Approximation by family of max-min sampling operators in function spaces

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***Abstract*—The approximation of functions using sampling-based operators has emerged as a prominent research area within approximation theory, with potential applications in signal analysis and image reconstruction. This note aims to introduce study the approximation capabilities of a novel family of sampling operators, namely generalized max-min sampling operators and Kantorovich type max-min sampling operators. This study has been carried out in the space of continuous functions, classical Lebesgue spaces and Orlicz spaces. Following the theoretical groundwork, we illustrate the practical performance of our operators through numerical examples, graphical representations.**

***Index Terms*—Approximation of functions; Max-min operators; Orlicz space; Modulus of Continuity.**

I. Introduction

Sampling and reconstruction is a mathematical tool that enables addressing the question of recovering the missing information from the sampled data. By virtue of this, it has a wide range of applications in signal analysis and image processing (see [2], [10], [16], [11]). To reconstruct a continuous function $f : \mathbb{R} \to \mathbb{R}$, Butzer and Stens in [4] constructed a family of generalized sampling operators given by the formula

$$(G_n^{\chi} f)(x) = \sum_{k \in \mathbb{Z}} \chi(nx - k)\ f\left(\frac{k}{n}\right);\ x \in \mathbb{R}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a general kernel satisfying certain assumptions. Later, in order to treat the functions that are not necessarily continuous, Bardaro in [6] proposed the Kantorovich-type generalized sampling series by replacing $f(k/w)$ with the mean value $n \int_{k/w}^{(k+1)/w} f(u)du$, that is,

$$(K_n^{\chi} f)(x) = \sum_{k \in \mathbb{Z}} \chi(nx - k)\ n \int_{k/n}^{(k+1)/n} f(u)du.$$

In practice, the approach of considering average sample values proved to be more effective since information stored in neighborhood is usually known than precisely at sampled points. This procedure simultaneously reduces the time-jitter errors. The Kantorovich-type modifications of various operators have been studied in various directions (see [6], [8], [13]).

In research fields like mathematical physics, decision analysis and automatic control, where non-linearity and uncertainty arises, pseudo-linear structures turn out to be more suitable (see [17], [15]). To bridge the gap for one to benefit from the other, the authors in [3] initiated the study of pseudo-linear operators by introducing class of pseudo-linear operators using ordered semi-ring structure as underlying algebraic structure. It was observed that these provide better approximation results. One of the approaches in this direction is to transform any linear operator into non-linear *max-min operator* by replacing *summation* and *product* in linear operator with *supremum* and *infimum* respectively.

A. Prior Work.

The max-min type Bernstein operators, Shepard operators, and their Kantorovich versions based on polynomial kernels have been analyzed in ([9], [13], [14]). The application of max-min type operator in image processing is given in ([5], [1])

B. Contibution.

We introduce and analyze a family of max-min sampling operators, namely *generalized max-min operators* and *Kantorovich type max-min sampling operators* in different function spaces, namely the space of all continuous functions, Lebesgue space and Orlicz space.

C. Notations.

For $[a, b] \subset \mathbb{R}$, $C([a, b])$ denotes the space of all continuous functions on $[a, b]$ endowed with the norm $\|f\|_\infty := \sup\{|f(x)| : x \in [a, b]\}$. The collection of all Lebesgue measurable functions $f : [a, b] \to [0, 1]$ is represented by $M([a, b])$. Moreover, $L^p([a, b])$, $1 \leq p < \infty$, denotes the space of p-integrable functions in Lebesgue sense equipped with usual p-norm.

D. Outline.

In Section II we discuss the framework. In Section III, we define and study the pointwise and uniform convergence results for our proposed family of operators within $C([a, b])$. Further we extend the study of the Kantorovich type max-min sampling operators to the general framework of Orlicz spaces $L^{\phi}([a, b])$ in Section IV. Section V consists of some numerical illustrations of our presented theory through graphical representation.

II. Framework

The following framework allows us to define our proposed family of operators effectively. Let $\mathcal{I}$ be an index set. The operations $\bigvee_{k\in\mathcal{I}}$ and $\bigwedge_{k\in\mathcal{I}}$, when operated on the set $\{a_k : k \in \mathcal{I}\}$ of real numbers, are assigned the following meaning:

$$\bigvee_{k\in\mathcal{I}} a_k = \sup\{a_k : k \in \mathcal{I}\} \text{ and } \bigwedge_{k\in\mathcal{I}} a_k = \inf\{a_k : k \in \mathcal{I}\}.$$

The structure $([0,1], \vee, \wedge)$ is a semi-ring. In addition to these operations, we will also employ classical operations of addition and multiplication. Additionally, we equip $[0,1]$ with usual order and Euclidean metric.

Having outlined idea behind the construction, we now turn to discussing the class of admissible kernels.

A bounded and measurable function $\chi : \mathbb{R} \to \mathbb{R}^+$ is said to be a kernel if it satisfies the following conditions:

(χ_1) $\chi \in L^1(\mathbb{R})$

(χ_2) there exists $\beta > 0$ such that,

$$m_\beta(\chi) := \sup_{x\in\mathbb{R}^+} \bigvee_{k\in\mathbb{Z}} |\chi(u-k)||k-u|^\beta < \infty,$$

(χ_3) For $c_\chi > 0$, we have $\inf\limits_{x\in[-3/2,3/2]} \chi(x) =: c_\chi$.

We have the following results related to χ.

Lemma 1: ([12], Lemma 2.4) Assume that a bounded function χ satisfies $\chi(x) = \mathcal{O}(|x|^{-\alpha})$ as $|x| \to +\infty$. Then $m_\alpha(\chi) < +\infty$ for all $0 \le \beta \le \alpha$.

Lemma 2: ([7], Lemma 2.2) Let χ be a bounded function which satisfies (χ_2) for some $\beta > 0$. Then for every $0 < \alpha \le \beta$, $m_\alpha(\chi) < +\infty$.

Lemma 3: ([7], Lemma 2.5) Assume that $\chi : \mathbb{R} \to \mathbb{R}^+$ is a function which satisfies (χ_2) with $\beta > 0$, then for every $\gamma > 0$, we have

$$\bigvee_{\{k\in\mathbb{Z}:|u-k|>n\gamma\}} \chi(u-k) = \mathcal{O}(n^{-\beta})$$

as $n \to \infty$, uniformly on $\mathbb{R}$.

Lemma 4: ([7], Lemma 2.3) Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be the given function which satisfies (χ_3). Then for every $n \in \mathbb{N}$ and $x \in [a,b]$, we have

$$\bigvee_{k\in\mathcal{J}_n} \chi(nx-k) \ge c_\chi > 0,$$

where $\mathcal{J}_n := \{k \in \mathbb{Z} : k = \lceil na \rceil, ..., \lfloor nb \rfloor\}$.

In order to study the convergence of proposed family of max-min sampling operators, the interplay of metric structure and pseudo-linear operations on $[0,1]$ becomes a essential. This has been addressed in the following results.

Lemma 5: ([3], Lemma 4) For sequences $\{a_k\}_{k=1}^\infty, \{b_k\}_{k=1}^\infty$ in $\mathbb{R}$ with $\bigvee\limits_{k\in\mathbb{Z}} a_k < \infty$ or $\bigvee\limits_{k\in\mathbb{Z}} b_k < \infty$, we have

$$\left|\bigvee_{k\in\mathbb{Z}} a_k - \bigvee_{k\in\mathbb{Z}} b_k\right| \le \bigvee_{k\in\mathbb{Z}} |a_k - b_k| \, .$$

Lemma 6: ([3], Lemma 5) For all $x, y, z \in [0,1]$, we have $|x \wedge y - x \wedge z| \le x \wedge |y - z|$.

III. Basic Convergence in $C[a,b]$

A. Generalized max-min sampling operators.

Let χ be the kernel and $f : [a,b] \to [0,1]$. Then we define the family of generalized max-min sampling operators as

$$(\mathcal{G}_n^\chi f)(x) := \bigvee_{k\in\mathcal{J}_n} \left\{ \frac{\chi(nx_0-k)}{\vee_{d\in\mathcal{J}_n}\chi(nx_0-d)} \wedge f\left(\frac{k}{n}\right) \right\},$$

for $x \in [a,b]$, where $\mathcal{J}_n = \{k \in \mathbb{Z} : k = \{\lceil na \rceil ..., \lfloor nb \rfloor\}\}$. It can be observed that

$$(\mathcal{G}_n^\chi f)(x) \le \bigvee_{k\in\mathcal{J}_n} \left\{ \frac{\chi(nx-k)}{\vee_{d\in\mathcal{J}_n}\chi(nx-d)} \right\} \le 1, \text{ for } x \in [a,b].$$

This substantiates the well-definedness of $(\mathcal{G}_n^\chi f)$ for a continuous function f on $[a,b]$. The next result provides insights to some operational properties of the operator $(\mathcal{G}_n^\chi)$.

Lemma 7: Let χ be the kernel and $h, g : [a,b] \to [0,1]$ be two arbitrary functions. Then for each $n \in \mathbb{N}$, we have

(a) If $h(x) \le g(x)$ then $(\mathcal{G}_n^\chi f)(x) \le (\mathcal{G}_n^\chi g)(x)$, $\forall\ x \in [a,b]$,
(b) $(\mathcal{G}_n^\chi (h+g))(x) \le (\mathcal{G}_n^\chi h)(x) + (\mathcal{G}_n^\chi g)(x)$, $\forall\ x \in [a,b]$,
(c) $\left|(\mathcal{G}_n^\chi h)(x) - (\mathcal{G}_n^\chi g)(x)\right| \le (\mathcal{G}_n^\chi(|h-g|))(x)$, $\forall\ x \in [a,b]$.

In the next theorem, we have the pointwise and uniform approximation theorem for $(\mathcal{G}_n^\chi)$.

Theorem 1: Under the assumptions on kernel χ and $f : [a,b] \to [0,1]$, $(\mathcal{G}_n^\chi f)$ converges to f at the point of continuity of f, that is,

$$\lim_{n\to\infty} (\mathcal{G}_n^\chi f)(x_0) = f(x_0).$$

at each point x_0 of continuity of f. Further, if $f \in C([a,b])$ then

$$\lim_{n\to\infty} \|(\mathcal{G}_n^\chi f) - f\|_\infty = 0.$$

Proof 1: The proof follows by using continuity of f at x_0 and Lemma 3.
In case of discontinuous but locally integrable functions on $[a,b]$, the family of operator $(\mathcal{G}_n^\chi f)$ is not suitable. Hence we define and analyze the family of Kantorovich type max-min sampling operators in the following subsection.

B. The Kantorovich type max-min sampling operators.

For the kernel χ and a locally integrable function $f : [a,b] \to [0,1]$, we define the Kantorovich type max-min sampling operator as

$$(\mathcal{K}_n^{\chi} f)(x) := \bigvee_{k \in \mathcal{J}_n} \left\{ \frac{\chi(nx-k)}{\vee_{d \in \mathcal{J}_n} \chi(nx-d)} \wedge n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u) du \right\}$$

for $x \in [a,b]$, where $\mathcal{J}_n = \{k \in \mathbb{Z} : k = \{\lceil na \rceil, ..., \lfloor nb \rfloor - 1\}\}$. Since $\sup\{f(u) : u \in [a,b]\} \leq 1$, we have

$$|(\mathcal{K}_n^{\chi} f)(x)| \leq \bigvee_{k \in \mathcal{J}_n} \left\{ \frac{\chi(nx-k)}{\vee_{d \in \mathcal{J}_n} \chi(nx-d)} \right\} \leq 1.$$

Hence the operator $(\mathcal{K}_n^{\chi})$ is well-defined for locally integrable functions on $[a,b]$. Some useful properties of $(\mathcal{K}_n^{\chi})$ are given in next lemma.

Lemma 8: For a kernel χ and locally integrable functions $h, g : [a,b] \to [0,1]$, the following holds for all $n \in \mathbb{N}$,:

(i) If $h(x) \leq g(x)$ then $(\mathcal{K}_n^{\chi} h)(x) \leq (\mathcal{K}_n^{\chi} g)(x)$, $\forall\, x \in [a,b]$,
(ii) $(\mathcal{K}_n^{\chi}(h+g))(x) \leq (\mathcal{K}_n^{\chi} h)(x) + (\mathcal{K}_n^{\chi} g)(x)$, $\forall\, x \in [a,b]$,
(iii) $|(\mathcal{K}_n^{\chi} h)(x) - (\mathcal{K}_n^{\chi} g)(x)| \leq (\mathcal{K}_n^{\chi} |h-g|)(x)$, $\forall\, x \in [a,b]$.

In the next result, we have uniform convergence of $(\mathcal{K}_n^{\chi} f)$ for $f \in C([a,b])$.

Theorem 2: Let χ be the kernel and $f : [a,b] \to [0,1]$ be continuous at $x_0 \in [a,b]$. Then we have

$$\lim_{n \to \infty} (\mathcal{K}_n^{\chi} f)(x_0) = f(x_0).$$

In addition, if $f \in C([a,b])$, then

$$\lim_{n \to \infty} \|(\mathcal{K}_n^{\chi} f) - f\|_{\infty} = 0.$$

Proof 2: The proof follows by using continuity of f at x_0 and Lemma 3.

IV. Convergence in $L^{\phi}([a,b])$

A. Orlicz Space

A convex function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is known as *Orlicz function* if it satisfies the following conditions:

(O1) $\phi(0) = 0$,
(O2) ϕ is non-decreasing and continuous from the left,
(O3) $\lim_{u \to \infty} \phi(u) = \infty$.

For Orlicz function ϕ, the modular functional $I^{\phi} : M([a,b]) \to [0, \infty]$ defined as

$$I^{\phi}[f] := \int_a^b \phi(|f(x)|)\, dx, \text{ for } f \in M([a,b]).$$

The modular space corresponding to $I^{\phi}([a,b])$ is known as *Orlicz space* generated by ϕ and is defined as

$$L^{\phi}([a,b]) := \{f \in M([a,b]) : I^{\phi}[\lambda f] < +\infty \text{ for some } \lambda > 0\}.$$

It is important to note that $L^{\phi}([a,b])$ is a vector space over $\mathbb{R}$. Moreover, the space $L^{\phi}([a,b])$ is a normed linear space with respect to *Luxemburg norm*, which is defined as

$$\|f\|_{\phi} := inf\{\lambda > 0 : I^{\phi}[\lambda f] < \infty\}, \quad \text{for } f \in L^{\phi}([a,b]).$$

To study the convergence in Orlicz space, the notion of modular convergence is important. A sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $L^{\phi}([a,b])$ is said to be modularly convergent to $f \in L^{\phi}([a,b])$ if there exists $\lambda > 0$ such that

$$\lim_{n \to \infty} I^{\phi}[\lambda(f_n - f)] = 0.$$

This induces modular topology on $L^{\phi}([a,b])$. In general, the concept of norm convergence is stronger than modular convergence.
In the next result we prove the modular convergence of $(\mathcal{K}_n^{\chi} f)$ for $f \in C([a,b])$.

Theorem 3: If f is continuous on $[a,b]$ then for every $\lambda > 0$, we have

$$\lim_{n \to \infty} I^{\phi}[\lambda((\mathcal{K}_n^{\chi} f) - f)] = 0.$$

Proof 3: First we observe that for any $f \in C([a,b])$, we have

$$\begin{aligned} &|(\mathcal{K}_n^{\chi} f)| \\ &\leq \bigvee_{k \in \mathcal{J}_n} \left\{ \frac{\chi(nx-k)}{\vee_{d \in \mathcal{J}_n} \chi(nx-d)} \right\} \\ &\leq \frac{m_0(\chi)}{c_{\chi}}. \end{aligned}$$

To prove the assertion, we use *Vitali convergence theorem*. Let $\epsilon > 0$ be given and $\mathcal{B}$ be a measurable subset of $[a,b]$ with

$$\mu(\mathcal{B}) < 2\epsilon \left(\phi \left[\frac{m_0(\chi)}{c_{\chi}} \right] + \phi[2\lambda f(x)] \right)^{-1}.$$

Now in view convexity of ϕ, we have

$$\begin{aligned} &\int_{\mathcal{B}} \phi\left[\lambda |(\mathcal{K}_n^{\chi} f)(x) - f(x)|\right] dx \\ &\leq \int_{\mathcal{B}} \phi\Big[\frac{2\lambda}{2} |K_n^{\chi} f(x)| + \frac{2\lambda}{2} |f(x)|\Big] dx \\ &\leq \int_{\mathcal{B}} \frac{1}{2} \phi\left[2\lambda |(\mathcal{K}_n^{\chi} f)(x)|\right] dx + \int_{\mathcal{B}} \frac{1}{2} \phi\left[2\lambda |f(x)|\right] dx. \end{aligned}$$

Using non-decreasing nature of ϕ, we get

$$\begin{aligned} &\int_{\mathcal{B}} \phi\left[\lambda |(\mathcal{K}_n^{\chi} f)(x) - f(x)|\right] dx \\ &\leq \frac{1}{2} \int_{\mathcal{B}} \phi\left[2\lambda \frac{m_0(\chi)}{c_{\chi}}\right] dx + \frac{1}{2} \int_{\mathcal{B}} \phi[2\lambda f(x)] dx \\ &\leq \frac{\mu(\mathcal{B})}{2} \left(\phi\left[\frac{m_0(\chi)}{c_{\chi}}\right] + \phi[2\lambda f(x)] \right) \\ &< \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have established the result.

The following modular convergence can be proved using modular density of continuous functions in $L^{\phi}([a,b])$.

Theorem 4: For every $f \in L^{\phi}([a,b])$, there exists $\lambda > 0$ such that

$$\lim_{n\to\infty} I^{\phi}[\lambda((\mathcal{K}_n^{\chi} f) - f)] = 0.$$

V. Example

The B-spline kernel: The B-spline kernel of order $n \in \mathbb{N}$ is defined by (see [4])

$$B_n(x) := \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(\frac{n}{2}+x-j\right)_+^{n-1} \quad x \in \mathbb{R},$$

where $(x)_+ := \max\{x, 0\}$. Since B-spline kernel has compact support, $B_n \in L^1(\mathbb{R})$, for each $n \in \mathbb{N}$. Also, we have

$$\inf_{x\in[-3/2,3/2]} B_n(x) = B_n(0) > 0, \text{ for } n \geq 4.$$

Now we use B_4 to demonstrate the convergence of $(\mathcal{G}_n^{\chi})$. Consider $f_1 : [-1, 2] \to [0, 1]$ defined as

$$f_1(x) = \frac{1}{15}\left(sin(2\pi x) + 2sin\left(\frac{\pi x}{2}\right)\right).$$

Fig. 1 illustrates the approximation of f_1 by $(\mathcal{G}_n^{B_4} f_1)$.

The Fejér's kernel: The Fejér's kernel is defined as (see [6])

$$F(x) = \frac{1}{2} sinc^2\left(\frac{x}{2}\right).$$

Clearly $F \in L^1(\mathbb{R})$ and

$$\inf_{x\in[-3/2,3/2]} F(x) = F(1/2) > 0.$$

Moreover we have $F(x) = \mathcal{O}(|x|^{-2})$ as $x \to \infty$. Therefore by Lemma 2, we conclude that the condition (χ_2) is satisfies for $\beta = 2$.

In Fig. 2, we illustrate the behavior of $(\mathcal{K}_n^F f_2)$ for $n = 5, 10$ as compared to $f_2 : [1, 8] \to [0, 1]$:

$$f_2(x) = \begin{cases} \dfrac{1}{5(x-1/2)^2}, & 1 \leq x < 3 \\ 0.2, & 3 \leq x < 5 \\ 0.8, & 5 \leq x < 6 \\ \dfrac{-100}{x^3} + 1, & 6 \leq x < 8. \end{cases}$$

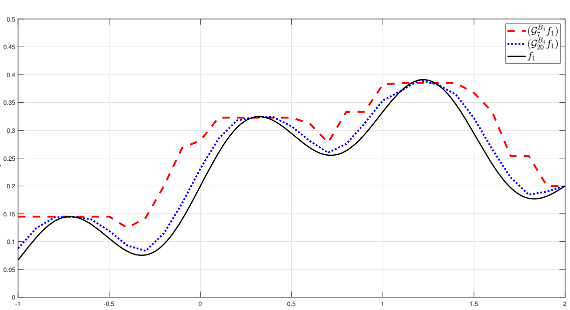

Fig. 1. Approximation of f_1 by $(\mathcal{G}_n^{B_4} f_1)$ based on B_4 kernel.

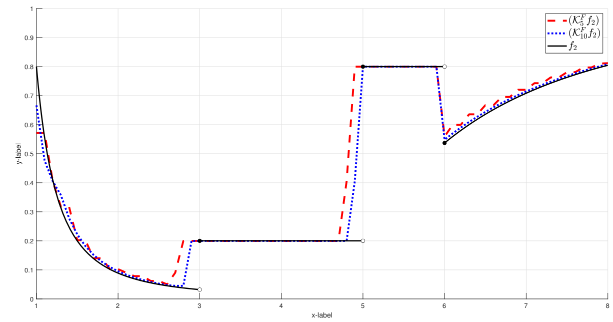

Fig. 2. Approximation of f_2 by $(\mathcal{K}_n^F f_2)$ based on Fejér's kernel.

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