reconstruction formula is $f(z) =$

$$f(0)\frac{G(z)}{4\pi\varphi z} + f'(0)\frac{G(z)}{4\pi\varphi z^2} + \sum_{\lambda\in\Lambda_\sigma} f(\lambda)\frac{G(z)}{G'(\lambda)(z-\lambda)} +$$
$$\sum_{\lambda_k,\lambda_{k'}\in\Lambda_\eta} [f(\lambda_k)\frac{G(z)}{G'(\lambda_k)(z-\lambda_k)} + f(\lambda_{k'})\frac{G(z)}{G'(\lambda_{k'})(z-\lambda_{k'})}].$$

Remark: The result generalizes. Given $\{r_i\}_{i=1}^{\ell}$ such that (r_i/r_j) is irrational for $i \neq j$, let $R = \sum_k r_k$, and let $\Lambda_k = \left\{\frac{\pm n}{2r_k}\right\}$ for $n \in \mathbb{N}$. Let $\Lambda = \bigcup_{k=1}^{\ell} \Lambda_k \cup \{0\}$. Then,

$$G(z) = \prod_{k=1}^{\ell} \sin(2\pi r_k z) \quad (15)$$

is a generating function for the multi-rate sampling scheme. We let $z = 0$ be a sample point of multiplicity ℓ. By a generalization of the lemma above Λ will be a set of uniqueness for $\mathbb{PW}_R$. Clusters of sample point will occur in different combinations, from clusters of k points requiring the data $f^j(\lambda)$, $0 \leq j \leq k-1$, down to pairs of points. For example, if $k = 3$, we will have clusters of three points containing $\frac{n}{2r_1}, \frac{m}{2r_2}, \frac{p}{2r_3}$, for some $n, m, p \in \mathbb{Z}$, and three different set of clusters of two points generated by different pairs of rates r_i, r_j, $i \neq j$. Convergence again is in the sense of a generalized basis. Once again, develop $\Lambda = \Lambda_\sigma \cup \Lambda_\eta$, and construct $f \in \mathbb{PW}_R$ with $z = 0$ being a sample point of multiplicity ℓ, single element generalized basis elements on Λ_σ, and appropriate clusterings on Λ_η.

V. Radial Sampling

Let $x \in \mathbb{R}^2$, and let $f \in L^2(\mathbb{R}^2)$ be radial, i.e. $f(x) = f(|x|) = f(r)$. Then

$$\begin{aligned}\widehat{f}(\omega) &= \int_{\mathbb{R}^2} f(x)e^{-2\pi i x\cdot\omega}dx \\ &= \int_0^\infty f(r)r\,dr\int_0^{2\pi} e^{-2\pi i|\omega|r\cos(\theta)}d\theta \\ &= 2\pi\int_0^\infty f(r)J_0(2\pi|\omega|r)\,r\,dr\end{aligned}$$

J_0 is a Bessel function, and can generate other Bessel functions. In particular, for $\nu > -1$, the Bessel function of order ν is given by

$$J_\nu(t) = \frac{t^\nu}{2^\nu\Gamma(\nu+1)}[1 + \sum_{n=1}^\infty \frac{(-1)^n}{n!(1+\nu)\dots(n+\nu)}(\frac{t}{2})^{2n}]\,.$$

Let λ_n be the nth positive zero of $J_\nu(t)$. For radial functions in $L^2(0,1)$, the Fourier-Bessel set $\{\sqrt{x}J_\nu(x\lambda_n)\}_{n=1}^\infty$ is an ON basis. We also have that, for fixed $t > 0$

$$\sqrt{xt}J_\nu(xt) = \sum_{n=1}^\infty \frac{2\sqrt{t\lambda_n}J_\nu(t)}{(J'_\nu(\lambda_n)(t^2-\lambda_n^2)}\sqrt{x}J_\nu(x\lambda_n)\,, \qquad (16)$$

converging in $L^2(0,1)$. Let

$$G(z) = \sqrt{z}J_\nu(z)\,. \qquad (17)$$

Applying the Cauchy-Jacobi machinery gives the sampling formula

$$f(t) = \sum_{n=1}^\infty f(\lambda_n)\frac{2\sqrt{t\lambda_n}J_\nu(t)}{J'_\nu(\lambda_n)(t^2-\lambda_n^2)}\,. \qquad (18)$$

VI. Radial Multi-Rate Sampling

We finish with a conjecture. We can use tools from Levin's multi-rate sampling in rectangular coordinates to get multi-rate sampling in radial coordinates. Given $\{r_i\}_{i=1}^\ell$ such that (r_i/r_j) is **rational** for $i \neq j$, then the conjectured generating function is

$$G(z) = \prod_{k=1}^\ell \sqrt{r_k z}J_\nu(r_k z)\,.$$

The first thing to notice is that the criterion on the zeroes is considerably *less* rigid then the one for rectangular coordinates. The reason lies in the following. This goes back to work of Berenstein *et al.* on multichannel deconvolution.

Multi-channel deconvolution utilizes information recovery from a given signal by taking several "looks" at the signal, each of which recovers information possibly missed by one of the other "looks." The "looks" are sensors, and can be modeled as a collection of compactly supported distributions $\{\mu_i\}_{i=1}^m \subseteq \mathcal{E}'(\mathbb{R}^d)$. The first set in this recovery process is how one chooses $\{\mu_i\}_{i=1}^m$. The framework of how this is done is given in a theorem of Hörmander [10]. This first step gives conditions on the sensors $\{\mu_i\}_{i=1}^m$ which allow for this reconstruction. This step gives us the discrete sets which will act as our sampling sets. Creation of these sets requires clever manipulation of decay and zeroes in the transform domain. *The tricks for these manipulations come from number theory.*

Cubes in $\mathbb{R}^d$ (Petersen and Meisters) Let $0 < r_1 < \cdots < r_m$, $m \geq d+1$ satisfy r_i/r_j is poorly approximated by rationals whenever $i \neq j$. Then the collection $\{\chi_{[-r_i,r_i]^d}\}_{i=1}^m$ is a strongly coprime set.

Balls in $\mathbb{R}^d$ (Berenstein and Yger) Let μ_1 and μ_2 be the characteristic functions of the disks $B(0,r_1)$ and $B(0,r_2) \subseteq \mathbb{R}^2$ respectively. Then the system $\{\mu_1, \mu_2\}$ is strongly coprime if and only if there is a constant $A > 0$ such that

$$|r_2/r_1 - \xi/\eta| \geq (1/A)\,|\eta|^{-A}$$

for any pair ξ, $\eta > 0$ with $J_1(\xi) = J_1(\eta) = 0$ where J_1 is the Bessel function of order 1. *This is true if* $r_2/r_1 \in \mathbb{Q}$.

References

[1] J. J. Benedetto, *Harmonic Analysis and Applications*, CRC Press, Boca Raton, FL (1997).

[2] C.A. Berenstein and A. Yger: "A local version of the two circles theorem," *Isr. J. Math.*, **55**, 267–288, 1986.

[3] J.R. Higgins: "Five short stories about the cardinal series," *Bull. AMS*, **12** (1), 45–89, 1985.

[4] S.D. Casey and D.F. Walnut: "Systems of convolution equations, deconvolution, Shannon sampling, and the wavelet and Gabor transforms," *SIAM Review*, **36** (4), 537–577, 1994.

[5] S.D. Casey and D.F. Walnut: "Residue and sampling techniques in deconvolution," Chapter 9 in *Modern Sampling Theory: Mathematics and Applications*, Springer-Birkhäuser, Applied and Numerical Harmonic Analysis Series, ed. by P. Ferreira and J.J. Benedetto, Birkhauser, Boston, 193–217, 2001.

[6] S.D. Casey: "Shannon sampling via Poisson, Cauchy, Jacobi and Levin," to appear in Springer-Birkhäuser, Applied and Numerical Harmonic Analysis Series 43 pp., (2023).

[7] H. Dym and H.P. McKean: *Fourier Series and Integrals*. Academic Press, Orlando, FL, 1972.

[8] Y.C. Eldar: *Sampling Theory (Beyond Bandlimited Systems)*. Cambridge University Press, Cambridge, 2015.

[9] K. Gröchenig: *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2000.

[10] L. Hörmander: "Generators for some rings of analytic functions," *Bull. Amer. Math. Soc.*, **73**, 943–949, 1967.

[11] A.J. Jerri: "The Shannon sampling theorem - its various extensions and applications: a tutorial review," *Proc. IEEE* **65** (11), 1565–1596, 1977.

[12] B.Ya. Levin: *Lectures on Entire Functions*. American Mathematical Society, Providence, RI, 1996.

[13] B.Ya. Levin: "On bases of exponential functions on L^2," *Zap. Mekh.-Mat. Fak. i Khar'kov. Mat, Obshch.*, **27**, 39–48, 1961.

[14] A. Papoulis: "Generalized sampling expansion," *IEEE Trans. Circuits and Systems*, **24** (11), 652–654, 1977.

[15] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge (2010).

[16] E. Petersen and G. Meisters: "Non-Liouville numbers and a theorem of Hörmander," *Jour. Funct. Anal.*, **29**, 142–150, 1978.

[17] B. Rom and D. Walnut: "Sampling on unions of shifted lattices in one dimension," in *Harmonic Analysis and Applications*, C. Heil, Ed., 289-323, 2006.

[18] R. Young: *An Introduction to Nonharmonic Fourier Series* Academic Press, New York (1980).