BIL: Bandit Inference Learning for Online Representational Similarity Test

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Abstract

Similarity analysis is commonly used to determine the size of the discrepancy between two representations of a distribution pattern. In contrast to classical representational similarity analysis, which identifies disparate types of representations based on their shared similarity structures in distance matrices, this article proposes an online hypothesis testing procedure that will be able to determine whether a representation's difference from a constant is more significant than a predefined margin for streaming data. As a basic reinforcement learning model, two-armed bandits (TAB) are used to construct test statistics that update online. To achieve the most efficient testing results, an optimal strategy is developed for the TAB process. Asymptotic test statistics are discussed in theory, as are its corresponding explicit density functions, which are more accumulated than the normal distribution commonly applied in classical statistical analysis. Since the power of the proposed representative similarity test (RST) method is higher than that of the classical test, simulation studies support the validity of the proposed method.

1 Introduction

Similarity analysis has been widely used in some scientific fields, such as biological pharmacy aims at checking the biosimilarity between an innovative biological product and a reference product that share similar pharmacokinetic or pharmacodynamic profiles; transfer or meta learning transforms similar information from the source domain to the target domain through similarity analysis; representational similarity analysis (RSA) usually focuses on integrating disparate types of data based on shared structure in their similarity (or distance) matrices. This article first considers a statistical inference of similarity test with streaming data set continuously generated from different sources, such as file data generated by mobile or web applications, online shopping data and telemetry data from connected devices in the data center, etc. Such data should be processed incrementally using stream processing techniques. Stream processing analyzes and performs actions on real-time data through the use of continuous queries, which enables the applications to update and integrate the processed information.

Recently, some literature explore the stream processing techniques for data mining by the tool of statistical inference. The popularly used methods among them includes aggregated estimating equation (Lin & Xi, 2011), cumulatively updating estimating equation (Schifano et al., 2016) and renewable estimator (Luo & Song, 2020). Furthermore, the diffusion approximation method to analyze the exact dynamics of online regression was proposed by Fan et al. (2018). An online inference in high-dimensional linear models with streaming data was developed by Han et al. (2021), Deshpande et al. (2021) and Sun & Barbu (2021), specially the variable selection in high-dimensional linear model based on the continuously updated statistics was argued by (Sun & Barbu, 2021). An novel inference framework for high-dimensional generalized linear models by recursive online-score estimation was developed by Shi et al. (2021). Stochastic gradient descent algorithm (SGD), as currently popular stream processing technique, has been widely used in various fields of artificial intelligence as well as a prototype of online learning algorithms (Chen et al., 2020; Zhu et al., 2021). And a variant of the truncated SGD in online settings was proposed by Liu et al. (2022), Langford et al. (2009).

However, above stream processing techniques focus on online learning procedure to conduct statistical inference, but no consideration of similarity analysis. Meanwhile, all studies must use the historical information to update the current estimation, which needs detailed deduction for the updated statistics in case of wrong convergence. This article admits this general challenge but turn to open a novel and simple path. Specifically, we just obtain a sequence of estimators and summarize the batch-specific estimators by simple average-based aggregation and additional strategy, which tells us how to integrate all sequential estimators. The proposed strategic aggregation is essentially motivated by the "two armed bandit" (TAB) problem, as the prototype of a slot TAB machine (Bellman, 1956) with two arms L(left) and R(right), which behaves like two working models under two treatment groups. Classical TAB is a special type of sequential random sampling-based clinical trials (Feldman, 1962) where one would need to select one treatment from several treatments to be used for the next patient based on the performances already observed. Popular bandit models include Bayesian bandit of bandit with the mathematical formulation (Bradt et al., 1956; Gittins et al., 2011), the Markovian bandits (Gittins, 1979); Bayesian framework examples (Whittle, 1988); Frequentist setup of bandits can refer to Robbins (1952), the extended frequentist examples contain but not limited to the non-i.i.d. rewards (Perchet & Rigollet, 2013), the combinatorial bandit problem (Chen et al., 2013), contextual multi-armed bandit (Chen et al., 2021), and upper-confidence-bound strategy problem by Lai et al. (1985). All applications of bandit model in a wide range of areas including clinical trials, biological modelling, data processing, internet, and machine learning, see examples Thompson (1933); Sutton & Barto (2018); Lattimore & Szepesvári (2020); Slivkins et al. (2019); And Jacko (2019) is based on the sequential dataset where researchers would need to select one treatment (or action) from several treatments to be used for the next patient based on the performances already observed.

This article devises the similarity analysis by a test in (1) aiming to examine how far one target task away from a goal (i.e., the constant c), which is meaningful in online learning for streaming data, because if we have enough confidence to believe that the goal has been achieved, we can stop the task in advance. We use models with interpretability, such as individual feature-based methods (linear regression), supervised models (logistic regression), unsupervised models, which learn the similarity of the representations and accordingly improve the interpretability of the subsequent algorithms. In terms of algorithm, this work proposes strategic aggregation learning to summarize all sequential estimators averagely by a strategy designed by TAB process, where we use the the estimator and its opposite to denote the respective reward of arm L or arm R at each time, i.e, playing arm L(left) to obtain the estimator and achieve its opposite form by playing arm R(right). The device of optimal strategy can refer to Chen et al. (2022), because they propose a asymptotic distribution for the strategically and averagely aggregated statistics and studied that the strategy must be goal-specific under a statistical issue, such as estimation or hypothesis testing. Therefore, the task of this work is the considered one-sided test and aims at explore an optimal strategy for the aggregated test statistic to attain the most power or lowest type I error.

We list the main contributions and the merits as follows:

- This is the first consideration about the combination of knowledge-driven and data-driven learning procedures based on a simple reinforcement learning model to construct the test statistic.
- The proposed online learning is actually speedy, because we summarize the online updates by the simple average-aggregation and a strategy without access to the historical data.
- The reconstructed representational similarity testing framework is general, because it includes all models of statistical learning.
- The developed representational similarity test statistic improves the testing power largely than classical normal distribution-based test statistic.

2 Methodology

2.1 Online Representational Similarity Test

This article considers a online RST problem for streaming data with a sequence of B data batches, each arrives sequentially at time point $b = 1, \dots, B$ and the corresponding batch-specific sample size is n_b. Denote the b-th data batch as $D_b = \{Z_{bi}, i = 1, \dots, n_b\}$. Unlike the classical offline studies, the total sample size $N = \sum_{b=1}^{B} n_b$ can grow to infinity as B increases and break the constraint of data storage capacity. Therefore, it may not be feasible to store all the historical data up to each time point, instead, we aim to figure out a way to store the information in historical data with an acceptable storage cost and be able to update the result according to the current arriving data batch.

The online RST problem in this article would like to evaluate how far one task away from the the goal the experimenter aims at attaining. Let θ be the true parameter for the representation population of the task, and c is a known constant denoting pre-specified goal, where the goal c is not smaller than the true value θ. Then we consider the following RST:

$$\mathbf{H}_0 : c-\theta \geq d_0 \quad \mathbf{H}_1 : 0 < c-\theta < d_0 \tag{1}$$

where $d_0 \geq 0$ is a given equivalence margin to determine the distance between the unknown θ and the goal c. Certainly, the null hypothesis $\mathbf{H}_0$ is also equivalent to $\theta \leq c - d_0$, but the reason that we still use the translational form $c - \theta \geq d_0$ is the technical consideration about the strategic construction of the test statistic as shown in (2) and (3). The constructed RST is meaningful for the task of streaming data, because we can stop the online learning procedure in advance if we have enough confidence to accept a minimum equivalent margin d_0 with a significant level α.

2.2 Bandit Inference Learning

Different from classical online learning concentrating on accumulating the historical information into one statistics, bandit process as a simple model of reinforcement learning introduces a strategic variable remembering historical information to decide how to integrate current estimation into the previously summarize statistics, which designs a decision rule for online learning and brings a simple and speed computation.

Specifically, let $\widehat{\theta}_b$ be the unbiased estimator of θ based on the data set D_b and we use the TAB process to play the arm L to obtain the estimator (i.e., the reward) $c - \widehat{\theta}_b$ or the arm R to achieve its opposite $\widehat{\theta}_b - c$, which can been denoted by

$$\widehat{Z}_b^{\xi} = \begin{cases} \widehat{W}_b^L = c - \widehat{\theta}_b, \text{ if } \vartheta_b = 1. \\ \widehat{W}_b^R = \widehat{\theta}_b - c, \text{ if } \vartheta_b = 2. \end{cases} \tag{2}$$

Then the TAB process generates a sequence of statistics (i.e., the observed rewards) $\{\widehat{Z}_1^{\xi}, \cdots, \widehat{Z}_B^{\xi}\}$ under the strategy $\xi = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_B\}$, where ϑ_b is constructed as $\vartheta_b = 2 - I\{T_{b-1}^{\xi} \leq 0\}$ considered by Chen et al. (2022). The proposed bandit inference learning aggregates the statistics $\{\widehat{Z}_1^{\xi}, \cdots, \widehat{Z}_B^{\xi}\}$ under the defined strategy ξ by the statistics T_b^{ξ} (Chen et al., 2022):

$$T_b^{\xi} = \frac{1}{B}\sum_{l=1}^{b} \widehat{Z}_l^{\xi} + \frac{1}{\sqrt{B}}\sum_{l=1}^{b} \frac{\widehat{Z}_l^{\xi} - \mu_l^{\xi}}{\widehat{\sigma}_l}, \quad 1 \leq b \leq B. \tag{3}$$

where $\mu_l^{\xi} = I(\vartheta_l = 1)(d_0) + I(\vartheta_l = 2)(-d_0)$ and $\widehat{\sigma}_l^2 = \frac{1}{l}\sum_{j=1}^{l} \widehat{\mathrm{var}}(\widehat{\theta}_j)$ that is a valid estimator of σ^2 . T_b^{ξ} can be regarded as a historical statistics determining the decision rule to construct the online statistics under the strategy ξ. Intuitively, if T_{b-1}^{ξ} is positive (negative), TAB process applies the arm R to generate a negative value with large probability and enforce the test statistic at next time point b to be close to 0. Next we proceed to state how to update the above test statistic construction sequentially with historical information properly stored in terms of summary statistics, which is popularly used in current online learning (Schifano et al., 2016; Luo & Song, 2020). Considering the structure of T_b^{ξ}, we define two summary statistics S_1^b and S_2^b:

$$S_1^b = \sum_{l=1}^{b} \widehat{Z}_l^{\xi} = S_1^{b-1} + \widehat{Z}_b^{\xi}, \quad S_2^b = \sum_{l=1}^{b}\left(\widehat{Z}_l^{\xi} - \mu_l^{\xi}\right)/\widehat{\sigma}_l = S_2^{b-1} + (\widehat{Z}_b^{\xi} - \mu_b^{\xi})/\widehat{\sigma}_b. \tag{4}$$

The online statistics T_b^{ξ} is updated as follows:

$$T_b^{\xi} = \frac{1}{B}S_1^b + \frac{1}{\sqrt{B}}S_2^{b-1} = T_{b-1}^{\xi} + \frac{1}{B}\left\{\widehat{Z}_b^{\xi} + \frac{\sqrt{B}(\widehat{Z}_b^{\xi} - \mu_b^{\xi})}{\widehat{\sigma}_b}\right\} \tag{5}$$

Then our statistics T_b^ξ can be constructed sequentially as follows. When a new data batch $D_b, b \geq 2$ arrives, we calculate the current reward $\widehat{Z}_b^\xi$ and estimator $\widehat{\sigma}_b$ based on the current data set D_b, so T_b^ξ can be updated by (5). The current strategy $\vartheta_{b+1} = 2 - I\{T_b^\xi \leq 0\}$ can be got too.

And then ϑ_{b+1} determines the next reward $\widehat{Z}_{b+1}^\xi$ and the margin μ_{b+1}^ξ the null hypothesis used. Therefore the proposed online strategic test statistic in (5) aligns with the classical online accumulation framework, as it only requires the currently batch-specific estimator and the previously stored statistics T_{b-1}^ξ to form the current statistics T_b^ξ. Then our final test statistic is

$$T_B^\xi = \frac{1}{B}\sum_{l=1}^{B} \widehat{Z}_l^\xi + \frac{1}{\sqrt{B}}\sum_{l=1}^{B} \frac{\widehat{Z}_l^\xi - \mu_l^\xi}{\widehat{\sigma}_l}$$

and the proposed bandit inference learning for obtaining this final test statistic can be displayed as follows:

Algorithm 1 Test Statistic T_B^ξ of Bandit Inference Learning

Input: Sequential data D_b ($b = 1, \ldots, B$), d_0, c.
Output: T_B^ξ

```
Set T_0^ξ = 0
for b = 1, ..., B do
    Obtaining a estimator θ̂_b using data set D_b
    Set reward function Ŵ_b^L = c − θ̂_b and Ŵ_b^R = θ̂_b − c
    if T_{b−1}^ξ ≤ 0 then
        Ẑ_b^ξ = Ŵ_b^L and μ_b^ξ = d_0. Update T_{b−1}^ξ to T_b^ξ based on (5).
    else
        Ẑ_b^ξ = Ŵ_b^R and μ_b^ξ = −d_0. Update T_{b−1}^ξ to T_b^ξ based on (5).
    end if
end for
return T_B^ξ
```

Next we use two practical examples to introduce the representational similarity test construction and the proposed bandit inference learning procedure. The related simulation results of the two examples are shown in the Appendix.

2.3 Some Examples

Example 2.1 (Unsupervised learning) Based on this example, we can determine whether the distribution of independent samples from the data stream is close to a constant. Suppose that the observation is $\{Y_{bi}\}, i = 1, \ldots, n_b, b = 1, \ldots, B$ with distribution function F_0. The testing problem can be expressed as

$$\mathbf{H}_0 : F_0(y) - c > d_0 \quad \mathbf{H}_1 : 0 < c - F_0(y) < d_0$$

given the time point $b, b = 1, \ldots, B$, the natural estimators of F_0 is given by $\widehat{\theta}_b = \widehat{F}_b(y) = \sum_{i=1}^{n_b} I(Y_{bi} \leq y)/n_b$ and the variance can be estimated by $\widehat{\sigma}_b = \widehat{F}_0(y)\left(1 - \widehat{F}_0(y)\right)/n_b$.

Example 2.2 (Supervised learning) Suppose that the observations arrive sequentially i.e. $\{\boldsymbol{Z}_{bi}\}$, $i = 1, \ldots, n_b$, $b = 1, \ldots, B$ at a time point b where $\boldsymbol{Z}_{bi} = (Y_{bi}, \boldsymbol{X}_{bi}^\top)^\top$. $Y_{bi} \in \{0, 1\}$ denotes the response variable, $\boldsymbol{X}_{bi} \in \mathbb{R}^p$ stands for the associated covariates. We consider the logistic regression model:

$$P(Y = 1|\boldsymbol{X}) = g(\boldsymbol{X}^\top\boldsymbol{\beta}) = \exp(\boldsymbol{X}^\top\boldsymbol{\beta})/\{1 + \exp(\boldsymbol{X}^\top\boldsymbol{\beta})\},$$

with $\boldsymbol{\beta} \in \mathbb{R}^p$. Then, we test the value $P(Y_{bi} = 1|\boldsymbol{X}_{bi} = \boldsymbol{x}) = g(\boldsymbol{x}^\top\boldsymbol{\beta})$ ($\boldsymbol{x}$ is a constant when the value is determined), i.e.

$$\mathbf{H}_0 : g(\boldsymbol{x}^\top\boldsymbol{\beta}) - c > d_0 \qquad \mathbf{H}_1 : 0 < c - g(\boldsymbol{x}^\top\boldsymbol{\beta}) < d_0.$$

The natural estimators of $g(\boldsymbol{x}^\top \boldsymbol{\beta})$ is given by $\widehat{g}(\boldsymbol{x}^\top \widehat{\boldsymbol{\beta}}_b)$ with the corresponding MLE estimators $\widehat{\boldsymbol{\beta}}_b$ based on the current bth data batch $\mathcal{D}_b$. Moreover, for the bth data batch, $\mathrm{var}(g(\boldsymbol{x}^\top \widehat{\boldsymbol{\beta}}_b))$ can be approximated by $\{\exp \boldsymbol{x}^\top \widehat{\boldsymbol{\beta}}_b\}^2/\{1+\exp(\boldsymbol{x}^\top \widehat{\boldsymbol{\beta}}_b)\}^4 \mathrm{var}(\boldsymbol{x}^\top \widehat{\boldsymbol{\beta}}_b)$ with

$$\mathrm{var}(\boldsymbol{x}^\top \widehat{\boldsymbol{\beta}}_b) = \boldsymbol{x}^\top \left(\sum_{i=1}^{n_b} \{1 - g(\boldsymbol{X}_{bi}^\top \widehat{\boldsymbol{\beta}}_b)\}^2 \boldsymbol{X}_{bi} \boldsymbol{X}_{bi}^\top \right)^{-1} \boldsymbol{x}.$$

3 Theoretical Results

3.1 Asymptotic Distribution Under the Null Hypothesis

The design of T_b^ξ in (3) combines the expressions of the large number law and classical central limit theorem has been proved to be a key technique to explore the central limit theorem because of the simultaneous consideration of location and scale. More details can refer to Peng (2008) and Chen & Epstein (2022). Despite using the strategy-driven limit theorem in this paper, which refers to Chen & Epstein (2022), it is not necessary to consider the center of symmetry of the function. Additionally, we skillfully construct an opposite reward function based on randomness so that the proposed strategy is incorporates the "knowledge" from the null hypothesis, which is that the expectation of the left arm under our proposed strategy is greater than 0 when the null hypothesis is true. We have applied it for online representative similarity test problem under streaming data. This is the novelty of Theorem 3.1.

THEOREM 3.1 *Let $\varphi \in C(\overline{\mathbb{R}})$ be the set of all continuous functions on $\mathbb{R}$ with finite limits at $\pm\infty$, a even function and monotone on $(0, \infty)$. We have*

$$\lim_{B\to\infty} \left\{ E\left[\varphi\left(T_B^\xi\right)\right] - E\left[\varphi\left(\sigma_d \eta_B\right)\right] \right\} = 0 \tag{6}$$

and

$$\alpha_B = \sqrt{B}\left(d_0 - (c-\theta)\right)/\sigma - (c-\theta), \tag{7}$$
$$\sigma_d = \sqrt{1 + \left((c-\theta) - d_0\right)^2/\sigma^2}.$$

where $\eta_B \sim \mathcal{S}(\alpha_B, 0)$. If $Y \sim \mathcal{S}(\kappa, 0)$, Y has the density function

$$f^\kappa(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(|y|-\kappa)^2}{2}} - \kappa e^{2\kappa|y|} \Phi(-|y| - \kappa).$$

According to Chen et al. (2022), the distribution of $\mathcal{S}(\kappa, 0)$ has the following conclusions:

(1) If $\kappa < 0$, the image of this distribution is more spike than the corresponding normal distribution.

(2) The distribution is similar to two normal distributions hand in hand under $\kappa > 0$.

(3) The distribution becomes the standard normal distribution with $\kappa = 0$. In particular, for any $b \in \mathbb{R}$, we have

$$\lim_{B\to\infty} P\left(\left|T_B^\xi\right| \le b\right) = \lim_{B\to\infty} \left[\Phi(\alpha_B + \frac{b}{\sigma_d}) - e^{\frac{2\alpha_B b}{\sigma_d}} \Phi(\alpha_B - \frac{b}{\sigma_d})\right]. \tag{8}$$

Specially, when $c - \theta = d_0$, we have $\lim_{B\to\infty}\{E\left[\varphi(T_B^\xi)\right] - E\left[\varphi(\eta_B)\right]\} = 0$, where $\eta_B \sim \mathcal{S}(-d_0, 0)$, which is spike with $\kappa = -d_0 < 0$. For any $0 < \alpha < 1/2$, let $z_{\alpha/2}$ be critical value of distribution $\mathcal{S}(-d_0, 0)$ and satisfies $\lim_{B\to\infty} P\left(\left|T_B^\xi\right| > z_{\alpha/2}\right) = \alpha$, then $z_{\alpha/2}$ can be calculated from the following equation

$$\Phi\left(z_{\alpha/2} + d_0\right) - e^{-2d_0 z_{\alpha/2}} \Phi\left(-z_{\alpha/2} + d_0\right) = 1 - \alpha \tag{9}$$

where Φ denotes the distribution function of the standard normal distribution.

Under the hypothesis $\mathbf{H}_0$, i.e. $c - \theta \ge d_0$, we conclude that the limit distribution of T_B^ξ becomes more spike than $S(-d_0, 0)$ and $\mathbf{H}_0$ will be rejected at the significance level α if the condition $(|T_B^\xi| > z_{\alpha/2})$ is satisfied.

3.2 Asymptotic Distribution under the Alternative Hypothesis

To check the power of the proposed test statistic, we consider the alternative hypothesis $\mathbf{H}_1$ with $c - \theta < d_0$ in the following corollary. The proposed test statistic improves the testing power largely than classical normal distribution-based test, which is theoretically examined and illustrated in the Corollary 3.1 and intuitively showed in Figure 2.

COROLLARY 3.1 *Under the same assumptions as Theorem 3.1, we have results:*

Under the hypothesis $\mathbf{H}_1$ i.e. $c - \theta < d_0$, let $d = c - \theta$ below. For a fixed large enough B, from (7), we know $\alpha_B > 0$ and the limit distribution of T_B^{ξ} is a binormal distribution i.e. the distribution is similar to two normal distributions hand in hand. The power of this test is given by $P(|T_B^{\xi}| > z_{\alpha/2} \mid \mathbf{H}_1)$, which can be approximately calculated by

$$1 - \gamma_1 = P(|T_B^{\xi}| > z_{\alpha/2} \mid \mathbf{H}_1) \approx 1 - \Phi(\frac{z_{\alpha/2}}{\sigma_d} - \alpha_B) + e^{\frac{2\alpha_B z_{\alpha/2}}{\sigma_d}} \Phi(-\frac{z_{\alpha/2}}{\sigma_d} - \alpha_B)$$

where

$$\alpha_B = \sqrt{B}(d_0 - d)/\sigma - d, \sigma_d = \sqrt{1 + (d - d_0)^2/\sigma^2}.$$

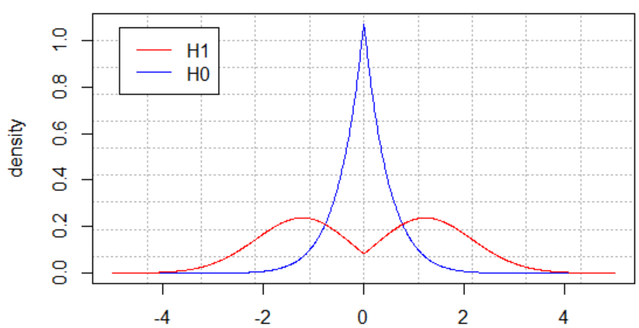


Figure 1: The blue line denotes the density plot of distribution $\mathcal{S}(-1, 0)$. The red line denotes the induced distribution of test statistic T_B^{ξ} with $d_0 = 1, d = 0.5, B = 9$ and $\sigma = 1$ on the correctness of alternative hypothesis $\mathbf{H}_1$ in Corollary 3.1.

According to the traditional method of hypothesis testing, one usually uses the strategy $\xi' = \{1, 1, 1, \cdots\}$ to obtain a sequence of data $\{Z_1^{\xi'}, Z_2^{\xi'}, \cdots\}$. The test statistic is

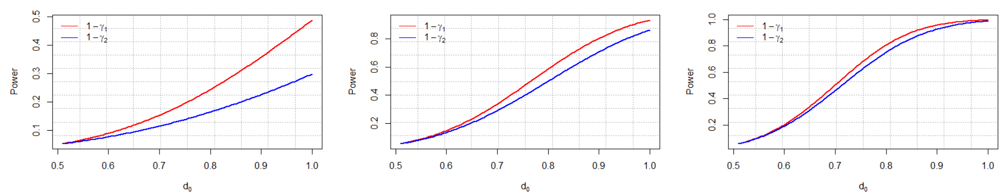
$$M_B^{\xi'} = \frac{1}{\sqrt{B}} \sum_{l=1}^{B} \frac{(Z_l^{\xi'} - d_0)}{\widehat{\sigma}_l}$$

Given a significance level $\alpha > 0$, the occurrence of $M_B^{\xi'} < -u_\alpha$ for large enough B will lead to the rejection of $\mathbf{H}_0$ at the significance level α where $\Phi(u_\alpha) = 1 - \alpha(u_\alpha > 0)$. Denote $d = c - \theta$, then for a fixed large enough B, the distributions of $M_B^{\xi'}$ are similar to $\mathcal{N}(0, 1) + (d - d_0)\sqrt{B}/\sigma$, that is,

$$1 - \gamma_2 = P(M_B^{\xi'} < -u_\alpha \mid \mathbf{H}_1) \approx \Phi(\frac{d_0 - d}{\sigma}\sqrt{B} - u_\alpha)$$

Correspondingly, if $\mathbf{H}_1$ is true, the limit distribution of T_B^{ξ} has two peak with the rejection region $(-\infty, -z_{\alpha/2}) \cup (z_{\alpha/2}, \infty)$, inducing a larger power than the distribution of $M_B^{\xi'}$. The following simulation shows the property.

Under different values of d_0 with $d = 0.5$ and $\sigma = 1$, Figure 2 plots the approximated powers of $1 - \gamma_1$ and $1 - \gamma_2$ corresponding to the proposed strategy test statistic and classical normal-distribution based test statistic respectively, which confirms the power improvement.



(a) $B = 10$ (b) $B = 30$ (c) $B = 60$

Figure 2: The approximated power of the proposed test statistic T_B^{ξ} and classical normal distribution based test statistic $M_B^{\xi'}$, i.e., $1 - \gamma_1$ and $1 - \gamma_2$ respectively, are shown above under different values of d_0 with $d = 0.5$ and $\sigma = 1$.

4 Simulation Studies

4.1 Data Generation Design

The paper contributes to the development of a generalized procedure, which can be applied to a variety of complex models, for inferring statistical information from streaming data. In this section, numerical experiments of linear regression model are conducted to check the similarity testing performance of the proposed strategic statistics, as well as examining the correctness of the concluded theoretical results and other arguments.

Specifically, by the following simulations, we aim to test:

1.Whether the constructed statistics T_B^{ξ} based streaming data can convergent in distribution to the induced distribution;

2.Whether the proposed test statistics can enhance the testing power in any settings compared with classical normal distribution-based test statistic (normal statistics (Kang & Kim, 2014)).

Example 4.1 In this example, we consider mean model with

$$Y_i = \theta + \epsilon_i$$

where ϵ_i from normal distribution with zero mean and $E{\epsilon_i}^2 = \sigma^2$, so $\theta = E(Y_i)$, i.e. $c - \theta = c - E(Y_i)$ and $\mathrm{var}(Y_i) = \sigma^2$.

Next, we generate B group-specific samples $(D_1, D_2, \cdots, D_B)$ under the policy $\xi = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_B\}$ as $\vartheta_b = 2 - I\{T_{b-1}^{\xi} \leq 0\}$. And the b-th streaming data samples are denoted as $D_b = \{Y_{bi} : i = 1, \cdots, n_b\}$.

$$\widehat{Z}_b^{\xi} = \begin{cases} \widehat{W}_b^L = c - \sum_{i=1}^{n_b} Y_{bi}/n_b, \text{ if } \vartheta_b = 1. \\ \widehat{W}_b^R = \sum_{i=1}^{n_b} Y_{bi}/n_b - c, \text{ if } \vartheta_b = 2. \end{cases}$$

Then we generate the test statistic T_b^{ξ} by

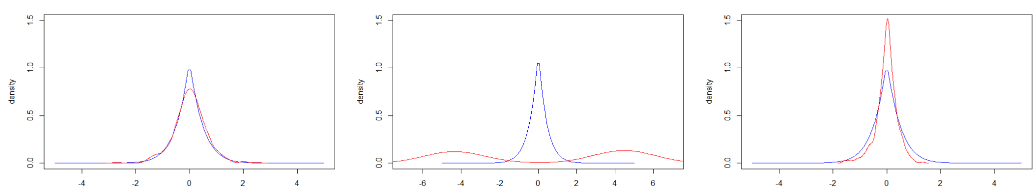
$$T_b^{\xi} = \frac{1}{B} \sum_{l=1}^{b} \widehat{Z}_l^{\xi} + \frac{1}{\sqrt{B}} \sum_{l=1}^{b} \frac{\widehat{Z}_l^{\xi} - \mu_l^{\xi}}{\widehat{\sigma}_l}, \quad 1 \leq b \leq B,$$

where $\mu_l^{\xi} = I(\vartheta_l = 1)(d_0) + I(\vartheta_l = 2)(-d_0)$, $\widehat{\sigma}_l^2 = \sum_{i=1}^{n_l} (Y_{li} - \widehat{\theta}_l)^2/n_l$, and $\widehat{\theta}_l = \sum_{i=1}^{n_l} Y_{li}/n_l$. Without loss of generality, in simulation we can use the same sample size across each data batch, that is, $n_b \equiv n$ for $b = 1, \ldots, B$.

4.2 Evaluation of the Results

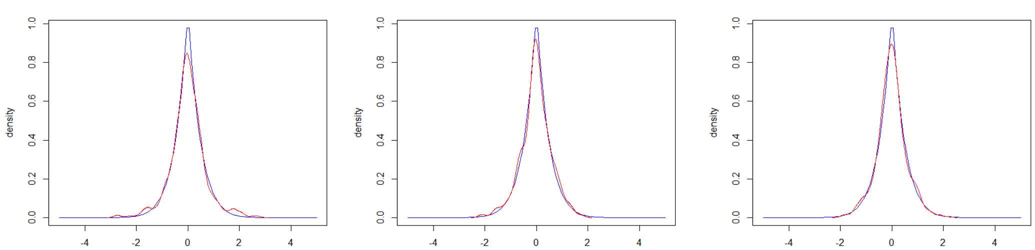
Next, we analyze the simulated results of Examples 4.1. With $c - \theta = d_0$, Figure 3 (a) elucidates that the density plot of T_B^{ξ} after 500 replicates nearly approaches that of the distribution $\mathcal{S}(-d_0, 0)$, implying the correctness of the deduced asymptotic distribution in Theorem 3.1 under different settings of $d_0 = 1$; with $c - \theta < d_0$, Figure 3 (b) elucidates that the limit distribution of T_B^{ξ} is a binormal distribution; with $c - \theta > d_0$, Figure 3 (c) elucidates that the limit distribution of T_B^{ξ} is more spike than the distribution $S(-d_0, 0)$. With $c - \theta = d_0$, Figure 4 elucidates how the empirical distribution of the test statistic T_B^{ξ} changes as the sample size of each data batch n increases, in terms of the density plot of the asymptotic distribution consistently approaches the distribution $\mathcal{S}(-d_0, 0)$.

Tables 1 reports the estimated proportions $P(|T_B^{\xi}| > z_{\alpha/2})$ after 500 replicates under the considered critical value $z_{\alpha/2}$ with significant level $\alpha = 0.05$, which implies that: the estimated proportions $P(|T_B^{\xi}| > z_{\alpha/2})$ can consistently approach that critical value solved by the (9) at the selected significant level α and it further shows proposed test statistic is distributed asymptotically with the distribution $\mathcal{S}(-d_0, 0)$;



(a) $d_0 = 1$ (b) $d_0 = 1.1$ (c) $d_0 = 0.99$

Figure 3: The estimated density plot of the statistic T_B^{ξ} after 500 replicates (the red line) and the density plot of distribution $\mathcal{S}(-d_0, 0)$ (the blue line) are shown above under different d_0 in Example 4.1 with $c = 0.5$, $\theta = -0.5$, $n = 100$, $\sigma = 1$. Specifically, in the plot (a), we have $c - \theta = d_0$, $B = 5000$; in the plot (b), $c - \theta < d_0$, $B = 25$; in the plot (c), $c - \theta > d_0$, $B = 5000$.



(a) $n = 20$ (b) $n = 50$ (c) $n = 200$

Figure 4: The estimated density plot of the statistic T_B^{ξ} after 500 replicates (the red line) and the density plot of distribution $\mathcal{S}(-d_0, 0)$ (the blue line) are shown above under different sample size n in Example 4.1 with $c = 0.5$, $\theta = -0.5$, $d_0 = c - \theta = 1$, $\sigma = 1$, $B = 5000$.

Table 1: The null rejection ratio(or size) of testing statistic under 5% significance level in Example 4.1: The estimated propositions $P(|T_B^{\xi}| > z_{\alpha/2})$ under 500 replications in Example 4.1 with $B = 5000$, $c = 0.5$.

n	θ	$\sigma = 1$	$\sigma = 0.5$
	-0.5	0.052	0.058
50	-1	0.056	0.052
	-1.5	0.058	0.066
	-0.5	0.046	0.060
100	-1	0.056	0.064
	-1.5	0.056	0.060

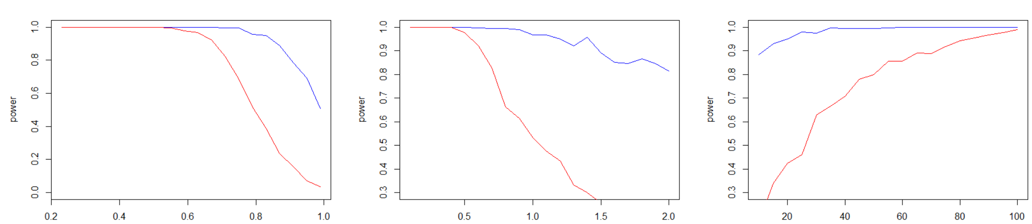
Then, we check the power performance of the proposed test statistic compared with normal statistic (Kang & Kim, 2014) in Example 4.1. We consider three designs including various θ corresponding to different alternative hypothesis $\mathbf{H}_1$, distinct values of standard deviations of error σ, and various sample size n, and present the simulated results of power performance in Figures 5.

We summarize the following conclusions:

1.Conditional on the alternative hypothesis $\mathbf{H}_1$ being true with $d_0 = 2$, $c = 1$, $n = 100$ and $\sigma = 1$, the estimated probability of rejecting $\mathbf{H}_0$ holds in large values along with various values of $-\theta$ under the proposed test statistic in Figure 5 (a), whilst the normal statistic behaves worse along with value between θ and c of $\mathbf{H}_1$ close to d_0;

2.Figure 5 (b) shows a robust performance of proposed online strategy inference about the various values of σ under hypothesis with $c = 1, \theta = -1$ and $n = 100$;

3.Figure 5 (c) elucidates that the test statistic is still powerful in the small group with a little observations under hypothesis with $c = 1, \theta = -1$ and $\sigma = 0.5$.



(a) $-\theta$ (b) σ (c) n

Figure 5: The power plots of the proposed test statistic T_B^{ξ} (the blue line) and normal statistic (the red line) in Example 4.1 are shown above under various expectation θ (a), distinct variance σ (b) and various sample size n (c).

5 Discussion

In this paper, we consider an representational similarity test for online learning and utilize the idea of TAB model to construct an online updating test statistic. The best testing performance is attained by making an optimal strategy by the TAB process. Our proposed method is shown to be more powerful than existing methods based on normal distributed test statistic via theoretical results and numerical experiments. Intuitively, we gain the statistical power by using a test statistic with more accumulated density function. Our simulation study and real data analysis demonstrate that the proposed estimator outperforms normal statistic (Kang & Kim, 2014) or other online-updated estimators in terms of lower type I and II errors. The developed online-updating strategic test statistic and inferences are applicable for all statistical models.

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A Appendix

The following numerical results show the simulated performance of the strategic test statistic in unsupervised learning and supervised learning models, i.e. Example 2.1 and Example 2.2. Figures 6 and 7 display the images of the distribution of the proposed strategic test statistic T_B^{ξ} under different sample size compared with the truly induced distribution $\mathcal{S}(-d_0, 0)$ in the Theorem 3.1, which implies that the empirical distribution (red line) consistently approximate the population distribution (blue line) of test statistic. Table 2 and 3 show the estimated significant level of the test statistic in Examples 2.1 and Examples 2.2 under the true significance level 5%.

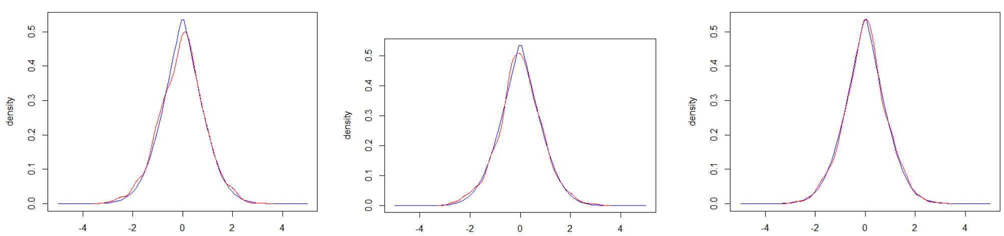
Table 2: The null rejection ratio (or size) of testing statistic under 5% significance level in Example 2.1: The estimated propositions $P\left(\left|T_B^{\xi}\right| > z_{\alpha/2}\right)$ under 500 replications in Example 2.1 with $c = 0.5$, $B = 1000$. $F_0(y)$ is normal distribution function with a mean of μ and a variance of σ_0.

n	μ	$\sigma_0 = 2$	$\sigma_0 = 1$
50	0	0.053	0.055
	-0.5	0.053	0.049
	-1	0.067	0.067
100	0	0.053	0.051
	-0.5	0.058	0.051
	-1	0.047	0.057

.

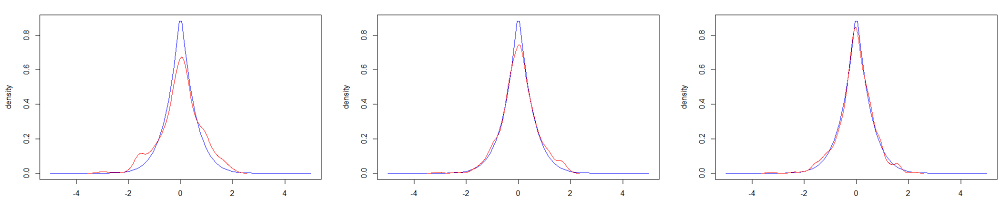
Table 3: The null rejection ratio (or size) of testing statistic under 5% significance level in Example 2.2: The estimated propositions $P\left(\left|T_B^{\xi}\right| > z_{\alpha/2}\right)$ under 500 replications in Example 2.2 with $c = 0.1$, $B = 1000$, $\boldsymbol{\beta} = (\mu, 2), \boldsymbol{x} = (1, 1)$.

n	μ	$\boldsymbol{\beta} = (\mu, 2)$
	1	0.050
800	1.2	0.048
	1.4	0.058
	1	0.068
1000	1.2	0.052
	1.4	0.068



(a) $n = 50$ (b) $n = 100$ (c) $n = 200$

Figure 6: The estimated density plot of the statistic T_B^{ξ} after 500 replicates (the red line) and the density plot of distribution $\mathcal{S}(-d_0, 0)$ (the blue line) are shown above under different sample size n in Example 2.1 with $c = 0.5$, $B = 1000$, $F_0(x)$ is normal distribution function with a mean of -1.5 and a variance of 2.



(a) $n = 500$ (b) $n = 800$ (c) $n = 1000$

Figure 7: The estimated density plot of the statistic T_B^{ξ} after 500 replicates (the red line) and the density plot of distribution $\mathcal{S}(-d_0, 0)$ (the blue line) are shown above under different sample size n in Example 2.2 with $c = 0.1$, $B = 1000$.

Proof of Theorem 3.1: Let $\{B_s\}_{s\geq 0}$ be the standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_s^*)_{s\geq 0}$ be the natural filtration generated by $\{B_s\}_{s\geq 0}$. For any integer $m \geq 1$, let $C_b^m(\mathbb{R})$ denote the set of functions on $\mathbb{R}$ that have bounded derivatives up to order m. Let $\varphi \in C_b^3(\mathbb{R})$ be a even function, for any $\alpha \in \mathbb{R}, \beta > 0$ and $t \in [0, 1)$, we define $H_1(x) = \varphi(x)$, and

$$H_t(x) = \int_{\mathbb{R}} \varphi(z) q_{\alpha,\beta}(t, x, z) dz \tag{10}$$

where

$$q_{\alpha,\beta}(t,x,z)=\frac{1}{\beta\sqrt{2\pi(1-t)}}e^{-\frac{(x-z)^2-2\alpha\beta(1-t)(|z|-|x|)+\alpha^2\beta^2(1-t)^2}{2(1-t)\beta^2}}-\frac{\alpha}{\beta}e^{\frac{2\alpha|z|}{\beta}}\Phi\left(-\frac{|z|+|x|}{\beta\sqrt{1-t}}-\alpha\sqrt{1-t}\right).$$

Here the dependence of H_t on φ,α,β and c is not explicitly noted for simplicity. It is clear from the definition that

$$H_0(0)=E[\varphi(\beta\eta)]$$

where $\eta\sim\mathcal{B}(\alpha,0)$ is a spike distribution. The following lemma lists some analytic properties of the family $\{H_t(x)\}_{t\in[0,1]}$.

Lemma 1.1. Let the number of dots on top of a function denote the same order derivatives with respect to x.

(1) For each fixed t, $H_t(x)\in C_b^2(\mathbb{R})$. In addition, the first and second order derivatives of $H_t(x)$ are uniformly bounded for all $0\le t\le 1$ and x.

(2) The family $\left\{\ddot{H}_t(x)\right\}_{t\in[0,1]}$ is uniformly Lipschitz, i.e., there exists a constant L, independent with t, such that

$$\left|\ddot{H}_t(x_1)-\ddot{H}_t(x_2)\right|\le L|x_1-x_2|,\quad x_1,x_2\in\mathbb{R}.$$

(3) $H_t(x)$ is an even function. Furthermore, if for any $x\in\mathbb{R}$,

$$\operatorname{sgn}(\dot{\varphi}(x))=\pm\operatorname{sgn}(x),$$

then

$$\operatorname{sgn}\left(\dot{H}_t(x)\right)=\pm\operatorname{sgn}(x),x\in\mathbb{R}$$

(4) If $\operatorname{sgn}(\dot{\varphi}(x))=\pm\operatorname{sgn}(x)$ for all $x\in\mathbb{R}$, then

$$\lim_{n\to\infty}\sum_{m=1}^{n}\sup_{x\in\mathbb{R}}\left|H_{\frac{m-1}{n}}(x)-H_{\frac{m}{n}}(x)\mp\frac{\alpha}{n}|\dot{H}_{\frac{m}{n}}(x)|-\frac{\beta^2}{2n}\ddot{H}_{\frac{m}{n}}(x)\right|=0.$$

Proof: We prove the lemma in numerical order.

(1) For $t=1$, $H_1(x)\equiv\varphi(x)$ and the result is trivial. Next we assume that $0\le t<1$. Since φ is an even function, with the definition of $H_t(x)$, it follows by direct calculation that

$$\dot{H}_t(x)=\int_0^\infty\frac{\operatorname{sgn}(x)}{\beta\sqrt{2\pi(1-t)}}\dot{\varphi}(z)e^{-\frac{(z-\alpha\beta(1-t)-|x|)^2}{2(1-t)\beta^2}}\left[1-e^{-\frac{2|x|z}{(1-t)\beta^2}}\right]dz \tag{11}$$

$$\begin{aligned}\ddot{H}_t(x)=&\int_0^\infty\frac{1}{\beta\sqrt{2\pi(1-t)}}\ddot{\varphi}(z)e^{-\frac{(z-\alpha\beta(1-t)-|x|)^2}{2(1-t)\beta^2}}\left[1+e^{-\frac{2|x|z}{(1-t)\beta^2}}\right]dz\\&+\int_0^\infty\frac{2\alpha}{\beta^2\sqrt{2\pi(1-t)}}\dot{\varphi}(z)e^{-\frac{(z+\alpha\beta(1-t)+|x|)^2}{2(1-t)\beta^2}}e^{\frac{2\alpha z}{\beta}}dz\\=&\int_0^\infty\frac{1}{\beta\sqrt{2\pi(1-t)}}\ddot{\varphi}(z)e^{-\frac{(z-\alpha\beta(1-t)-|x|)^2}{2(1-t)\beta^2}}\left[1+e^{-\frac{2|x|z}{(1-t)\beta^2}}\right]dz\\&+\int_0^\infty\frac{2\alpha}{\beta^2\sqrt{2\pi(1-t)}}\dot{\varphi}(z)e^{-\frac{(z-\alpha\beta(1-t)+|x|)^2}{2(1-t)\beta^2}}e^{-\frac{2\alpha|x|}{\beta}}dz\end{aligned}$$

Since $\varphi\in C_b^3(\mathbb{R})$, we conclude that $H_t(x)\in C_b^2(\mathbb{R})$, and the first and second order derivatives of $H_t(x)$ are uniformly bounded for all t and x.

(2) For $x<0$, we have

$$\begin{aligned}
\dddot{H}_t(x) &= \int_0^\infty \frac{1}{\beta\sqrt{2\pi(1-t)}}\dddot{\varphi}(z)e^{-\frac{(z-\alpha\beta(1-t)+x)^2}{2(1-t)\beta^2}}\left[e^{\frac{2xz}{(1-t)\beta^2}}-1\right]dz \\
&\quad + \int_0^\infty \frac{4\alpha}{\beta^3\sqrt{2\pi(1-t)}}[\alpha\dot{\varphi}(z)+\beta\ddot{\varphi}(z)]e^{-\frac{(z+\alpha\beta(1-t)-x)^2}{2(1-t)\beta^2}}e^{\frac{2\alpha z}{\beta}}dz \\
&= \int_0^\infty \frac{1}{\beta\sqrt{2\pi(1-t)}}\dddot{\varphi}(z)e^{-\frac{(z-\alpha\beta(1-t)+x)^2}{2(1-t)\beta^2}}\left[e^{\frac{2xz}{(1-t)\beta^2}}-1\right]dz \\
&\quad + \int_0^\infty \frac{4\alpha}{\beta^3\sqrt{2\pi(1-t)}}[\alpha\dot{\varphi}(z)+\beta\ddot{\varphi}(z)]e^{-\frac{(z-\alpha\beta(1-t)-x)^2}{2(1-t)\beta^2}}e^{\frac{2\alpha x}{\beta}}dz.
\end{aligned}$$

For $x>0$, we have

$$\begin{aligned}
\dddot{H}_t(x) &= \int_0^\infty \frac{1}{\beta\sqrt{2\pi(1-t)}}\dddot{\varphi}(z)e^{-\frac{(z-\alpha\beta(1-t)-x)^2}{2(1-t)\beta^2}}\left[1-e^{-\frac{2xz}{(1-t)\beta^2}}\right]dz \\
&\quad - \int_0^\infty \frac{4\alpha}{\beta^3\sqrt{2\pi(1-t)}}[\beta\ddot{\varphi}(z)+\alpha\dot{\varphi}(z)]e^{-\frac{(z+\alpha\beta(1-t)+x)^2}{2(1-t)\beta^2}}e^{\frac{2\alpha z}{\beta}}dz \\
&= \int_0^\infty \frac{1}{\beta\sqrt{2\pi(1-t)}}\dddot{\varphi}(z)e^{-\frac{(z-\alpha\beta(1-t)-x)^2}{2(1-t)\beta^2}}\left[1-e^{-\frac{2xz}{(1-t)\beta^2}}\right]dz \\
&\quad - \int_0^\infty \frac{4\alpha}{\beta^3\sqrt{2\pi(1-t)}}[\beta\ddot{\varphi}(z)+\alpha\dot{\varphi}(z)]e^{-\frac{(z-\alpha\beta(1-t)+x)^2}{2(1-t)\beta^2}}e^{-\frac{2\alpha x}{\beta}}dz
\end{aligned}$$

Since $\varphi\in C_b^3(\mathbb{R})$, it follows that $\dddot{H}_t(x)$ is uniformly bounded for all t and $x\neq 0$. For $x=0$, the third order left and right derivatives of $H_t(x)$ can be shown to exist and are also bounded uniformly in t. Thus by the mean value theorem one can find a constant L, independent with t, such that for any $x_1,x_2\in\mathbb{R}$,

$$\left|\ddot{H}_t(x_1)-\ddot{H}_t(x_2)\right|\le L\,|x_1-x_2|\,.$$

(3) It follows by direct calculation that for any $x\in\mathbb{R}$,

$$\begin{aligned}
H_t(x) &= \int_{\mathbb{R}}\varphi(z)q_{\alpha,\beta}(t,x,z)dz = \int_{\mathbb{R}}\varphi(z)q_{\alpha,\beta}(t,-x,-z)dz \\
&= \int_{\mathbb{R}}\varphi(z)q_{\alpha,\beta}(t,-x,z)dz \\
&= H_t(-x)
\end{aligned}$$

That is H_t is an even function. By (11) we have that for any $x\in\mathbb{R}$,

$$\operatorname{sgn}\left(\dot{H}_t(x)\right)=\pm\operatorname{sgn}(x)\text{ when }\operatorname{sgn}(\dot{\varphi}(x))=\pm\operatorname{sgn}(x)$$

(4) We only prove the case $\operatorname{sgn}(\dot{\varphi}(x))=\operatorname{sgn}(x)$. The other case follows by similar arguments. For any $(t,x)\in[0,1]\times\mathbb{R}$, let $\{Y_s^{t,x}\}_{s\in[t,1]}$ denote the solution of the SDE

$$\begin{cases} dY_s^{t,x}=\alpha\operatorname{sgn}\left(Y_s^{t,x}\right)ds+dB_s, \quad s\in[t,1] \\ Y_t^{t,x}=x \end{cases}$$

Although the drift coefficient is discontinuous, this equation does have a unique strong solution (Mel'nikov, 1979). Fortunately, $\{Y_s^{t,x}\}_{s\in[t,1]}$ has an explicit probability density function, which can be denoted by

$$q_\alpha(t,x;s,z)=\frac{1}{\sqrt{2\pi(s-t)}}e^{-\frac{(x-z)^2-2\alpha(s-t)(|z|-|x|)+\alpha^2(s-t)^2}{2(s-t)}}-\alpha e^{2\alpha|z|}\int_{|x|+|z|+\alpha(s-t)}^\infty\frac{1}{\sqrt{2\pi(s-t)}}e^{-\frac{u^2}{2(s-t)}}du$$

Then the basic function H_t can also be denoted by

$$H_t(x)=E\left[\varphi\left(\beta Y_1^{t,\frac{x}{\beta}}\right)\right]. \tag{12}$$

Follows from the Markov property of $(Y_s^{t,x})$, we have for any $h \in [0, 1-t]$,

$$H_t(x) = E\left[\varphi\left(\beta Y_1^{t,\frac{x}{\beta}}\right)\right] = E\left[E\left[\varphi\left(\beta Y_1^{t,\frac{x}{\beta}}\right) \mid \mathcal{F}_{t+h}^*\right]\right] = E\left[H_{t+h}\left(\beta Y_{t+h}^{t,\frac{x}{\beta}}\right)\right].$$

Applying the Markov property and (12), we have for any $1 \le m \le n$,

$$H_{\frac{m-1}{n}}(x) = E\left[H_{\frac{m}{n}}\left(\beta Y_{\frac{m}{n}}^{\frac{m-1}{n},\frac{x}{\beta}}\right)\right].$$

By Itô's formula, we have

$$H_{\frac{m}{n}}\left(\beta Y_{\frac{m}{n}}^{\frac{m-1}{n},\frac{x}{\beta}}\right) = H_{\frac{m}{n}}(x)+\int_{\frac{m-1}{n}}^{\frac{m}{n}} \dot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right)\beta dY_s^{\frac{m-1}{n},\frac{x}{\beta}}+\frac{\beta^2}{2}\int_{\frac{m-1}{n}}^{\frac{m}{n}} \ddot{H}_{\frac{m}{n}}\left(Y_s^{\frac{m-1}{n},x}\right)ds.$$

This combined with **(3)** implies that

$$\begin{aligned}
&H_{\frac{m-1}{n}}(x)\\
=&E\left[H_{\frac{m}{n}}(x) + \int_{\frac{m-1}{n}}^{\frac{m}{n}} \dot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right)\beta dY_s^{\frac{m-1}{n},\frac{x}{\beta}} + \frac{\beta^2}{2}\int_{\frac{m-1}{n}}^{\frac{m}{n}} \ddot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right)ds\right]\\
=&E\left[H_{\frac{m}{n}}(x) + \int_{\frac{m-1}{n}}^{\frac{m}{n}} \alpha\dot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right)\operatorname{sgn}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right)ds + \frac{\beta^2}{2}\int_{\frac{m-1}{n}}^{\frac{m}{n}} \ddot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right)ds\right]\\
=&E\left[H_{\frac{m}{n}}(x) + \int_{\frac{m-1}{n}}^{\frac{m}{n}} \alpha\left|\dot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right)\right|ds + \frac{\beta^2}{2}\int_{\frac{m-1}{n}}^{\frac{m}{n}} \ddot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right)ds\right]
\end{aligned}$$

Taking the supremum over x, we obtain

$$\begin{aligned}
&\sum_{m=1}^{n}\sup_{x\in\mathbb{R}}\left|H_{\frac{m-1}{n}}(x) - H_{\frac{m}{n}}(x) - \frac{\alpha}{n}\left|\dot{H}_{\frac{m}{n}}(x)\right| - \frac{\beta^2}{2n}\ddot{H}_{\frac{m}{n}}(x)\right|\\
&\le \sum_{m=1}^{n}\sup_{x\in\mathbb{R}} E\left[\int_{\frac{m-1}{n}}^{\frac{m}{n}} |\alpha|\left|\dot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right) - \dot{H}_{\frac{m}{n}}(x)\right|ds\right.\\
&\left.+\frac{1}{2}\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left|\ddot{H}_{\frac{m}{n}}\left(\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}}\right) - \ddot{H}_{\frac{m}{n}}(x)\right|ds\right]\\
&\le \sum_{m=1}^{n}\sup_{x\in\mathbb{R}}\frac{C}{n}E\left[\sup_{s\in\left[\frac{m-1}{n},\frac{m}{n}\right]}\left|\beta Y_s^{\frac{m-1}{n},\frac{x}{\beta}} - x\right|\right]\\
&\le \sum_{m=1}^{n}\frac{C\beta}{n}E\left[\frac{|\alpha|}{n} + \sup_{s\in\left[\frac{m-1}{n},\frac{m}{n}\right]}\left|B_s - B_{\frac{m-1}{n}}\right|\right]\\
&\le C\beta\left(\frac{|\alpha|}{n} + \frac{1}{\sqrt{n}}\right)
\end{aligned}$$

where C is a constant depending only on α, L and the bound of $\ddot{H}_t(x)$. This concludes the proof of the lemma.

Lemma 1.2. Let $\varphi \in C_b^3(\mathbb{R})$ be an even function, and $\{H_t(x)\}_{t\in[0,1]}$ be defined as in **Lemma 1.1**. Define functions $\{L_{m,B}(x)\}_{m=1}^{B}$ and $\left\{\widehat{L}_{m,B}(x)\right\}_{m=1}^{B}$ by

$$L_{m,B}(x) = H_{\frac{m}{B}}(x) + \frac{d}{B}\left|\dot{H}_{\frac{m}{B}}(x)\right| + \frac{1}{2B}\ddot{H}_{\frac{m}{B}}(x), \quad x \in \mathbb{R}, \tag{13}$$

$$\widehat{L}_{m,B}(x) = H_{\frac{m}{B}}(x) - \frac{d}{B}\left|\dot{H}_{\frac{m}{B}}(x)\right| + \frac{1}{2B}\ddot{H}_{\frac{m}{B}}(x), \quad x \in \mathbb{R}. \tag{14}$$

Under the assumption that $\mu_1 - \mu_0 = d > d_0$, let $\{\xi : B \geq 1\}$ be the strategies defined by section 3, then the followings hold.

(1) If $\operatorname{sgn}(\dot{\varphi}(x)) = -\operatorname{sgn}(x)$ for all $x \in \mathbb{R}$, then

$$\lim_{B\to\infty} \sum_{m=1}^{B} \left| E\left[H_{\frac{m}{B}}\left(T_{m,B}^{\xi}\right)\right] - E\left[L_{m,B}\left(T_{m-1,B}^{\xi}\right)\right]\right| = 0,$$

(2) If $\operatorname{sgn}(\dot{\varphi}(x)) = \operatorname{sgn}(x)$ for all $x \in \mathbb{R}$, then

$$\lim_{B\to\infty} \sum_{m=1}^{B} \left| E\left[H_{\frac{m}{B}}\left(T_{m,B}^{\xi}\right)\right] - E\left[\widehat{L}_{m,B}\left(T_{m-1,B}^{\xi}\right)\right]\right| = 0$$

Proof: Since the proofs of **(1)** and **(2)** are similar, we only give the proof of **(1)**. For any $\xi \in \Theta, T \in \mathbb{N}^{+}$and $1 \leq m \leq B$, set

$$\Gamma(m,B,\xi) = H_{\frac{m}{B}}\left(\mathcal{T}_{m-1,B}^{\xi}\right) + \dot{H}_{\frac{m}{T}}\left(\mathcal{T}_{m-1,B}^{\xi}\right)\left(\frac{Z_m^{\xi}}{B} + \frac{\bar{Z}_m^{\xi}}{\sqrt{B}}\right) + \ddot{H}_{\frac{m}{B}}\left(\mathcal{T}_{m-1}^{\xi}\right)\frac{\left(\bar{Z}_m^{\xi}\right)^2}{2B}$$

where $\bar{Z}_m^{\xi} = \left(Z_m^{\xi} - \mu_i^{\xi}\right)/\widehat{\sigma}_f$. Now we prove **(1)** in two steps.

Step 1: for any $\xi \in \Theta$, we have

$$\lim_{B\to\infty} \sum_{m=1}^{B} \left| E\left[H_{\frac{m}{B}}\left(\mathcal{T}_{m,B}^{\xi}\right)\right] - E[\Gamma(m,T,\xi)]\right| = 0. \tag{15}$$

In fact, by **(1)** and **(2)** of **Lemma 1.1**, there exists a constant $C > 0$ such that

$$\sup_{t\in[0,1]}\sup_{x\in\mathbb{R}}\left|\ddot{H}_t(x)\right| \leq C, \quad \sup_{t\in[0,1]}\sup_{x,y\in\mathbb{R},x\neq y}\frac{\left|\ddot{H}_t(x)-\ddot{H}_t(y)\right|}{|x-y|} \leq C.$$

It follows from Taylor's expansion that for any $\varepsilon > 0$, there exists $\delta > 0$ (depends only on C and ε), such that for any $x, y \in \mathbb{R}$, and $t \in [0,1]$,

$$\left|H_t(x+y) - H_t(x) - \dot{H}_t(x)y - \frac{1}{2}\ddot{H}_t(x)y^2\right| \leq \varepsilon I_{\{|y|<\delta\}} + C|y|^2 I_{\{|y|\geq\delta\}}. \tag{16}$$

For any $1 \leq m \leq B$, taking

$$x = \mathcal{T}_{m-1,B}^{\xi}, y = Z_m^{\xi}B + \bar{Z}_m^{\xi}/\sqrt{B}$$

in (16), we obtain

$$\begin{aligned}
&\sum_{m=1}^{B} E\left[\left|H_{\frac{m}{B}}\left(\mathcal{T}_{m,B}^{\xi}\right) - \Gamma(m,B,\xi)\right|\right] \\
&\leq \varepsilon + \frac{C}{2}\sum_{m=1}^{B} E\left[\left|\frac{Z_m^{\xi}}{B}\right|^2 + 2\left|\frac{Z_m^{\xi}}{B}\right|\left|\frac{\bar{Z}_m^{\xi}}{\sqrt{B}}\right|\right] \\
&\leq \varepsilon + \frac{3C}{2}\sum_{m=1}^{B} E\left[\left|\frac{Z_m^{\xi}}{B}\right|^2 + 2\left|\frac{Z_m^{\xi}}{B}\right|\left|\frac{\bar{Z}_m^{\xi}}{\sqrt{B}}\right|\right] \\
&+C\sum_{m=1}^{B} E\left[\left|\frac{\bar{Z}_m^{\xi}}{\sqrt{B}}\right|^2 I_{\left\{\left|\frac{Z_m^{\xi}}{B}+\frac{\bar{Z}_m^{\xi}}{\sqrt{B}}\right|\geq\delta\right\}}\right] \\
&=: \varepsilon + \Delta_B^1 + \Delta_B^2
\end{aligned}$$

For each $m \geq 1$, we define the σ-field

$$\mathcal{H}_m = \sigma\left\{W_1^L, W_1^R, \cdots, W_m^L, W_m^R\right\}$$

and set

$$\mathcal{H}_0 = \{\emptyset, \Omega\}$$

then we have

$$\begin{aligned} E\left[\left|Z_m^\xi\right|^2\right] &= E\left[\left|w_m^L\right|^2 I_{\{\vartheta_m=1\}} + \left|w_m^R\right|^2 I_{\{\vartheta_m=2\}}\right] \\ &= E\left[I_{\{\vartheta_m=1\}} E\left[\left|W_m^L\right|^2 \mid \mathcal{D}, \mathcal{H}_{m-1}\right] + I_{\{\vartheta_m=2\}} E\left[\left|W_m^R\right|^2 \mid \mathcal{D}, \mathcal{H}_{m-1}\right]\right] \\ &= E\left[\left(\widehat{\mu}_1 - \widehat{\mu}_0\right)^2 + \widehat{\sigma}_f^2\right], \end{aligned}$$

and

$$E\left[\left|\bar{Z}_m^\xi\right|^2\right] = E\left[E\left[\left|\bar{Z}_m^\xi\right|^2 \mid \mathcal{D}, \mathcal{H}_{m-1}\right]\right] = 1$$

As a result, we conclude that

$$\begin{aligned} \Delta_B^1 &= \frac{3C}{2} \sum_{m=1}^{B} E\left[\left|\frac{Z_m^\xi}{B}\right|^2 + 2\left|\frac{Z_m^\xi}{B}\right|\left|\frac{\bar{Z}_m^\xi}{\sqrt{B}}\right|\right] \\ &= \frac{3C}{2B} E\left[\left(\widehat{\mu}_1 - \widehat{\mu}_0\right)^2 + \widehat{\sigma}_f^2\right] + \frac{3C}{\sqrt{B}} E\left[\left|Z_m^\xi\right|\left|\bar{Z}_m^\xi\right|\right] \\ &\leq \frac{3C}{2B} E\left[\left(\widehat{\mu}_1 - \widehat{\mu}_0\right)^2 + \widehat{\sigma}_f^2\right] + \frac{3C}{\sqrt{B}} E\left[\left|Z_m^\xi\right|^2\right]^{\frac{1}{2}} \\ &= \frac{3C}{2B} E\left[\left(\widehat{\mu}_1 - \widehat{\mu}_0\right)^2 + \widehat{\sigma}_f^2\right] + \frac{3C}{\sqrt{B}} E\left[\left(\widehat{\mu}_1 - \widehat{\mu}_0\right)^2 + \widehat{\sigma}_f^2\right]^{\frac{1}{2}} \\ &\to 0, \text{ as } n, B \to \infty \end{aligned}$$

On the other hand, we have

$$\Delta_B^2 = C \sum_{m=1}^{B} E\left[\left|\frac{\bar{Z}_m^\xi}{\sqrt{B}}\right|^2 I_{\left\{\left|\frac{Z_m^\xi}{B} + \frac{Z_m^\xi}{\sqrt{B}}\right| \geq \delta\right\}}\right]$$

$$\leq 2CE\left[\left|\frac{W_1^L - \widehat{\mu}_1 + \widehat{\mu}_0}{\widehat{\sigma}_f}\right|^2 I^{\left|\frac{W_1^L}{B} + \frac{W_1^L - \widehat{\mu}_1 + \widehat{\mu}_0}{\sqrt{\bar{\sigma}\widehat{\sigma}_f}}\right| \geq \delta}\right]$$

$\to 0$, as $n, B \to \infty$.

Then we complete the proof of (15).

Step 2: for the strategy ξ given in section 2.2, we have

$$\lim_{B \to \infty} \sum_{m=1}^{B} \left|E\left[\Gamma\left(m, B, \xi\right)\right] - E\left[L_{m,B}\left(T_{m-1}^\xi\right)\right]\right| = 0.$$

Note that

$$\begin{aligned} &E\left[Z_m^\xi \mid \mathcal{D}, \mathcal{H}_{m-1}\right] = I_{\{\vartheta_m=1\}}\left(\widehat{\mu}_1 - \widehat{\mu}_0\right) + I_{\{\vartheta_m=2\}}\left(\widehat{\mu}_0 - \widehat{\mu}_1\right) \\ &E\left[\left(\bar{Z}_m^\xi\right)^2 \mid \mathcal{D}, \mathcal{H}_{m-1}\right] = 1 \end{aligned}$$

then we have

$$
\begin{aligned}
&\sum_{m=1}^{B}\left|E\left[\Gamma\left(m,B,\xi\right)\right]-E\left[L_{m,B}\left(T_{m-1}^{\xi}\right)\right]\right| \\
=&\sum_{m=1}^{B}\mid E\left[E\left[\Gamma\left(m,B,\xi\right)-L_{m,B}\left(T_{m-1}^{\xi}\right)\mid \mathcal{D},\mathcal{H}_{m-1}\right]\right]\mid \\
=&\frac{1}{B}\sum_{m=1}^{B}\left|E\left[\dot{H}_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)\left(I_{\{\vartheta_m=1\}}\left(\widehat{\mu}_1-\widehat{\mu}_0\right)+I_{\{\vartheta_m=2\}}\left(\widehat{\mu}_0-\widehat{\mu}_1\right)\right)-d\left|\dot{H}_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)\right|\right]\right| \\
=&\frac{1}{B}\sum_{m=1}^{B}\left|E\left[\left(\widehat{\mu}_1-\widehat{\mu}_0-d\right)\left|\dot{H}_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)\right|\right]\right| \\
\leq&CE\left[\left|\widehat{\mu}_1-\widehat{\mu}_0-d\right|\right]\to 0,\ \text{as } n\to\infty
\end{aligned}
$$

Lemma 1.3. Let $\varphi\in C_b^3(\mathbb{R})$ be an even function, and $\{H_t(x)\}_{t\in[0,1]}$ be defined as in (10). Define functions $\left\{L_{m,B}^*(x)\right\}_{m=1}^{B}$ and $\left\{\widehat{L}_{m,B}^*(x)\right\}_{m=1}^{B}$ *by*

$$
L_{m,B}^*(x)=H_{\frac{m}{B}}(x)+\frac{\widehat{d}+\sqrt{B}\left(\widehat{d}-d_0\right)/\widehat{\sigma}_f}{B}\left|\dot{H}_{\frac{m}{B}}(x)\right|+\frac{\widehat{\sigma}_f^2+\left(\widehat{d}-d_0\right)^2}{2B\widehat{\sigma}_f^2}\ddot{H}_{\frac{m}{B}}(x),\quad x\in\mathbb{R},
$$

$$
\widehat{L}_{m,B}^*(x)=H_{\frac{m}{B}}(x)-\frac{\widehat{d}+\sqrt{B}\left(\widehat{d}-d_0\right)/\widehat{\sigma}_f}{B}\left|\dot{H}_{\frac{m}{B}}(x)\right|+\frac{\widehat{\sigma}_f^2+\left(\widehat{d}-d_0\right)^2}{2B\widehat{\sigma}_f^2}\ddot{H}_{\frac{m}{B}}(x),\quad x\in\mathbb{R}.
$$

Let $\{\xi:T\geq 1\}$ be the strategies defined by section 2.2, then the followings hold.

(1) If $\operatorname{sgn}(\dot{\varphi}(x))=-\operatorname{sgn}(x)$ for all $x\in\mathbb{R}$, then

$$
\lim_{B\to\infty}\sum_{m=1}^{B}\left|E\left[H_{\frac{m}{B}}\left(T_m^{\xi}\right)\right]-E\left[L_{m,B}^*\left(T_{m-1}^{\xi}\right)\right]\right|=0
$$

(2) If $\operatorname{sgn}(\dot{\varphi}(x))=\operatorname{sgn}(x)$ for all $x\in\mathbb{R}$, then

$$
\lim_{B\to\infty}\sum_{m=1}^{B}\left|E\left[H_{\frac{m}{B}}\left(T_m^{\xi}\right)\right]-E\left[\widehat{L}_{m,B}^*\left(T_{m-1}^{\xi}\right)\right]\right|=0
$$

Proof: Since the proofs of **(1)** and **(2)** are similar, we only give the proof of **(1)**. We continue use the notations in the proof of **Lemma 1.2.** With the result of (15), which still holds under the assumption that $\mu_1-\mu_0=d<d_0$, we only need to show

$$
E\left[\Gamma\left(m,B,\xi\right)\right]=E\left[L_{m,B}^*\left(T_{m-1}^{\xi}\right)\right],\ \text{for any } 1\leq m\leq B
$$

Under the hypothesis $\mu_1-\mu_0=d<d_0$, we have

$$
E\left[Z_m^{\xi}\mid\mathcal{D},\mathcal{H}_{m-1}\right]=I_{\{\vartheta_m=1\}}\widehat{d}+I_{\{\vartheta_m=2\}}\widehat{d}
$$

$$
E\left[\left(\bar{Z}_m^{\xi}\right)^2\mid\mathcal{D},\mathcal{H}_{m-1}\right]=1+\frac{\left(\widehat{d}-d_0\right)^2}{\widehat{\sigma}_f^2}
$$

then we obtain

$$
\begin{aligned}
&E\left[\Gamma\left(m,B,\xi\right)\right]-E\left[L_{m,B}^*\left(T_{m-1}^{\xi}\right)\right] \\
=&E\left[H_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)+\dot{H}_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)\left(\frac{Z_m^{\xi,n}}{B}+\frac{\bar{Z}_m^{\xi}}{\sqrt{B}}\right)+\frac{1}{2}\ddot{H}_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)\left(\frac{\bar{Z}_m^{\xi}}{\sqrt{B}}\right)^2\right] \\
&-E\left[H_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)+\frac{\widehat{d}+\sqrt{B}\left(\widehat{d}-d_0\right)/\widehat{\sigma}_f}{B}\left|\dot{H}_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)\right|+\frac{\widehat{\sigma}_f^2+\left(\widehat{d}-d_0\right)^2}{2B\ddot{\sigma}_f^2}\left(\ddot{H}_{\frac{m}{B}}\left(T_{m-1}^{\xi}\right)\right)\right] \\
=&0.
\end{aligned}
$$

Proof of Theorem 3.1: We only give the proof of part (a) here. Part (b) can be obtained similarly.

Let $\varphi \in C(\overline{\mathbb{R}})$ be an even function. The result is clear if φ is globally constant. Thus we assume that φ is not a constant function. We only give the proof for the case that φ is decreasing on $(0,\infty)$, when φ is increasing on $(0,\infty)$ it can be proved similarly. Assume that φ is decreasing on $(0,\infty)$. For any $h>0$, define the function φ_h by

$$\varphi_h(x)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\varphi(x+hy)e^{-\frac{y^2}{2}}\,dy.$$

By the Approximation Lemma in Feller (2008), we have that

$$\lim_{h\to 0}\sup_{x\in\mathbb{R}}|\varphi(x)-\varphi_h(x)|=0. \tag{17}$$

It follows from direct calculation that

$$\begin{aligned}\varphi_h(x)&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\varphi(x+hy)e^{-\frac{y^2}{2}}\,dy\\&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\varphi(-x-hy)e^{-\frac{y^2}{2}}\,dy\\&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\varphi(-x+hy)e^{-\frac{y^2}{2}}\,dy\\&=\varphi_h(-x)\end{aligned}$$

Thus φ_h is also an even function. In addition, we have

$$\begin{aligned}\dot{\varphi}_h(x)&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}h^3}\varphi(x+y)ye^{-\frac{y^2}{2h^2}}\,dy\\&=\int_{0}^{\infty}\frac{1}{\sqrt{2\pi}h^3}\varphi(y+x)ye^{-\frac{y^2}{2h^2}}\,dy+\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}h^3}\varphi(y+x)ye^{-\frac{y^2}{2h^2}}\,dy\\&=\int_{0}^{\infty}\frac{1}{\sqrt{2\pi}h^3}(\varphi(y+x)-\varphi(y-x))ye^{-\frac{y^2}{2h^2}}\,dy\end{aligned}$$

Since φ is decreasing on $(0,\infty)$, it follows that

$$\operatorname{sgn}\left(\dot{\varphi}_h(x)\right)=-\operatorname{sgn}(x)$$

In the remaining proof of this theorem, we continue to use $\{H_t(x)\}_{t\in[0,1]}$ to denote the functions defined in (10) with φ_h in place of φ and $\alpha=-d_0,\beta=1$ there. Let $\{L_{m,B}(x)\}_{m=1}^{T}$ be functions defined in (13) with $\{H_t(x)\}_{t\in[0,1]}$ here. Let $\eta\sim\mathcal{B}(-d,0)$ be a spike distribution,

by direct calculation we obtain

$$\begin{aligned}&E\left[\varphi_h\left(T_B^{\xi}\right)\right]-E\left[\varphi_h(\eta)\right]\\=&E\left[H_1\left(T_B^{\xi}\right)\right]-H_0(0)\\=&\sum_{m=1}^{B}\left\{E\left[H_{\frac{m}{B}}\left(T_{m,B}^{\xi}\right)\right]-E\left[H_{\frac{m-1}{B}}\left(T_{m-1}^{\xi}\right)\right]\right\}\\=&\sum_{m=1}^{B}\left\{E\left[H_{\frac{m}{B}}\left(T_{m,B}^{\xi}\right)\right]-E\left[L_{m,B}\left(T_{m-1,B}^{\xi}\right)\right]\right\}\\&+\sum_{m=1}^{B}\left\{E\left[L_{m,B}\left(T_{m-1,B}^{\xi}\right)\right]-E\left[H_{\frac{m-1}{B}}\left(T_{m-1,B}^{\xi}\right)\right]\right\}\\=&I_{B,n}^1+I_{B,n}^2.\end{aligned}$$

An application of **Lemma 1.2** implies that $\left|I_{B,n}^1\right| \to 0$ as $n \to \infty$ and $B \to \infty$. It follows from **(4)** in Lemma **Lemma 7.1** that

$$
\begin{aligned}
\left|I_{B,n}^2\right| &\le \sum_{m=1}^{B} E\left[\left|L_{m,B}\left(T_{m-1,B}^{\xi}\right) - H_{\frac{m-1}{B}}\left(T_{m-1,B}^{\xi}\right)\right|\right] \\
&\le \sum_{m=1}^{B} \sup_{x\in\mathbb{R}} \left|L_{m,B}(x) - H_{\frac{m-1}{B}}(x)\right| \\
&= \sum_{m=1}^{B} \sup_{x\in\mathbb{R}} \left|H_{\frac{m-1}{B}}(x) - H_{\frac{m}{B}}(x) - \frac{d}{T}\left|\dot{H}_{\frac{m}{B}}(x)\right| - \frac{1}{2B}\ddot{H}_{\frac{m}{B}}(x)\right| \\
&\to 0, \quad \text{as } B \to \infty
\end{aligned}
$$

which implies that

$$
\lim_{h\to 0}\lim_{B\to\infty}\left|E\left[\varphi_h\left(T_B^{\xi}\right)\right] - E\left[\varphi_h(\eta)\right]\right| = 0. \tag{18}
$$

Putting together (17) and (18), we have

$$
\begin{aligned}
&\lim_{B\to\infty}\left|E\left[\varphi\left(T_B^{\xi}\right)\right] - E[\varphi(\eta)]\right| \\
\le &\lim_{h\to 0}\lim_{B\to\infty}\left|E\left[\varphi\left(T_B^{\xi}\right)\right] - E\left[\varphi_h\left(T_B^{\xi}\right)\right]\right| \\
&+ \lim_{h\to 0}\lim_{B\to\infty}\left|E\left[\varphi_h\left(T_B^{\xi}\right)\right] - E\left[\varphi_h(\eta)\right]\right| \\
&+ \lim_{h\to 0}\left|E\left[\varphi_h(\eta)\right] - E[\varphi(\eta)]\right| \\
=&0.
\end{aligned}
$$

Then we complete the proof.