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Bootstrap approaches for homogeneous test of location parameters under skew-normal settings

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ABSTRACT

When the scale parameters and skewness parameters are unknown, we consider the problem of homogeneous test of location parameters in several skew-normal populations. First, the conditional test statistic is constructed and its approximate distribution is proved. Second, we estimate the unknown parameters based on the methods of moment and maximum likelihood estimation. Then we construct the Bootstrap test statistics and generalize the results of Xu from normal population to skew-normal population. Further, the Monte-Carlo simulation results indicate that in terms of controlling the Type I error probability, the Bootstrap test statistic based on the moment estimator performs better than that based on the maximum likelihood estimator. Finally, the above approaches are illustrated with two real examples of gross domestic product and turbine bearing performance.

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1. Introduction

In recent years, the real data of economics, physics and epidemiology often show obvious skewed and asymmetric characteristics (e.g. Lin, Lee, and Yen 2007; Basso et al. 2010; Fruhwirth-Schnatter and Pyne 2010). For this, Azzalini (1985) first proposed the skew-normal distribution, whose density function can be expressed as

$$f(y;\xi,\eta^2,\alpha) = 2\phi(y;\xi,\eta^2)\Phi\left[\alpha\eta^{-1}(y-\xi)\right], \tag{1}$$

where $\xi \in R$ denotes the location parameter, $\eta^2 \in R^+$ denotes the scale parameter, $\alpha \in R$ denotes the skewness parameter, $\phi(y;\xi,\eta^2)$ is the normal density function with mean ξ and variance η^2, and $\Phi(\cdot)$ is the standard normal distribution function. Denote $Y \sim SN(\xi,\eta^2,\alpha)$. When $\xi = 0$ and $\eta^2 = 1$, Equation (1) is degenerated to the standard skew-normal distribution $SN(\alpha)$. When $\alpha = 0$, Equation (1) is degenerated to the normal distribution $N(\xi,\eta^2)$. In short, the alteration of the skewness parameter allows for a continuous variation from normality to skew-normality.

In view of the extensive applications of the skew-normal distribution, many scholars have made indepth studies on it and its properties. Azzalini and Genton (2007), Balakrishnan and Scarpa (2012) gave statistical properties such as moment generation function, marginal distribution and conditional distribution, then applied them to discriminant analysis, regression analysis and graph model analysis. Within a Bayesian framework, Maleki and Wraith (2019) combined the skew-normal distribution with the factor analysis model to derive the mixture of skew-normal factor analysis model and its parameter estimation. Based on the functional principal component analysis, Hu et al. (2020) discussed the maximum likelihood (ML) estimation of the skew-normal

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partial functional linear model. Arellano-Valle et al. (2019) used the forward filtering and backward sampling method to study the Bayesian inference of the skew-normal dynamic linear model, and applied this methodology to the analysis of the condition factor index of male and female anchovies off northern Chile. Based on the penalized ML estimation and penalized EM-type algorithm, Jin et al. (2019) proposed the ML estimation of skew-normal mixture model. Said, Ning, and Tian (2017) derived a detection procedure based on the likelihood ratio test for the change point problem of skew-normal distribution, and applied it to the stock return problem.

At present, the statistical inference of location parameter in skew-normal distribution has become one of the hot topics in statistics research. For example, Azzalini, Genton, and Scarpa (2010) thought that it is of interest to estimate the location parameter and scale parameter, and used the invariance-based estimating equations to obtain the estimation of them. Wang, Wang, and Wang (2016) discussed the interval estimation of location parameter when the variation coefficient and skewness parameter were known. Ye et al. (2020) studied the interval estimation and hypothesis testing of location parameter with unknown scale parameter and skewness parameter. Gui and Guo (2018) derived the explicit estimators of location parameter and scale parameter based on the approximate likelihood equations. The inferences on the location parameter vector in the multivariate skew-normal distribution with different conditions on scale parameter and shape parameter were studied by Ma et al. (2019) and Ma et al. (2020). However, the homogeneous test of location parameters in several populations is more common in practical applications. For example, the comparison of therapeutic effects of multiple drugs with the same function, the comparison of life spans of multiple batches of the same product, and the comparison of detection results of multiple institutions on the same material, and so on. But most existing researches on homogeneous test of location parameters assume that the populations follow the normal distribution, inverse Gaussian distribution or exponential distribution (e.g. Tian 2006; Krishnamoorthy, Lu, and Mathew 2007; Ma and Tian 2009; Shi and Lv 2012; Kharrati-Kopaei and Eftekhar 2017; Eftekhar, Sadooghi-Alvandi, and Kharrati-Kopaei 2018; Eftekhar and Kharrati-Kopaei 2019). However, the real data commonly show the characteristic of skew-normal distribution. In view of that, this paper studies the problem of homogeneous test of location parameters in several skew-normal populations with unknown scale parameters and skewness parameters.

This article is organized as follows. In Sec. 2, we construct the conditional test statistic for homogeneous test of location parameters in several skew-normal populations and prove its approximate distribution. In Sec. 3, the moment estimation and ML estimation of unknown parameters are given. In Sec. 4, the Bootstrap test statistics are established, which generalize the results given by Xu (2016) under the normal distribution. Section 5 shows Monte-Carlo simulation results based on two different approaches. In Sec. 6, the proposed approaches are used in two real examples of regional GDP of China and performance data of high-speed turbine bearings. Section 7 provides a summary of this paper.

2. Conditional test statistic

For convenience, let $\overset{asy}{\sim}$ denote the approximate distribution, and $D(X)$ denote the variance of random variable X. In this section, the conditional test statistic for homogeneous test of location parameters in k skew-normal populations is constructed, and then the relevant properties of skew-normal distribution are given.

Lemma 1. *Suppose* $X_j \overset{asy}{\sim} N(0,1)$, $j = 1, \dots, k$, *and* $X_1, \dots, X_k$ *are mutually independent to each other, then* $\sum_{j=1}^{k} X_j^2 \overset{asy}{\sim} \chi^2(k)$.

The proof of Lemma 1 is given in Appendix.

Suppose $Y_{i1}, \dots, Y_{in_i}$ is a group of random samples from the skew-normal distribution $SN(\xi_i, \eta_i^2, \alpha_i), i = 1, \dots, k$. The sample mean, the second-order, and third-order central moments can be expressed as

$$\bar{Y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij},\ \ S_{2i} = \frac{1}{n_i}\sum_{j=1}^{n_i}(Y_{ij}-\bar{Y}_i)^2,\ \ S_{3i} = \frac{1}{n_i}\sum_{j=1}^{n_i}(Y_{ij}-\bar{Y}_i)^3,\ \ i = 1,...,k. \tag{2}$$

Obviously, the moment generating function of Y_i is

$$M_{Y_i}(t) = 2\exp\left(t\xi_i + \frac{t^2\eta_i^2}{2}\right)\Phi(t\eta_i\delta_i),\ \ i = 1,...,k. \tag{3}$$

From (3), we can obtain

$$\begin{aligned}
\mathrm{E}(\mathrm{Y}_i) &= \mathrm{M}'_{Y_i}(t)|_{t=0} = \xi_i + b\eta_i\delta_i,\\
\mathrm{E}(\mathrm{Y}_i^2) &= \mathrm{M}''_{Y_i}(t)|_{t=0} = \xi_i^2 + 2b\xi_i\eta_i\delta_i + \eta_i^2,\\
\mathrm{E}(\mathrm{Y}_i^3) &= \mathrm{M}'''_{Y_i}(t)|_{t=0} = \xi_i^3 + 3b\xi_i^2\eta_i\delta_i + 3\xi_i\eta_i^2 + 3b\eta_i^3\delta_i - b\eta_i^3\delta_i^3,
\end{aligned}$$

where $\delta_i = \alpha_i/(1+\alpha_i^2)^{1/2}$, and $b = (2/\pi)^{1/2}$, $i = 1,...,k$. For a given (η_i,δ_i), the estimator of ξ_i and its variance can be expressed as

$$\hat{\xi}_i|_{(\eta_i,\delta_i)} = \bar{Y}_i - b\eta_i\delta_i,\quad D(\hat{\xi}_i|_{(\eta_i,\delta_i)}) = \frac{\eta_i^2(1-b^2\delta_i^2)}{n_i},\quad i = 1,...,k. \tag{4}$$

For $i = 1,...,k$, let $\xi_i = \xi + \Delta\xi_i$, where $\Delta\xi_i$ denotes the difference between the i-th population location parameter ξ_i and the common location parameter ξ. Then the homogeneous test of location parameters is equal to

$$H_0 : \Delta\xi_1 = \Delta\xi_2 = ... = \Delta\xi_k = 0 \qquad \text{vs} \qquad H_1 : \exists\ i,\ \Delta\xi_i \neq 0. \tag{5}$$

When the null hypothesis H_0 is true and (η_i,δ_i) is known, using the idea of Graybill–Deal estimation (Graybill and Deal 1959), the estimator of the common location parameter ξ is

$$\hat{\xi} = \frac{\sum_{i=1}^k \frac{1}{D(\hat{\xi}_i|_{(\eta_i,\delta_i)})}\hat{\xi}_i|_{(\eta_i,\delta_i)}}{\sum_{i=1}^k \frac{1}{D(\hat{\xi}_i|_{(\eta_i,\delta_i)})}} = \frac{\sum_{i=1}^k \frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}\hat{\xi}_i|_{(\eta_i,\delta_i)}}{\sum_{i=1}^k \frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}} = \sum_{i=1}^k \omega_i\hat{\xi}_i|_{(\eta_i,\delta_i)}, \tag{6}$$

where $\omega_i = \frac{\frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}}{\sum_{i=1}^k \frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}}$, $i = 1,...,k$.

Then, define a statistic

$$Z_i = \frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}\left(\hat{\xi}_i|_{(\eta_i,\delta_i)} - \sum_{i=1}^k \omega_i\hat{\xi}_i|_{(\eta_i,\delta_i)}\right),\quad i = 1,...,k. \tag{7}$$

Under H_0 in (5), the expectation and variance of Z_i are

$$\begin{aligned}
E(Z_i) &= \frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}\left(E(\hat{\xi}_i|_{(\eta_i,\delta_i)}) - \sum_{i=1}^k \omega_i E(\hat{\xi}_i|_{(\eta_i,\delta_i)})\right) = 0,\\
D(Z_i) &= \left(\frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}\right)^2 D\left((1-\omega_i)\hat{\xi}_i|_{(\eta_i,\delta_i)} + \sum_{j\neq i}^k \omega_j\hat{\xi}_j|_{(\eta_j,\delta_j)}\right)\\
&= \left(\frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}\right)^2\left[(1-\omega_i)^2\frac{\eta_i^2(1-b^2\delta_i^2)}{n_i} + \sum_{j\neq i}^k \omega_j^2\frac{\eta_j^2(1-b^2\delta_j^2)}{n_j}\right]\\
&= \left(\frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}\right)^2\left(\frac{\eta_i^2(1-b^2\delta_i^2)}{n_i} - \frac{1}{\sum_{i=1}^k \frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}}\right),\quad i = 1,...,k.
\end{aligned}$$

By the central limit theorem, we have

$$\frac{Z_i - E(Z_i)}{\sqrt{D(Z_i)}} = \frac{\xi_i|_{(\eta_i,\delta_i)} - \sum_{i=1}^{k}\omega_i\hat{\xi}_i|_{(\eta_i,\delta_i)}}{\sqrt{\frac{\eta_i^2(1-b^2\delta_i^2)}{n_i}(1-\omega_i)}}, \quad i = 1, \ldots, k.$$

When $n_i \to \infty$, the approximate distribution of $\frac{Z_i - E(Z_i)}{\sqrt{D(Z_i)}}$ is standard normal distribution $N(0,\ 1)$, $i = 1, \ldots, k$. Under H_0 in (5), define the conditional test statistic as

$$\begin{aligned} T &= \sum_{i=1}^{k}\left(\frac{\xi_i|_{(\eta_i,\delta_i)} - \sum_{i=1}^{k}\omega_i\hat{\xi}_i|_{(\eta_i,\delta_i)}}{\sqrt{\frac{\eta_i^2(1-b^2\delta_i^2)}{n_i}}(1-\omega_i)}\right)^2 \\ &= \sum_{i=1}^{k}\frac{n_i}{\eta_i^2(1-b^2\delta_i^2)}\frac{\left((\bar{Y}_i - b\eta_i\delta_i) - \sum_{i=1}^{k}\omega_i(\bar{Y}_i - b\eta_i\delta_i)\right)^2}{1-\omega_i}. \end{aligned} \tag{8}$$

Due to Lemma 1, we get $T \overset{asy}{\sim} \chi^2(k)$.

Denote

$$\bar{Y} = (\bar{Y}_1 - b\eta_1\delta_1, \bar{Y}_2 - b\eta_2\delta_2, \ldots, \bar{Y}_k - b\eta_k\delta_k)',$$

$$\Delta = diag\left(\frac{\eta_1^2(1-b^2\delta_1^2)}{n_1}, \frac{\eta_2^2(1-b^2\delta_2^2)}{n_2}, \ldots, \frac{\eta_k^2(1-b^2\delta_k^2)}{n_k}\right).$$

Obviously, the conditional test statistic T can be expressed as

$$T = (\Delta^{-1/2}\bar{Y})'AB^{-1}A(\Delta^{-1/2}\bar{Y}), \tag{9}$$

where $A = I_k - (\Delta^{-1/2}11'\Delta^{-1/2})/(1'\Delta^{-1}1)$ is an idempotent matrix with rank $k-1$, $B = I_k - \Delta^{-1}/(1'\Delta^{-1}1)$, and I_k is an identity matrix of order k. By the properties of identity matrix and orthogonal decomposition, we have $A = P\begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}$, where P is an orthogonal matrix. Accordingly, we obtain

$$AB^{-1}A = P\begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}B^{-1}P\begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}.$$

Let $Z = \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}B^{-1}P\begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}$. Based on the orthogonal decomposition, we have $Z = Q\Lambda Q^{-1}$, where $\Lambda = \text{diag}\ (\lambda_1, \ldots, \lambda_k)$, and Q is the eigenvector matrix corresponding to Λ. Thus, Equation (9) can be expressed as

$$T = (\Delta^{-1/2}\bar{Y})'PQ\Lambda Q^{-1}P^{-1}(\Delta^{-1/2}\bar{Y}). \tag{10}$$

Theorem 1. *Let $\tau = \zeta'\Delta^{-1}\zeta$, where $\zeta = (\xi_1, \ldots, \xi_k)'$. If $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$, then $T \sim \sum_{i=1}^{k}\lambda_i\chi_i^2(1,\tau)$, where $\chi_i^2(1,\tau)$ represents the non-central χ^2 distribution with degree of freedom 1 and non-centrality parameter τ, $i = 1, \ldots, k$. And $\chi_1^2(1,\tau), \ldots, \chi_k^2(1,\tau)$ are mutually independent to each other.*

The proof of Theorem 1 is given in Appendix.

Remark 1. If $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$, then T in Equation (8) can be expressed as $T = \sum_{i=1}^{k}\frac{n_i}{\eta_i^2}\frac{(\bar{Y}_i - \sum_{i=1}^{k}\omega_{1i}\bar{Y}_i)^2}{1-\omega_{1i}}$, where $\omega_{1i} = \left(\frac{n_i}{\eta_i^2}\right)/\left(\sum_{i=1}^{k}\frac{n_i}{\eta_i^2}\right)$, $i = 1, \ldots, k$. By Theorem 1, T can be used as the conditional test statistic of homogeneous test of means in several normal populations. Hence, T in Equation (8) is an extension of the result given by Xu (2016).

3. Parameter estimation

For $i = 1, ..., k$, it is well-known that (η_i^2, δ_i) is often unknown in practical issues. Then we consider the parameter estimation problem of (η_i^2, δ_i).

Theorem 2. *If $Y_i \sim SN(\xi_i, \eta_i^2, \alpha_i)$, then the moment estimators of $(\xi_i, \eta_i^2, \alpha_i)$ satisfy*

$$\hat{\xi}_i = \bar{Y}_i - cS_{3i}^{1/3}, \ \ \hat{\eta}_i^2 = S_{2i} + c^2 S_{3i}^{2/3}, \ \ \hat{\alpha}_i = \hat{\delta}_i/(1-\hat{\delta}_i^2)^{1/2}, \ i = 1, ..., k, \tag{11}$$

where $c = [2/(4-\pi)]^{1/3}$ and $\hat{\delta}_i = \frac{cS_{3i}^{1/3}}{b(S_{2i}+c^2S_{3i}^{2/3})^{1/2}}$.

The proof of Theorem 2 is given in Appendix.

Pewsey (2000) proved that the results of using numerical techniques to maximize the log-likelihood for direct parameters $(\xi_i, \eta_i^2, \alpha_i)$ may be highly misleading as for this case no unique solution exists. For this, we derive the ML estimators of the unknown parameters based on the method of centered parametrization by Azzalini (1985), Azzalini and Capitanio (2014) and Pewsey (2000). Denote

$$W_i = \frac{Y_i - \xi_i}{\eta_i} \sim SN(\alpha_i), \quad Y_{Ci} = \mu_i + \sigma_i\left(\frac{W_i - E(W_i)}{\sqrt{D(W_i)}}\right) \sim SN_C(\mu_i, \sigma_i^2, \gamma_i), \tag{12}$$

where $SN_C(\mu_i, \sigma_i^2, \gamma_i)$ represents the skew-normal distribution with mean $\mu_i \in R$, variance $\sigma_i^2 \in R^+$, and skewness coefficient γ_i, $i = 1, ..., k$. Pewsey (2006) introduced the following relationship between the direct parameter $(\xi_i, \eta_i^2, \alpha_i)$ and the centered one $(\mu_i, \sigma_i^2, \gamma_i)$

$$\xi_i = \mu_i - c\gamma_i^{1/3}\sigma_i, \quad \eta_i^2 = \sigma_i^2(1 + c_i^2\gamma_i^{2/3}), \quad \alpha_i = \frac{c\gamma_i^{1/3}}{\sqrt{b^2 + c^2(b^2-1)\gamma_i^{2/3}}}, \quad i = 1, ..., k. \tag{13}$$

Theorem 3. *Suppose $Y_i \sim SN(\xi_i, \eta_i^2, \alpha_i)$. Let*

$$W_i = \frac{Y_i - \xi_i}{\eta_i} \quad \text{and} \quad Y_{Ci} = \mu_i + \sigma_i\left(\frac{W_i - E(W_i)}{\sqrt{D(W_i)}}\right),$$

then $Y_{Ci} = Y_i$, $i = 1, ..., k$.

The proof of Theorem 3 is given in Appendix.

Remark 2. If $|\alpha_i| \to \infty$, then $|\delta_i| \to 1$. By (A5) in Appendix, we have $\gamma_i \in (-0.99527, 0.99527)$, $i = 1, ..., k$.

Now consider the ML estimators of the centered parameter $(\mu_i, \sigma_i^2, \gamma_i)$, $i = 1, ..., k$. Denote the observed value of $(\bar{Y}_{Ci}, S_{C2i}, S_{C3i})$ by $(\bar{y}_{Ci}, s_{C2i}, s_{C3i})$. Similarly, let $Y_{sij} = (Y_{Cij} - \bar{y}_{Ci})/\sqrt{s_{C2i}}$, where $Y_{si1}, ..., Y_{sin_i}$ is a group of samples from $Y_{si} \sim SN_C(\mu_{si}, \sigma_{si}^2, \gamma_i)$ with $\mu_{si} = (\mu_i - \bar{y}_{Ci})/\sqrt{s_{C2i}}$ and $\sigma_{si} = \sigma_i/\sqrt{s_{C2i}}$, $i = 1, ..., k$, $j = 1, ..., n_i$. The density function of Y_{si} is obtained as follows

$$\begin{aligned} f(y_{si}; \mu_{si}, \sigma_{si}^2, \gamma_i) &= \frac{2}{\sigma_{si}\sqrt{s_{C2i}\left(1 + c^2\gamma_i^{2/3}\right)}}\phi\left[\left(\frac{y_{si} - \mu_{si}}{\sigma_{si}} + c\gamma_i^{1/3}\right)\frac{1}{\sqrt{1 + c^2\gamma_i^{2/3}}}\right] \\ &\times \Phi\left\{\left(\frac{y_{si} - \mu_{si}}{\sigma_{si}} + c\gamma_i^{1/3}\right)\frac{c\gamma_i^{1/3}}{\sqrt{\left(1 + c^2\gamma_i^{2/3}\right)\left[b^2 + c^2\gamma_i^{2/3}(b^2-1)\right]}}\right\}. \end{aligned} \tag{14}$$

By (14), we derive the logarithmic likelihood function (without constant terms)

$$
\begin{aligned}
l(y_{si1},\dots,y_{sin_i};\mu_{si},\sigma_{si}^2,\gamma_i) &= -n_i\log\sigma_{si}-\frac{n_i}{2}\log\left(1+c^2\gamma_i^{2/3}\right)\\
&+\sum_{i=1}^{n_i}\log\phi\left[\left(\frac{y_{sij}-\mu_{si}}{\sigma_{si}}+c\gamma_i^{1/3}\right)\frac{1}{\left(1+c^2\gamma_i^{2/3}\right)^{1/2}}\right]\\
&+\sum_{i=1}^{n_i}\log\Phi\left\{\frac{\frac{(y_{sij}-\mu_{si})c\gamma_i^{1/3}}{\sigma_{si}}+c^2\gamma_i^{2/3}}{\left(1+c^2\gamma_i^{2/3}\right)^{1/2}\left[b^2+c^2\gamma_i^{2/3}(b^2-1)\right]^{1/2}}\right\},\quad i=1,\dots,k.
\end{aligned}
\tag{15}
$$

Therefore, let $(\tilde{\mu}_{si}^*,\tilde{\sigma}_{si}^{*^2},\tilde{\gamma}_i^*)$ denote the ML estimate of $(\mu_{si},\sigma_{si}^2,\gamma_i)$ in (15) with default starting value given by the moment estimates of $(\mu_{si},\sigma_{si}^2,\gamma_i)$. Namely

$$
\hat{\mu}_{si}=-cS_{C2i}^{-1/2}S_{C3i}^{1/3},\quad \hat{\sigma}_{si}^2=1+cS_{C2i}^{-1}S_{C3i}^{2/3},\quad \hat{\gamma}_i=\frac{b\hat{\delta}_i^3(2b^2-1)}{\left(1-b^2\hat{\delta}_i^2\right)^{3/2}},\quad i=1,\dots,k.
$$

Further, the ML estimates of (μ_i,σ_i^2) are available, namely

$$
\tilde{\mu}_i^*=\bar{y}_{Ci}+s_{C2i}^{1/2}\tilde{\mu}_{si}^*,\quad \tilde{\sigma}_i^{*^2}=s_{C2i}\tilde{\sigma}_{si}^{*^2},\quad i=1,\dots,k.
$$

By (13), we obtain the ML estimates of the direct parameter $(\xi_i,\eta_i^2,\alpha_i)$ as follows

$$
\tilde{\xi}_i^*=\tilde{\mu}_i^*-c\tilde{\gamma}_i^{*1/3}\tilde{\sigma}_i^*,\quad \tilde{\eta}_i^{*2}=\tilde{\sigma}_i^{*2}(1+c^2\tilde{\gamma}_i^{*^{2/3}}),\quad \tilde{\alpha}_i^*=\frac{c\tilde{\gamma}_i^{*^{1/3}}}{\sqrt{b^2+c^2(b^2-1)\tilde{\gamma}_i^{*^{2/3}}}}. \tag{16}
$$

Then the ML estimate of δ_i is $\tilde{\delta}_i^*=\tilde{\alpha}_i^*/(1+\tilde{\alpha}_i^{*2})^{1/2},\ i=1,\dots,k$. Accordingly, we have the following results.

Theorem 4. *Suppose $(\tilde{\mu}_{si},\tilde{\sigma}_{si}^2,\tilde{\gamma}_i)$ be the ML estimator corresponding to $(\tilde{\mu}_{si}^*,\tilde{\sigma}_{si}^{*^2},\tilde{\gamma}_i^*)$, then we get the ML estimators of the direct parameter $(\xi_i,\eta_i^2,\alpha_i)$*

$$
\tilde{\xi}_i=\tilde{\mu}_i-c\tilde{\gamma}_i^{1/3}\tilde{\sigma}_i,\quad \tilde{\eta}_i^2=\tilde{\sigma}_i^2(1+c^2\tilde{\gamma}_i^{2/3}),\quad \tilde{\alpha}_i=\frac{c\tilde{\gamma}_i^{1/3}}{\sqrt{b^2+c^2(b^2-1)\tilde{\gamma}_i^{2/3}}}, \tag{17}
$$

where $(\tilde{\mu}_i,\tilde{\sigma}_i^2)$ denotes the ML estimator corresponding to $(\tilde{\mu}_i^*,\tilde{\sigma}_i^{*^2})$. Further, the ML estimator of δ_i is $\tilde{\delta}_i=\tilde{\alpha}_i/(1+\tilde{\alpha}_i^2)^{1/2},\ i=1,\dots,k$.

4. Bootstrap test

For hypothesis testing problem (5), we can establish the Bootstrap test statistics. First, by replacing (η_i^2,δ_i) with its moment estimators and ML estimators in (8), we construct the following test statistics

$$
T_1=\sum_{i=1}^k\frac{n_i}{\hat{\eta}_i^2(1-b^2\hat{\delta}_i^2)}\frac{\left(\left(\bar{Y}_i-b\hat{\eta}_i\hat{\delta}_i\right)-\sum_{i=1}^k\hat{\omega}_i\left(\bar{Y}_i-b\hat{\eta}_i\hat{\delta}_i\right)\right)^2}{1-\hat{\omega}_i}, \tag{18}
$$

$$
T_2=\sum_{i=1}^k\frac{n_i}{\tilde{\eta}_i^2(1-b^2\tilde{\delta}_i^2)}\frac{\left(\left(\bar{Y}_i-b\tilde{\eta}_i\tilde{\delta}_i\right)-\sum_{i=1}^k\tilde{\omega}_i\left(\bar{Y}_i-b\tilde{\eta}_i\tilde{\delta}_i\right)\right)^2}{1-\tilde{\omega}_i}, \tag{19}
$$

where $\hat{\omega}_i=\frac{\frac{n_i}{\hat{\eta}_i^2(1-b^2\hat{\delta}_i^2)}}{\sum_{i=1}^k\frac{n_i}{\hat{\eta}_i^2(1-b^2\hat{\delta}_i^2)}}$, and the definition of $\tilde{\omega}_i$ is similar to $\hat{\omega}_i$.

Under H_0 in (5), denote $Y_{BMij} \sim SN(\hat{\xi}^*, \hat{\eta}_i^{*2}, \hat{\alpha}_i^*)$, $i = 1, ..., k$, $j = 1, ..., n_i$, where

$$\hat{\xi}^* = \sum_{i=1}^{k} \hat{\omega}_i^* \hat{\xi}_i^*|_{(\hat{\eta}_i^*, \hat{\delta}_i^*)}, \quad \hat{\omega}_i^* = \frac{\frac{n_i}{\hat{\eta}_i^{*2}(1-b^2\hat{\delta}_i^{*2})}}{\sum_{i=1}^{k} \frac{n_i}{\hat{\eta}_i^{*2}(1-b^2\hat{\delta}_i^{*2})}}, \quad \hat{\xi}_i^*|_{(\hat{\eta}_i^*, \hat{\delta}_i^*)} = \bar{y}_i - b\hat{\eta}_i^* \hat{\delta}_i^*. \tag{20}$$

The sample mean, the second-order and third-order central moments can be expressed as $(\bar{Y}_{BMi}, S_{BM2i}, S_{BM3i})$, $i = 1, ..., k$. By Theorem 2, we obtain the moment estimators of (η_i^2, δ_i) as

$$\hat{\eta}_{BMi}^2 = S_{BM2i} + c^2 S_{BM3i}^{2/3}, \quad \hat{\delta}_{BMi} = \frac{c S_{BM3i}^{1/3}}{b\sqrt{S_{BM2i} + c^2 S_{BM3i}^{2/3}}}, \quad i = 1, ..., k. \tag{21}$$

Similarly, suppose $Y_{BLij} \sim SN(\tilde{\xi}^*, \tilde{\eta}_i^{*2}, \tilde{\alpha}_i^*), i = 1, ..., k$, $j = 1, ..., n_i$, where the definition of $\tilde{\xi}^*$ is analogous to $\hat{\xi}^*$ in (20), and its sample mean is $\bar{Y}_{BLi}$. By theorem 4, $(\tilde{\eta}_{BLi}^2, \tilde{\delta}_{BLi})$ denotes the ML estimator of (η_i^2, δ_i), $i = 1, ..., k$. Furthermore, similar to (18) and (19), construct the Bootstrap test statistics

$$T_{B1} = \sum_{i=1}^{k} \frac{n_i}{\hat{\eta}_{BMi}^2 (1 - b^2 \hat{\delta}_{BMi}^2)} \frac{\left(\left(\bar{Y}_{BMi} - b\hat{\eta}_{BMi}\hat{\delta}_{BMi}\right) - \sum_{i=1}^{k} \hat{\omega}_{BMi}\left(\bar{Y}_{BMi} - b\hat{\eta}_{BMi}\hat{\delta}_{BMi}\right)\right)^2}{1 - \hat{\omega}_{BMi}}, \tag{22}$$

$$T_{B2} = \sum_{i=1}^{k} \frac{n_i}{\tilde{\eta}_{BLi}^2 (1 - b^2 \tilde{\delta}_{BLi}^2)} \frac{\left((\bar{Y}_{BLi} - b\tilde{\eta}_{BLi}\tilde{\delta}_{BLi}) - \sum_{i=1}^{k} \tilde{\omega}_{BLi}(\bar{Y}_{BLi} - b\tilde{\eta}_{BLi}\tilde{\delta}_{BLi})\right)^2}{1 - \tilde{\omega}_{BLi}}, \tag{23}$$

where $\hat{\omega}_{BMi} = \frac{\frac{n_i}{\hat{\eta}_{BMi}^{*2}(1-b^2\hat{\delta}_{BMi}^{*2})}}{\sum_{i=1}^{k} \frac{n_i}{\hat{\eta}_{BMi}^{*2}(1-b^2\hat{\delta}_{BMi}^{*2})}}$, and the definition of $\tilde{\omega}_{BLi}$ is similar to $\hat{\omega}_{BMi}$. Then, based on T_{B1} and T_{B2}, we have

$$p_i = 2\min\{P(T_{Bi} > t_i), P(T_{Bi} < t_i)\}, \quad i = 1, 2, \tag{24}$$

where t_1 and t_2 denote the observed values of T_1 and T_2, respectively. The null hypothesis H_0 in (5) is rejected whenever the above p-values are less than the nominal significance level of β, which means that there are at least two location parameters are unequal.

Remark 3. If $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$, then T_{B1} can be expressed as $T_{B1} = \sum_{i=1}^{k} \frac{n_i}{\hat{\eta}_{BMi}^2} \frac{(\bar{Y}_{BMi} - \sum_{i=1}^{k} \hat{\omega}_{1BMi}\bar{Y}_{BMi})^2}{1-\hat{\omega}_{1BMi}}$, where $\hat{\omega}_{1BMi} = \left(\frac{n_i}{\hat{\eta}_{BMi}^2}\right) / \left(\sum_{i=1}^{k} \frac{n_i}{\hat{\eta}_{BMi}^2}\right)$, $i = 1, ..., k$. Thus, T_{B1} can be used as the Bootstrap test statistic for homogeneous test of means in several normal populations, namely the result of Xu (2016).

5. Monte-Carlo simulation

In this section, the Monte-Carlo simulation is adopted to study numerically the Type I error probability and power of the above test approaches. For hypothesis testing problem (5), we only provide the steps of the Bootstrap approach based on the moment estimator in k skew-normal populations as follows.

Step 1: For a given $(n_i, \xi_i, \eta_i^2, \alpha_i)$, generate a group of random samples $Y_{ij} \sim SN(\xi_i, \eta_i^2, \alpha_i)$, and $(\bar{Y}_i, S_{2i}, S_{3i})$ is computed by (2), $i = 1, ..., k$, $j = 1, ..., n_i$.

Step 2: Using (20) and (A4) in Appendix, $(\hat{\xi}^*, \hat{\xi}_i^*, \hat{\eta}_i^{*2}, \hat{\alpha}_i^*)$, the feasible estimate of $(\xi, \xi_i, \eta_i^2, \alpha_i)$ is computed, $i = 1, ..., k$. Further, T_1 is computed by (18).

Table 1. Simulation results on Type I error probability as $k=2$ at the nominal significance level of 5%.

	N1		N2		N3		N4		N5	
$(\alpha_1,\alpha_2)=(3, 4)$										
	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
η_1^2	0.0416	0.0644	0.0364	0.0668	0.0300	0.0588	0.0240	0.0396	0.0328	0.0248
η_2^2	0.0416	0.0644	0.0364	0.0668	0.0300	0.0588	0.0240	0.0396	0.0328	0.0248
η_3^2	0.0476	0.0888	0.0472	0.0748	0.0476	0.0672	0.0328	0.0404	0.0340	0.0228
η_4^2	0.0532	0.0892	0.0452	0.0748	0.0476	0.0652	0.0336	0.0436	0.0304	0.0240
η_5^2	0.0532	0.0892	0.0452	0.0748	0.0476	0.0652	0.0336	0.0436	0.0304	0.0240
$(\alpha_1,\alpha_2)=(6, 7)$										
	N1		N2		N3		N4		N5	
	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
η_1^2	0.0356	0.0420	0.0304	0.0412	0.0304	0.0412	0.0304	0.0412	0.0304	0.0412
η_2^2	0.0356	0.0420	0.0304	0.0412	0.0304	0.0412	0.0304	0.0412	0.0304	0.0412
η_3^2	0.0336	0.0644	0.0304	0.0520	0.0304	0.0520	0.0304	0.0520	0.0304	0.0520
η_4^2	0.0356	0.0640	0.0340	0.0420	0.0340	0.0420	0.0340	0.0420	0.0340	0.0420
η_5^2	0.0356	0.0640	0.0340	0.0420	0.0340	0.0420	0.0340	0.0420	0.0340	0.0420

Note: $\eta_1^2=(0.2^2,0.6^2), \eta_2^2=(0.3^2,0.9^2), \eta_3^2=(0.5^2,0.7^2), \eta_4^2=(0.9^2,1.2^2), \eta_5^2=(1.5^2,2^2)$; $N1=(30,40), N2=(40,60), N3=(50,80), N4=(80,120), N5=(180,240)$.

Table 2. Simulation results on Type I error probability as $k=3$ at the nominal significance level of 5%.

	N1		N2		N3		N4		N5	
$(\alpha_1,\alpha_2,\alpha_3)$=(2–4)										
	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
η_1^2	0.0372	0.0784	0.0308	0.0632	0.0312	0.0548	0.0260	0.0404	0.0252	0.0180
η_2^2	0.0320	0.0756	0.0324	0.0648	0.0408	0.0592	0.0392	0.0420	0.0280	0.0184
η_3^2	0.0388	0.0748	0.0408	0.0652	0.0504	0.0636	0.0420	0.0472	0.0264	0.0364
η_4^2	0.0404	0.0704	0.0432	0.0660	0.0500	0.0624	0.0396	0.0464	0.0272	0.0400
η_5^2	0.0404	0.0704	0.0432	0.0660	0.0500	0.0624	0.0396	0.0464	0.0272	0.0400
$(\alpha_1,\alpha_2,\alpha_3)$=(5, 6)										
	N1		N2		N3		N4		N5	
	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
η_1^2	0.0376	0.0612	0.0304	0.0312	0.0268	0.0236	0.0172	0.0120	0.0208	0.0102
η_2^2	0.0380	0.0576	0.0240	0.0352	0.0228	0.0224	0.0148	0.0128	0.0260	0.0104
η_3^2	0.0344	0.0544	0.0264	0.0364	0.0268	0.0248	0.0232	0.0328	0.0252	0.0500
η_4^2	0.0352	0.0588	0.0292	0.0344	0.0244	0.0260	0.0244	0.0340	0.0252	0.0568
η_5^2	0.0352	0.0588	0.0292	0.0344	0.0244	0.0260	0.0244	0.0340	0.0252	0.0568

Note: $\eta_1^2=(0.2^2,0.4^2,0.6^2), \eta_2^2=(0.3^2,0.5^2,0.7^2), \eta_3^2=(0.5^2,0.7^2,0.9^2), \eta_4^2=(0.9^2,1.2^2,1.5^2), \eta_5^2=(1.5^2,2^2,2.5^2)$; $N1=(30,40,40), N2=(60,60,80), N3=(80,100,120), N4=(120,150,150), N5=(180,240,300)$.

Step 3: Under H_0, generate the Bootstrap samples $Y_{BMij}\sim SN(\hat{\xi}^*,\hat{\eta}_i^{*2},\hat{\alpha}_i^*)$, and compute $(\bar{Y}_{BMi},S_{2BMi},S_{3BMi})$ *for* $i=1,...,k,\ \ j=1,...,n_i$.

Step 4: From (A4), $(\hat{\eta}_{BMi}^{*2},\hat{\alpha}_{BMi}^*)$, the moment estimate of (η^2,α) from Bootstrap samples is computed, $i=1,...,k$. Then T_{B1} is obtained by (22).

Step 5: Repeat Steps 3-4 n_1 times and compute p_1 by (24). If $p_1<0.05$, then $Q=1$; otherwise, $Q=0$.

Step 6: Repeat Steps 1-5 n_2 times and we get $Q_1,...,Q_{n_2}$. Then the Type I error probability is $\frac{1}{n_2}\sum_{i=1}^{n_2}Q_i$. Based on the above steps, the power of hypothesis testing problem (5) under H_1 can be obtained similarly.

Table 3. Simulation results on Type I error probability as $k = 5$ at the nominal significance level of 5%.

$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$=(3–6)										
	N1		N2		N3		N4		N5	
	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
η_1^2	0.0408	0.0444	0.0364	0.0544	0.0224	0.0464	0.0220	0.0548	0.0220	0.0600
η_2^2	0.0384	0.0456	0.0320	0.0468	0.0248	0.0408	0.0268	0.0360	0.0240	0.0356
η_3^2	0.0392	0.0516	0.0328	0.0500	0.0244	0.0444	0.0312	0.0332	0.0280	0.0308
η_4^2	0.0400	0.0560	0.0368	0.0548	0.0252	0.0464	0.0332	0.0308	0.0308	0.0324
η_5^2	0.0424	0.0560	0.0364	0.0588	0.0276	0.0536	0.0324	0.0392	0.0284	0.0364
$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$=(5–8)										
	N1		N2		N3		N4		N5	
	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
η_1^2	0.0396	0.0440	0.0332	0.0560	0.0204	0.0476	0.0220	0.0560	0.0220	0.0616
η_2^2	0.0380	0.0436	0.0308	0.0468	0.0220	0.0300	0.0252	0.0296	0.0220	0.0284
η_3^2	0.0372	0.0492	0.0300	0.0420	0.0232	0.0316	0.0288	0.0284	0.0252	0.0236
η_4^2	0.0428	0.0516	0.0352	0.0476	0.0232	0.0336	0.0300	0.0256	0.0248	0.0268
η_5^2	0.0448	0.0540	0.0348	0.0552	0.0220	0.0396	0.0316	0.0340	0.0284	0.0328

Note: $\eta_1^2 = (0.1^2, 0.3^2, 0.5^2, 0.7^2, 0.9^2), \eta_2^2 = (0.2^2, 0.4^2, 0.6^2, 0.8^2, 1^2), \eta_3^2 = (0.3^2, 0.5^2, 0.7^2, 0.9^2, 0.9^2), \eta_4^2 = (0.4^2, 0.6^2, 0.8^2, 1^2, 1.2^2)$, $\eta_5^2 = (0.5^2, 0.7^2, 0.9^2, 1.1^2, 1.3^2)$; $N1 = (30, 30, 40, 40, 50)$, $N2 = (40, 50, 60, 60, 70), N3 = (60, 70, 80, 80, 90)$, $N4 = (80, 90, 90, 100, 120)$, $N5 = (90, 100, 100, 120, 150)$.

Table 4. Simulated powers of hypothesis testing problem (5) as $k = 2$ *and* $\xi_1 = 2$.

(η_1^2, η_2^2)=(0.3,0.8)												
	(α_1, α_2)=(3, 4)						(α_1, α_2)=(6, 7)					
	N1		N2		N3		N1		N2		N3	
ξ_2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
2.5	0.1928	0.1072	0.3592	0.0780	0.6292	0.0620	0.2388	0.0704	0.5096	0.0624	0.8096	0.0536
2.6	0.3464	0.1220	0.5532	0.0992	0.7696	0.0820	0.4428	0.0884	0.7076	0.0720	0.9208	0.1256
2.7	0.5044	0.1520	0.7356	0.1340	0.9128	0.1812	0.6328	0.1168	0.8748	0.1396	0.9892	0.2944
2.8	0.6880	0.2032	0.8844	0.2224	0.9832	0.3656	0.8124	0.1932	0.9684	0.2768	0.9992	0.5236
2.9	0.8268	0.2792	0.9624	0.3624	0.9968	0.5768	0.9184	0.2948	0.9956	0.4560	0.9996	0.7308
(η_1^2, η_2^2)=(0.5,0.7)												
	(α_1, α_2)=(3, 4)						(α_1, α_2)=(6, 7)					
	N1		N2		N3		N1		N2		N3	
ξ_2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
2.5	0.3344	0.1292	0.5364	0.1048	0.7684	0.1480	0.4344	0.0972	0.6976	0.0964	0.7684	0.1480
2.6	0.5092	0.1700	0.7196	0.1784	0.9144	0.3092	0.6560	0.1540	0.8760	0.2348	0.9144	0.3092
2.7	0.6808	0.2440	0.8632	0.3228	0.9776	0.5460	0.8216	0.2656	0.9692	0.4360	0.9776	0.5460
2.8	0.8092	0.3524	0.9408	0.5052	0.9908	0.7552	0.9236	0.4148	0.9928	0.6480	0.9908	0.7552
2.9	0.8980	0.4944	0.9732	0.6792	0.9972	0.8896	0.9732	0.5812	0.9980	0.8100	0.9972	0.8896
(η_1^2, η_2^2)=(0.6,0.6)												
	(α_1, α_2)=(3, 4)						(α_1, α_2)=(6, 7)					
	N1		N2		N3		N1		N2		N3	
ξ_2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
2.5	0.3944	0.1168	0.5640	0.2092	0.8384	0.3516	0.5500	0.1148	0.7492	0.2908	0.9548	0.5532
2.6	0.6416	0.1940	0.7952	0.3112	0.9572	0.4800	0.7892	0.2180	0.9260	0.4232	0.9972	0.7028
2.7	0.7984	0.3156	0.8944	0.4596	0.9724	0.6820	0.9148	0.3796	0.9756	0.6048	0.9992	0.8660
2.8	0.8612	0.4968	0.9248	0.6428	0.9764	0.8736	0.9488	0.6016	0.9880	0.7884	0.9996	0.9628
2.9	0.8860	0.6548	0.9424	0.8200	0.9876	0.9604	0.9612	0.7792	0.9924	0.9212	1.0000	0.9952

Note: $N1 = (20, 30), N2 = (30, 40), N3 = (50, 60)$.

In simulation, the parameters and sample sizes are set as follows. First, Let the nominal significance level be 5%, the number of inner loops n_1 and number of outer loops n_2 both be 2500, and

Table 5. Simulated powers of hypothesis testing problem (5) as $k = 3$ *and* $\xi_1 = \xi_2 = 2$.

$(\eta_1^2, \eta_2^2, \eta_3^2)$=(0.1,0.4,0.7)												
	$(\alpha_1, \alpha_2, \alpha_3)$=(2–4)						$(\alpha_1, \alpha_2, \alpha_3)$=(5, 6)					
	N1		N2		N3		N1		N2		N3	
ξ_3	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
2.6	0.3616	0.1112	0.6604	0.1152	0.8800	0.1960	0.5028	0.0780	0.8344	0.1636	0.9736	0.2516
2.7	0.5408	0.1556	0.8512	0.2416	0.9628	0.3920	0.6752	0.1296	0.9312	0.3376	0.9904	0.4692
2.8	0.6956	0.2320	0.9524	0.3880	0.9924	0.5576	0.7996	0.2376	0.9776	0.5012	0.9980	0.6976
2.9	0.8272	0.3260	0.9856	0.5092	0.9992	0.6924	0.8924	0.3708	0.9936	0.6388	0.9996	0.8356
3.0	0.9124	0.4356	0.9968	0.6140	1.0000	0.7860	0.9484	0.4960	0.9976	0.7368	1.0000	0.9144
$(\eta_1^2, \eta_2^2, \eta_3^2)$=(0.4,0.8,0.8)												
	$(\alpha_1, \alpha_2, \alpha_3)$=(2–4)						$(\alpha_1, \alpha_2, \alpha_3)$=(5, 6)					
	N1		N2		N3		N1		N2		N3	
ξ_3	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
2.6	0.1500	0.0812	0.4984	0.0516	0.8056	0.0584	0.2012	0.0548	0.7524	0.0540	0.9572	0.1048
2.7	0.2504	0.0928	0.7112	0.0640	0.9140	0.1132	0.3668	0.0596	0.8860	0.1180	0.9780	0.2100
2.8	0.4052	0.1112	0.8636	0.1408	0.9588	0.2580	0.5628	0.0924	0.9472	0.2480	0.9908	0.3796
2.9	0.5556	0.1436	0.9416	0.2644	0.9872	0.4256	0.7332	0.1356	0.9784	0.3992	0.9960	0.5708
3.0	0.7132	0.1972	0.9720	0.4140	0.9960	0.5896	0.8548	0.2116	0.9892	0.5628	0.9988	0.7260
$(\eta_1^2, \eta_2^2, \eta_3^2)$=(0.6,0.7,0.8)												
	$(\alpha_1, \alpha_2, \alpha_3)$=(2–4)						$(\alpha_1, \alpha_2, \alpha_3)$=(5, 6)					
	N1		N2		N3		N1		N2		N3	
ξ_3	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
2.6	0.1544	0.0768	0.4596	0.0540	0.7412	0.0912	0.2504	0.0596	0.7228	0.0964	0.9384	0.2436
2.7	0.2884	0.0924	0.6552	0.0788	0.8704	0.1480	0.4280	0.0676	0.8580	0.1832	0.9704	0.3856
2.8	0.4284	0.1108	0.8112	0.1264	0.9420	0.2308	0.6136	0.0984	0.9316	0.2880	0.9892	0.5572
2.9	0.5632	0.1484	0.8952	0.2156	0.9752	0.3696	0.7552	0.1508	0.9712	0.4200	0.9960	0.6892
3.0	0.6884	0.2028	0.9416	0.3316	0.9904	0.5068	0.8520	0.2292	0.9904	0.5536	0.9988	0.7852

Note: $N1 = (30, 40, 40), N2 = (60, 60, 80), N3 = (80, 100, 120)$.

$\xi = 2$. Second, for two populations, we set $\eta_1^2 = (0.2^2, 0.6^2), \eta_2^2 = (0.3^2, 0.9^2), \eta_3^2 = (0.5^2, 0.7^2)$, $\eta_4^2 = (0.9^2, 1.2^2)$, $\eta_5^2 = (1.5^2, 2^2)$, $(\alpha_1, \alpha_2) = (3, 4), (6, 7)$, and $(n_1, n_2) = (30, 40), (40, 60)$, $(50, 80)$, $(80, 120)$, $(180, 240)$.

For three populations, we set $\eta_1^2 = (0.2^2, 0.4^2, 0.6^2), \eta_2^2 = (0.3^2, 0.5^2, 0.7^2), \eta_3^2 = (0.5^2, 0.7^2, 0.9^2), \eta_4^2 = (0.9^2, 1.2^2, 1.5^2), \eta_5^2 = (1.5^2, 2^2, 2.5^2)$, $(\alpha_1, \alpha_2, \alpha_3) = (2, 3, 4), (5, 5, 6)$, and $(n_1, n_2, n_3) = (30, 40, 40), (60, 60, 80), (80, 100, 120), (120, 150, 150), (180, 240, 300)$.

For five populations, we set $\eta_1^2 = (0.1^2, 0.3^2 0.5^2, 0.7^2, 0.9^2), \eta_2^2 = (0.2^2, 0.4^2, 0.6^2, 0.8^2, 1^2), \eta_3^2 = (0.3^2, 0.5^2, 0.7^2, 0.9^2, 0.9^2)$, $\eta_4^2 = (0.4^2, 0.6^2, 0.8^2, 1^2, 1.2^2), \eta_5^2 = (0.5^2, 0.7^2, 0.9^2, 1.1^2, 1.3^2)$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (3, 4, 4, 5, 6), (5, 5, 6, 7, 8)$, and $(n_1, n_2, n_3, n_4, n_5) = (30, 30, 40, 40, 50)$, $(40, 50, 60, 60, 70)$, $(60, 70, 80, 80, 90), (80, 90, 90, 100, 120)$, $(90, 100, 100, 120, 150)$.

For hypothesis testing problem (5), Tables 1–3 present the simulated Type I error probabilities of the proposed two approaches p_1 and p_2. No matter in two populations, three populations or five populations, the actual levels of p_1 are all strictly less than the nominal significance level of 5%, which can effectively control the Type I error probabilities under certain parameter settings. The performance of p_2 is liberal relatively when sample sizes and skewness parameters are small. With the increase of skewness parameters, p_2 appears conservative individually. However, in the case of five populations, the actual levels of p_2 are controlled around the nominal significance level of 5%, and perform well generally.

Tables 4–6 present the simulated powers of the proposed approaches. As ξ_i departs from the null hypothesis, the powers of the proposed approaches are significantly improved no matter in two populations, three populations or five populations. However, the power of the Bootstrap test statistic based on the moment estimator performs significantly better than that based on the ML estimator. In terms of the Bootstrap test statistic based on the ML estimator, the power rise more

Table 6. Simulated powers of hypothesis testing problem (5) as $k = 5$ *and* $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 2$.

	$(\eta_1^2, \eta_2^2, \eta_3^2, \eta_4^2, \eta_5^2)$=(0.1,0.3,0.5,0.7,0.9)											
	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$=(3–6)						$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$=(5–8)					
	N1		N2		N3		N1		N2		N3	
ξ_5	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
3.1	0.4940	0.0996	0.7552	0.1296	0.8148	0.1840	0.5544	0.0932	0.8200	0.1572	0.8628	0.2368
3.2	0.6448	0.1344	0.8500	0.1796	0.8848	0.2464	0.6952	0.1288	0.8948	0.2180	0.9232	0.3192
3.3	0.7700	0.1736	0.9180	0.2420	0.9444	0.3160	0.8080	0.1676	0.9452	0.2928	0.9612	0.4020
3.4	0.8620	0.2152	0.9580	0.3144	0.9768	0.3936	0.8916	0.2160	0.9740	0.3692	0.9860	0.4808
3.5	0.9240	0.2688	0.9824	0.3864	0.9896	0.4680	0.9412	0.2712	0.9900	0.4388	0.9936	0.5476
	$(\eta_1^2, \eta_2^2, \eta_3^2, \eta_4^2, \eta_5^2)$=(0.2,0.4,0.6,0.8,1)											
	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$=(3–6)						$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$=(5–8)					
	N1		N2		N3		N1		N2		N3	
ξ_5	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
3.1	0.2096	0.0652	0.4368	0.0592	0.5716	0.0568	0.2348	0.0552	0.5476	0.0612	0.6812	0.0652
3.2	0.3064	0.0716	0.5692	0.0760	0.6808	0.0784	0.3416	0.0636	0.6712	0.0824	0.7696	0.0972
3.3	0.4172	0.0852	0.6976	0.1004	0.7704	0.1252	0.4780	0.0768	0.7828	0.1112	0.8364	0.1572
3.4	0.5372	0.1040	0.7952	0.1376	0.8492	0.1752	0.5960	0.0972	0.8480	0.1500	0.8924	0.2232
3.5	0.6508	0.1332	0.8636	0.1736	0.9012	0.2288	0.7112	0.1312	0.9072	0.2012	0.9332	0.2892
	$(\eta_1^2, \eta_2^2, \eta_3^2, \eta_4^2, \eta_5^2)$=(0.3,0.5,0.7,0.9,0.9)											
	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$=(3–6)						$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$=(5–8)					
	N1		N2		N3		N1		N2		N3	
ξ_5	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
3.1	0.5048	0.0944	0.7792	0.1128	0.8408	0.1328	0.5628	0.0876	0.8348	0.1204	0.8892	0.1648
3.2	0.6396	0.1252	0.8636	0.1640	0.9068	0.2016	0.7020	0.1212	0.9108	0.1880	0.9420	0.2520
3.3	0.7636	0.1704	0.9260	0.2276	0.9568	0.2788	0.8188	0.1680	0.9508	0.2616	0.9732	0.3376
3.4	0.8612	0.2192	0.9652	0.3132	0.9820	0.3576	0.8940	0.2248	0.9800	0.3444	0.9912	0.4408
3.5	0.9224	0.2732	0.9860	0.3872	0.9940	0.4572	0.9464	0.2772	0.9912	0.4380	0.9964	0.5220

Note: $N1 = (30, 30, 40, 40, 50), N2 = (40, 50, 60, 60, 70), N3 = (60, 70, 80, 80, 90)$.

slowly as ξ_i departs from the null hypothesis when scale parameters are small. With the increase of scale parameters and sample sizes, the power is improved significantly.

Tables 7 and 8 respectively present the simulated powers under the condition of $\xi_1 \neq \xi_2 \neq \cdots \neq \xi_k$ in three and five populations. The simulated results from Tables 7 and 8 are, respectively, similar to those of Tables 5 and 6. However, the powers of the proposed approach based on the ML estimator decrease slightly as ξ_i departs from the null hypothesis with $k = 5$.

Remark 4. To compare the proposed approaches with the likelihood ratio test, we do the simulations of Type I error probability and power of likelihood ratio test. The simulation results show that the likelihood ratio test performs poorly in the Type I error probability and power, so they are not given in this section.

6. Illustrative examples

To verify the reasonableness and effectiveness of the proposed approaches in this section, two real examples of regional GDP of China and performance data of high-speed turbine bearings are presented.

Example 1. The above approaches are applied to the GDP data of Tianjin and Chongqing Municipalities from 1996 to 2018. As in Figures 1 and 2, the distributions of the GDPs of Tianjin and Chongqing don't follow the normal distribution but show asymmetric and right-skewed characteristics. To confirm the conclusion, we first conduct the normality test for these data. It turns out that the p-values of Shapiro–Wilk test and Kolmogorov–Smirnov test for Tianjin's GDP are 0.001 and 0.016, and for Chongqing's GDP are 0.001 and 0.005. Hence, the GDPs of Tianjin and Chongqing are not

Table 7. Simulated powers of hypothesis testing problem (5) as $k = 3, \xi_1 = 1 and \xi_2 = 1.1$.

	$(\eta_1^2, \eta_2^2, \eta_3^2)$=(0.4,0.4,0.8)											
	$(\alpha_1, \alpha_2, \alpha_3)$=(2–4)						$(\alpha_1, \alpha_2, \alpha_3)$=(5, 6)					
	N1		N2		N3		N1		N2		N3	
ξ_3	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
1.6	0.1616	0.0852	0.4948	0.0776	0.6868	0.0876	0.3240	0.0740	0.8632	0.1040	0.9692	0.2440
1.7	0.2760	0.1028	0.7000	0.1056	0.8680	0.1308	0.5116	0.0840	0.9472	0.1884	0.9932	0.3816
1.8	0.4252	0.1224	0.8432	0.1484	0.9612	0.2140	0.6852	0.1192	0.9780	0.3124	0.9996	0.5396
1.9	0.5832	0.1664	0.9160	0.2320	0.9856	0.3096	0.8092	0.1764	0.9948	0.4648	0.9996	0.6744
2.0	0.7148	0.2136	0.9600	0.3400	0.9952	0.4276	0.8944	0.2496	0.9988	0.5880	1.0000	0.7860
	$(\eta_1^2, \eta_2^2, \eta_3^2)$=(0.5,0.7,0.8)											
	$(\alpha_1, \alpha_2, \alpha_3)$=(2–4)						$(\alpha_1, \alpha_2, \alpha_3)$=(5, 6)					
	N1		N2		N3		N1		N2		N3	
ξ_3	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
1.6	0.1444	0.0840	0.4768	0.0548	0.7516	0.0620	0.2164	0.0560	0.7472	0.0524	0.9452	0.1352
1.7	0.2460	0.0920	0.6804	0.0660	0.8576	0.1072	0.3696	0.0692	0.8712	0.1208	0.9732	0.2684
1.8	0.3644	0.1108	0.8196	0.1144	0.9264	0.2104	0.5456	0.1000	0.9388	0.2296	0.9932	0.4448
1.9	0.4992	0.1412	0.8980	0.2104	0.9732	0.3532	0.6980	0.1456	0.9740	0.3932	0.9984	0.5952
2.0	0.6304	0.1956	0.9400	0.3448	0.9876	0.5040	0.8096	0.2212	0.9920	0.5412	1.0000	0.7420
	$(\eta_1^2, \eta_2^2, \eta_3^2)$=(0.6,0.8,0.8)											
	$(\alpha_1, \alpha_2, \alpha_3)$=(2–4)						$(\alpha_1, \alpha_2, \alpha_3)$=(5, 6)					
	N1		N2		N3		N1		N2		N3	
ξ_3	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
1.6	0.1396	0.0744	0.4476	0.0580	0.7252	0.0812	0.2208	0.0548	0.7288	0.0848	0.9384	0.2116
1.7	0.2428	0.0904	0.6484	0.0732	0.8420	0.1224	0.3720	0.0716	0.8664	0.1584	0.9720	0.3552
1.8	0.3588	0.1076	0.7996	0.1204	0.9144	0.2044	0.5396	0.1092	0.9408	0.2712	0.9928	0.5132
1.9	0.4860	0.1424	0.8864	0.2044	0.9676	0.3312	0.6880	0.1596	0.9744	0.4160	0.9984	0.6524
2.0	0.6100	0.1932	0.9276	0.3180	0.9824	0.4836	0.8012	0.2384	0.9920	0.5460	1.0000	0.7688

Note: $N1 = (30, 40, 40), N2 = (60, 60, 80), N3 = (80, 100, 120)$.

normally distributed at the nominal significance level of 5%. In addition, we should prove whether the distributions of the GDPs of Tianjin and Chongqing are skew-normal by the Chi-square goodness-of-fit test. By calculation, the fitted value of Tianjin is $\chi_t^2 = 5.9149 < \chi_2^2(0.95) = 5.99$ with p-value 0.0520, and the one of Chongqing is $\chi_c^2 = 4.2243 < \chi_2^2(0.95) = 5.99$ with p-value 0.1210. Therefore, the GDPs of Tianjin and Chongqing from 1996 to 2018 follow the skew-normal distributions $SN(\xi_t, \eta_t^2, \alpha_t)$ and $SN(\xi_c, \eta_c^2, \alpha_c)$ respectively at the nominal significance level of 5%.

Consider the hypothesis testing problem

$$H_0 : \xi_t = \xi_c \ vs \ H_1 : \xi_t \neq \xi_c.$$

The p-values of the Bootstrap test statistics based on the moment estimator and ML estimator are 0.5666 and 0.6694, respectively. Thus, the null hypothesis H_0 cannot be rejected at the nominal significance level of 5%, which means that there is no significant difference between the location parameters of the GDPs of Tianjin and Chongqing from 1996 to 2018.

Example 2. Consider comparing the performance of high-speed turbine bearings made of two different compounds. In this study, 10 bearings of each type were tested, and the failure times of each bearing were recorded in units of millions of cycles. Similar to Example 1, we conduct the normality test for the bearings made of two different compounds, named X and Y. It turns out that the p-values of Shapiro–Wilk test and Kolmogorov-Smirnov test for X are 0.014 and 0.026 and for Y are 0.001 and 0.011. Hence, the failure times of each type are not normally distributed at the nominal significance level of 5%. Furthermore, to verify the skew-normality of the failure times of each type, we intend to test the null hypothesis H_0: the failure times of X and Y are skew-normally distributed. It can be obtained by calculation that the fitted values of X is $\chi_x^2 = 1.5498 < \chi_1^2(0.95) = 3.84$ with p-value 0.2132, and the one of Y is $\chi_y^2 = 3.5732 < \chi_2^2(0.95) = 5.99$ with p-value 0.1675. Therefore, the

Table 8. Simulated powers of hypothesis testing problem (5) as $k = 5, \xi_1 = 1, \xi_2 = 1.1, \xi_3 = 1.2,$ and $\xi_4 = 1.3$.

$(\eta_1^2,\eta_2^2,\eta_3^2,\eta_4^2,\eta_5^2)$=(0.2,0.4,0.6,0.8,0.9)												
	$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$=(3–6)						$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$=(5–8)					
	N1		N2		N3		N1		N2		N3	
ξ_5	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
1.4	0.3160	0.1952	0.6904	0.1792	0.9248	0.2264	0.3940	0.1760	0.8344	0.1916	0.9780	0.2516
1.5	0.4184	0.1760	0.7960	0.1600	0.9612	0.2012	0.5180	0.1528	0.8952	0.1672	0.9912	0.2184
1.6	0.5488	0.1652	0.8736	0.1352	0.9812	0.1608	0.6632	0.1332	0.9396	0.1384	0.9968	0.1776
1.7	0.6824	0.1476	0.9336	0.1100	0.9924	0.1224	0.7780	0.1204	0.9700	0.1008	0.9992	0.1300
1.8	0.7904	0.1432	0.9692	0.0912	0.9968	0.0940	0.8708	0.1128	0.9900	0.0828	0.9996	0.1024
$(\eta_1^2,\eta_2^2,\eta_3^2,\eta_4^2,\eta_5^2)$=(0.3,0.4,0.5,0.8,0.9)												
	$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$=(3–6)						$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$=(5–8)					
	N1		N2		N3		N1		N2		N3	
ξ_5	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
1.4	0.2524	0.1252	0.5248	0.1060	0.8504	0.1088	0.3216	0.1112	0.7248	0.0984	0.9500	0.1004
1.5	0.3328	0.1216	0.6484	0.0972	0.9092	0.0984	0.4280	0.1056	0.8136	0.0852	0.9726	0.0864
1.6	0.4496	0.1232	0.7588	0.0820	0.9440	0.0832	0.5572	0.0976	0.8832	0.0684	0.9884	0.0740
1.7	0.5744	0.1224	0.8484	0.0696	0.9676	0.0688	0.6972	0.0984	0.9384	0.0544	0.9956	0.0604
1.8	0.7048	0.1196	0.9064	0.0644	0.9836	0.0620	0.8152	0.0988	0.9712	0.0512	0.9980	0.0640
$(\eta_1^2,\eta_2^2,\eta_3^2,\eta_4^2,\eta_5^2)$=(0.3,0.5,0.6,0.8,1.0)												
	$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$=(3–6)						$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$=(5–8)					
	N1		N2		N3		N1		N2		N3	
ξ_5	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2	p1	p2
1.4	0.1104	0.1748	0.1944	0.2092	0.5240	0.2764	0.0920	0.1788	0.3408	0.2324	0.7492	0.3336
1.5	0.1224	0.1700	0.2624	0.1988	0.6156	0.2588	0.1232	0.1716	0.4376	0.2216	0.8124	0.3120
1.6	0.1540	0.1600	0.3620	0.1780	0.7076	0.2368	0.1800	0.1556	0.5412	0.1924	0.8688	0.2764
1.7	0.2096	0.1456	0.4756	0.1524	0.7956	0.1960	0.2616	0.1372	0.6644	0.1636	0.9140	0.2308
1.8	0.3000	0.1364	0.6100	0.1208	0.8684	0.1532	0.3708	0.1152	0.7700	0.1280	0.9444	0.1836

Note: $N1 = (30, 30, 40, 40, 50), N2 = (40, 50, 60, 60, 70), N3 = (60, 70, 80, 80, 90)$.

failure times of X and Y follow the skew-normal distributions $SN(\xi_x, \eta_x^2, \alpha_x)$ and $SN(\xi_y, \eta_y^2, \alpha_y)$ respectively at the nominal significance level of 5%.

Consider the hypothesis testing problem

$$H_0 : \xi_x = \xi_y \qquad vs \qquad H_1 : \xi_x \neq \xi_y \quad .$$

The p-values of the Bootstrap test statistics based on the moment estimator and ML estimator are 0.8480 and 0.8432, respectively. Thus, the null hypothesis H_0 cannot be rejected at the nominal significance level of 5%, which means that there is no significant difference between high-speed turbine bearings made of these two different compounds.

7. Conclusion

When the scale parameters and skewness parameters are unknown, we consider the problem of homogeneous test of location parameters in several skew-normal populations. First, we construct the conditional test statistic and prove its approximate distribution. Second, the moment estimation and ML estimation of unknown parameters are given. Then we construct the Bootstrap test statistics, which generalize the results of Xu (2016) from normal population to skew-normal population. Further, the Monte-Carlo simulation results show that the Bootstrap test statistic based on the moment estimator performs better than that based on the ML estimator in most cases. Finally, the above approaches are used in two real examples of regional GDP of China and performance data of high-speed turbine bearings to verify the reasonableness and effectiveness of the proposed approaches.

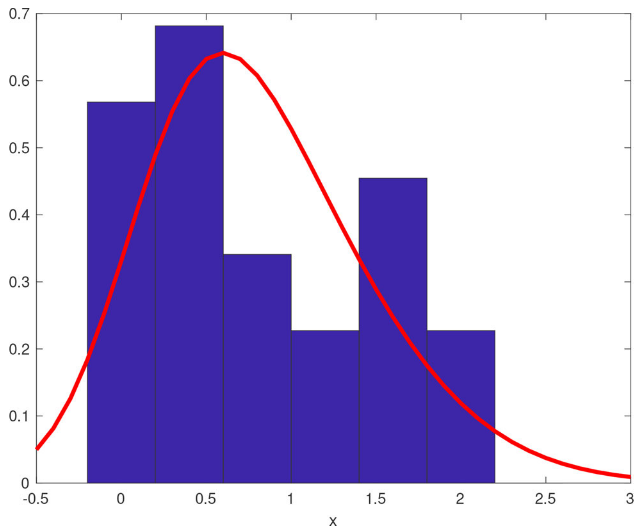

Figure 1. GDP histogram and probability density curve of Tianjin (1995–2018).

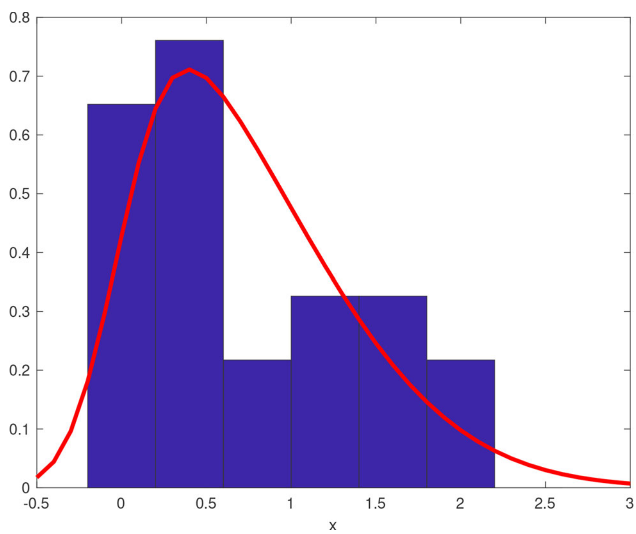

Figure 2. GDP histogram and probability density curve of Chongqing (1995–2017).

In summary, the Bootstrap test statistic based on the moment estimator is preferentially suggested to be used for homogeneous test of location parameters in several skew-normal populations.

Appendix

Proof of Lemma 1. Suppose the distribution function of X_j is $F_j(x)$, $j = 1, \ldots, k$. And let $F(x)$ denote the standard normal distribution function. From the weak convergence theorem (Hu 2009), we get

$$\int e^{itx^2} dF_j(x) \to \int e^{itx^2} dF(x), \quad j = 1, \ldots, k,$$

namely

$$E(e^{itx_j^2}) \to E(e^{itx^2}), \quad j = 1, \ldots, k.$$

By the continuity theorem (Wu et al. 1979), we obtain

$$X_j^2 \overset{asy}{\sim} \chi^2(1), \quad j = 1, ..., k.$$

Since $X_1, ..., X_k$ are mutually independent to each other, we have by Zhong (2010)

$$\sum_{j=1}^{k} X_j^2 \overset{asy}{\sim} \chi^2(k).$$

Therefore, the proof of Lemma 1 is completed.

Proof of Theorem 1. If $\alpha_1 = \alpha_2 = ... = \alpha_k = 0$, then

$$\Delta^{-1/2}\bar{Y} \sim N_k(\Delta^{-1/2}\zeta, I_k),$$
$$(\Delta^{-1/2}\bar{Y})' PQQ^{-1}P^{-1}(\Delta^{-1/2}\bar{Y}) \sim \chi_k^2(\zeta'\Delta^{-1}\zeta).$$

Further,

$$T = (\Delta^{-1/2}\bar{Y})' PQ\Lambda Q^{-1}P^{-1}(\Delta^{-1/2}\bar{Y}) \sim \sum_{i=1}^{k} \lambda_i \chi_i^2(1, \tau).$$

Therefore, the proof of Theorem 1 is completed.

Proof of Theorem 2. Let $(\bar{y}_i, s_{2i}, s_{3i})$ be the observed value of $(\bar{Y}_i, S_{2i}, S_{3i})$. Denote $X_{i1}, ..., X_{in_i}$ as the standardized samples, where $X_{ij} = (Y_{ij} - \bar{y}_i)/\sqrt{s_{2i}}$ from $X_i \sim SN(\xi_{si}, \eta_{si}^2, \alpha_i)$, $i = 1, ..., k$, $j = 1, ..., n_i$. For convenience, we denote

$$\xi_{si} = (\xi_i - \bar{y}_i)/\sqrt{s_{2i}}, \eta_{si} = \eta_i/\sqrt{s_{2i}}, i = 1, ..., k. \tag{A1}$$

Obviously, the moment generating function of X_i is

$$M_{X_i}(t) = 2\exp\left(t\xi_{si} + \frac{t^2\eta_{si}^2}{2}\right)\Phi(t\eta_{si}\delta_i), \ i = 1, ..., k. \tag{A2}$$

From (A2), we can obtain

$$\begin{aligned} M'_{X_i}(t)|_{t=0} &= \xi_{si} + b\eta_{si}\delta_i = 0, \\ M''_{X_i}(t)|_{t=0} &= \xi_{si}^2 + 2b\xi_{si}\eta_{si}\delta_i + \eta_{si}^2 = 1, \\ M'''_{X_i}(t)|_{t=0} &= \xi_{si}^3 + 3b\xi_{si}^2\eta_{si}\delta_i + 3\xi_{si}\eta_{si}^2 + 3b\eta_{si}^3\delta_i - b\eta_{si}^3\delta_i^3 = s_{2i}^{-3/2}s_{3i}. \end{aligned} \tag{A3}$$

By (A1) and (A3), we get the moment estimates of $(\xi_i, \eta_i^2, \alpha_i)$ as follows

$$\hat{\xi}_i^* = \bar{y}_i - cs_{3i}^{1/3}, \ \hat{\eta}_i^{*2} = s_{2i} + c^2 s_{3i}^{2/3}, \ \hat{\alpha}_i^* = \frac{\hat{\delta}_i^*}{\sqrt{1 - \hat{\delta}_i^{*2}}}, \tag{A4}$$

where $\hat{\delta}_i^* = \frac{cs_{3i}^{1/3}}{b(s_{2i} + c^2 s_{3i}^{2/3})^{1/2}}$, $i = 1, ..., k$. Then the proof of Theorem 2 is completed.

Proof of Theorem 3. By (3), we can derive the skewness coefficient γ_i as follows

$$\gamma_i = \frac{E\left[(Y_i - EY_i)^3\right]}{\left[E(Y_i - EY_i)^2\right]^{3/2}} = \frac{b^3\delta_i^3}{c^3(1 - b^2\delta_i^2)^{3/2}}. \tag{A5}$$

From (13) and (A5), we have

$$\sigma_i = \eta_i\sqrt{1 - b^2\delta_i^2}, \quad \mu_i = \xi_i + b\eta_i\delta_i, \quad i = 1, ..., k. \tag{A6}$$

For $W_i \sim SN(\alpha_i)$, it easy to see that

$$E(W_i) = b\delta_i, \quad D(W_i) = 1 - b^2\delta_i^2, \quad i = 1, ..., k.$$

Then,

$$Y_{Ci} = \xi_i + b\eta_i\delta_i \quad + \quad \eta_i\sqrt{1 - b^2\delta_i^2}\left(\frac{\frac{Y_i - \xi_i}{\eta_i} - b\delta_i}{\sqrt{1 - b^2\delta_i^2}}\right) = Y_i.$$

Therefore, the proof of Theorem 3 is completed.

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