Neural Incremental Data Assimilation

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Abstract

Data assimilation is a central problem in many geophysical applications, such as weather forecasting. It aims to estimate the state of a potentially large system, such as the atmosphere, from sparse observations, supplemented by prior physical knowledge. The size of the systems involved and the complexity of the underlying physical equations make it a challenging task from a computational point of view. Neural networks represent a promising method of emulating the physics at low cost, and therefore have the potential to considerably improve and accelerate data assimilation. In this work, we introduce a deep learning approach where the physical system is modeled as a sequence of coarse-to-fine Gaussian prior distributions parametrized by a neural network. This allows us to define an assimilation operator, which is trained in an end-to-end fashion to minimize the reconstruction error on a dataset with different observation processes. We illustrate our approach on chaotic dynamical physical systems with sparse observations, and compare it to traditional variational data assimilation methods.

1. Introduction

Artificial intelligence is transforming many fields, and has a growing number of applications in industry. In the sciences, it has the potential to considerably accelerate the scientific process. Geophysics and weather forecasting are areas where deep learning is particularly active, with recent months seeing an explosion in the number of large neural models for the weather forecasting problem (Pathak et al., 2022; Lam et al., 2022; Hoyer et al., 2023). In this work, we focus on the data assimilation problem that underpins weather forecasting: tomorrow's weather forecast is based on today's weather conditions, which are not directly measured, but are estimated from few observations. Data assimilation is the inverse problem of estimating the geophysical state of the globe on the basis of these sparse observations and of prior knowledge of the physics. The estimated state then serves as the starting point for forecasting. While deep learning models are revolutionizing the forecasting problem, they have yet to be applied operationally to data assimilation.

Related work The application of neural networks to inverse problems is an active area of research. For the data assimilation problem, several approaches have been proposed to incorporate a deep learning in the loop. (Arcucci et al., 2021) propose a sequential scheme where a neural network is trained at regular time steps to combine data assimilation and the forecasting model. Recently, the success of diffusion models for imaging (Ho et al., 2020) has led to the development of so-called ”plug and play” methods, where the neural network is trained to learn a prior (Laumont et al., 2022). Once trained, the neural prior can be used to solve a large number of inverse problems. In this line of work, (Rozet & Louppe, 2023) proposed a data assimilation method based on a diffusion model. An other type of approaches called "end-to-end" aim at directly training a neural network to minimize the reconstruction error. They have the benefit of training the network directly on the task of interest, but the versatility of the trained model with respect to the different observational processes is challenging. An end-to-end neural reconstruction algorithm is proposed in (Fablet et al., 2021), and aims at learning the prior distribution of the signal by defining the reconstruction as a maximum a posterior estimate, leading to a bi-level optimization problem. However, the complex prior induced by the neural network may hamper the convergence of this estimate, as it relies on non-convex optimization. Instead, we explore a model where the prior has a sufficiently simple structure to guarantee a convex posterior distribution.

Contributions In this work, we present a neural method for data assimilation. We introduce a data assimilation operator parametrized by a neural Gaussian prior, that is designed to locally improve the likelihood of an estimate. Our model is trained to minimize the reconstruction error in an end-to-end fashion. We show how this operator may be iterated to reconstruct complex signals. The effectiveness of our method is demonstrated on simulated nonlinear physical

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systems. We also show how our method may be used to enhance traditional data assimilation methods.

2. The data assimilation inverse problem

The aim of data assimilation is to reconstruct a state $x \in \mathbb{R}^d$ from partial noisy measurements $y = h(x) + \xi \in \mathbb{R}^m$ of that state, with $h(x)$ the observation process (Bouttier & Courtier, 2002; Bocquet et al., 2014). For meteorological applications, for instance, the state x represents the physical quantities on a grid representing the globe, and the observations y are partial measurements, from different sources: in situ measurements, weather balloons, satellites, . . . As these measurements are very sparse, we cannot generally hope to recover the state as a function of the data alone. Indeed, for a given observation vector y, a large number of states are compatible, making data assimilation an inverse problem. To reconstruct the state, we need to supplement the partial observations with another source of prior information on the state, which comes from our physical or statistical knowledge of the problem.

The data assimilation problem is then as follows. Given partial observations y and prior information on the state, the aim is to estimate the most probable underlying state x. The Bayesian probabilistic framework lends itself well to the mathematical formalization of the problem : the theoretical information about the state physics is captured by a prior distribution $x \sim p(x)$, and the noisy, partial observations of x can be modeled as $y|x \sim h(x) + \xi$, with a known observation process h and an unbiased additive noise that is typically assumed to be Gaussian $\xi \sim \mathcal{N}(0, R)$ and independent of x. Then, data assimilation can be seen as the estimation of the state maximizing the state posterior distribution $p(x|y) = p(x)p(y|x)/p(y)$. Under the assumption of Gaussian observational noise, this can be formulated as the following minimization problem

$$\underset{x \in \mathbb{R}^d}{\text{minimize}} \quad U(x) + \frac{1}{2}\|h(x) - y\|_R^2, \tag{2.1}$$

with $U(x) = -\log p(x)$ up to additive constants, and where we have adopted the notation $\|z\|_B = z^\top B^{-1} z$ for a positive definite matrix B.

Problem size For weather prediction, the state x represents the geophysical variables on a large spatial grid. It is hence a signal of very high dimension with typically $d \sim 10^6$ or even $d \sim 10^9$. The size of the data assimilation problem makes the computations and memory costs very heavy, severely limiting the computational budget of any numerical method. In the development of new learning-based methods, it is essential to keep this computational constraint in mind if we hope to scale up to real-size systems.

2.1. Least-square Gaussian interpolation

The first approach considered for data assimilation is naturally that of a linear-quadratic model. Assuming a Gaussian a priori on the state $x \sim \mathcal{N}(\mu, P)$ and a linear observation function $h(x) = Hx$, with $H \in \mathbb{R}^{m \times d}$, the variational Bayesian formulation for data assimilation (2.1) becomes a quadratic least-square problem:

$$\underset{x \in \mathbb{R}^d}{\text{minimize}} \quad \frac{1}{2}\|x - \mu\|_P^2 + \frac{1}{2}\|Hx - y\|_R^2 \tag{2.2}$$

whose maximum a posteriori solution takes the form

$$\text{MAP}(y; \mu, P) := \mu + K(y - H\mu), \tag{2.3}$$

with the H-dependent Kalman gain

$$K = PH^\top(HPH^\top + R)^{-1} \; \in \mathbb{R}^{m \times d}. \tag{2.4}$$

In the remained of this work, the dependence with respect to H is implicitly assumed in all quantities that depend on the observation vector y.

For meteorological applications, the state x that is optimized for is a snapshot of the set of geophysical variables at a given time, when the observations have been collected. The background term μ is the forecast of this state from the past observations.

Computational cost For large-scale applications, solving (2.2) by computing the closed-form expression (2.3) yields a $\mathcal{O}(m^3 + dm)$ complexity in general, as it involves solving a $m \times m$ linear system and computing a matrix-vector products of size $d \times m$. In operational geophysical applications, this cost may be prohibitive as d and m may reach prohibitively large values. To avoid such costs, (2.2) is solved by such as conjugate gradient (Fletcher & Reeves, 1964). In the data assimilation community, this variational approach for the estimation of a large-scale geophysical spatial state is called 3D-Var (Courtier et al., 1998).

2.2. Spatiotemporal data assimilation

So far, the prior knowledge of the state has taken the form of a Gaussian distribution, which can capture the proximity of the searched state to an estimate, and the correlations of one state variable to another. Least squares interpolation then searches for the state most faithful to the data, within a fluctuation zone around the estimate. Although simple and analytically solvable, this approach does not use signal physics equations as prior information.

In the 1990s, the quality of data assimilation analyses improved significantly by incorporating a physical model to the reconstruction algorithm, leading to the state-of-the-art variational assimilation algorithm 4D-Var (Le Dimet &

Talagrand, 1986). This algorithm is a generalization of 3D-Var to time-distributed observations, where the estimated signal x is a temporal sequence of the spatial geophysical state on a time window, *i.e.* a trajectory, rather than one single snapshot. It is applied sequentially on a sliding time window, in combination with a forecasting model, to produce regularly updated estimates of the meteorological variables. Alongside 4D-Var, other algorithms exist for data assimilation of dynamical systems, including sequential methods such as the celebrated Kalman filter, and its extensions to nonlinear models (Jazwinski, 2007), and to ensembling (Evensen, 2003). In this work, we focus on the so-called weak-constraint 4D-Var algorithm (Fisher et al., 2012), which we briefly explain next. Weak-constraint 4D-Var has the advantage being naturally related to the Bayesian formulation (2.1), and is used in operational systems.

For simplicity, we abstract from the time dimension in our mathematical formalism, and still denote the spatiotemporal signal as $x \in \mathbb{R}^d$. The knowledge of a physical dynamical model materializes as knowledge of a prior distribution $U(x)$ in (2.1), which can be computed and differentiated through with respect to x. In geophysics, this model is typically a fluid dynamics simulator, and its gradients are computed using the adjoint method (Talagrand & Courtier, 1987). Hence, the resulting $U(x)$ is more complex and more informative than a Gaussian prior, but comes with heavy computational costs. In the remained of this work, we assume that the observational processes are linear: $h(x) = Hx$. In practice, h is nonlinear and is sequentially approximated by its linear approximation. We argue that linearizing the physical model is computationally far more expensive than linearizing the observational process, and hence that considering only linear observations does not severly restrict the problem generality.

The weak-constraint 4D-Var algorithm aims at minimizing (2.1) by a Gauss-Newton descent algorithm (Gauss, 1877), with line-serach correction (Nocedal & Wright, 1999). More precisely, a sequence of estimates $z_k \in \mathbb{R}^d, 1 \leq k \leq \ell$ approximating the reconstruction signal is iteratively computed. At each iteration k, the objective function is approximated by its quadratic expansion in the vicinity of z_k. Specifically, the prior term is approximated as

$$U(x) \simeq U(z) + \nabla U(z)^\top (x - z) + \frac{1}{2}(x - z)^\top \nabla^2 U(z)(x - z). \tag{2.5}$$

We may express expansion (2.5) as a Gaussian log-likelihood:

$$U(x) \simeq \frac{1}{2}\|x - \mu(z)\|^2_{P(z)}, \tag{2.6}$$

with

$$P(z) \simeq \nabla^2 U(z), \tag{2.7a}$$

$$\mu(z) = z - P(z)^{-1} U(z), \tag{2.7b}$$

the approximation above refering to the gradient-hessian approximation.

Weak-constraint 4D-Var is described in Algorithm 1. We see that the sequence of estimates (z_k) is iterated with a recursion taking the form

$$x_k = A(z_k, y), \tag{2.8}$$

where assimilation operator A improves the current estimate z using the observations and the local approximation of the model, by performing a local optimal interpolation:

$$A(z, y) = \mathrm{MAP}(y; \mu(z), P(z)). \tag{2.9}$$

Computational cost The 4D-Var algorithm represents the state of the art for data assimilation in geophysics, and is deployed in operational meteorological centers. Its main limitation is the high computational cost of simulating and differentiating through the physical model. In Algorithm 1, each computation of P_k and μ_k comes with a large cost in addition to the cost of computing (2.9), hence limiting the method's accuracy. Additionally, the complexity of U may lead to a complex minimization landscape, making the descent algorithm likely to be stuck in local minima (Gratton et al., 2007).

In the next section, we propose to overcome these limitations by learning operator A from data.

3. Neural data assimilation

Deep neural networks hold great promise for solving inverse problems (Bai et al., 2020), as they can help recover the corrupted signal by using the large amount statistical information acquired on a training dataset. For the data assimilation problem in meteoroloy or oceanography, the ground truth signals x are not available as the geophysical systems are not observed. However, a promising research direction consists in training a deep neural network to learn a prior on high-resolution simulations, or on the reanalysis datasets such as ERA5, like neural weather models (Ben Bouallègue et al., 2024).

Deep learning approaches to inverse problems may be separated in two categories (Mukherjee et al., 2021). A first category of algorithms aims at learning a prior $U(x)$ from a training dataset, using a neural network, independently of the inverse problem. Once trained, the learned prior can be adapted to a reconstruction algorithm to reconstruct the signal. These algorithms are often called “plug-and-play”, as the trained neural prior can be used for any downstream inverse problem. In a second category of algorithms, refered to as “end-to-end” learning algorithms, the neural network

Algorithm 1 Incremental weak-constraint 4D-Var

input observation vector $y \in \mathbb{R}^m$, observation matrix H, iteration number ℓ, initial estimate z_0, tangent linear physical model μ, P
output state estimation z_ℓ
initialize $z_0 := x_0$
for $0 \leq k \leq \ell - 1$ **do**
  compute $P_k := P(z_k)$, $\mu_k = \mu(z_k)$
  estimate $x_k = \mathrm{MAP}(y; \mu_k, P_k)$
  compute line search parameter α_k
  update $z_{k+1} = z_k + \alpha_k(x_k - z_k)$
end for

Algorithm 2 Incremental neural data assimilation

input observation vector $y \in \mathbb{R}^m$, observation matrix H, iteration number ℓ, initial estimate z_0, neural models μ, P, trained parameter θ
output state estimation z_ℓ
initialize $z_0 := x_0$
for $0 \leq k \leq \ell - 1$ **do**
  compute $P_k := P(z_k; \theta, s_k)$, $\mu_k = \mu(z_k; \theta, s_k)$
  estimate $x_k = \mathrm{MAP}(y; \mu_k, P_k)$
  compute temperature parameter s_k
  update $z_{k+1} = z_k + s_k(x_k - z_0)$
end for

is explicitly trained to solve the inverse problem. In this case, the training consists of minimizing the neural network's reconstruction error, based on a dataset of state and observations pairs $(x^{(i)}, y^{(i)})$.

One challenge in training end-to-end algorithms is the multiplicity of possible observation processes: the trained neural network must be compatible with all possible (x, y), and hence with varying observation processes H, with different dimensions m for the observations. It should therefore model only the prior distribution $U(x)$, and not depend on the observation process H.

3.1. Neural assimilation operator

We adopt an end-to-end learning approach and we aim at learning a neural assimilation algorithm by minimizing a reconstruction error. We observe that, unlike other inverse problems such as image impainting, data assimilation often starts with a first physically plausible estimate $z$of the unknown state. Therefore, rather than learning to interpolate the observations from scratch, we train a neural network to improve the state estimate given z. Drawing inspiration from the 4D-Var algorithm, we learn an assimilation operator $A(z, y; \theta)$, where θ denotes the parameter vector of a neural network. As in (2.6), we model the local prior distribution conditioned on z as a Gaussian prior

$$x|z \sim \mathcal{N}(\mu(z; \theta), P(z; \theta)), \tag{3.1}$$

where $\mu(z; \theta)$ and $P(z; \theta)$ are trainable neural networks. Given this Gaussian prior, the observations are incorporated by solving the least-square interpolation (2.2):

$$A(z, y; \theta) = \mathrm{MAP}(y; \mu(z; \theta), P(z; \theta)). \tag{3.2}$$

By formulating it as the solution of a y-dependent interpolation problem, our assimilation operator (3.2) is defined for any observation process (H, y), although the underlying neural network models only the prior distribution. Given a dataset $(x^{(i)}, y^{(i)}, z_0^{(i)})$ consisting of signals $x^{(i)}$ and partial observations $y^{(i)}$, supplemented with coarse estimates $z^{(i)}$ of the signal, the neural prior (3.1) is trained to minimize the reconstruction error with the following objective:

$$\begin{aligned} \underset{\theta \in \mathbb{R}^n}{\text{minimize}} \quad & \sum_{i=1}^{N} \frac{1}{2} \| A(z^{(i)}, y^{(i)}; \theta) - x^{(i)} \|^2 \\ \text{subject to} \quad & A(z, y; \theta) = \mathrm{MAP}(y; \mu(z; \theta), P(z; \theta)). \end{aligned} \tag{3.3}$$

This training objective takes the form of a bi-level optimization problem. It is similar to that of (Fablet et al., 2021), where a neural interpolator called 4DVarNet is used to learn both the global prior $U(x)$ and the minimization algorithm of (2.1), rather than a local operator $A(z, y) \mapsto x$. In our case, however, the inner optimization problem (2.1) can be solved explicitly because the cost is quadratic. In contrast, it is only approximatively solved in the case of 4DVarNet, due to the non-convexity of the inner cost.

Training We train our model by minimizing (3.3) using stochastic gradient descent, with the ADAM optimizer (Kingma & Ba, 2015). Computing the optimal interpolation involves solving a linear systems of size m, and we need to propagate the gradients with respect to θ through this no-trivial operation during training. This may be handled by implicit differentiation, allowing to compute the gradients of the solution with respect to θ, without explicitly inverting the system's matrices (Johnson, 2012).

3.2. Incremental neural data assimilation

Since our assimilation operator is trained to reconstruct the signal from coarse approximation, a one-shot reconstruction is likely to yield blurry results. To improve reconstruction, we may iterate this operator, with the aim of progressively improving the reconstruction signal. Building on the recent advances of cold diffusion (Bansal et al., 2024), we propose an iterative strategy aiming at reconsctruting the signal in a coarse-to-fine fashion. We introduce a scalar temperature parameter $0 \leq s \leq 1$ modelling the coarseness of the reconstruction, and we allow our neural prior to depend on s as $\mu(z;\theta,s)$, $P(z;\theta,s)$. Intuitively, the prior should be coarser for larger values of s, and become sharper and more local as $s \to 0$. We provide initial estimates $z_k^{(i)}$ at different temperature levels $\{s_1 \geq \cdots \geq s_\ell\}$ as linear interpolations between $z_0^{(i)}$ and $z_k^{(i)}$:

$$z_k^{(i)} = s_k z_0^{(i)} + (1-s_k) z_0^{(i)}. \tag{3.4}$$

Our training objective is adapted as

$$\underset{\theta\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{k=1}^{\ell}\sum_{i=1}^{N} \|A(z_k^{(i)}, y^{(i)}; \theta, s_k) - x^{(i)}\|^2. \tag{3.5}$$

At prediction time, the signal is reconstructed by iteratively applying $A(z,y;\theta,s)$ following the sampling algorithm introduced in (Bansal et al., 2024). We provide a detailed description of our iterative reconstruction method in Algorithm 2.

4. Experiments on physical systems

In order to evaluate the performances of our data assimilation algorithm, we experiment on two simulated dynamical systems: the pendulum and the Lorenz 63 dynamical systems. We train our neural model on a dataset generated from the dynamical system with different trajectories x sampled from random initial conditions, and different observation processes, leading to various (x,y) pairs for the same x. Our JAX implementation of our neural assimilation algorithm is available online at `https://anonymous.4open.science/r/assimilation-3F9E`.

Architecure We take for $\mu(z;\theta,s)$ and $P(z;\theta,s)$ two fully-connected neural networks of depth 4 and width 32. The dependence with respect to s is implemented as a positional embedding. The $d \times d$ matrix P is modeled as a block-diagonal matrix, hence limiting the computational cost and imposing a temporal structure in the signal.

4.1. Pendulum

We start with the pendulum, which is is arguably one of the simplest nonlinear physical systems.

Baseline Importantly, the pendulum is simple enough to be decently approximated by linear dynamics. It can be shown that a linear dynamical model with Gaussian model noise yields a Gaussian prior distribution for the trajectory x. Therefore, a natural data assimilation baseline for the pendulum consists in the quadratic least-square estimator $z_0 := \mathrm{MAP}(y;\mu_0,P_0)$, where μ_0 and P_0 can be computed analytically as a function of the initial condition distribution and the pendulum's linear model.

Data We generate discrete trajectories $x^{(i)}$ of $T = 100$ time steps from the nonlinear pendulum dynamics with random initial conditions sampled in phase space, which is of dimension 2, hence $d = 2 \times 100 = 200$. The observations are generated by observing the pendulum's position at sparse time steps, with Gaussian observation noise $\xi \sim \mathcal{N}(0,\rho^2 I_m)$, with $\rho = 0.01$.

Experimental setup We train an adaptation operator to reconstruct the signal in one shot from z_0, following (3.3). At prediction time, we apply the trained neural assimilation map $A(z;y;\theta)$ to z_0 on a separate independent dataset.

Results Reconstruction samples are presented in Figure 1. While the linear model fails at reconstructing the trajectories outside of the linearization zone (angle and momentum close to 0), one application of our neural assimilation operator accurately reconstructs the signal.

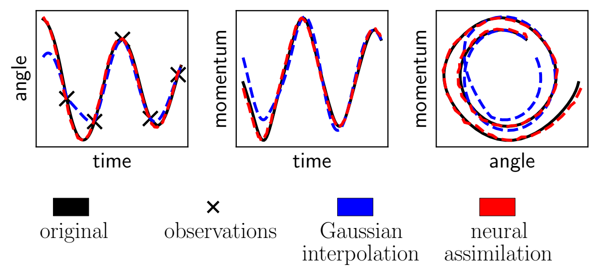

Figure 1. Reconstructed trajectories for the pendulum.

4.2. Lorenz 63

We now turn to a more complex system. The Lorenz system is a simplified physical model for for atmospheric convection (Lorenz, 1963). Three variables are governed by the following set of coupled nonlinear ordinary differential equations:

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}t} &= \sigma(u_2 - u_1)\\ \frac{\mathrm{d}u_2}{\mathrm{d}t} &= \rho u_1 - u_2 - u_1 u_3\\ \frac{\mathrm{d}u_3}{\mathrm{d}t} &= u_1 u_2 - \beta u_3.\end{aligned} \tag{4.1}$$

We set $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$, values for which the system is known to exhibit chaotic solutions. We sample the initial conditions in the system's stationary distribution, following the experimental setup of (Rozet & Louppe, 2023).

Data We generate datasets of trajectories by integrating (4.1) between time steps of length $\mathrm{d}t = 0.025$, and adding a small amount of Gaussian noise $\eta \sim \mathcal{N}(0, \mathrm{d}t I_3)$ at each time step. The number of time steps is $T = 32$,

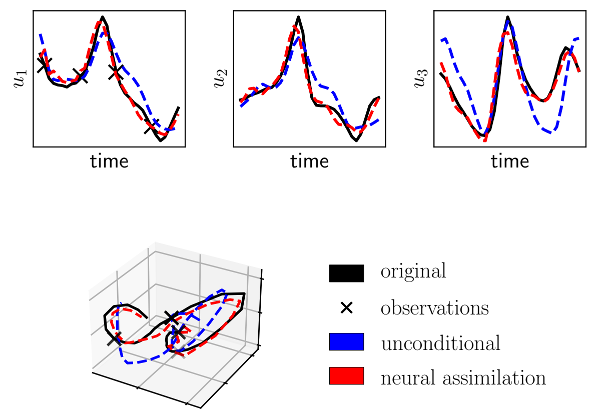

Figure 2. Reconstructed trajectories for the Lorenz 63 system.

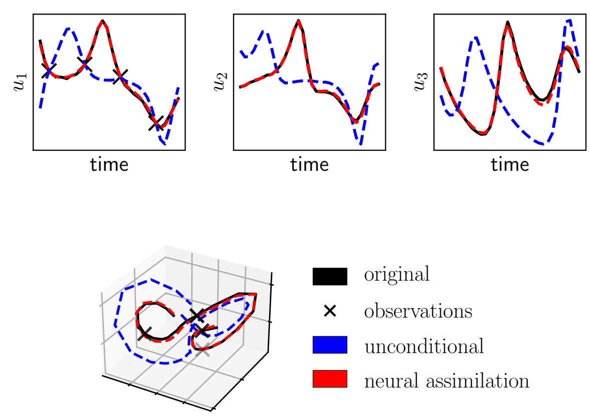

Figure 3. Output of 4D-Var from various initializations.

hence $d = 96$. We normalize each component of the trajectory to have zero mean and unit variance. The observations are sparse samples from the first component u_1 only, with observation noise of size 0.05. We take for the initial state estimate $z^{(i)}$ the maximum likelihood interpolation of $y^{(i)}$ under the moment-matching Gaussian distribution of $x^{(i)}$, which is the coarse Gaussian approximate of $p(x)$. More precisely, $z_0^{(i)} = \mathrm{MAP}(y^{(i)}; \hat{\mu}, \hat{P})$, with $\hat{\mu}$ and $\hat{P}$ the empirical mean and the empirical covariance of $\{x^{(i)}\}$. We define $\{z_k^{(i)}\}$ as in (3.4) with regular spacing $s_k = 1 - k/(\ell + 1)$. We take $\ell = 5$.

Baseline As a baseline, an “unconditional” neural network $F(z; \theta, s)$ is trained to restore the signal from the $z_k^{(i)}$ without the information brought by the observations. More specifically, $F(z; \theta, s)$ is a function of z only and not of y, and it is trained as a cold diffusion model to minimize objective (3.5) without the information provided by the observations.

Table 1. Average reconstrucition error for the different approaches.

Method	Gaussian	Unconditional	Neural assimilation
Error	1.5	1.1	0.5

Experimental setup We train our neural assimilation operator to reconstruct the sginal at different temperatures following (3.5). At prediction time, we apply Algorithm 2. Furthermore, in order to establish a link between our new neural method and traditional assimilation methods, we investigate how the output of the neural method, which is a priori uninterpretable, may be transformed into a plausible physical signal. To do this, we correct these estimates with several iterations of 4D-Var on top of the neural estimate of the signal, until the objective function (2.1) becomes lower than 0.05. As a result, the new output is constrained to satisfy the physical model, but potentially at a lower cost than if we had started from scratch because the initialization that we provided is already close to the true signal.

Results Figure 2 shows reconstruction samples from the baseline and from our method. We can see that our neural data assimilation algorithm can reconstruct the signal while staying close to the observations. In contrast, the unconditional baseline cannot efficiently improve both the signal likelihood and the data fidelity. Table 1 shows the average reconstruction error for the various methods. Further, we compare the reconstructed signals corrected by 4D-Var for an observation sample in Figure 3. The initialization provided by our method allows to recover the original signal with very high accuracy by running few steps of 4D-Var on top of the neural estimate, while the Gaussian initialization leads to an inaccurate local minimum. Importantly, the improvement with respect to a Gaussian initialization is significant, both in terms of reconstruction error and in terms of number of iterations, as the 4D-var algorithm converged after 4 iterations from the neural initialization and 23 iterations from the Gaussian initialization.

5. Conclusion

In this work, we have shown how deep learning methods may be applied to the data assimilation problem. Our neural method models in a coarse-to-fine fashion and is trained to minimize the reconstruction error. Importantly, we have shown how such a deep learning method may be used in combination with a traditional data assimilation method to enhance the reconstruction accuracy and reduce the computational time.

In future work, it would be interesting to apply our method to physical systems of larger scale, and to explore how the computational burden of data assimilation may be further reduced on such high-dimensional systems. Another important aspect that is crucial for data assimilation is uncertainty quantification.

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