Optimality of Stationary Policies in Risk-averse Total-reward MDPs with EVaR

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Abstract

The risk-neutral discounted objective is popular in reinforcement learning, in part due to existence of stationary optimal policies and convenient analysis based on contracting Bellman operators. Unfortunately, for some common risk-averse discounted objectives, such as Value at Risk (VaR) and Conditional Value at Risk (CVaR), optimal policies must be history-dependent and must be computed using complex state augmentation. In this paper, we show that the risk-averse *total reward* objective, under the Entropic Risk Measure (ERM) and Entropic Value at Risk (EVaR), can be optimized by a stationary policy, an important property for practical implementations. In addition, an optimal policy can be efficiently computed using value iteration, policy iteration, and even linear programming. Importantly, our results only require the relatively mild condition of transient MDPs, and allow for *both* positive and negative rewards, unlike prior work requiring assumptions on the sign of the rewards. Overall, our results suggest that the total reward criterion may be preferable to the discounted criterion in a broad range of risk-averse reinforcement learning problems.

1 Introduction

The literature on Markov decision processes (MDP) [Puterman, 2005] has seen a growing interest in risk-averse objectives [Kastner et al., 2023, Marthe et al., 2023, Lam et al., 2022, Li et al., 2022, Bäuerle and Glauner, 2022, Hau et al., 2023b,a]. Risk-averse objectives penalize the variability of returns and prefer policies with stronger guarantees on the probability of catastrophic losses. As a result, risk-averse objectives are important in critical applications, such as healthcare, autonomous driving, or finance, where avoiding disastrous failures is essential. In modern work, the most common metric in risk-averse objectives is to use a *monetary risk measure*, which generalizes the expectation operator and assigns a real value to any random variable [Follmer and Schied, 2016].

Most reinforcement learning (RL) algorithms, risk-neutral and risk-averse alike, are designed for the *discounted* objective, which computes a weighted sum of rewards over an infinite time horizon with weights that decrease geometrically with time according to a known discount rate [Puterman, 2005, Su and Petrik, 2023]. In financial applications of RL, discounting future rewards accounts for inflation or the option to invest gains. In non-financial applications, the justification for discounting is

more complex and often driven by its algorithmic convenience—discounting guarantees the Bellman operator is a contraction.

The *total reward criterion* (TRC), also known as the stochastic shortest path, is an alternative objective to discounting [Puterman, 2005, Kallenberg, 2021]. In TRC, the horizon is infinite and future rewards are not discounted. While the undiscounted sum of rewards may be unbounded in general, a common assumption on the model is that the model is transient. In transient MDPs, there is some positive probability that the process terminates in a bounded number of steps and reaches an absorbing *sink state* with instantaneous reward 0. In the risk-neutral settings, transience guarantees that the total sum of rewards remains finite under any policy and, consequently, that an optimal policy exists [Kallenberg, 2021, Filar and Vrieze, 2012].

In this paper, we analyze the foundations of *risk-averse MDPs under the TRC objective* and propose algorithms for solving it. We focus on risk aversion modeled by the Entropic Value-at-Risk (EVaR) and Entropic Risk Measures (ERM) risk measures. As our *main contribution*, we show that stationary deterministic optimal policies always exist for TRC with EVaR and ERM risk-averse objectives. We also show that these stationary policies and value functions can be computed using value iteration, policy iteration and linear programming. The implementation of these algorithms is simple and closely resembles the algorithms for solving MDPs.

Transient MDPs with the TRC criterion are a particularly salient model in risk-averse reinforcement learning. In reinforcement learning, it is common to adopt discounted objectives to account for a probability of termination (transition to a sink state) [Sutton and Barto, 2018]. It is well known that in risk-neutral settings there is an equivalence between the probability of terminating and using a discount factor. However, as we show in this work, no such correspondence exists with risk-averse objectives and the difference between them may be arbitrarily large.

Our results also show that EVaR is a particularly interesting risk measure in reinforcement learning. ERM and the closely related exponential utility functions have been popular in sequential decision-making problems because they admit dynamic programming decompositions [Patek and Bertsekas, 1999, de Freitas et al., 2020, Smith and Chapman, 2023, Denardo and Rothblum, 1979, Hau et al., 2023b,a]. Unfortunately, ERM is difficult to interpret, its risk level is scale dependent, and it is difficult to relate it to popular risk measures like VaR and CVaR. Because EVaR reduces to an optimization over ERM, it preserves most of the computational advantages of ERM. Because EVaR closely approximates CVaR and VaR at the same risk level, its value is also much easier to interpret. Finally, EVaR is also a coherent risk measure unlike ERM [Ahmadi-Javid, 2012, Ahmadi-Javid and Pichler, 2017].

While we are not aware of prior work on the TRC objective with ERM or EVaR risk-aversion, the ERM risk measure is closely related to exponential utility functions. All prior works on TRC with exponential utility functions impose some constraints on the sign of the instantaneous rewards, e.g. positive rewards [Blackwell, 1967] or negative rewards [Bertsekas and Tsitsiklis, 1991, Freire and Delgado, 2016, de Freitas et al., 2020, Fei et al., 2021a,b]. Note that this significantly limits the modeling power of prior works, since assuming positive (resp. negative) rewards mean that all states are more desirable (resp. detrimental) than the sink state. Our analysis allows rewards that are negative as well as positive.

The remainder of the paper is organized as follows. Section 2 describes basic properties of transient MDPs and monetary risk measures. Section 3 establishes the main properties of ERM-TRC and designs algorithms to compute its optimal stationary policy. Section 4 establishes the main properties of EVaR-TRC and adapts the algorithm(Algorithm 2 in [Hau et al., 2023b]) to solve EVaR-TRC objective. Section 5 compares the ERM and EVaR returns with discounted and TRC criteria.

Notation. We use a tilde to mark random variables, e.g. $\tilde{x}$. Bold lower-case letters represent vectors, and upper-case bold letters represent matrices. Sets are either calligraphic or upper-case Greek letters. The symbol $\mathbb{X}$ represents the space of real-valued random variables. When a function is defined over an index set, such as $z \colon \{1, 2, \ldots, N\} \to \mathbb{R}$, we also treat it interchangeably as a vector $\boldsymbol{z} \in \mathbb{R}^n$ such that $z_i = z(i), \forall i = 1, \ldots, n$.

2 Background on risk-averse MDPs

Markov Decision Processes We focus on solving Markov decision processes (MDPs) [Puterman, 2005], modeled by a tuple $(\bar{S}, \mathcal{A}, p, r, \mu)$, where $\bar{S} = \{1, 2, \ldots, S, S+1\}$ is the finite set of states and $\mathcal{A} = \{1, 2, \ldots, A\}$ is the finite set of actions. The transition function $p\colon \bar{S} \times \mathcal{A} \to \Delta_{\bar{S}}$ represents the probability $p(s, a, s')$ of transitioning to $s' \in \bar{S}$ after taking $a \in \mathcal{A}$ in $s \in \bar{S}$ and $\boldsymbol{p}_{sa} \in \Delta_{\bar{S}}$ is such that $(\boldsymbol{p}_{sa})_{s'} = p(s, a, s')$. The function $r\colon \bar{S} \times \mathcal{A} \times \bar{S} \to \mathbb{R}$ represents the reward $r(s, a, s') \in \mathbb{R}$ associated with transitioning from $s \in \bar{S}$ and $a \in \mathcal{A}$ to $s' \in \bar{S}$. The vector $\boldsymbol{\mu} \in \Delta_{\bar{S}}$ is the initial state distribution. We designate the state $e := S + 1$ as a *sink state* and use $S = \{1, \ldots, S\}$ to denote the set of all non-sink states. The sink state e must satisfy that $p(e, a, e) = 1$ and $r(e, a, e) = 0$ for each $a \in \mathcal{A}$, and $\mu_e = 0$.

The solution to an MDP is a *policy*. Given a horizon $t \in \mathbb{N}$, a history-dependent policy in the set Π_{HR}^t maps the history of states and actions to a distribution over actions. A *Markov policy* $\pi \in \Pi_{\mathrm{MR}}^t$ is a sequence of decision rules $\pi = (\boldsymbol{d}_0, \boldsymbol{d}_1, \ldots, \boldsymbol{d}_{t-1})$ with $\boldsymbol{d}_k\colon S \to \Delta_{\mathcal{A}}$ the decision rule for taking actions at time k. The set of all *randomized decision rules* is $\mathcal{D} = (\Delta_{\mathcal{A}})^S$. *Stationary policies* Π_{SR} are of Markov policies with $\pi = (\boldsymbol{d}, \boldsymbol{d}, \ldots) := (\boldsymbol{d})_\infty$ with the identical decision rule in every timestep. We treat decision rules and stationary policies interchangeably. The sets of *deterministic* Markov and stationary policies are denoted by Π_{MD}^t and Π_{SD}. Finally, we omit the subscript t to indicate infinite horizon definitions of policies for histories of any length.

Optimizing the risk-neutral Total Reward Criterion (TRC) involves solving for

$$\sup_{\pi \in \Pi_{\mathrm{HR}}} \liminf_{T \to \infty} \mathbb{E}^{\pi, \boldsymbol{\mu}} \left[\sum_{t=0}^{T} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right], \tag{1}$$

where $\tilde{s}_t$ and $\tilde{a}_t$ represent the state from $\bar{S}$ and action from $\mathcal{A}$ at time t. The superscript π denotes the policy that governs the actions $\tilde{a}_t$ when visiting $\tilde{s}_t$ and $\boldsymbol{\mu}$ denotes the initial distribution. Finally, note that $\liminf$ gives a conservative estimate of a policy's return since the limit does not necessarily exist for non-stationary policies.

In risk-neutral objectives, TRC is more challenging to optimize than the discounted criterion. Without any additional assumptions, it is known that TRC may be unbounded, optimal policies may not exist or may be non-stationary [Bertsekas and Yu, 2013, James and Collins, 2006]. A common assumption that guarantees that the total return is well-behaved is that all policies have a positive probability of eventually transitioning to the sink state. Such MDPs are referred to as being transient [Kallenberg, 2021].

Definition 2.1 (Transient MDP). *An MDP is* transient *if for any $\pi \in \Pi_{\mathrm{SD}}$:*

$$\sum_{t=0}^{\infty} \mathbb{P}^{\pi, s} \left[\tilde{s}_t = s' \right] < \infty, \qquad \forall s, s' \in S. \tag{2}$$

Transient MDPs are important because their optimal policies exist and can be chosen stationary deterministic [Kallenberg, 2021, theorem 4.12]. An important tool in their analysis is the *spectral radius* $\rho\colon \mathbb{R}^{n \times n} \to \mathbb{R}$ which is defined for each $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ as the maximum absolute eigenvalue: $\rho(\boldsymbol{A}) := \max_{i=1, \ldots, n} |\lambda_i|$ where λ_i is the i-th eigenvalue [Horn and Johnson, 2013].

Lemma 2.2 (Theorem 4.8 in Kallenberg [2021]). *An MDP is transient if and only if $\rho(\boldsymbol{P}^\pi) < 1$ for all $\pi \in \Pi_{\mathrm{SD}}$.*

Monetary risk measures Monetary risk measures aim to generalize the expectation operator to account for the spread of the random variable. *Entropic risk measure* (ERM) is a popular risk measure, defined for any risk level $\beta > 0$ and $\tilde{x} \in \mathbb{X}$ as [Follmer and Schied, 2016]

$$\mathrm{ERM}_\beta [\tilde{x}] = -\beta^{-1} \cdot \log \mathbb{E} \exp (-\beta \cdot \tilde{x}). \tag{3}$$

and extended to $\beta \in [0, \infty]$ as $\mathrm{ERM}_0[\tilde{x}] = \lim_{\beta \to 0^+} \mathrm{ERM}_\beta [\tilde{x}] = \mathbb{E}[\tilde{x}]$ and $\mathrm{ERM}_\infty [\tilde{x}] = \lim_{\beta \to \infty} \mathrm{ERM}_\beta [\tilde{x}] = \mathrm{ess\,inf}[\tilde{x}]$. ERM plays a special role in sequential decision-making because it is the only law-invariant risk measure that satisfies the tower property shown in Proposition A.1 [Kupper and Schachermayer, 2006, Marthe et al., 2023], which is is essential in constructing dynamic programs [Hau et al., 2023b].

Unfortunately, two major limitations of ERM hinder its practical applications. First, it is not positively homogenous and, therefore, the risk value depends on the scale of the rewards and ERM is not coherent [Follmer and Schied, 2016, Hau et al., 2023b, Ahmadi-Javid, 2012]. Second, the risk parameter β is difficult to interpret and does not relate well to other common risk measures, like VaR or CVaR.

For these reasons, we focus on the *Entropic Value at Risk* (EVaR), defined as, for a given $\alpha \in (0, 1)$,

$$\text{EVaR}_\alpha [\tilde{x}] = \sup_{\beta > 0} -\beta^{-1} \log \left(\alpha^{-1} \mathbb{E} \exp \left(-\beta \tilde{x} \right) \right) = \sup_{\beta > 0} \text{ERM}_\beta [\tilde{x}] + \beta^{-1} \log \alpha, \quad (4)$$

and is extended to $\text{EVaR}_0 [\tilde{x}] = \operatorname{ess\,inf}[\tilde{x}]$ and $\text{EVaR}_1 [\tilde{x}] = \mathbb{E} [\tilde{x}]$ [Ahmadi-Javid, 2012]. EVaR addresses the limitations of ERM while preserving its main benefits [Ahmadi-Javid, 2012, Hau et al., 2023b].

3 Analysis of ERM Total Reward Criterion

In this section, we analyze the ERM-TRC problem. We show that an optimal stationary policy exists for this criterion, and we describe several algorithms for computing it. We do not assume that the rewards are positive or negative. We make no additional assumptions on the value of β.

Our objective $l \colon \mathbb{R}_{++} \to \bar{\mathbb{R}}$ in this section is to maximize the ERM of the infinite-horizon total sum of rewards, which is formally defined as

$$l(\beta) := \sup_{\pi \in \Pi_{\text{HR}}} \liminf_{T \to \infty} \text{ERM}_\beta^{\pi, \mu} \left[\sum_{t=0}^{T} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right], \quad (5)$$

where $\mathbb{R}_{++}$ is the set of positive real numbers.

Surprisingly, adding risk aversion to the TRC can result in unbounded returns even when the MDP is transient, as we show below. In the remainder of the section, we generally assume that the risk level $\beta > 0$ is fixed and omit it in notations when its value is unambiguous from the context.

3.1 Existence of Optimal Value Functions and Stationary Policies

To prove our results, we first study the finite-horizon ERM objective and then treat the total reward criterion as a limiting case as the horizon tends to infinity.

For the finite-horizon ERM objective, there always exists a Markov deterministic optimal policy (see Appendix B.1), and we therefore define finite-horizon time-dependent value functions for such policies. The finite-horizon value and optimal value functions, $\boldsymbol{v}^t(\pi) \in \mathbb{R}^{\bar{S}}$ and $\boldsymbol{v}^{t,\star} \in \mathbb{R}^{\bar{S}}$ respectively, are defined for each horizon $t = 0, \ldots$ and policy $\pi \in \Pi_{\text{MD}}$, $s \in \bar{S}$ as

$$v_s^t(\pi) := \text{ERM}_\beta^{\pi, s} \left[\sum_{k=0}^{t-1} r(\tilde{s}_k, \tilde{a}_k, \tilde{s}_{k+1}) \right], \qquad v_s^{t,\star} = \max_{\pi \in \Pi_{\text{MD}}} v_s^t(\pi). \quad (6)$$

Instead of value functions, it will be convenient to consider their exponential transformation that will linearize the corresponding Bellman operators. The *exponential value function* $w_s^\pi \in \mathbb{R}^{\bar{S}}$ for $\pi \in \Pi_{\text{MD}}$, $t = 0, 1, \ldots$, and $s \in \bar{S}$ is defined as

$$w_s^t(\pi) := -\exp \left(-\beta \cdot v_s^t(\pi) \right) = -\mathbb{E}^{\pi, s} \left[\exp \left(-\beta \cdot \sum_{k=0}^{t-1} r(\tilde{s}_k, \tilde{a}_k, \tilde{s}_{k+1}) \right) \right]. \quad (7)$$

The optimal exponential value function $\boldsymbol{w}^{t,\star} \in \mathbb{R}^{\bar{S}}$ is defined analogously. Note that the exponential value functions satisfy $\boldsymbol{w}^t < \boldsymbol{0}$ (componentwise) and $\boldsymbol{w}^0(\pi) = \boldsymbol{w}^{t,\star} = -\boldsymbol{1} = -(1, ..., 1)$ for any $\pi \in \Pi_{\text{MD}}$. The value function can be recovered as

$$v_s^t(\pi) = -\beta^{-1} \log(-w_s^t(\pi)), \qquad \forall s \in \bar{S}, t = 0, 1, \ldots. \quad (8)$$

As is usual in MDPs, we employ dynamic programming to compute exponential value functions. The *exponential Bellman operator* for $\boldsymbol{w} \in \mathbb{R}^S$ is defined as

$$L^d \boldsymbol{w} := \boldsymbol{B}^d \boldsymbol{w} - \boldsymbol{b}^d, \qquad L^\star \boldsymbol{w} := \max_{d \in \mathcal{D}} L^d \boldsymbol{w} = \max_{d \in \text{ext}\,\mathcal{D}} L^d, \quad (9)$$

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where $\operatorname{ext} \mathcal{D}$ is the set of extreme points of $\mathcal{D}$ corresponding to deterministic decision rules. The exponential transition matrix $\boldsymbol{B}^{\boldsymbol{d}} \in \mathbb{R}_+^{S \times S}$ and vector $\boldsymbol{b}^{\boldsymbol{d}} \in \mathbb{R}_+^S$ are defined for $s, s' \in \mathcal{S}$ and $\boldsymbol{d} \in \mathcal{D}$ as

$$B_{s,s'}^{\boldsymbol{d}} := \sum_{a \in \mathcal{A}} p(s, a, s') \cdot d_a(s) \cdot \exp\left(-\beta \cdot r(s, a, s')\right), \tag{10a}$$

$$b_s^{\boldsymbol{d}} := \sum_{a \in \mathcal{A}} p(s, a, e) \cdot d_a(s) \cdot \exp\left(-\beta \cdot r(s, a, e)\right). \tag{10b}$$

The following theorem builds on previous results for MDPs with ERM [Hau et al., 2023b] and exponential utility functions [Patek, 1997] to show that exponential value functions can be computed by applying the exponential Bellman operator iteratively. We use the shorthand notation $\pi_{1:t-1} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{t-1}) \in \Pi_{\mathrm{MR}}^{t-1}$ to denote the tail of π that starts with $\boldsymbol{d}_1$ instead of $\boldsymbol{d}_0$.

Theorem 3.1. *The exponential value functions $\boldsymbol{w}^t(\pi)$ in (7) for $\pi = (\boldsymbol{d}_0, \ldots, \boldsymbol{d}_{t-1}) \in \Pi_{\mathrm{MR}}^t$ and $\boldsymbol{w}^{t,\star}$ can be computed from $\boldsymbol{w}^t(\pi_{1:t-1})$ and $\boldsymbol{w}^{t,\star}$ respectively for $t = 1, \ldots$ as:*

$$\boldsymbol{w}^t = L^{\boldsymbol{d}_t} \boldsymbol{w}^{t-1}(\pi_{1:t-1}), \qquad \boldsymbol{w}^{t,\star} = L^\star \boldsymbol{w}^{t-1,\star},$$

and $\boldsymbol{w}^0(\pi) = \boldsymbol{w}^{0,\star} = -\mathbf{1}$. Moreover, there exists $\pi_t^\star \in \Pi_{\mathrm{MD}}^t$ such that $\boldsymbol{w}^t(\pi_t^\star) = \boldsymbol{w}^{t,\star}$ for each $t = 0, \ldots$.

The proof of Theorem 3.1 is shown in Appendix B.3.

Using the notation and results above, we now turn to constructing infinite-horizon optimal policies as a limiting case of the finite horizon. The ERM-TRC objective defined in (5) can be expressed for the initial distribution $\boldsymbol{\mu} \in \Delta_{\bar{\mathcal{S}}}$ as

$$\sup_{\pi \in \Pi_{\mathrm{HR}}} \liminf_{t \to \infty} \boldsymbol{\mu}^\top \boldsymbol{v}^t(\pi). \tag{11}$$

As in the finite-horizon case, it will be beneficial to define an exponential transformation of the value function for each $\pi \in \Pi_{\mathrm{MR}}$ as elementwise limits:

$$\boldsymbol{w}^\infty(\pi) := \liminf_{t \to \infty} \boldsymbol{w}^t(\pi), \qquad \boldsymbol{w}^{\infty,\star} := \liminf_{t \to \infty} \boldsymbol{w}^{t,\star}.$$

The following theorem shows the main results of this section. It establishes the existence of an optimal exponential value function, attained by a stationary deterministic policy, and shows that it is the fixed point of the exponential Bellman operator.

Theorem 3.2. *Assume the MDP is transient, $\boldsymbol{\mu} > 0$, and $\boldsymbol{\mu}^\top \boldsymbol{w}^{\infty,\star} > -\infty$. Then there exists $\pi^\star = (\boldsymbol{d}^\star)_\infty \in \Pi_{\mathrm{SD}}$ such that*

$$\boldsymbol{w}^{\infty,\star} = \boldsymbol{w}^\infty(\pi^\star) = L^{\boldsymbol{d}^\star} \boldsymbol{w}^{\infty,\star}.$$

Moreover, $\boldsymbol{w}^{\infty,\star}$ is the unique fixed point of $L^{\boldsymbol{d}^\star}$.

Before discussing the proof of Theorem 3.2, we state its immediate corollary. That is, there exists an optimal stationary policy that solves the ERM-TRC objective.

Corollary 3.3. *Assume a transient MDP and $\boldsymbol{\mu} > \mathbf{0}$. Then:*

$$\boldsymbol{\mu}^\top \boldsymbol{v}^{\infty,\star} = \max_{\pi \in \Pi_{\mathrm{SD}}} \boldsymbol{\mu}^\top \boldsymbol{v}^\infty(\pi).$$

The proof of Corollary 3.3 is shown in Appendix B.5.

It is important to note that our results are somewhat stronger than purely showing the existence of optimal stationary policies in infinite-horizon objectives. Our results show that there exists an optimal stationary policy whenever the planning horizon t is sufficiently large. This property mirrors *turnpikes* in discounted MDPs [Puterman, 2005].

Finally, we use the properties above to discuss the impact of β on the objective function.

Proposition 3.4. *There exists a transient MDP and a risk level $\beta > 0$ such that $l(\beta) = -\infty$.*

The proof of Proposition 3.4 is shown in Appendix B.6.

Although the TRC may be unbounded, for each transient MDP, there exists a β such that the TRC is bounded. This result will be important in the analysis of the EVaR objective.

Lemma 3.5. *Assume that the MDP is transient. Then there exists $\beta > 0$ such that $\infty > l(\beta) > -\infty$.*

The proof of Lemma 3.5 is shown in Appendix B.7.

3.2 Outline of the Convergence Proof of Theorem 3.2

We now outline the proof of Theorem 3.2; please see Appendix B.4 for details. To establish Theorem 3.2, we show that $\boldsymbol{w}^{t,\star}$ converges to a fixed point as $t \to \infty$.

Note that standard discounted infinite-horizon arguments do not apply to our ERM-TRC setting: in discounted objectives, one would usually use the contraction property of the Bellman operator under the L_∞ norm to establish the existence of a single fixed point. However, under the TRC, the Bellman operator is not a L_∞-contraction. There are two common techniques in TRC with transient MDPs. The first one is to argue that the Bellman operator is a contraction under specially weighted L_∞ norm [Bertsekas, 2018]. The second one is to argue that $(T^{\boldsymbol{d}})^k$ is an L_∞ contraction for a sufficiently large k where T is the Bellman operator [Bertsekas, 2017].

The risk-neutral TRC proof techniques rely on the linearity of the Bellman evaluation operator [Kallenberg, 2021] and cannot be applied to the nonlinear ERM Bellman operator. To overcome this nonlinearity, we consider the exponential Bellman operator $L^{\boldsymbol{d}}$ which is linear for each $\boldsymbol{d} \in \mathcal{D}$. Although $\boldsymbol{B}^{\boldsymbol{d}}$ is linear, it may be a non-contraction with $\rho(\boldsymbol{B}^{\boldsymbol{d}}) \geq 1$ when the MDP is a transient. This is because the transformation in (10) can increase the transition probabilities leading to rows sums greater than 1 [Horn and Johnson, 2013, theorem 8.1.22]. This precludes us from using standard fixed-point arguments to argue that the limit exists (does not oscillate with time) and is bounded.

Our main contribution is to show that whenever the exponential value functions are bounded, they must be contractions and the limit exists. To facilitate the analysis, we define $\boldsymbol{w}^t \colon \Pi_{\mathrm{SR}}^t \times \mathbb{R}^{\bar{\mathcal{S}}} \to \mathbb{R}^{\bar{\mathcal{S}}}, t = 0, \dots$ for $\boldsymbol{z} \in \mathbb{R}^{\bar{\mathcal{S}}}$ as, for $\pi \in \Pi_{\mathrm{SR}}^t$,

$$\boldsymbol{w}^t(\pi, \boldsymbol{z}) \;=\; L^{\boldsymbol{d}}\boldsymbol{w}(\pi_{1:t-1}) \;=\; L^{\boldsymbol{d}}L^{\boldsymbol{d}}\dots L^{\boldsymbol{d}}(-\boldsymbol{z}) \;=\; -(\boldsymbol{B}^{\boldsymbol{d}})^t \boldsymbol{z} - \sum_{k=0}^{t-1}(\boldsymbol{B}^{\boldsymbol{d}})^k \boldsymbol{b}^{\boldsymbol{d}}. \tag{12}$$

The value $\boldsymbol{z}$ can be interpreted as the exponential value function at the termination of the process following π for t periods, with exponential value at termination $\boldsymbol{z}$. Note that $\boldsymbol{w}^t(\pi) = \boldsymbol{w}^t(\pi, \mathbf{1})$, $\forall \pi \in \Pi_{\mathrm{MR}}, t = 0, \dots$. An important technical result we show is that the only way a *stationary* policy's return can be bounded is if the policy's matrix has a contracting spectral radius.

Lemma 3.6. *Assume a transient MDP and $\pi = (\boldsymbol{d})_\infty \in \Pi_{\mathrm{SR}}$. Then for each $\boldsymbol{\mu} > 0$ and $\boldsymbol{z} \geq 0$*

$$\boldsymbol{\mu}^\top \boldsymbol{w}^\infty(\pi, \boldsymbol{z}) > -\infty \qquad \Rightarrow \qquad \rho(\boldsymbol{B}^{\boldsymbol{d}}) < 1.$$

The proof of Lemma 3.6 is shown in Appendix B.8.

Lemma 3.6 uses the transience property to show that Perron vector $\boldsymbol{f}$ of $\boldsymbol{B}^{\boldsymbol{d}}$ satisfies that $\boldsymbol{f}^\top \boldsymbol{b}^{\boldsymbol{d}} > 0$. Recall that the Perron vector of a non-negative matrix is the eigenvector with the maximum absolute eigenvalue [Horn and Johnson, 2013]. Therefore, $\rho(\boldsymbol{B}^{\boldsymbol{d}}) < 1$ is necessary for the series in (12) to be bounded.

3.3 Algorithms for Computing Value Functions

We now briefly summarize the algorithms that can be used to compute the optimal ERM-TRC policies.

Value iteration is the most direct method for computing the optimal value function [Puterman, 2005]. It iteratively applies the ERM Bellman operator (see (20) in the appendix) or the exponential Bellman operator. Theorem 3.2 shows that value iteration will converge to the optimal value function (if it is bounded). *Policy iteration* (PI) starts by initializing with a stationary policy $\pi_0 = (\boldsymbol{d}_0) \in \Pi_{\mathrm{SR}}$. Then, for each iteration k, PI alternates between policy evaluation step: $\boldsymbol{w}_k = -(\boldsymbol{I} - \boldsymbol{B}^{\boldsymbol{d}_k})\boldsymbol{b}^{\boldsymbol{d}_k}$ and policy improvement, i.e., choosing $\boldsymbol{d}_{k+1}$ such that $L^{\boldsymbol{d}_{k+1}}\boldsymbol{w}_k = L^\star \boldsymbol{w}_k$. The convergence of policy iteration can be shown because it monotonically improves the value functions when initialized with a policy with bounded return [Patek, 2001]. We are not aware, however, of a practical approach to find such an initial policy. Finally, the optimal exponential value function can be computed using the following linear program

$$\begin{aligned}
\min_{\boldsymbol{w} \in \mathbb{R}^{\bar{\mathcal{S}}}} \quad & \mathbf{1}^\top \boldsymbol{w} \\
\text{subject to} \quad & w_s \geq -b_s^a + \boldsymbol{B}_{s,\cdot}^a \cdot \boldsymbol{w}, \quad \forall s \in \bar{\mathcal{S}}, a \in \mathcal{A}.
\end{aligned} \tag{13}$$

Where $\boldsymbol{w} = (w_1, \cdots, w_{|\bar{\mathcal{S}}|})$, $\boldsymbol{B}_{s,\cdot}^a = (\boldsymbol{B}_{s,s_1}^a, \cdots, \boldsymbol{B}_{s,s_{|\bar{\mathcal{S}}|}}^a)$, $\boldsymbol{B}_{s,s'}^a$ and b_s^a are calculated in (10).

4 Reduction of EVaR TRC to ERM TRC

In this section, we analyze the EVaR-TRC objective and show that it can be reduced to a sequence of ERM-TRC problems. This reduction is inspired by a reduction proposed for discounted MDPs [Hau et al., 2023b]. Using this reduction, we show that the optimal EVaR-TRC policy is stationary.

The objective in this section is to compute a policy that maximizes the EVaR of the random return $\sum_{t=0}^{\infty} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1})$ at some given risk level $\alpha \in (0, 1)$ as

$$\rho^\star = \sup_{\pi \in \Pi_{\mathrm{HR}}} \mathrm{EVaR}_\alpha^{\pi,\boldsymbol{\mu}} \left[\sum_{t=0}^{\infty} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right], \tag{14}$$

where π denotes the policy that governs the actions $\tilde{a}_t$ when visiting $\tilde{s}_t$ and $\boldsymbol{\mu}$ denotes the initial state distribution. In (14), we interpret the EVaR of the infinite sum as

$$\mathrm{EVaR}_\alpha^{\pi,\boldsymbol{\mu}} \left[\sum_{t=0}^{\infty} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] = \sup_{\beta > 0} \lim_{T \to \infty} \mathrm{ERM}_\beta^{\pi,\boldsymbol{\mu}} \left[\sum_{t=0}^{T} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] + \beta^{-1} \cdot \log \alpha.$$

Note that the objective in (14) differs from prior work on EVaR in discounted MDPs in that it considers an undiscounted criterion and a static risk measure [Ahmadi et al., 2021, Hau et al., 2023b].

To compute an optimal policy for EVaR-TRC objective, we define a proxy objective function $h \colon \mathbb{R} \to \mathbb{R}$ as

$$h(\beta) := \max_{\pi \in \Pi_{\mathrm{SD}}} \left(\lim_{T \to \infty} \mathrm{ERM}_\beta^{\pi,\boldsymbol{\mu}} \left[\sum_{t=0}^{T} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] + \beta^{-1} \cdot \log(\alpha) \right). \tag{15}$$

The main result of this section is given by Theorem 4.1.

Theorem 4.1. *Assume a transient MDP and approximation error δ, for each $\beta > 0$, $h(\beta)$ is bounded, then:*

1) if $\sup_{\beta > 0} h(\beta)$ is attained, then there exists $\beta^\star > 0$ and $\pi^\star \in \Pi_{\mathrm{SD}}$ such that

$$h(\beta^\star) \geq \sup_{\beta > 0} h(\beta) - \delta, \quad \pi^\star \in \arg\max_{\pi \in \Pi_{\mathrm{SD}}} \left(\lim_{T \to \infty} \mathrm{ERM}_{\beta^\star}^{\pi,\boldsymbol{\mu}} \left[\sum_{t=0}^{T} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] \right), \tag{16}$$

then $\pi^\star$ is $\delta-$optimal policy for EVaR-TRC objective in (14).

2) if $\sup_{\beta > 0} h(\beta)$ is not attained, then

$$\pi^\star \in \arg\max_{\pi \in \Pi_{\mathrm{SD}}} \mathrm{ess\,inf} \left[\sum_{t=0}^{\infty} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right],$$

$\pi^\star$ is the optimal policy for EVaR-TRC objective in Equation (14).

The proof of Theorem 4.1 is shown in Appendix C.1.

Now, we reduce the EVaR-TRC problem to a specific sequence of ERM-TRC problems. Given an approximation error δ, we use a discrete grid $\{\beta_k\}_{k=1}^K$ to search over the risk level β that can maximize the function h. $\{\beta_k\}_{k=1}^K$ can be constructed for $k = 1, \ldots, K-2$ as

$$\beta_1 = \frac{8\delta}{\Delta_{r_{\max}}^2}, \quad \beta_{k+1} = \frac{\beta_k \log(\alpha)}{\beta_k \delta + \log(\alpha)}. \quad \beta_K = \frac{-\log(\alpha)}{\delta}, \tag{17}$$

where $\Delta_{r_{\max}}$ denotes the range of the immediate return. See Equations (19) and (20) in [Hau et al., 2023b] for more details, derivation, and guarantees. Find the $k^\star$ value, $k^\star \in \arg\max_{k=1:K} h(\beta_k)$, and then $\pi^{k^\star}$ is the δ-sub-optimal policy to EVaR-TRC objective in (14).

Table 1: ERM and EVaR returns with discounted and TRC criteria.

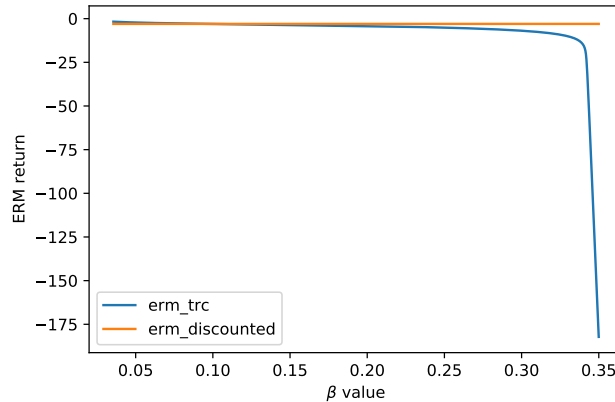|	Discounted criterion			TRC criterion			
	ERM	EVaR	$\beta^\star$	ERM	EVaR	$\beta^\star$	$\rho(\boldsymbol{B^d})$
Example 3	-3	-3	1053.6	-2.9	-4.07	0.09	0.96
Example 4	3	3	1053.6	2.88	2.14	0.142	0.929

Figure 1: Unbounded ERM values with TRC criterion for Example 3

5 Numerical Illustration

In this section, we compute the EVaR and ERM returns with discounted and TRC criteria. We show that the ERM return in a discounted MDP and the ERM return in a converted transient MDP diverge. Then, in TRC, we show that $\rho(\boldsymbol{B^d}) < 1$ is the necessary condition for the ERM return to be bounded.

Before computing the EVaR and ERM returns, we show how to construct a transient MDP from a discounted MDP. For a discounted MDP, there always exists an equivalent transient MDP constructed using Remark 1 in Appendix C.2. It is important to note that However, there exists a transient MDP that can not be converted to a discounted MDP; please see Example 2 in Appendix C.3 for details. The intuition is that the discount factors of different states can have different values.

We run our algorithm on Example 3 and Example 4. Please see Example 3 and Example 4 in Appendix C.3 for details. Given $\alpha = 0.9$ and $\delta = 0.0001$, we generate a sequence of $\{\beta_k\}_{k=1}^K$ using Equation (17). For ERM risk, Table 1 shows that ERM and EVaR returns with discounted and TRC criteria diverge. This is different from the neutral-risk case. Proposition C.1 shows that the expected return with TRC criterion is equal to the expected return with discounted criterion.

In TRC, the ERM return can be unbounded. Figure 1 shows that the ERM return dramatically decreases after β greater than 0.342. When $\beta = 0.3149$, we have $\rho(\boldsymbol{B^d}) = 1$. $\rho(\boldsymbol{B^d}) < 1$ is necessary for the ERM return to be bounded. However, in discounted criterion, ERM return is always bounded due to the discount factor.

6 Conclusion

We analyze transient MDPs with two risk measures: ERM and EVaR. We establish the necessary and sufficient conditions of the existence of stationary optimal polices. We allow rewards that are negative as well as positive. We prove the convergence of value iteration and show that the optimal stationary policy can be computed using value iteration, policy iteration and linear program. Our numerical illustration shows that TRC criterion is preferable to the discounted criterion under ERM and EVaR risk measures.

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A   Background

Proposition A.1. *Tower Property for Expectation(Proposition 3.4 in [Ross and Peköz, 2023],Proposition B.1. in [Hau et al., 2023b]) Any two random variables $X_1, X_2 \in \mathbb{X}$, we have*

$$\mathbb{E}[X_1] = \mathbb{E}[\mathbb{E}[X_1|X_2]]$$

B   Proofs for Section 3

B.1   Optimality of Markov Policies

The equivalence to solving finite-horizon MDPs with exponential utility functions gives us the following result.

Theorem B.1. *For each $\beta > 0$, there exists an optimal deterministic Markov policy $\pi^{t,\star} \in \Pi_{\mathrm{MD}}$ for each horizon $t = 0, \dots$:*

$$\max_{\pi \in \Pi_{\mathrm{MR}}} \mathrm{ERM}_\beta^{\pi,s} \left[\sum_{k=0}^{t-1} r(\tilde{s}_k, \tilde{a}_k, \tilde{s}_{k+1}) \right] = \max_{\pi \in \Pi_{\mathrm{HR}}} \mathrm{ERM}_\beta^{\pi,s} \left[\sum_{k=0}^{t-1} r(\tilde{s}_k, \tilde{a}_k, \tilde{s}_{k+1}) \right].$$

See [Hau et al., 2023b, Corollary 4.2] for a proof. The result can also be derived from the optimality of Markov deterministic policies in MDPs with exponential utility functions [Chung and Sobel, 1987, Patek, 2001].

B.2   Bellman Operator

Lemma B.2. *The exponential Bellman operator is monotone. That is for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^S$*

$$\boldsymbol{x} \geq \boldsymbol{y} \quad \Rightarrow \quad L^{\boldsymbol{d}}\boldsymbol{x} \geq L^{\boldsymbol{d}}\boldsymbol{y}, \qquad \forall \boldsymbol{d} \in \mathcal{D} \tag{18}$$
$$\boldsymbol{x} \geq \boldsymbol{y} \quad \Rightarrow \quad L^{\star}\boldsymbol{x} \geq L^{\star}\boldsymbol{y}. \tag{19}$$

Proof. The property in (18) follows immediately from non-negattivity of $\boldsymbol{B}^{\boldsymbol{d}}$. The property in (19) then follows from the monotonicity of the max operator. □

Lemma B.3. *The exponential Bellman operators $L^{\boldsymbol{d}}$, $\forall \boldsymbol{d} \in \mathcal{D}$ and $L^{\star}$ are continuous.*

Proof. The lemma follows directly from the continuity of linear operators and from the fact that the pointwise maximum of a finite number of continuous functions is continuous. See also [Patek, 2001, lemma 5] □

B.3   Proof of Theorem 3.1

Proof of Theorem 3.1. To construct the value function, we can define a Bellman operator $T^{\boldsymbol{d}} \colon \mathbb{R}^S \to \mathbb{R}^S$ for any decision rule $\boldsymbol{d} \colon \mathcal{S} \to \Delta_A$ and the optimal Bellman operator $T^{\star} \colon \mathbb{R}^S \to \mathbb{R}^S$ for a *value vector* $\boldsymbol{v} \in \mathbb{R}^S$ as

$$\begin{aligned}
(T^{\boldsymbol{d}}\boldsymbol{v})_s &:= \mathrm{ERM}_\beta^{\boldsymbol{d},s}\left[r(s, \tilde{a}_0, \tilde{s}_1) + v_{\tilde{s}_1} \right], \\
T^{\star}\boldsymbol{v} &:= \max_{\boldsymbol{d} \in \mathcal{D}} T^{\boldsymbol{d}}\boldsymbol{v} = \max_{\boldsymbol{d} \in \mathrm{ext}\,\mathcal{D}} T^{\boldsymbol{d}}\boldsymbol{v}.
\end{aligned} \tag{20}$$

It is easy to see that $\boldsymbol{d}$ can be chosen independently for each state to maximize $\boldsymbol{v}$ uniformly across states. The optimality of deterministic decision rules, $\boldsymbol{d} \in \mathrm{ext}\,\mathcal{D}$, follows because ERM is a mixture quasi-convex function [Delage et al., 2019].

The existence of value function for finite-horizon problem under the ERM objective has been analyzed previously [Hau et al., 2023b] including in the context of exponential utility functions [Chung and Sobel, 1987].

To derive the exponential Bellman operator for the exponential value function for $\boldsymbol{d}\colon \mathcal{S} \to \Delta_A$ we concatenate the Bellman operator with the transformations to and from the exponential value function:

$$
\begin{aligned}
(L^{\boldsymbol{d}}\boldsymbol{w})_s &:= -\exp\left(-\beta \cdot T^{\boldsymbol{d}}(-\beta^{-1}\log(-\boldsymbol{w}))\right) \\
&= -\mathbb{E}^{\boldsymbol{d},s}\left[\exp\left(-\beta \cdot r(s, \tilde{a}_0, \tilde{s}_1) + \log(-w_{\tilde{s}_1})\right)\right] \\
&= -\mathbb{E}^{\boldsymbol{d},s}\left[\exp\left(-\beta \cdot r(s, \tilde{a}_0, \tilde{s}_1)\right) \cdot (-w_{\tilde{s}_1})\right] \\
&= \sum_{s' \in \bar{\mathcal{S}}}\sum_{a \in \mathcal{A}} p(s, a, s') \cdot d_a(s) \cdot \exp\left(-\beta \cdot r(s, a, s')\right) \cdot w_{s'} \\
&= \sum_{s' \in \mathcal{S}}\sum_{a \in \mathcal{A}} p(s, a, s') \cdot d_a(s) \cdot \exp\left(-\beta \cdot r(s, a, s')\right) \cdot w_{s'} \\
&\qquad\qquad - \sum_{a \in \mathcal{A}} p(s, a, e) \cdot d_a(s) \cdot \exp\left(-\beta \cdot r(s, a, e)\right).
\end{aligned}
\tag{21}
$$

The derivation above uses the fact that $w_e = -1$ since $v_e = 0$ by definition. The statement of the theorem then follows by algebraic manipulation of $\boldsymbol{B}^{\boldsymbol{d}}, \boldsymbol{b}^{\boldsymbol{d}}$ and by induction on t. The base case hold by the definition of $\boldsymbol{w}^0(\pi) = \boldsymbol{w}^{0,\star} = -\boldsymbol{1}$.

The existence of an optimal $\pi^\star$ follows by choosing the maximum in the definition of $L^\star$ which is attained by compactness and continuity of the objective. $\qquad\square$

B.4 Proof of Theorem 3.2

Lemma B.4. *Assume some $\pi = (\boldsymbol{d})_\infty \in \Pi_{\mathrm{SR}}$ such that $\rho(\boldsymbol{B}^{\boldsymbol{d}}) < 1$. Then for all $\boldsymbol{z} \in \mathbb{R}^S$*

$$
\boldsymbol{w}^\infty(\pi) = \boldsymbol{w}^\infty(\pi, \boldsymbol{z}) = L^{\boldsymbol{d}}\boldsymbol{w} > -\infty.
$$

Proof. The result follows by algebraic manipulation from (12) and basic matrix analysis. When $\rho(\boldsymbol{B}^{\boldsymbol{d}}) < 1$, we get from Neumann series [Horn and Johnson, 2013, problem 5.6.P26]

$$
\sum_{k=0}^{t-1}(\boldsymbol{B}^{\boldsymbol{d}})^k \boldsymbol{b}^{\boldsymbol{d}} = (\boldsymbol{I} - \boldsymbol{B}^{\boldsymbol{d}})^{-1}\boldsymbol{b}^{\boldsymbol{d}}
$$

and a consequence of Gelfand's formula [Kallenberg, 2021, theorem 4.5]

$$
\lim_{k \to \infty}(\boldsymbol{B}^{\boldsymbol{d}})^k \boldsymbol{z} = \boldsymbol{0}.
$$

$\qquad\square$

Proof of Theorem 3.2. When $\boldsymbol{\mu}^\top \boldsymbol{v}^{\infty,\star} = -\infty$ then the result follows immediately because $-\infty = \sup_{\pi \in \Pi_{\mathrm{MR}}} \boldsymbol{\mu}^\top \boldsymbol{v}^\infty(\pi) \geq \max_{\pi \in \Pi_{\mathrm{SR}}} \boldsymbol{\mu}^\top \boldsymbol{v}^\infty(\pi) \geq -\infty$.

For the remainder of the proof, suppose that $\boldsymbol{\mu}^\top \boldsymbol{v}^\infty(\pi_{\mathrm{M}}^\star) > -\infty$. Then, the exponential value function $\boldsymbol{w}^{t,\star} = \boldsymbol{w}^t(\pi_{\mathrm{M}}) \in \mathbb{R}^S$ of π_{M} satisfies by Theorem 3.1 that

$$
\boldsymbol{w}^{0,\star} = -\boldsymbol{1}, \qquad \boldsymbol{w}^t(\pi_{\mathrm{M}}^\star) = L^\star \boldsymbol{w}^{t-1}(\pi_{\mathrm{M}}^\star), \quad t = 1, \ldots.
$$

We show that $\lim_{t \to \infty}\boldsymbol{w}^{t,\star}$ exists and that it is attained by a stationary policy. We construct a sequence $\boldsymbol{w}_{\mathrm{u}}^t \in \mathbb{R}^S, t = 0, \ldots$ as

$$
\boldsymbol{w}_{\mathrm{u}}^0 = \boldsymbol{0}, \qquad \boldsymbol{w}_{\mathrm{u}}^t = L^\star \boldsymbol{w}_{\mathrm{u}}^{t-1}, \quad t = 1, \ldots.
$$

First, we show by induction that

$$
\boldsymbol{w}_{\mathrm{u}}^t \quad \geq \quad \boldsymbol{w}^{t,\star}, \qquad t = 0, \ldots. \tag{22}
$$

The base case $t = 0$ follows immediately from the definitions of $\boldsymbol{w}_{\mathrm{u}}^0$ and $\boldsymbol{w}^{0,\star}$. Next, suppose that (22) holds for some $t > 0$, then it also holds for $t + 1$:

$$
\boldsymbol{w}_{\mathrm{u}}^{t+1} \quad = \quad L^\star \boldsymbol{w}_{\mathrm{u}}^t \quad \geq \quad L^\star \boldsymbol{w}^{t,\star} \quad = \quad \boldsymbol{w}^{t+1,\star},
$$

12

where the inequality follows from the inductive assumption and from Lemma B.2. Second, we show by induction that

$$
\boldsymbol{w}_{\mathrm{u}}^{t+1} \quad \leq \quad \boldsymbol{w}_{\mathrm{u}}^{t}, \qquad t = 0, \dots . \tag{23}
$$

The base case for $t = 0$ holds as

$$
\boldsymbol{w}_{\mathrm{u}}^{1} \;=\; L^{\star}\boldsymbol{w}_{\mathrm{u}}^{0} \;=\; L^{\star}\boldsymbol{0} \;=\; \max_{\boldsymbol{d}\in\mathcal{D}} -\boldsymbol{b}^{\boldsymbol{d}} \;\leq\; \boldsymbol{0} \;=\; \boldsymbol{w}_{\mathrm{u}}^{0},
$$

where the inequality holds because $\boldsymbol{b}^{\boldsymbol{d}} \geq 0$ from its construction. To prove the inductive step, assume that (23) holds for $t > 0$ and prove it for $t + 1$:

$$
\boldsymbol{w}_{\mathrm{u}}^{t+1} \;=\; L^{\star}\boldsymbol{w}_{\mathrm{u}}^{t} \;\leq\; L^{\star}\boldsymbol{w}_{\mathrm{u}}^{t-1} \;=\; \boldsymbol{w}_{\mathrm{u}}^{t},
$$

where the inequality follows from the inductive assumption and Lemma B.2.

Then, using the Monotone Convergence Theorem [Johnsonbaugh and Pfaffenberger, 1981, theorem 16.2], finite $\mathcal{S}$, and $\inf_{t=0,\dots} \boldsymbol{w}_{\mathrm{u}}^{t} \geq \inf_{t=0,\dots} \boldsymbol{w}^{t,\star} > -\infty$, we get that there exists $\boldsymbol{w}_{\mathrm{u}}^{\star} \in \mathbb{R}^{S}$ such that

$$
\boldsymbol{w}_{\mathrm{u}}^{\star} = \lim_{t\to\infty} \boldsymbol{w}_{\mathrm{u}}^{t},
$$

and the limit exists. Then, taking the limit of both sides of $\boldsymbol{w}_{\mathrm{u}}^{t} = L^{\star}\boldsymbol{w}_{\mathrm{u}}^{t-1}$, we have that

$$
\lim_{t\to\infty} \boldsymbol{w}_{\mathrm{u}}^{t} = \lim_{t\to\infty} L^{\star}\boldsymbol{w}_{\mathrm{u}}^{t-1}
$$
$$
\boldsymbol{w}_{\mathrm{u}}^{\star} = \lim_{t\to\infty} L^{\star}\boldsymbol{w}_{\mathrm{u}}^{t-1}
$$
$$
\boldsymbol{w}_{\mathrm{u}}^{\star} = L^{\star} \lim_{t\to\infty} \boldsymbol{w}_{\mathrm{u}}^{t-1}
$$
$$
\boldsymbol{w}_{\mathrm{u}}^{\star} = L^{\boldsymbol{d}^{\star}} \boldsymbol{w}_{\mathrm{u}}^{\star},
$$

where $\boldsymbol{d}^{\star} = \operatorname{argmax}_{\boldsymbol{d}\in\mathcal{D}} L^{\boldsymbol{d}}\boldsymbol{w}_{\mathrm{u}}$. Above, we can exchange the operators $L^{\star}$ and $\lim$ by the continuity of $L^{\star}$ (Lemma B.3).

Now, define $\boldsymbol{w}_{\mathrm{l}}^{t} \in \mathbb{R}^{S}, t = 0, \dots$ as

$$
\boldsymbol{w}_{\mathrm{l}}^{0} = -\boldsymbol{1}, \qquad \boldsymbol{w}_{\mathrm{l}}^{t} = L^{\boldsymbol{d}^{\star}} \boldsymbol{w}_{\mathrm{l}}^{t-1}, \quad t = 1, \dots .
$$

From the definition of $L^{\star}$ and by induction on t we have that

$$
\boldsymbol{w}_{\mathrm{l}}^{t} \leq \boldsymbol{w}^{t,\star}.
$$

By Lemma 3.6 for $\boldsymbol{z} = \boldsymbol{0}$, we have that $\rho(\boldsymbol{B}^{\boldsymbol{d}^{\star}}) < 1$ and therefore from Lemma B.4

$$
\lim_{t\to\infty} \boldsymbol{w}_{\mathrm{l}}^{t} = \lim_{t\to\infty} \boldsymbol{w}_{\mathrm{u}}^{t} = \boldsymbol{w}_{\mathrm{u}}^{\star}.
$$

In addition, because

$$
\boldsymbol{w}_{\mathrm{u}}^{t} \;\geq\; \boldsymbol{w}^{t,\star} \;\geq\; \boldsymbol{w}_{\mathrm{l}}^{t}, \qquad t = 0, \dots
$$

The Squeeze Theorem [Johnsonbaugh and Pfaffenberger, 1981, theorem 14.3] then shows that

$$
\lim_{t\to\infty} \boldsymbol{w}^{t,\star} = \boldsymbol{w}_{\mathrm{u}}^{\star},
$$

and $\boldsymbol{d}^{\star}$ is a stationary policy that attains the return of $\pi_{\mathrm{M}}^{\star}$. $\qquad\square$

B.5 Proof of Corollary 3.3

Proof of Corollary 3.3. From the existence of an optimal stationary policy $\pi^{\star} \in \Pi_{\mathrm{SD}}$ from for a sufficiently large horizon t from Theorem 3.2 and Appendix B.1, we get that

$$
\boldsymbol{\mu}^{\top}\boldsymbol{v}^{\infty}(\pi^{\star}) \leq \sup_{\pi\in\Pi_{\mathrm{HR}}} \liminf_{t\to\infty} \boldsymbol{\mu}^{\top}\boldsymbol{v}^{t}(\pi) \leq \liminf_{t\to\infty} \sup_{\pi\in\Pi_{\mathrm{HR}}^{t}} \boldsymbol{\mu}^{\top}\boldsymbol{v}^{t}(\pi) \leq \boldsymbol{\mu}^{\top}\boldsymbol{v}^{\infty}(\pi^{\star}),
$$

which implies that all inequalities above hold with equality. $\qquad\square$
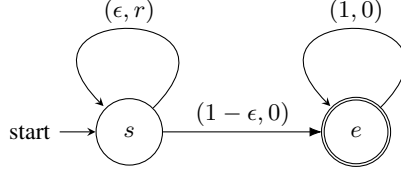
Figure 2: A transient MDP used to show that risk-averse TRC may be unbounded/bounded.

B.6 proof of Proposition 3.4

Example 1. Consider the transient MDP in Figure 2 with $\mathcal{S} = \{s\}$, $\mathcal{A} = \{a\}$, $\mu(s) = 1$, and e is the sink state. Only one action a is available in both states. The rewards are $r(s, a, s) = r$, $r(s, a, e) = 0$, $r(e, a, e) = 0$. The transition probabilities are $p(s, a, s) = \epsilon$, $p(s, a, e) = 1 - \epsilon$, and $p(e, a, e) = 1$. The tuple on the edge represents the transition probability and reward respectively.

For the transient MDP in Figure 2, let us derive a general equation shown in (24) to compute ERM value functions and $\tilde{\chi}$ is the random variable of return.

$$
\begin{aligned}
&\mathrm{ERM}_\beta^\pi \left[\tilde{\chi} \right] \\
&= -\frac{1}{\beta} \log(\mathbb{E}[\exp(-\beta \cdot \chi)]) \\
&= -\frac{1}{\beta} \log((1 - \epsilon) \cdot \epsilon \cdot \exp(-\beta r) + (1 - \epsilon) \cdot \epsilon^2 \cdot \exp(-\beta 2r) + \cdots + (1 - \epsilon) \cdot \epsilon^n \cdot \exp(-\beta n r)) \\
&= -\frac{1}{\beta} \log((1 - \epsilon) \cdot \epsilon \cdot \exp(-\beta r) + (1 - \epsilon) \cdot \epsilon^2 \cdot (\exp(-\beta r))^2 + \cdots + (1 - \epsilon) \cdot \epsilon^n \cdot (\exp(-\beta r))^n) \\
&= -\frac{1}{\beta} \log((1 - \epsilon) \cdot \epsilon \cdot \exp(-\beta r) + (1 - \epsilon) \cdot (\epsilon \exp(-\beta r))^2 + \cdots + (1 - \epsilon) \cdot (\epsilon \exp(-\beta r))^n) \\
&= -\frac{1}{\beta} \log((1 - \epsilon) \cdot \epsilon \cdot \exp(-\beta r) \frac{1 - (\epsilon \exp(-\beta r))^{n+1}}{1 - \epsilon \exp(-\beta r)})
\end{aligned}
$$

(24)

In Example 1, when $r = -1$ and $\beta \geq log(\frac{1}{\epsilon})$, we have $\epsilon \cdot \exp(-\beta r) > 1$, then $\mathrm{ERM}_\beta^\pi \left[\tilde{\chi} \right] = -\infty$ for the policy $\pi : s \mapsto a, \forall s \in \mathcal{S}$.

B.7 Proof of Lemma 3.5

The result follows from an identical argument to [Patek and Bertsekas, 1999, lemma 1].

B.8 Proof of Lemma 3.6

We use $\boldsymbol{p^d} \in \mathbb{R}_+^S$ to represent the probability of terminating in any state for each $\boldsymbol{d} \in \mathcal{D}$:

$$
p_s^d = \sum_{a \in \mathcal{A}} d_a(s) \cdot \bar{p}(s, a, e), \quad \forall s \in \mathcal{S}.
$$

The following lemma establishes a convenient representation of the termination probabilities.

Lemma B.5. *Assume a transient MDP and a policy $\pi = (\boldsymbol{d})_\infty \in \Pi_{\mathrm{SR}}$. Then, the probability of termination in $t = 0, 1, \ldots$ or fewer steps is*

$$
\sum_{k=0}^{t} \boldsymbol{\mu}^\top (\boldsymbol{P^d})^k \boldsymbol{p^d} = \boldsymbol{\mu}^\top (\boldsymbol{I} - (\boldsymbol{P^d})^{t+1}) \mathbf{1}.
$$

Proof. We have by algebraic manipulation that

$$
\boldsymbol{p^d} = (\boldsymbol{I} - \boldsymbol{P}) \mathbf{1}.
$$

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The probability of terminating in step $t \geq 0$ exactly is

$$\boldsymbol{\mu}^\top (\boldsymbol{P^d})^t \boldsymbol{p^d}.$$

Using algebraic manipulation and recognizing a telescopic sum, we have that the probability of terminating in $k \leq t$ steps is

$$\sum_{k=0}^{t} \boldsymbol{\mu}^\top (\boldsymbol{P^d})^k \boldsymbol{p^d} = \sum_{k=0}^{t} \boldsymbol{\mu}^\top (\boldsymbol{P^d})^k (\boldsymbol{I} - \boldsymbol{P^d}) \mathbf{1} = \boldsymbol{\mu}^\top (\boldsymbol{I} - (\boldsymbol{P^d})^{t+1}) \mathbf{1}.$$

$\square$

Lemma B.6. *For any $\boldsymbol{d} \in \mathcal{D}$, the exponential transition matrix is monotone:*

$$\boldsymbol{x} \geq \boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{B^d x} \geq \boldsymbol{B^d y}, \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^S.$$

Proof. The result follows immediately from the fact that $\boldsymbol{B^d}$ is a non-nonnegative matrix. $\square$

Lemma B.7. *For each $t = 0, \dots$ and each policy $\pi = (\boldsymbol{d})_\infty \in \Pi_{\mathrm{SR}}$ and each $\boldsymbol{\mu} \in \Delta_S$:*

$$\boldsymbol{\mu}^\top (\boldsymbol{B^d})^t \boldsymbol{b^d} = 0 \qquad \Leftrightarrow \qquad \boldsymbol{\mu}^\top (\boldsymbol{P^d})^t \boldsymbol{p^d} = 0. \tag{25}$$

Proof. Algebraic manipulation from the definition in (10) shows that

$$\begin{array}{rcccl}
c_\mathrm{l} \cdot \boldsymbol{p^d} & \leq & \boldsymbol{b^d} & \leq & c_\mathrm{u} \cdot \boldsymbol{p^d}, \\
c_\mathrm{l} \cdot \boldsymbol{P^d x} & \leq & \boldsymbol{B^d x} & \leq & c_\mathrm{u} \cdot \boldsymbol{P^d x}, \qquad \forall \boldsymbol{x} \in \mathbb{R}^S,
\end{array} \tag{26}$$

where

$$c_\mathrm{l} := \min_{s, s' \in \bar{\mathcal{S}}, a \in \mathcal{A}} \exp\left(-\beta \cdot r(s, a, s')\right), \qquad c_\mathrm{u} := \max_{s, s' \in \bar{\mathcal{S}}, a \in \mathcal{A}} \exp\left(-\beta \cdot r(s, a, s')\right).$$

Note that $\infty > c_\mathrm{u} > c_\mathrm{l} > 0$.

We now extend the inequalities in (26) to multiple time steps. Suppose that $\boldsymbol{y}_\mathrm{l} \leq \boldsymbol{y} \leq \boldsymbol{y}_\mathrm{u}$, then, for $t = 0, 1, \dots$:

$$c_\mathrm{l}^t \cdot (\boldsymbol{P^d})^t \boldsymbol{y}_\mathrm{l} \; \leq \; (\boldsymbol{B^d})^t \boldsymbol{y} \; \leq \; c_\mathrm{u}^t \cdot (\boldsymbol{P^d})^t \boldsymbol{y}_\mathrm{u}. \tag{27}$$

For the left inequality in (27), the induction proceeds as follows. The base case $t = 0$ holds immediately. For the inductive step, suppose that the left inequality in (27) property holds for $t = 0, \dots$ then it also holds for $t + 1$ for each $\boldsymbol{y} \in \mathbb{R}^S$ as

$$(\boldsymbol{B^d})^{t+1} \boldsymbol{y} \; = \; \boldsymbol{B^d}(\boldsymbol{B^d})^t \boldsymbol{y} \; \geq \; c_\mathrm{l}^t \boldsymbol{B^d}(\boldsymbol{P^d})^t \boldsymbol{y}_\mathrm{l} \; \geq \; c_\mathrm{l}^{t+1} \boldsymbol{P^d}(\boldsymbol{P^d})^t \boldsymbol{y}_\mathrm{l} \; = \; c_\mathrm{l}^{t+1}(\boldsymbol{P^d})^{t+1} \boldsymbol{y}_\mathrm{l}.$$

Above, the first inequality follows from Lemma B.6 and from the inductive assumption, and the second inequality follows from (27) by setting $\boldsymbol{x} = \boldsymbol{P^d y}$. The right inequality in (27) follows analogously.

Exploiting the fact that $\boldsymbol{\mu} \geq \mathbf{0}$ and substituting $\boldsymbol{y} = \boldsymbol{b^d}$, $\boldsymbol{y}_\mathrm{l} = c_\mathrm{l} \cdot \boldsymbol{p^d}$, $\boldsymbol{y}_\mathrm{u} = c_\mathrm{u} \cdot \boldsymbol{p^d}$ into (27) and using the bounds in (26), we get that

$$0 \; \leq \; c_\mathrm{l}^{t+1} \cdot \boldsymbol{\mu}^\top (\boldsymbol{P^d})^t \boldsymbol{p^d} \; \leq \; \boldsymbol{\mu}^\top (\boldsymbol{B^d})^t \boldsymbol{b^d} \; \leq \; c_\mathrm{u}^{t+1} \cdot \boldsymbol{\mu}^\top (\boldsymbol{P^d})^t \boldsymbol{p^d},$$

where the terms are non-negative because all constants, matrices, and vectors are non-negative. Therefore,

$$\boldsymbol{\mu}^\top (\boldsymbol{B^d})^t \boldsymbol{b^d} = 0 \quad \Leftrightarrow \quad \boldsymbol{\mu}^\top (\boldsymbol{P^d})^t \boldsymbol{p^d} = 0.$$

$\square$

Lemma B.8. *Assume a transient MDP and a $\pi = (\boldsymbol{d})_\infty \in \Pi_{\mathrm{SR}}$. Then there exists $\boldsymbol{f} \in \mathbb{R}^S$ such that $\boldsymbol{f}^\top \boldsymbol{B^d} = \rho(\boldsymbol{B^d}) \cdot \boldsymbol{f}^\top$ and $\boldsymbol{f} \geq \mathbf{0}, \boldsymbol{f} \neq \mathbf{0}$ and*

$$\boldsymbol{f}^\top \boldsymbol{b^d} > 0.$$

Proof. Because $\boldsymbol{B^d}$ is non-negative, the required vector $\boldsymbol{f}$ exists from the Perron-Frobenius theorem, e.g. [Horn and Johnson, 2013, Theorem 8.3.1]. Therefore,

$$\boldsymbol{f}^\top \boldsymbol{b^d} \geq \boldsymbol{0},$$

since $\boldsymbol{b^d} \geq \boldsymbol{0}$.

It remains to show that $\boldsymbol{f}^\top \boldsymbol{b^d} \neq 0$, which we do by deriving a contradiction. Without loss of generality, assume that $\boldsymbol{1}^\top \boldsymbol{f} = 1$ and suppose that $\boldsymbol{f}^\top \boldsymbol{b^d} = 0$. Then:

$$
\begin{aligned}
\boldsymbol{f}^\top \boldsymbol{b^d} &= 0 \\
\boldsymbol{f}^\top (\boldsymbol{B^d})^t \boldsymbol{b^d} &= 0, \forall t = 0, 1, \dots && \Downarrow \text{ from } \boldsymbol{f}^\top \boldsymbol{B^d} = \rho(\boldsymbol{B^d}) \cdot \boldsymbol{f}^\top \\
\boldsymbol{f}^\top (\boldsymbol{P^d})^t \boldsymbol{p^d} &= 0, \forall t = 0, 1, \dots && \Downarrow \text{ from Lemma B.7} \\
\sum_{k=0}^{t} \boldsymbol{f}^\top (\boldsymbol{P^d})^k \boldsymbol{p^d} &= 0, \forall t = 0, 1, \dots && \Downarrow \text{ by summing elements} \\
\boldsymbol{f}^\top (\boldsymbol{I} - (\boldsymbol{P^d})^{t+1}) \boldsymbol{1} &= 0, \forall t = 0, 1, \dots && \Downarrow \text{ from Lemma B.5} \\
\lim_{t \to \infty} \boldsymbol{f}^\top (\boldsymbol{I} - (\boldsymbol{P^d})^{t+1}) \boldsymbol{1} &= 0, && \Downarrow \text{ limit} \\
\boldsymbol{f}^\top \boldsymbol{1} &= 0, && \Downarrow \text{ from Lemma 2.2}
\end{aligned}
$$

which is a contradiction with $\boldsymbol{1}^\top \boldsymbol{f} \neq 0$. The last step in the derivation follows from $\rho(\boldsymbol{P^d}) < 1$ and therefore $\lim_{t \to \infty} (\boldsymbol{P^d})^{t+1} = \boldsymbol{0}$ [Kallenberg, 2021, Theorem 4.5]. $\square$

Proof of Lemma 3.6. From Lemma B.8, there exists an $\boldsymbol{f} \in \mathbb{R}_+^S$ that $\boldsymbol{f}^\top \boldsymbol{B^d} = \rho(\boldsymbol{B^d}) \cdot \boldsymbol{f}^\top$ and $\boldsymbol{f} \geq \boldsymbol{0}, \boldsymbol{f} \neq \boldsymbol{0}$. Then from (12):

$$-\infty < \boldsymbol{\mu}^\top \boldsymbol{w}^t(\pi) = -\boldsymbol{\mu}^\top (\boldsymbol{B^d})^t \boldsymbol{z} - \boldsymbol{\mu}^\top \sum_{k=0}^{t-1} (\boldsymbol{B^d})^k \boldsymbol{b} \leq - \sum_{k=0}^{t-1} \rho(\boldsymbol{B^d})^k \boldsymbol{\mu}^\top \boldsymbol{b}.$$

The second inequality follows because $\boldsymbol{z} \geq \boldsymbol{0}$ and $\boldsymbol{B^d}$ is non-negative. Since $\boldsymbol{\mu}^\top \boldsymbol{b} > 0$ from Lemma B.8, we can cancel it from the inequality getting that

$$\sum_{k=0}^{t-1} \rho(\boldsymbol{B^d})^k < \infty.$$

Then $\rho(\boldsymbol{B^d}) < 1$ because $\rho(\boldsymbol{B^d}) \geq 0$ the geometric series is bounded. $\square$

C   Proofs for Section 4

C.1   Proof of Theorem 4.1

Proof. Assume an approximation error $\delta \geq 0$, let us define a function $\psi(\beta, \pi) = \lim_{T \to \infty} \mathrm{ERM}_\beta^{\pi, \mu} \left[\sum_{t=0}^{T} r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right]$. when $\sup_{\beta > 0} h(\beta)$ is attained, do some algebraic manip-

ulation, we have

$$\max_{\pi \in \Pi_{\mathrm{SD}}} \mathrm{EVaR}_\alpha^{\pi,\mu} \left[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right]$$

$$= \max_{\pi \in \Pi_{\mathrm{SD}}} \sup_{\beta > 0} (\psi(\beta, \pi) + \beta^{-1} \cdot \log(\alpha))$$

$$\leq \sup_{\beta > 0} \max_{\pi \in \Pi_{\mathrm{SD}}} (\psi(\beta, \pi) + \beta^{-1} \cdot \log(\alpha))$$

$$= \sup_{\beta > 0} h(\beta)$$

$$\leq h(\beta^\star) + \delta$$

$$\leq \psi(\beta^\star, \pi^\star) + {\beta^\star}^{-1} \cdot \log(\alpha) + \delta$$

$$\leq \sup_{\beta > 0} (\psi(\beta, \pi^\star) + \beta^{-1} \cdot \log(\alpha)) + \delta$$

$$= \mathrm{EVaR}_\alpha^{\pi^\star,\mu} \left[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] + \delta$$

The second case is that $\sup_{\beta > 0} h(\beta)$ is not attained, then $\sup_{\beta > 0} \psi(\beta, \pi) + \beta^{-1} \cdot \log(\alpha)$ is also not attained for any optimal policy $\hat{\pi} \in \arg\max_{\pi \in \Pi_{\mathrm{SD}}} \mathrm{EVaR}_\beta^{\pi,\mu}[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1})]$. We prove the statement by contradiction. Suppose that there exists an optimal policy $\hat{\pi}$ and a $\hat{\beta}$ such that

$$\psi(\hat{\beta}, \hat{\pi}) + \hat{\beta}^{-1} \cdot \log(\alpha) = \max_{\pi \in \Pi_{\mathrm{SD}}} \sup_{\beta > 0} (\psi(\beta, \pi) + \beta^{-1} \cdot \log(\alpha))$$

Note that

$$\mathrm{EVaR}_\alpha^{\hat{\pi},\mu} \left[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] = \sup_{\beta > 0} (\psi(\beta, \hat{\pi}) + \beta^{-1} \cdot \log(\alpha)) \leq \sup_{\beta > 0} h(\beta) \qquad (28)$$

There are two possibilities: equality and strict inequality for (28). When the equality holds, then

$$\mathrm{EVaR}_\alpha^{\hat{\pi},\mu} \left[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] = \sup_{\beta > 0} h(\beta)$$

in which case $\hat{\beta}$ attains the supremum. When the strict inequality holds, then there exists a $\bar{\beta}(\bar{\beta} \neq \hat{\beta})$ and a policy $\bar{\pi} \in \arg\max_{\pi \in \Pi_{\mathrm{SD}}} (\psi(\bar{\beta}, \pi) + \bar{\beta}^{-1} \cdot \log(\alpha)), \bar{\pi} \neq \hat{\pi}$ such that

$$\mathrm{EVaR}_\alpha^{\bar{\pi},\mu} \left[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] > \mathrm{EVaR}_\alpha^{\hat{\pi},\mu} \left[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right]$$

which contradicts the optimality of $\hat{\pi}$.

If $\sup_{\beta > 0} (\psi(\beta, \hat{\pi}) + \beta^{-1} \cdot \log(\alpha))$ is unattained for some $\hat{\pi} \in \Pi_{\mathrm{SD}}$, then according to proposition 2.11 in [Ahmadi-Javid and Pichler, 2017], we have

$$\mathrm{EVaR}_\alpha^{\hat{\pi},\mu} \left[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] = \mathrm{ess\,inf} \left[\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right]$$

Therefore, $\pi^\star \in \arg\max_{\pi \in \Pi_{\mathrm{SD}}} \mathrm{ess\,inf} [\sum_{t=0}^\infty r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1})]$ is the optimal policy for EVaR-TRC objective. $\qquad \square$

C.2 Transient MDP construction

In the risk-neutral setting, it is well-known that the discounted objective can be interpreted as TRC. The discounted infinite-horizon objective for a factor $\gamma \in (0, 1)$ is [Puterman, 2005]

$$\max_{\pi \in \Pi_{\mathrm{SD}}} \rho_\gamma(\pi) := \mathbb{E}^{\pi,\mu} \left[\sum_{t=0}^\infty \gamma^t \cdot r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right]. \qquad (29)$$
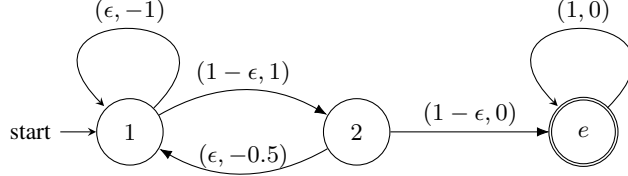
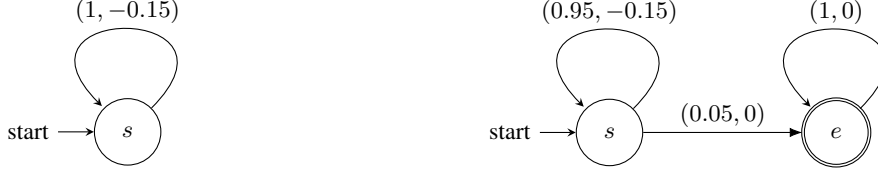Figure 3: Transient MDP without an equivalent discounted MDP

Figure 4: A discounted MDP with reward -0.15   Figure 5: A transient MDP converted from Figure 4

Here, $\tilde{s}_t$ and $\tilde{a}_t$ are random variables for the state $\tilde{s}_t$ and action $\tilde{a}_t$ at time t distributed according to the transition probabilities p. The superscript of $\mathbb{E}$ denotes the policy that governs the distribution of $\tilde{a}_t$ and the distribution over the initial state $\tilde{s}_0$. We replace the distribution by a specific state $s \in \mathcal{S}$ when $\mu_s = 1$.

We describe how to construct a transient MDP from a discounted MDP as follows.

Remark 1 (Transient MDP construction). Given an MDP $\mathcal{M} = (\mathcal{S}, \mathcal{A}, p, r, \mu)$ and a discount factor $\gamma \in [0, 1)$, construct an MDP $\bar{\mathcal{M}}_\gamma = (\bar{\mathcal{S}}, \bar{\mathcal{A}}, \bar{p}, \bar{r}, \bar{\mu})$ such that $\bar{\mathcal{S}} = \mathcal{S} \cup \{g\}$, $\bar{\mathcal{A}} = \mathcal{A}$, and $\bar{\mu}(s) = \mu(s), \forall s \in \mathcal{S}$ and $\mu(g) = 0$. The transition function is defined as

$$\bar{p}(s, a, s') = \begin{cases} \gamma \cdot p(s, a, s') & \text{if} \quad s, s' \in \mathcal{S}, a \in \mathcal{A}, \\ 1 - \gamma & \text{if} \quad s \in \mathcal{S}, s' = g, a \in \mathcal{A}, \\ 1 & \text{if} \quad s = s' = g, a \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

When the rewards $r \colon \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ are independent of the next state, then $\bar{r} \colon \bar{\mathcal{S}} \times \bar{\mathcal{A}} \to \mathbb{R}$ are defined as $\bar{r}(s, a) = r(s, a)$ when $s \in \mathcal{S}, a \in \mathcal{A}$ and $r(s, a) = 0$ otherwise. The model can be readily extended to account for the target state dependence by constructing an $\bar{\mathcal{M}}_\gamma$ with a random reward function.

It is well-known that discounted MDPs can be reduced to TRCs [Altman, 1998, Section 1.10]. The construction can readily be shown using standard dynamic programming techniques to satisfy the following property [Feinberg and Huang, 2019].

Proposition C.1. *For each MDP $\mathcal{M}$, discount factor $\gamma \in [0, 1)$, and $\pi \in \Pi_{\mathrm{SD}}$ we have that*

$$\rho_\gamma(\pi, \mathcal{M}) = \rho(\bar{\pi}, \bar{\mathcal{M}}_\gamma),$$

where ρ_γ, ρ are model-dependent and $\bar{\pi}$ extends π to $\bar{\mathcal{S}}$.

C.3   Examples

Example 2. For the transient MDP in Figure 3, $\mathcal{S} = \{1, 2\}$, $\mathcal{A} = \{a\}$, $\mu(s) = 1$, and e is the sink state. The parameters are $r(1, a, 1) = -1$, $r(1, a, 2) = 1$, $r(2, a, 1) = -0.5$, $r(2, a, e) = 0$, $r(e, a, e) = 0$, $p(1, a, 1) = \epsilon$, $p(1, a, 2) = 1 - \epsilon$, $p(2, a, 1) = \epsilon$, $p(2, a, e) = 1 - \epsilon$, $p(e, a, e) = 1$ and $\epsilon \in [0, 1]$. In state 1, the discount factor is $1 - (1-\epsilon)^2$, and in state 2, the discount factor is ϵ. When $1 - (1-\epsilon)^2 \neq \epsilon$, there is no equivalent discounted MDP because the future rewards at state 1 and 2 are discounted differently.

For Example 3 and Example 4, we compute ERM value function for Figure 5 and Figure 7 using Equation (24). $\tilde{\chi}$ is the random variable of return, $\epsilon = p(s, a, s)$, $r = r(s, a, s)$ and β is the risk level of ERM.

Example 3. Consider a discounted MDP $\mathcal{M}$ with discount factor $\gamma = 0.95$ in Figure 4. $\mathcal{S} = \{s\}$, $\mathcal{A} = \{a\}$. The transition probabilities are $p(s, a, s) = 1$, and the reward is $r(s, a, s) = -0.15$. Given
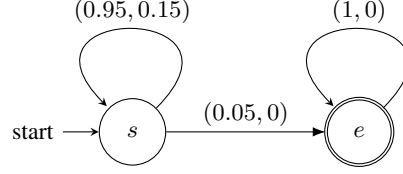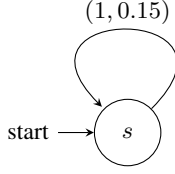
Figure 6: A discounted MDP with reward 0.15   Figure 7: A transient MDP converted from Figure 6

$\alpha = 0.9$ and $\delta = 0.0001$, we generate a sequence of $\{\beta_k\}_{k=1}^K$ using Equation (17) and compute the ERM and EVaR returns.

The only policy π is to take action a at each time step, and π is deterministic. Because the policy is deterministic, for any risk level β ,as $n \to \infty$, $\mathrm{ERM}_\beta^\pi[\tilde{\chi}] = \mathbb{E}[\tilde{\chi}] = -3$. The EVaR value is -3, and the optimal β value is 1053.6.

Convert the discounted MDP in Figure 4 to a transient MDP $\mathcal{M}_\gamma$ in Figure 5 by Remark 1.With TRC criterion, the EVaR value is -4.07, and the optimal β value is 0.09. When $\beta^\star = 0.09$, we have $\rho(\boldsymbol{B^d}) = \epsilon \exp(-\beta r) = 0.95 \exp(-0.09(-0.15)) = 0.96 < 1$, as $n \to \infty$, $\mathrm{ERM}_\beta^\pi[\tilde{\chi}] = -2.9$. Therefore, $\rho_\gamma(\pi, \mathcal{M}) \neq \rho(\pi, \mathcal{M}_\gamma)$.

Example 4. We have a discounted MDP $\mathcal{M}$ with discount factor $\gamma = 0.95$ in Figure 6. $\mathcal{S} = \{s\}$, $\mathcal{A} = \{a\}$. The transition probabilities are $p(s, a, s) = 1$, and the reward is $r(s, a, s) = 0.15$. Given $\alpha = 0.9$ and $\delta = 0.0001$, we generate a sequence of $\{\beta_k\}_{k=1}^K$ using Equation (17) and compute the ERM and EVaR returns.

The only policy π is to take action a at each time step. Because the policy is deterministic, for any risk level β, as $n \to \infty$, $\mathrm{ERM}_\beta^\pi[\tilde{\chi}] = \mathbb{E}[\tilde{\chi}] = 3$.T he EVaR value is 3, and the optimal β value is 1053.6.

Convert the discounted MDP in Figure 6 to a transient MDP $\mathcal{M}_\gamma$ in Figure 7 by Remark 1. With TRC criterion, the EVaR value is 2.14, and the optimal β value is 0.142. When $\beta^\star = 0.09$, we have $\rho(\boldsymbol{B^d}) = \epsilon \exp(-\beta r) = 0.95 \exp(0.142(-0.1)) = 0.929 < 1$, as $n \to \infty$, $\mathrm{ERM}_\beta^\pi[\tilde{\chi}] = 2.88$.