

Nonlinear Differential Equations with External Forcing

Paul Pukite (puk@umn.edu)

What happens when training data does not readily reveal the underlying patterns? Or if the pattern in the data is so long that the algorithm cannot deduce it? That can happen with a complex natural behavior such as the El Nino Southern Oscillation. As the forcing for the El Nino cycles is governed by an externally driving source of luni-solar tides, and the highly erratic oceanic sloshing response is obscured by the physics of fluid dynamics, we should not expect a non-physics-based network to automatically find the pattern.

In fact, the correct CFD solution to the El Nino time-series involves topological constraints to the partial differential equations, and that indeed generates a highly nonlinear transfer function. To the climatologist, this represents an analytic solution to Laplace's Tidal Equations, a type of climate model that provides a shallow-water approximation to the full Navier-Stokes formulation. This naturally requires tidal factors as an input, with the full derivation as described in our text *Mathematical GeoEnergy*.

The complexity of the transfer function is evident as it matches that of Mach-Zehnder modulation – which is of sufficient nonlinearity that it has been used as a practical encryption scheme for analog signals. And as most are familiar, without a decryption key, any underlying pattern is almost impossible to decode.

Fortunately, there are many sources of data for constraining, aligning, and calibrating the forcing. One such key is the length of day deviation in the Earth's rotation. This delta perturbation is directly related to the cyclic tidal forcing and thus can be used to calibrate the tidal periods and amplitudes – the only distinction being that one acts on an inertial solid while the other acts on an inertial fluid. The fluid sensitivity is enhanced as the inertial response is amplified by the reduced gravity that exists along the Pacific Ocean thermocline – that is between the denser cooler water and the lighter warmer water that sits above the thermocline. (The length of day variation is undetectable by a human but the fluid inertial response is significant as subsurface tides and resultant El Nino events)

The schematic for computational processing involves the factored and roughly calibrated tidal input as a forcing, which is then fed into an annual impulse modulation and lag integrator to simulate a seasonal barrier. This filtered forcing is then fed into the Laplace's Tidal Equation transfer modulation. At least two wavenumber modulations are applied – a low wavenumber corresponding to the main ENSO standing-wave dipole; and a higher wavenumber solution corresponding to a tropical instability wave train.

As an important aside, routine signal processing approaches can be used to help validate the analysis. A Fourier series of the El Nino data shows a double-sideband suppressed-carrier modulation that is strongly indicative of an annual impulse modulating the external forcing input. This is carried through by the nonlinear transform, which only generates harmonics of the fundamental periods.

The fitting of the model based on the initial calibration relies on slight deviations of the tidal factors, including the perigee factor related to the moon's proximity to the earth. This isn't as strong a gravitational forcing to the Earth's solid rotational moment of inertia as it is to the fluid volume.

The periodic complexity of just the nodal cycle is 18.6 years, which is seen in the length of day repeat pattern, but adding the perigeon anomalistic tidal factor along with the annual cycle boosts it up to a longer 57-year eclipse cycle. The overlay demonstrates how complex the input factor is and how long a time series is required for an accurate fit and a good predictive model.

The most straightforward model fitting approach requires a choice of spectral or temporal domain, a choice of target metric such as correlation coefficient or least squared error, and a search algorithm such as gradient descent. There are a variety of cross-validation approaches that one can apply to check for overfitting.

This nonlinear analysis is in its infancy. Lin and Qian from the Ohio State atmospheric science program have compelling evidence that the tidal forcing is driving subsurface waves and the subsequent El Nino to La Nina cycling. There is also strong evidence that the basic mechanism also applies to other climate dipoles and the equatorial atmospheric behavior known as the quasi-biennial oscillation.

The overall approach is conducive to numerical analysis at any level. Since a closed-form solution is available, the amount of iterative solving is reduced to an integrated lagged response, making the computational effort modest and on the level of conventional tidal analysis. The compute cycles spent are mainly in the gradient descent search or a genetic algorithm, as the nonlinear response precludes direct inversion. The key here though is to leverage the known geophysics to constrain the solution.