Synchronisation Detection Using Spatial Ordinal Partitions in Networks

Keywords: Ordinal Partitioning, Synchronisation, Dynamical Networks, Forbidden Sequences, Permutation Entropy

Extended Abstract

Our method begins with the use of ordinal patterns to describe the spatial configuration of neighbouring oscillators at each time point. Ordinal analysis provides a representation that is noise-resistant and independent of the scaling of the underlying dynamics by focusing on the ordering of values rather than their raw amplitudes [1]. This symbolic representation allows us to determine whether neighbouring nodes in a network behave synchronously over a given time period. We extend this framework using permutation entropy and the cardinality of forbidden sequences as complementary indicators of collective dynamics. Permutation entropy measures the diversity of ordinal patterns observed over time, indicating dynamical complexity [2]. Forbidden sequences, on the other hand, highlight patterns that do not appear in symbolic dynamics. It demonstrates dynamical constraints imposed by the system's fundamental characteristics when applied to a time series [3]. The idea behind this presentation is that if we apply it to the neighbour nodes in a network, it can reveal structural constraints imposed by synchrony. Together, these two measures provide a powerful and computationally efficient method for categorising various regimes of collective behaviour, ranging from fully synchronised states to partially synchronised clusters and fully desynchronised dynamics.

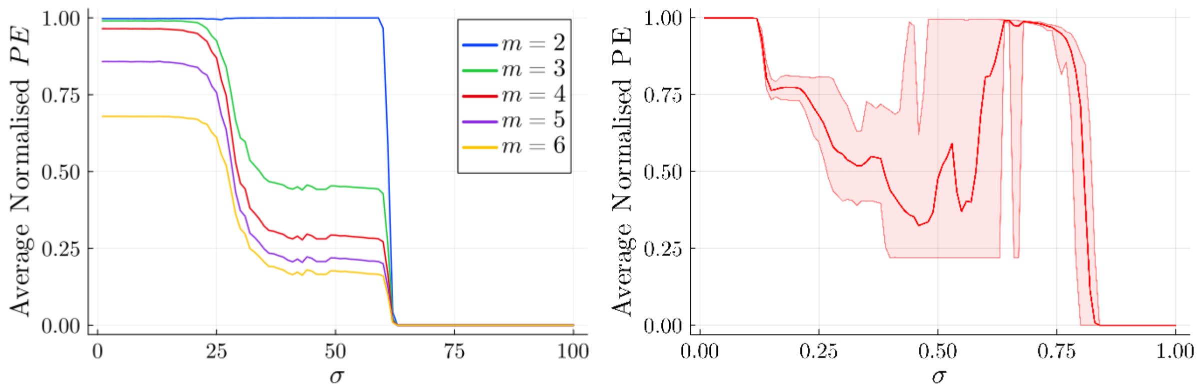
To demonstrate the effectiveness of the proposed methodology, we first apply it to a time series generated by a network of coupled identical logistic maps arranged on a ring. This classical system provides a benchmark for testing synchronisation detection techniques. Our findings not only confirm findings in the previous literature [4,5,6], but also provide new insights by identifying the boundaries of synchronous regions, particularly when oscillators are only partially synchronised. As a result, the method allows for a more detailed analysis of spatial organisation within synchronised clusters. We then extend the analysis to networks of logistic maps with random connections, where traditional spatio-temporal visualisation methods fail because the network lacks a simple geometric embedding. In this more complex case, our ordinal-partition-based method still detects synchronised groups and classifies the network's collective behaviour. Figure 1 shows the average of normalised permutation entropy for a network of $N = 100$ coupled logistic maps, that each node has 64 neighbours which are chosen (a) regularly on a ring network (b) randomly, with distinct sets of random initial conditions. In both cases of the ring network (a) and the random network (b), the value of the normalised permutation entropy can detect the boundaries of coupling strength that cause full synchronisation and full asynchronisation.

The findings presented here establish ordinal analysis as a general-purpose synchronisation detection tool when combined with entropy-based measures. Unlike many other approaches, it does not require explicit knowledge of the system's governing equations and is robust in the presence of noise. Beyond its validation on logistic map networks, the method has the potential to be applied to empirical data from a wide range of domains where understanding synchronous behaviour is critical. In summary, this work presents a conceptually simple yet powerful framework for detecting synchronous regions in networks of coupled oscillators and classifying their collective dynamics using symbolic time-series methods. Its demonstrated efficacy on

both ring and random networks suggests that it has applicability in many fields where detecting synchrony is still a major challenge.

References

[1] A. M. Unakafov and K. Keller, Conditional entropy of ordinal patterns, Physica D: Nonlinear Phenomena 269, 94 (2014).
[2] C. Bandt, G. Keller, and B. Pompe, Entropy of interval maps via permutations, Nonlinearity 15, 1595 (2002).
[3] J. M. Amigo´, Ordinal patterns, in Permutation Complexity in Dynamical Systems: Ordinal Patterns, Permutation Entropy and All That (Springer, 2010) pp. 49–67.
[4] A.-K. Malchow, I. Omelchenko, E. Scho¨ll, and P. Ho¨vel, Robustness of chimera states in nonlocally coupled networks of nonidentical logistic maps, Physical Review E 98, 012217 (2018).
[5] I. Omelchenko, Y. Maistrenko, P. Ho¨vel, and E. Scho¨ll, Loss of coherence in dynamical networks: spatial chaos and chimera states, Physical Review Letters 106, 234102 (2011).
[6] I. Omelchenko, B. Riemenschneider, P. Ho¨vel, Y. Maistrenko, and E. Scho¨ll, Transition from spatial coherence to incoherence in coupled chaotic systems, Physical Review E 85, 026212 (2012).



(a) (b)

Figure 1. **The average of normalised permutation entropy between all the ordinal windows in a network of $N = 100$ coupled logistic maps.** (a) in a ring network with 32 neighbours from the right and 32 neighbours from the left, with different lengths of ordinal windows from $m = 2$ to $m = 6$, for 25 different initial conditions, with error bars intentionally removed to avoid overcrowding the plot's areas of interest. (b) in a random network with 64 neighbours for each node. The ordinal window length is $m = 4$, and 50 distinct sets of random initial conditions are used. The red line is the mean of the ensemble of simulations, and the minimum and maximum observed depict the spread of results. In the regions where the coupling coefficient is extremely low (high), the entropy value is exactly 1 (0), indicating complete asynchrony (complete synchrony). In these regions, the network behaviour remains constant across different sets of initial conditions. For intermediate σ values, the network dynamics are dependent on the initial conditions. In this region, subsets of the network can synchronise (group synchrony) as evident from the lower but non-zero average normalised PE.