RELEVANCE-BASED EMBEDDINGS FOR EFFICIENT RELEVANCE RETRIEVAL

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ABSTRACT

In many machine learning applications, the most relevant items for a particular query should be efficiently extracted. The relevance function is usually an expensive similarity model making the exhaustive search infeasible. A typical solution to this problem is to train another model that separately embeds queries and items to a vector space, where similarity is defined via the dot product or cosine similarity. This allows one to search the most relevant objects through fast approximate nearest neighbors search at the cost of some reduction in quality. To compensate for this reduction, the found candidates are re-ranked by the expensive similarity model. In this paper, we investigate an alternative approach that utilizes the relevances of the expensive model to make *relevance-based embeddings* (RBE). The idea is to describe each query (item) by its relevance for a set of support items (queries) and use these new representations to obtain query (item) embeddings. We theoretically prove that relevance-based embeddings are powerful enough to approximate any complex similarity model (under mild conditions). An important ingredient of RBE is the choice of support items. We investigate several strategies and demonstrate that significant improvements can be obtained compared to random choice. Our experiments on diverse datasets illustrate the power of relevance-based embeddings.

1 INTRODUCTION

Finding the most relevant element (item) i to a query q among a large set of candidates I is a key task for a wide range of machine learning problems, for example, information retrieval, recommender systems, question-answering systems, or search engines. In such problems, the final score (relevance) is often predicted by a pairwise function $R : I \times Q \to \mathbb{R}$, where Q is a query space and R approximates some ground truth relevance such as click probability, time spent or something else. Depending on the task, the relevance function R can utilize query attributes (e.g., the text of the query or a set of numerical features describing the user, such as age, time spent on the service, etc.), item attributes, or attributes describing the query-item pair (e.g., statistics based on counts of each query term in the document in information retrieval tasks).

The problem of relevance retrieval for a query q can be written as $\arg\max_{i \in I} R(i, q)$. For practical applications, it is usually required to return not one but K best items (for directly displaying to the user or further re-ranking). Most recommender systems are characterized by a large size of the item space I (millions to hundreds of millions), so an exhaustive search is not feasible. This problem is often solved by training an auxiliary model $\tilde{R}$, called a Siamese, two-tower, or dual encoder (DE), in which late binding is used: $\tilde{R}(i, q) = S(F_I(i), F_Q(q))$, where $F_I : I \to \mathbb{R}^d$, $F_Q : Q \to \mathbb{R}^d$, and S is some lightweight similarity measure, usually dot product or cosine similarity.

While a lot of effort has been put into developing dual-encoder models, the cross-encoder (CE) ones are generally more powerful (Wu et al., 2019; Yadav et al., 2022). Moreover, as mentioned above, in practice it is typical to also have features that describe a query-item pair: e.g., counts of query terms in the document (information retrieval), information about previous user-item interactions (recommender systems), and so on. Such features cannot be used by dual encoders.

In a recent paper, Yadav et al. (2022) suggested an alternative approach: to approximate the relevance of a given query to all the items using the relevance of this query to a fixed set of randomly chosen support items. In more detail, the authors apply the matrix factorization to the query-item relevance matrix to represent it as a product of its submatrix containing only a few columns (relevances to support items) and some other, explicitly computable.

Motivated by the above-mentioned work, we propose and analyze the concept of *relevance-based embeddings* (RBE). The main idea is to describe users by their relevance to some pre-selected support items and describe items by their relevance to some support users. Then, such representations can be used in various ways: as in Yadav et al. (2022), they can be multiplied by a certain matrix to obtain relevance approximations, or they can be passed into a neural network to potentially obtain better approximations (trained for a desired loss), or they can be additionally combined with the original node features to get even more powerful embeddings.

We theoretically prove the power of relevance-based embeddings. Namely, we show that (under mild conditions) they are informative enough to approximate any continuous similarity function. In particular, when the similarity function utilizes pairwise features, dual encoders based on the individual user and item features inevitably lose this information; however, relevance-based representations contain this information and thus can approximate the desired function.

From a practical perspective, an important aspect of our approach is how to properly choose support elements. Previous studies (Morozov & Babenko, 2019; Yadav et al., 2022) sampled them uniformly at random, while we show that there is significant room for improvement. We investigate different options: from simple heuristics (e.g., popular or diverse elements) to methods directly optimizing the relevance approximation quality. Surprisingly, even very simple strategies like clustering the elements and choosing the cluster centers as support items already give significant improvements that can be further improved by more advanced and theoretically justified strategies.

To evaluate the performance of RBE, we conduct experiments on textual and recommendation datasets. We compare our approach with dual encoders and with Yadav et al. (2022) and get an average improvement of 33% over this baseline for various datasets (from 8% to 69%, see Table 2).

2 Related work

In this section, we discuss research areas and representative papers related to our study.

Relevance retrieval problem is widespread in the context of building information retrieval systems (Kowalski, 2007), such as text search engines (Huang et al., 2013), image search (Gordo et al., 2016), entertainment recommender systems (Covington et al., 2016), question answering systems (Karpukhin et al., 2020), e-commerce systems (Yu et al., 2018), and other practical applications.

Usually, such problems are solved by learning **query and item embeddings** into a certain space and then searching for approximate nearest elements in this space, followed by rearrangement using a heavier ranker. In particular, Covington et al. (2016); Huang et al. (2013) explicitly use this approach, offering two-tower models (a.k.a. dual encoders). Note that there are simple alternatives to dual encoders that use, e.g., BM25 scores applicable to texts (Logeswaran et al., 2019; Zhang & Stratos, 2021) or other cheaper or more expensive alternatives (Humeau et al., 2019; Luan et al., 2021). However, there is usually trading-off complexity for quality.

It is also worth mentioning the works trying to facilitate the training of the dual encoder through distilling a heavier ranker model (Wu et al., 2019; Hofstätter et al., 2020; Lu et al., 2020; Qu et al., 2020; Liu et al., 2021). Although these works aim at simplifying the learning of light ranking using the heavy one, they differ from our approach in two aspects. First, distillation means that there are still two heavy (comparable in orders of magnitude of trainable parameters) models. Second, this approach requires two different complex model architectures that can be a significant disadvantage for, e.g., recommender systems with a wide variety of image/textual/statistical/sequential features.

As for the **nearest neighbors search** in a common query-item space, a wide variety of algorithms exist, including locality-sensitive hashing (LSH) (Indyk & Motwani, 1998; Andoni & Indyk, 2008), partition trees (Bentley, 1975; Dasgupta & Freund, 2008; Dasgupta & Sinha, 2013), and similarity graphs (Navarro, 2002). LSH-based and tree-based methods provide strong theoretical guarantees, however, it has been shown that graph-based methods usually perform better (Malkov & Yashunin, 2018; Aumüller et al., 2020), which explains their widespread use in practical applications.

Another research direction is methods that combine nearest neighbors search with heavy ranker calls (Morozov & Babenko, 2019; Chen et al., 2022) instead of separately embedding queries and items in a common space where the search for the nearest items can be efficiently performed. Such methods show better quality in comparison with separate embeddings, however, their practical application may be limited due to a significant change in the structure of the search index. In particular, in practice, microservices with neural networks and microservices with document indexes are different services, which allows for increasing GPU utilization on the one hand and using specialized (including sharded) solutions with a large amount of memory on the other. Therefore, in this paper, we focus on the basic scenario with a separate investment in space and a separate search for the nearest elements in it.

A recent paper by Yadav et al. (2022) is the most relevant for our research. The idea is to apply the matrix factorization to the query-item relevance matrix in order to represent it as a product of its submatrix containing only a few columns (relevances for random support items) and some other, explicitly computable. Despite the simplicity of the idea and implementation, the authors have shown in detail the superiority of their algorithm over more complex approaches, such as dual encoders. Our work is motivated by this study: we show that the approach of Yadav et al. (2022) has theoretical guarantees and suggest several improvements that significantly boost the performance. As an enhancement of the original approach, in another paper, Yadav et al. (2023) proposed the idea of selecting support items per each query independently, but the time complexity of each query processing becomes linear in the number of elements, which makes the approach infeasible in most practical applications. On the contrary, our support items selection is performed in the pre-processing stage and does not increase the query time.

Finally, the most recent article (Yadav et al., 2024) proposes an alternative solution to the problem. Their AXN algorithm dynamically learns the difference between the DE and CE (or other query-document embeddings) predictions for each query independently (at the query time). Learning the difference is carried out through iteratively choosing a set of anchor (supporting) elements, calculating CE scores for them, and learning linear regression with embeddings of these elements as features and CE scores as targets. Some disadvantages of this method are: 1) it requires previously trained embeddings of queries and documents; 2) it has increased query processing time due to the need for iterative refining of all item relevances for each query. For completeness of our study, we use the AXN algorithm as one of our baselines.

3 Relevance-based embeddings

In general, the information (attributes) used to calculate the ground-truth item-to-query relevances $R(i, q)$ can be divided into three types: depending only on the query q, only on the item i, and on both of them. The key problem when constructing separate embeddings of items and queries in the common space (that can be used for searching for the nearest elements) is the inability to use information that depends on both query and item, which lowers the quality of relevance search.

To address the above-mentioned issue, we introduce relevance-based representations that describe each query by its relevance to a pre-selected set of items and, vice-versa, each item by its relevance to a pre-selected set of queries. We prove that, under certain conditions, any relevance function can be well estimated using only such individual vectors.

3.1 Preliminaries

Let Q and I be compact topological spaces of queries and items, respectively. Assume that we are given a relevance function $R : I \times Q \to \mathbb{R}$. In practice, R is our relevance model which may be computationally expensive and rely on pairwise features.

Let $S_I \subset I$ and $S_Q \subset Q$ be some finite ordered sets of *support items* and *support queries*: $S_I = \{i_1, \ldots, i_m\}, S_Q = \{q_1, \ldots, q_n\}$. Let $R(i, S_Q)$ be a *relevance vector* of the item i w.r.t. the set of support queries S_Q: $R(i, S_Q) = (R(i, q_1), \ldots, R(i, q_n))$. Similarly, $R(S_I, q)$ is a relevance vector of the query q w.r.t. the set of support items S_I: $R(S_I, q) = (R(i_1, q), \ldots, R(i_m, q))$. By $R(S_I, S_Q)$ we denote a relevance matrix composed in a similar way.

3.2 CUR APPROXIMATION

Relevance vectors can be utilized in different ways and one of the possible approaches is to use the CUR decomposition (this approach was used by Yadav et al. (2022)). Using our notation, the relevance for a query q is approximated as:

$$\tilde{R}(I, q) := \langle\, R(I, S_Q) \times \mathrm{pinv}(R(S_I, S_Q)),\; R(S_I, q)\,\rangle, \tag{1}$$

where $\mathrm{pinv}(X)$ is the pseudo-inverse matrix of X. As support queries, Yadav et al. (2022) take the set of train queries: $S_Q = Q_{train} \subset Q$.

Regarding the computational complexity, we note that the CUR approximation requires computing the matrix $R(I, Q_{train})$ which takes $O(M \cdot |S_Q|) = (M \cdot |Q_{train}|)$ CE calls, where M is the total number of items in a database, which can be infeasible for large databases.

Our first theoretical result provides guarantees for the CUR approximation (1). Namely, we show that its regularized version approximates the true relevance arbitrarily precisely in L_2. Formally, let $\mathrm{pinv}_\lambda(A) = (A^T A + \lambda E)^{-1} A^T$ with E being the identity matrix. Then, the regularized CUR approximation CUR$_\lambda$ is defined by (1) with pinv_λ instead of pinv. For CUR$_\lambda$, we prove the following result (see Appendix A.3 for the proof).

Theorem 3.1. *Suppose that I and Q are equipped with the structure of a measure space and the integral of $R^4(i, q)$ over $I \times Q$ is finite. Then, the CUR$_\lambda$ approximation of R can be chosen arbitrarily close to the true R in $L_2(I \times Q)$ provided enough uniformly sampled support items and queries and sufficiently small λ.*

Our result gives theoretical support for the good performance of the CUR decomposition demonstrated in Yadav et al. (2022). In the next section, we show that stronger theoretical guarantees can be provided if we allow transforming query and item vectors into more powerful embeddings.

3.3 RELEVANCE-BASED EMBEDDINGS

In this section, we propose extending the CUR-based approximation by allowing transformations of relevance vectors, e.g., with a neural architecture. With such embeddings, we prove a stronger result: that any continuous relevance function can be uniformly approximated.

We say that a function on I is a *relevance-based embedding* if it has a representation of the form $e_I(i) = f_I(R(i, S_Q), \theta_I)$ where S_Q is a set of support queries and f_I is some ML-architecture with parameters θ_I which parametrizes a mapping $\mathbb{R}^n \to \mathbb{R}^d$. Analogously, $e_Q(q) = f_Q(R(S_I, q), \theta_Q)$ is a relevance-based embedding of the query q.

The following theorem holds (the proof can be found in Appendix A.1 and the guarantees for the RBE approach on a sphere are discussed in Appendix A.2).

Theorem 3.2. *Let I and Q be compact topological spaces, and $R : I \times Q \to \mathbb{R}$ be a continuous function. Then, R can be uniformly approximated up to an arbitrarily small absolute error by a function $\tilde{R}(i, q)$:*

$$\tilde{R}(i, q) = \langle\, f_I(R(i, S_Q), \theta_I),\; f_Q(R(S_I, q), \theta_Q)\,\rangle, \tag{2}$$

where $S_I \in I$ and $S_Q \in Q$ are some finite sets of support items and queries and f_I, f_Q are neural architectures with the universal approximation property (e.g., MLPs).

Note that compared to Theorem 3.1, this theorem has weaker requirements and stronger convergence. Trainable embeddings allow us to soften the requirements (R can be any continuous function) and prove uniform convergence instead of L_2 convergence).

This theorem shows that the true relevance function can be uniformly approximated with arbitrary precision by some functions of the relevance vectors. Our relevance-based framework is visualized in Figure 1. Note that the CUR

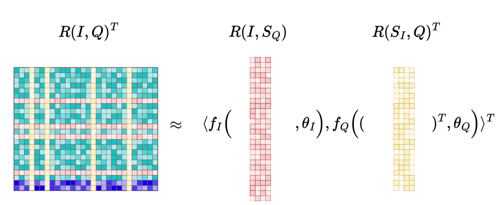

Figure 1: RBE visualization: support queries are red, support items are yellow; test queries are blue, and their relevance scores for the support items are used to approximate the remaining values

approximation discussed in Section 3.2 fits our framework with $\theta_I := \mathrm{pinv}(R(S_I, S_Q))$ and $f_I(R(i, S_Q), \theta_I) := R(i, S_Q) \times \theta_I$, $f_Q(R(S_I, q), \theta_Q) := R(S_I, q)$. Extending the CUR approximation to arbitrary embeddings allows us to obtain even stronger approximation guarantees. Moreover, such trainable embeddings are more flexible as they can adapt to a particular quality function one aims to optimize.

Regarding the computation complexity, the pre-processing step requires $O(M \cdot |S_Q|) + N$ computations of CE, where N is the size of the training set (this part is similar to the training of dual encoders). An advantage of RBE is that we may have a significantly smaller set of support queries $|S_Q| \ll |Q_{train}|$, while still utilizing the remaining queries for training the mappings f_Q and f_I. Also, the number of CE computations can be further reduced to just $O(N)$ if we replace the transformation $f_I(R(i, S_Q), \theta_I)$ with trainable embeddings of items $\theta_I(i) \in R^d$, see Section 3.5 for the discussion.

To sum up, the main outcome of our analysis is that even though the relevance function $R(\cdot, \cdot)$ is arbitrary (and, in particular, can rely on pairwise features only available for query-item pairs), it can be well approximated by relevance-based embeddings of individual users and items. This is a clear advantage of RBE over conventional dual-encoder models.

3.4 Selecting support items

Let us revise the statement of Theorem 3.2. The theorem states that there exist such sets S_Q, S_I that the relevance function R could be effectively approximated by separated embedders f_I, f_Q. In the related works (Morozov & Babenko, 2019; Yadav et al., 2022), this selection is random, implicitly assuming that the elements are in some sense equivalent. However, for example, when building a recommender service, the popularity of different objects has a strongly skewed distribution, which is why more information is known about a small set of highly popular items than about a large set of unpopular ones. Thus, it is natural to assume that the choice of support items may have a significant effect on performance. We investigate this direction and compare several approaches from simple heuristics to more complex ones optimizing the approximation quality.

Simple heuristics Our heuristic strategies include:

- **Random** Support items are chosen uniformly at random (Morozov & Babenko, 2019; Yadav et al., 2022). For better reproducibility, we also present the results of using the first $|S_I|$ items as support ones, assuming that the order of queries is pseudorandom.
- **Popular** As mentioned above, a recommender service usually has a small set of very popular elements that many users interact with. As a result, a lot of information can be collected from these interactions thus making the popular elements more informative. Since it is not always possible to get popularity explicitly, we consider the following surrogate: choose the objects with the highest average relevance for the training set.
- **Clusters centers** When it comes to the allocation of a representative subset of vectors, it is reasonable to consider the allocation of clusters. We consider various clustering algorithms and select the cluster centers as support elements. The number of clusters is set to the number of required support elements.
- **Most diverse** This strategy is a greedy algorithm maximizing the minimum distance between the support elements. We first choose the element furthest from the center (by Euclidean distance) and then, at each step, an element is selected whose minimum distance to the current support elements is maximal.

The approaches that require item representations can be applied to their relevance vectors.

l_2-greedy approach Let us now discuss a more theoretically justified approach that we call l_2-greedy. It aims at selecting the key elements that allow for a better approximation of the relevance matrix $R(i, q)$. In this strategy, we greedily select items so that the MSE error of the CUR approximation (Mahoney & Drineas, 2009) is minimized for the train queries.[1]

[1] We use the CUR approximation here since it optimizes the MSE error and is deterministically defined as in (1). In contrast, while the RBE approximation is more flexible, it requires fitting f_I and f_Q before estimating the approximation quality for each choice of the support queries and items.

We note that the CUR approximation replaces every item with a linear combination of support items so that the MSE between the true relevances and their CUR approximations on the train set of queries is minimized (see Appendix A.3 for more details). Our goal is to optimize the overall MSE for all items, which is:

$$\sum_{i} \|R(i, S_Q)^T - R(S_I, S_Q)^T \times \mathrm{pinv}(R(S_I, S_Q)^T) \times R(i, S_Q)^T\|_2^2. \tag{3}$$

We minimize this expression over all possible choices of S_I, $|S_I| = m$. We propose a greedy approach in which support items are selected one by one optimizing (3) at each step. Due to space constraints, the implementation details are placed in Appendix B.

Computation complexity In their default implementation, support items selection strategies *popular*, *cluster* centers, *most diverse*, and l_2-*greedy* require computing the relevance scores of all items to support queries that can be done in $O(M \cdot n)$. If this is infeasible, one can use downsampling to reduce the number of candidates for support items. Our preliminary experiments (see Appendix E) show that even significant downsampling gives reasonable performance of the obtained support items. Moreover, for heuristic approaches, instead of the relevance vectors, one can use cheaper embeddings (e.g., the original feature vectors) when they are available. Moreover, for *cluster centers*, there can be predefined clusters in data. For instance, in recommender services, it is typical that items are annotated with their categories that can be used as clusters without any additional cost.

3.5 Additional practical considerations

Dynamic set of items We note that relevance-based embeddings can naturally handle scenarios where the set of items frequently changes. The embeddings $f_I(R(i, S_Q), \theta_I)$ can be easily calculated for the new items without the need of re-training the embedding model f_I (similarly to feature-based dual encoders). On the other hand, if the set of items I is finite or changes not so frequently (e.g., the set of movies currently available in a recommender service), the transformation $f_I(R(i, S_Q), \theta_I)$ can be replaced with trainable embeddings $\theta_I(i) \in R^d$. In contrast to items, the query set Q cannot be assumed to be finite: queries can be represented by texts of unlimited length or characterized by real-valued features.

Improving approximation quality As discussed above, it is natural to assume the heavy ranker R to be the most expensive part in terms of computational complexity. Thus, during the calculation of $\tilde{R}$, we are mainly limited by the sizes of the support sets S_Q and S_I. In contrast, calculating f_Q, f_I, or the dot product between them is assumed to be significantly cheaper. Thus, f_Q and f_I may embed the relevance vectors in a higher-dimensional space. In particular, this may help to eliminate the disadvantages of the dot product, in comparison with other (Shevkunov & Prokhorenkova, 2021) ways of measuring the distances or similarities between objects.

While relevance-based embeddings have theoretical guarantees, they hold in the limit, when the sets of support queries and items are sufficiently large. In practice, to improve the approximation performance and reduce the number of CE calls per query, the mapping f_Q (or f_I) could be extended by enriching it with the features of the original query (or item).

Scalability The cost of the pre-processing and the ways to scale it are discussed in Section 3. At the inference, we need to compute the query representation, which requires m relevance computations. Then, the inference is similar to dual encoders: the item representations are pre-calculated and placed in the Approximate Nearest Neighbours index like HNSW (Malkov & Yashunin, 2018), which takes the embedding of the query as input. These m additional computations are taken into account in our experiments: when comparing with dual encoders, we reduce the final re-ranking budget by m.

Other applications Although the goal of RBEs is to approximate the relevance function, relevance vectors can also be considered good general representations. Thus, their application is not limited to relevance predictions, as shown below in Appendix D.2.

4 EXPERIMENTAL SETUP

Let us describe our experimental setup.[2] In all datasets that we use in this work, there is a heavy ranker that provides relevance, which we consider to be close to the ground truth. With this ranker, we build the complete table ($R : I \times Q \to \mathbb{R}$) of the relevance scores. We denote the predicted scores by $\hat{R}$.

The task is to find the most relevant items for a given query. The quality is evaluated as

$$\text{HitRate}(P, T) := \sum_{q_i \in Q_{test}} \frac{|\text{Best}_P(\hat{R}, q_i) \cap \text{Best}_T(R, q_i)|}{|\text{Best}_T(R, q_i)|\,|Q_{test}|},\ \text{HitRate}(K) := \text{HitRate}(K, K),$$

where $\text{Best}_K(\mathcal{R}, q) \subset I$ is defined as the set of K items $i_1, \ldots, i_K$ with the highest relevances $\mathcal{R}(i_1, q), \ldots, \mathcal{R}(i_K, q)$ to a given query $q \in Q$; Q_{test} is a set of test queries that do not participate in the training: $S_Q \subset Q_{train}$, $Q_{train} \bigsqcup Q_{test} = Q$. For all our experiments, $|Q_{test}| \approx 0.3|Q|$, $|S_I| = 100$, $S_Q = Q_{train}$.

4.1 DATASETS

ZESHEL The Zero-Shot Entity Linking (ZESHEL) dataset was constructed by Logeswaran et al. (2019) from Wikia. The task of zero-shot entity linking is to link mentions of objects in the text to an object from the list of entities with related descriptions. The dataset consists of 16 different domains. Each domain contains disjoint sets of entities, and during testing, mentions should be associated with invisible entities solely based on entity descriptions. We run the experiments on five domains from ZESHEL selected by Yadav et al. (2022). As a heavy ranker R, we use the cross-encoder trained by Yadav et al. (2022) and publicly available. Table 7 in Appendix shows the dataset statics for the domains used in this paper.

Question-Answering To additionally cover the question-answering domain, we conducted experiments on the MsMarco (Nguyen et al., 2016) based dataset provided by huggingface.[3] As a heavy ranker, we use *all-mpnet-base-v2* from the SentenceTransformers library (Reimers & Gurevych, 2019), which is trained, among other datasets, on the MsMarco data. We took ~10K test queries and 0.8M passages corresponding to them (QA). For the experiments in Table 1, we used the smaller (QA.Small) version with 82K passages (only test passages).

RecSys To evaluate the generalization of the proposed approach to other tasks and domains, we collected a dataset from a production service providing recommendations of items to users.[4] As a heavy ground-truth ranker R, we use the CatBoost gradient boosting model (Prokhorenkova et al., 2018) trained on a wide range of features, including categories and other static attributes of items, social information (age, language, etc.) of users, simple item statistics, user statistics, real-time statistics on user and item interaction, factors derived from the matrix factorizations, and multiple two-tower neural networks, receiving the features listed above as their features.

Two versions of the dataset are presented. In the first one, CatBoost was trained to predict the time that a user is going to spend on a given item immediately after the click (in one session). In the second version, CatBoost was trained on the pairwise PairLogit target to predict the item with the longest time spent for some long time after the click (including new sessions). The first version is denoted in the tables as RecSysLT and the second as RecSys. This dataset allows us to evaluate the generalizability of our approach across different domains and different types of heavy rankers since gradient-boosting models differ significantly from neural approaches. [5]

RecSys2 To further increase diversity of the considered domains, we collected a dataset from *another* production recommender service. Here, the CatBoost gradient boosting model is used as a

[2]The code and experimental data will be made publicly available after the blind review due to anonymity considerations.

[3]v1.1 version from `https://huggingface.co/datasets/microsoft/ms_marco`

[4]Not specified to preserve anonymity.

[5]We plan to publish both CE relevances and DE embeddings for RecSys, RecSysLT, RecSys2 in the final version of the paper (this requires legal approval).

Table 1: Support item selection applied to the CUR relevance approximation, HitRate(100) (larger is better) is reported

Support items	Yugioh	P.Wrest.	StarTrek	Dr.Who	Military	RecSys	RecSysLT	RecSys2	QA.Small
random (i.e. AnnCUR)	0.4724	0.4280	0.2287	0.1919	0.2455	0.6697	0.5842	0.1478	**0.5828**
first 100	0.4845	0.4182	0.2489	0.1975	0.2599	0.6490	0.5609	0.1551	0.5491
popular	0.2429	0.3001	0.1154	0.1197	0.1907	**0.7623**	**0.6695**	0.1422	0.5536
KMeans	0.5083	**0.4850**	0.3226	**0.2517**	**0.3042**	0.7070	0.6184	**0.1661**	0.5578
BisectingKMeans	0.4825	0.4592	0.2839	0.2159	0.2752	0.7035	0.6213	0.1483	0.5394
MiniBatchKMeans	0.5077	0.4737	0.2912	0.2365	**0.2826**	0.7033	0.5981	**0.1721**	0.5529
AgglomerativeClustering	**0.5105**	**0.4911**	**0.3264**	**0.2531**	0.2448	0.7050	**0.6265**	0.1557	0.5666
SpectralCoclustering	0.4618	0.4443	0.2540	0.2076	0.2551	0.6998	0.6094	0.1594	0.5415
SpectralBiclustering	0.4654	0.4708	0.2628	0.1845	0.2533	**0.7409**	0.5972	0.1607	0.5358
SpectralClusteringNN	0.5087	0.4690	0.2742	0.2048	0.2507	0.6936	0.5740	**0.2343**	0.5741
ByMin	**0.5333**	0.4290	**0.3325**	0.2278	0.2483	0.6504	0.6182	0.1329	**0.5925**
l_2-greedy	**0.5618**	**0.5119**	**0.3677**	**0.2960**	**0.3357**	**0.7197**	**0.6565**	0.1478	**0.6100**

Table 2: Evaluating neural relevance-based embeddings, HitRate(100) (larger is better) is reported

Model	Yugioh	P.Wrest.	StarTrek	Dr.Who	Military	RecSys	RecSysLT	RecSys2	QA
Popular	0.0917	0.2410	0.0884	0.0821	0.1127	0.5077	0.2886	0.0142	0.0001
AnnCUR	0.4724	0.4280	0.2287	0.1919	0.2455	0.6697	0.5842	0.1478	0.5522
AnnCUR+KMeans	0.5083	0.4850	0.3226	0.2517	0.3042	0.7070	0.6184	0.1661	0.5027
RBE+KMeans	0.5431	0.4979	0.3399	0.2539	0.3019	0.7137	0.6300	0.3729	0.5505
AnnCUR+l_2-greedy	0.5618	0.5119	0.3677	0.2960	**0.3357**	0.7197	0.6565	0.1478	0.5700
RBE+l_2-greedy	**0.5849**	**0.5249**	**0.3867**	**0.2992**	0.3349	**0.7234**	**0.6682**	**0.3964**	**0.6022**

heavy ranker and a dual encoder as a baseline. Both CE and DE are trained on a large set of external data and features. A smaller subsample of this data was used to train RBE, since we believe that RBE is able to show good quality even on a significantly smaller size of the training data.

4.2 Baselines

As our main baseline, we consider the **AnnCUR** (Yadav et al., 2022) recommendation algorithm that approximates relevances with the CUR decomposition as discussed in Section 3.2. What is important for further discussion, a broad comparison of this method with different basic approaches, including various dual encoders, is carried out by Yadav et al. (2022). In most of our experiments, we rely on these results, comparing only with AnnCUR. However, we explicitly provide the comparison with **dual encoders** for the new datasets RecSysLT and RecSys in Section 5. For better interpretability of the results, we also provide metrics for a baseline that always selects the most popular items (not to be confused with Table 1, where the “popular” refers to selecting support items).

4.3 RBE implementation

Following Theorem 3.2, we train lightweight neural networks f_I and f_Q, which are significantly faster than the heavy ranker R, and independently transform the relevance vectors $R(i, S_Q)$, $R(S_I, q)$ into embeddings. The training is performed by the Adam algorithm on sampled batches with a listwise loss function (see Appendix C), similar to the training of various DEs.

As follows from Section 3.2, the CUR decomposition gives a reasonably good approximation of the relevance function. Hence, we split the RBE representation into the CUR representation and the trainable prediction of its error. In the experiments, such decomposition improves the convergence and training stability. Technical implementation details are placed in Appendix C.

5 Experimental results

Support items selection Following Section 3.4, we check various ways of choosing support elements as opposed to the existing approaches that use random selection. All clustering algorithms are taken from the scikit-learn (Pedregosa et al., 2011) library, SpectralClusteringNN is a SpectralClustering with “nearest neighbors” affinity. The algorithms are used with their default parameters since even this simple setting already allows us to get significant improvements over the random selection.

The results are shown in Table 1, where the best three results for each dataset are highlighted. Clearly, there is a significant superiority of almost any approach based on clustering or diversity over the random selection. The theoretically justified l_2-greedy algorithm is the clear winner, second and third places are taken by KMeans and AgglomerativeClustering. However, due to the significantly worse quality of AgglomerativeClustering on the Military dataset, KMeans will be used in further experiments. Another observation is that on RecSys and RecSysLT, there is a clear superiority of the choice of popular items as the support ones. It is worth mentioning that for this dataset, the elements extracted by popularity are also quite stratified by their categories and an explicit restriction on the number of elements from one category changes the top slightly. However, this may not be true for other data (e.g., we do not observe a similar feature for RecSys2).

Neural relevance-based embeddings Following the description in Section 4.3, we also apply non-trivial trainable relevance mappings $f_I(R(I, S_Q), \theta_I)$, $f_Q(R(S_I, q), \theta_Q)$ to check whether this modification improves prediction quality in practice. The results are shown in Table 2. To better interpret the values, the quality of the constant output consisting of popular (in the same sense as in the previous paragraph) elements is also given. It can be seen that in most cases, except for one dataset (Military), trainable relevance mappings improve the final quality of the search for relevant elements and the improvements are obtained for both KMeans and l_2-greedy support elements selection. Improvement from the trainable mappings is most noticeable for the QA dataset. Note that the transformation that we use is not claimed to be optimal and is given rather to demonstrate that with the help of an easy transformation, one can get an increased quality on various datasets.

Table 3: Dual encoder embeddings vs support relevances, different tops, RecSysLT

X		100	200	300	400	500	600	700	800	900
Dual Encoder	HR(X+100, X)	0.7048	0.6803	0.6739	0.6739	0.6760	0.6792	0.6827	0.6868	0.6904
AXN_{DE}	HR(X+100, X)	**0.7065**	0.6769	0.6740	0.6769	0.6820	0.6883	0.6958	0.7054	0.7161
RBE+l_2-greedy	HR(X, X)	0.6682	**0.6955**	**0.7221**	**0.7406**	**0.7538**	**0.7639**	**0.7720**	**0.7792**	**0.7853**

Table 4: Dual encoder embeddings vs support relevances, different tops, RecSys2

X		100	200	300	400	500	600	700	800	900
Dual Encoder	HR(X+100, X)	0.3792	0.3198	0.2928	0.2784	0.2702	0.2661	0.2647	0.2652	0.2671
AXN_{DE}	HR(X+100, X)	0.3843	0.3333	0.3015	0.2879	0.2839	0.2835	0.2836	0.2855	0.2893
RBE+l_2-greedy	HR(X, X)	**0.3964**	**0.4471**	**0.4693**	**0.4833**	**0.4929**	**0.5008**	**0.5070**	**0.5119**	**0.5162**

Table 5: Dual encoder embeddings vs support relevances, RecSysLT

K		100	200	300	400	500	600	700	800	900
Dual Encoder	HR(K+100, 100)	0.7048	0.7977	0.8518	0.8855	0.9086	0.9258	0.9385	0.9484	0.9561
AXN_{DE}	HR(K+100, 100)	**0.7065**	0.7970	0.8538	0.8902	0.9153	0.9331	0.9465	0.9572	0.9660
RBE+l_2-greedy	HR(K, 100)	0.6682	**0.8359**	**0.9026**	**0.9342**	**0.9522**	**0.9632**	**0.9704**	**0.9760**	**0.9799**

Comparison with dual encoders In this part, we compare RBE with dual encoders. For RecSys and RecSys2 datasets, we consider the embeddings produced by the dual encoder used in the production of the service (i.e., the one that is proved to be the best in this task). To fairly compare the performance, we replace the relevance vectors in the RecSys experiments with embeddings obtained by the DE, keeping the transformation of the relevance vectors and the training parameters unchanged. The only difference between the pipelines for RBE and DE is that since we used $|S_I| = 100$ requests to a heavy ranker to obtain an RBE, we have to decrease the budget for RBE CE calls when calculating the top: compared to RBE, DE uses $|S_I|$ more CE calls for the re-ranking. We also applied the AXN[6] method in addition to DE-embedding as an alternative approach to improve DE-like methods using CE calls. Similarly to DE, AXN uses an increased budget for CE calls. Thus, for the dual encoder and AXN, we use the metric HitRate$(X + |S_I|, X)$ and for RBE — HitRate(X, X), which gives

[6]We have to note that, since the source code of the authors was not posted at the time of the experiment, we reproduced it ourselves, which may cause some differences in metrics.

Table 6: Dual encoder embeddings vs support relevances, RecSys2

K		100	200	300	400	500	600	700	800	900
Dual Encoder	HR(K+100, 100)	0.3792	0.4228	0.4514	0.4738	0.4927	0.5091	0.5239	0.5373	0.5493
AXN_{DE}	HR(K+100, 100)	0.3843	0.4296	0.4557	0.4834	0.5020	0.5198	0.5351	0.5485	0.5630
RBE+l_2-greedy	HR(K, 100)	**0.3964**	**0.5435**	**0.6253**	**0.6819**	**0.7253**	**0.7599**	**0.7888**	**0.8135**	**0.8343**

the former an advantage for small top sizes.[7] However, starting from $X = 200$, our algorithm is superior to both dual encoder and AXN_{DE} baselines, as can be seen from Table 3. As for Table 4, our algorithm is superior to both dual encoder and AXN_{DE} baselines from $X = 100$.

A second comparison, where the size of the desired top is fixed, while the number of extracted elements changes, is presented in Tables 5 and 6. Similarly to the previous comparison, RBE calls to the heavy ranker are taken into account: for the dual encoder and AXN, HitRate$(K + |S_I|, 100)$ is calculated, and for RBE — HitRate$(K, 100)$. In this comparison, RBE outperforms the dual encoder and AXN_{DE} starting at $K = 200$ for RecSysLT and at $K = 100$ for RecSys2. Let us note that in the production service, the actual size of the top used to select candidates before ranking exceeds the values indicated in the table.

A similar comparison with the dual encoder on the data from ZESHEL can be found in Yadav et al. (2022): it is shown that AnnCUR outperforms the dual encoder. Thus, on ZESHEL we compare only with AnnCUR.

Transfer Learning Often in practice, embeddings trained to solve one problem are also applied to other ones. Following this, we conducted an additional experiment on predicting the categories of items in the RecSys dataset and obtained that the relevance vectors are informative vector representations. Moreover, the proposed algorithms for selecting key elements improve the quality of the prediction. Due to space constraints, the detailed results are provided in Appendix D.2.

6 Conclusion

In this paper, we present the concept of relevance-based embeddings. We justify our approach theoretically and show its practical effectiveness on various datasets. We demonstrate that RBE allows one to obtain better quality compared with existing approaches. An important contribution of our work is the study of different strategies for choosing the support elements for RBE: we show that a proper choice of the support elements allows one to significantly boost performance.

Promising directions for future research include a deeper investigation of support element selection strategies as well as applying the proposed RBE to other algorithms, e.g., based on using a heavy ranker during the nearest neighbor search (Morozov & Babenko, 2019) or adaptive nearest neighbor search (Yadav et al., 2024). Regarding the latter approach, we note that AXN can be naturally combined with any query-item representations, e.g., RBE. The obtained AXN_{RBE} model has to perform at least as well as RBE due to the presence of the regularization that balances between the query-item representation and the proposed modification.

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[7]For all sizes of the top, both algorithms were trained once. By training algorithms with different loss functions for each top, taking into account different sizes of tops, the quality of both algorithms can be improved, which, however, is not essential for the current comparison, since the changes will affect both algorithms equally.

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A Theoretical analysis

A.1 Proof of Theorem 3.2

Queries as functions on items and vice versa Each query q defines a function r_q on items: $r_q(i) = R(i, q)$. Let us call two queries q and q' *R-equivalent* if $r_q = r_{q'}$ and write $q \sim_R q'$ to denote this relation. R-equivalent queries are interchangeable when it comes to measuring their relevance to any item. Let Q_R be the set of R-equivalence classes. Q_R may be considered as an image of Q in $C(I)$ under the mapping R_Q which maps query q to r_q. This point of view suggests a natural metric d_{Q_R} on Q_R induced by the uniform norm on $C(I)$: $d_{Q_R}(q, q') = \|r_q - r_{q'}\| = \sup_{i \in I} |R(i, q) - R(i, q')|$.

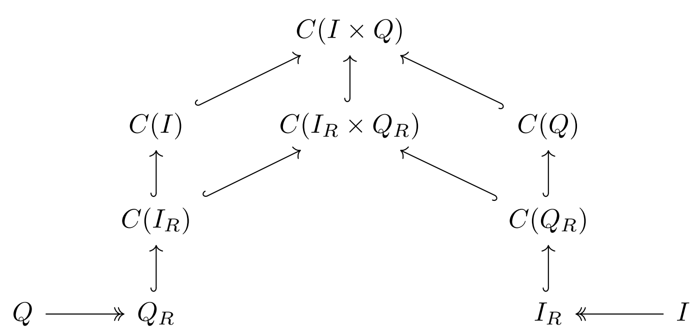
Now let us note that the map $R_Q : Q \to C(I)$ is continuous since R is continuous and I is compact. Therefore, Q_R is compact as an image of a compact space Q under a continuous mapping. And there is an (injective) embedding of $C(Q_R)$ to $C(Q)$ under which the function $f \in C(Q_R)$ goes to $f \circ R_Q \in C(Q)$. Simply speaking, a continuous function on the equivalence classes of queries is also a continuous function on the queries themselves.

Analogously, we define:

$$R_I : I \to C(Q), \quad R_I(i) = r_i, \quad r_i(q) = R(i, q), \quad R_I(I) = I_R,$$

$$d_{I_R}(i, i') = \|r_i - r_{i'}\| = \sup_{q \in Q} |R(i, q) - R(i', q)|.$$

For convenience, we will identify functions in $C(I_R)$ and $C(Q_R)$ with functions in $C(I_R \times Q_R)$ which are independent of one of their arguments. The relationships mentioned above and similar ones are shown in the diagram below (hooked arrows represent injective mappings, arrows with two heads stand for surjective ones):



Now, let us make several observations.

Claim 1. $r_i \in C(Q_R)$ and, similarly, $r_q \in C(I_R)$.

Proof. We note that $\|r_i(q) - r_i(q')\| = \|r_q(i) - r_{q'}(i)\| \leq \|r_q - r_{q'}\| = d_{Q_R}(q, q')$. So, $r_i(q) - r_i(q') = 0$ if $r_q = r_{q'}$ and the value $r_i(q)$ does not change if a query is replaced with an equivalent one. It means that r_i is a correctly defined function on the classes of equivalent queries, i.e., on Q_R. And finally the same inequality $\|r_i(q) - r_i(q')\| \leq d_{Q_R}(q, q')$ implies that the function r_i is 1-Lipschitz with respect to the metric d_{Q_R}. □

Claim 2. $R \in C(I_R, Q_R)$.

Proof. We have $|R(i, q) - R(i, q')| = |r_q(i) - r_{q'}(i)| \leq \|r_q - r_{q'}\| = d_{Q_R}(q, q')$ and $|R(i, q) - R(i', q)| \leq d_{I_R}(i, i')$. It follows that the value $R(i, q)$ does not change after replacement of a query-item pair (i, q) with some equivalent pair (i', q'). So, R can be considered as a function on $I_R \times Q_R$. And by the same inequality R is 1-Lipschitz with respect to the metric $d_R((i, q), (i', q')) = d_{I_R}(i, i') + d_{Q_R}(q, q')$ and hence is continuous. □

Stone-Weierstrass theorem Let us call all functions r_q and r_i *elementary*. Consider the family of all elementary functions $\mathcal{F} = \{r_q | q \in Q\} \cup \{r_i | i \in I\} \subset C(I_R \times Q_R)$.

Claim 3. The family $\mathcal{F}$ separates points in $I_R \times Q_R$, i.e., for each two different points $x, y \in I_R \times Q_R$, there is a function $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

Proof. Indeed, let (i_1, q_1) and (i_2, q_2) be any two different points in $I_R \times Q_R$. Then $i_1 \neq i_2$ or $q_1 \neq q_2$. Without loss of generality, we can assume that $i_1 \neq i_2$ (they are unequal as points in I_R). So, r_{i_1} and r_{i_2} are different functions on Q_R and there exists $q \in Q_R$ such that $r_{i_1}(q) \neq r_{i_2}(q) \Leftrightarrow R(i_1, q) \neq R(i_2, q) \Leftrightarrow r_q(i_1) \neq r_q(i_2) \Leftrightarrow r_q((i_1, q_1)) \neq r_q((i_2, q_2))$. Thus, we found a function (r_q) from our family that separates the two points. □

Next, consider the algebra of functions $\mathbb{R}[\mathcal{F}]$ generated by the family $\mathcal{F}$. This algebra consists of all polynomial combinations of functions in $\mathcal{F}$. More formally, each element of $\mathbb{R}[\mathcal{F}]$ has a representation of the form:

$$\mathbb{R}[\mathcal{F}] \ni f = \sum_{k=1}^{d} c_k \cdot r_{i_{k,1}} \cdot \ldots \cdot r_{i_{k,a_k}} \cdot r_{q_{k,1}} \cdot \ldots \cdot r_{q_{k,b_k}}.$$

In other words, there are d sets $S^1, \ldots, S^d$ of queries and items such that:

$$S^k = S_I^k \cup S_Q^k, \quad S_I^k = \{i_{k,1}, \ldots, i_{k,a_k}\} \subset I, \quad S_Q^k = \{q_{k,1}, \ldots, q_{k,b_k}\} \subset Q.$$

$$f = \sum_{k=1}^{d} c_k \cdot \left(\prod_{i \in S_I^k} r_i \right) \cdot \left(\prod_{q \in S_Q^k} r_q \right). \tag{4}$$

Products of the form $\prod_{i \in S_I^k} r_i$ may be empty and in this case the product equals 1. So, $\mathbb{R}[\mathcal{F}]$ contains constant functions and separates points of $I_R \times Q_R$ (because it contains $\mathcal{F}$). Hence, by the Stone-Weierstrass theorem, the algebra $\mathbb{R}[\mathcal{F}]$ is dense in $C(I_R \times Q_R)$. In particular, the function R can be approximated by an element of $\mathbb{R}[\mathcal{F}]$ up to an arbitrarily small absolute error.

Represent polynomials in $\mathbb{R}[\mathcal{F}]$ as products of query and item embeddings Consider an arbitrary function $f \in \mathbb{R}[\mathcal{F}]$ and its representation of the form (4). Denote the products $\prod_{i \in S_I^k} r_i(q)$ and $\prod_{q \in S_Q^k} r_q(i)$ by $\pi_{S_I^k}(q)$ and $\pi_{S_Q^k}(i)$ respectively. Consider two d-dimensional vectors:

$$e(q) = \left(c_1 \cdot \pi_{S_I^1}(q), \ldots, c_d \cdot \pi_{S_I^d}(q) \right),$$

$$e(i) = \left(\pi_{S_Q^1}(i), \ldots, \pi_{S_Q^d}(i) \right).$$

Then, $f(i, q) = \langle e(i), e(q) \rangle$. Let $S_I = \cup_{k=1}^{d} S_I^k$ and $S_Q = \cup_{k=1}^{d} S_Q^k$. Then $e(q)$ is a continuous (more specifically, polynomial) function of the vector $R(S_I, q)$ and $e(i)$ is a continuous function of the vector $R(i, S_Q)$. So, by the universality theorem for MLPs (Cybenko, 1989; Leshno et al., 1993), the vector $e(i)$ can be approximated up to arbitrarily small absolute error in the form $f_I(R(i, S_Q), \theta_I)$ where $f_I(\cdot, \theta_I)$ — a rich enough MLP architecture. Similarly, $e(q)$ can be approximated by $f_Q(R(S_I, q), \theta_Q)$. Hence, $\langle f_I(R(i, S_Q), \theta_I), f_Q(R(S_I, q), \theta_Q) \rangle$ approximates $f(i, q)$. Finally, we can consider $f \in \mathbb{R}[\mathcal{F}]$ such that $\|f - R\| < \frac{\varepsilon}{2}$ and then find such θ_I and θ_Q that $\|f - \langle f_I(R(i, S_Q), \theta_I), f_Q(R(S_I, q), \theta_Q) \rangle\| < \frac{\varepsilon}{2}$. These parameters will give us a desired ε-approximation of R in a form of the product of relevance-based embeddings.

A.2 RBE ON A SPHERE

The corollary below shows that the retrieval of the most R-relevant items with tolerance to ε-sized relevance loss can be reduced to the standard nearest neighbor search *on a sphere*.

Corollary A.1. *For each $\varepsilon > 0$, there is a multiplier $a \in \mathbb{R}$ such that $a\tilde{R}$ is an ε-approximation of R and $\tilde{R}$ uses embeddings scaled to the unit sphere.*

Proof. Let us take some $\frac{\varepsilon}{2}$-approximation of R of the form

$$R(i,q) \approx \langle e_I(q), e_Q(i)\rangle = \langle f_Q(R(S_I,q),\theta_Q), f_I(R(i,S_Q),\theta_I)\rangle$$

via the relevance-based embeddings $e_I(q)$ and $e_Q(i)$ of dimension d. Take a constant C such that $\|e_I(i)\| < C$ and $\|e_Q(Q)\| < C$ for all $q \in Q$ and $i \in I$ and consider the following vector embeddings of dimension $d+2$:

$$\tilde{e}_I(i) = \left(\frac{1}{C}e_I(i), \sqrt{1 - \|\frac{1}{C}e_I(i)\|^2}, 0\right),$$

$$\tilde{e}_Q(q) = \left(\frac{1}{C}e_q(q), 0, \sqrt{1 - \|\frac{1}{C}e_q(q)\|^2}\right).$$

Note that $\langle e_I(i), e_Q(q)\rangle = C^2 \cdot \langle \tilde{e}_I(i), \tilde{e}_q(q)\rangle$ and $\tilde{e}_I(i)$ and $\tilde{e}_q(q)$ lie on the $(d+1)$-dimensional unit sphere $S^{d+1} \subset \mathbb{R}^{d+2}$. We can take new powerful enough architectures $\tilde{f}_I$ and $\tilde{f}_Q$ with outputs normalized to unit sphere in $\mathbb{R}^{d+2}$ and fit for them parameters $\tilde{\theta}_I$ and $\tilde{\theta}_Q$ such that $\tilde{f}_I(R(i,S_Q),\tilde{\theta}_I) \approx \tilde{e}_I(i)$ and $\tilde{f}_Q(R(S_I,q),\tilde{\theta}_Q) \approx \tilde{e}_Q(q)$ and $\tilde{R}(i,q) = \langle \tilde{f}_I(R(i,S_Q),\tilde{\theta}_I), \tilde{f}_Q(R(S_I,q),\tilde{\theta}_Q)\rangle \approx \langle \tilde{e}_I(i), \tilde{e}_Q(q)\rangle$. More specifically, take $\tilde{\theta}_I$ and $\tilde{\theta}_Q$ such that:

$$|\langle \tilde{e}_I(i), \tilde{e}_Q(q)\rangle - \tilde{R}(i,q)| < \frac{\varepsilon}{2C^2} \Rightarrow |\langle e_I(i), e_Q(q)\rangle - C^2\tilde{R}(i,q)| < \frac{\varepsilon}{2}.$$

Given that $|R(i,q) - \langle e_I(i), e_Q(q)\rangle| < \frac{\varepsilon}{2}$, it yields $|R - a\tilde{R}| < \varepsilon$. Which means that the statement of the corollary is satisfied with $a = C^2$.

□

A.3 Proof of Theorem 3.1

Preliminaries Let us start with some notation. We assume that the spaces of items and queries I and Q are equipped with the structure of a measure space and probabilistic measures μ_I and μ_Q, respectively. For any item i, by r_i we denote a function on Q such that $r_i(q) = R(i,q)$. By $L_2(I) = L_2(I,\mu_I)$ we denote a Hilbert space of measurable functions whose square is integrable. We also denote $\mathbb{E}_i f(i) = \int_I f d\mu_I$ for $f \in L_2(I)$.

Recall that with CUR decomposition, the relevance $R(i,q)$ is approximated as:

$$\tilde{R}(i,q) = \langle\, R(i,S_Q) \times \operatorname{pinv}(R(S_I,S_Q)),\ R(S_I,q)\,\rangle = \sum_{t=1}^{|S_I|} c_t \cdot r_{i_t}(q) \tag{5}$$

with some coefficients c_t that are defined as:

$$\mathbf{c} = (c_1,\ldots,c_{|S_I|}) = [R(i,S_Q) \times \operatorname{pinv}(R(S_I,S_Q))]^T = \operatorname{pinv}(R(S_I,S_Q)^T) \times R(i,S_Q)^T.$$

The last equality holds because $\operatorname{pinv}(A)^T = \operatorname{pinv}(A^T)$.

For any vector $v \in \mathbb{R}^{|S_Q|}$:

$$\operatorname{pinv}(R(S_I,S_Q)^T)v \in \underset{x\in\mathbb{R}^{|S_I|}}{\arg\min} \|R(S_I,S_Q)^T x - v\|_2^2\,.$$

Thus, the coefficients $\mathbf{c}$ minimize the MSE between $R(i,S_Q)^T$ and $R(S_I,S_Q)^T\mathbf{c}$, i.e., between the relevances of queries from S_Q to the item i and $\sum_{t=1}^{|S_I|} c_t \cdot r_{i_t}$. Thus, calculating the item embedding $\mathbf{c}$ is merely solving a linear regression.

Then, CUR$_\lambda$ is the analog of the CUR approximation that uses l_2 regularization with coefficient λ while solving these multiple linear regression problems. Formally, recall that we define

$$\operatorname{pinv}_\lambda(A) = (A^TA + \lambda E)^{-1}A^T$$

with E being the identity matrix of a proper size. Then, CUR$_\lambda$ uses $\operatorname{pinv}_\lambda$ instead of pinv in (5).

Now, we are ready to prove the theorem. We do it in the following steps.

Step 1 We note that the function R can be represented in the form of a no more than countable sum:

$$R(i,q)=\sum_{k=0}^{K}\lambda_k f_k(i)h_k(q),$$

where $K\in\mathbb{N}\cup\infty$ and $\{f_0,f_1,\dots\}$, $\{h_0,h_1,\dots\}$ are orthonormal sets of vectors in $L_2(I)$ and $L_2(Q)$, respectively. Equality holds almost everywhere on $I\times Q$ with respect to the measure $\mu_I\times\mu_Q$. Note that this statement is a generalization of finite dimensional PCA. Without loss of generality, we can assume that $\lambda_k\geqslant\lambda_{k+1}$. Then, the approximation $R(i,q)\approx\sum_{k=1}^{n}\lambda_k f_k(i)h_k(q)$ is a function on $I\times Q$ of rank n closest to R in $L_2(I\times Q)$ (like in ordinary PCA).

To prove this, we consider an operator $A: L_2(I)\to L_2(Q)$, $A(f)(q)=E_{i\sim\mu_I}f(i)R(i,q)$. A is a Hilbert-Schmidt integral operator and hence it is compact. So, A^*A is compact self-adjoint positive semi-definite operator in $L_2(I)$. By the spectral theorem for compact operators, $A^*A(v)=\sum_k\lambda_k^2 f_k\langle f_k,v\rangle$, where $\{f_1,\dots\}$ is at most countable orthonormal set. Taking $h_k=\frac{1}{\lambda_k}A(f_k)$ completes the construction.

Step 2 Let $\{i_1,i_2,\dots\}$ be an infinite sequence of independently sampled items. Then, with probability one (with respect to sampling of $\{i_1,i_2,\dots\}$), there is N large enough that:

$$\mathrm{E}_i\left(\min_{i'\in\{i_1,\dots,i_N\}}\|r_i-r_{i'}\|_2^2\right)<\varepsilon.$$

In other words, for every item i, there is some replacement $r(i)\in\{i_1,\dots,i_N\}$ such that $\mathbb{E}_{i,q}|R(i,q)-R(r(i),q)|^2<\varepsilon$. To prove that, we need resolve some technical issues.

Substep 2.1 Let H be a countably dimensional (and hence separable) subspace of $L_2(Q)$ generated by $\{h_1,h_2,\dots\}$. Then, $r_i\in H$ with probability one (with respect to the measure μ_I).

We have $r_i(q)-\sum_{k=0}^{K}\lambda_k f_k(i)h_k(q)=0$ almost everywhere on $I\times Q$, so for almost every i: $r_i(q)=\sum_{k=0}^{K}\lambda_k f_k(i)h_k(q)$ for almost every q. Also, $\|r_i\|_2<\infty$ for almost every i (otherwise $\|R\|_2$ could not be finite). So, for i such that both $\|r_i\|_2<\infty$ and $r_i(q)=\sum_{k=0}^{K}\lambda_k f_k(i)h_k(q)$ holds that $r_i\in H$. Thus, further we can assume that $r_i\in H$ (ignoring the set of "bad" items of measure 0).

Substep 2.2 The mapping $e: i\to r_i\in H$ is measurable with respect to the Borel σ-algebra on H.

Let $B_\delta(h)$ be a ball of radius δ around $h\in H$. Then, it is sufficient to show that for every $h\in H$ and $\delta>0$ the set of items $e^{-1}(B_\delta(h))$ is measurable in I. Let $f(i)=\int_q|R(i,q)-h(q)|^2d\mu_Q$. The function $|R(i,q)-h(q)|^2$ is integrable over $I\times Q$. So by the Fubini's theorem, f is integrable over I. In particular, the set $\{i|f(i)<\delta^2\}=e^{-1}(B_\delta(h))$ is measurable.

Substep 2.3 $\forall\varepsilon>0$ for almost every $i\in I$: $P_{i'\sim\mu_I}(\|r_i-r_{i'}\|<\varepsilon)>0$.

Consider the Borel measure $e_*(\mu_I)$ on H (the image of μ_I under the mapping e: $e_*(\mu_I)(A)=\mu_I(e^{-1}(A))$ or, informally speaking, the distribution of all r_i in H). Let $X\subset H$ be a countable dense subset in H. Consider the set $X_{\varepsilon/2}=\{x\in X|e_*(\mu_I)(B_{\varepsilon/2}(x))=0\}$ and $Y=\cup_{x\in X_{\varepsilon/2}}B_{\varepsilon/2}(x)$. Y is the union of a countable set of balls of measure zero, so $e_*(\mu_I)(Y)=0$. For h such that $e_*(\mu_I)(B_\varepsilon(h))=0$ there is $x\in X$ such that $\|h-x\|<\varepsilon/2$. $B_{\varepsilon/2}(x)\subset B_\varepsilon(h)$, hence $e_*(\mu_I)(B_{\varepsilon/2}(x))=0$, hence $x\in X_{\varepsilon/2}$, hence $h\in B_{\varepsilon/2}(x)\subset Y$. So, $\{h\in H|P_i(\|r_i-h\|<\varepsilon)\}\subset Y$ and the measure of this set is zero which yields the required statement.

Substep 2.4 Take our infinite sequence of independently sampled items $\{i_1,i_2,\dots\}$. Consider the sequence of functions:

$$f_n(i,i_1,\dots,i_n)=\min_{i'\in\{i_1,\dots,i_n\}}\|r_i-r_{i'}\|_2^2.$$

We know that for every i: $P_{i'}(\|r_i-r_{i'}\|_2^2<\varepsilon^2)>0$. So with probability one some item i_k will fall into $B_\varepsilon(r_i)$ and $\forall n\geqslant k: f_n(i,i_1,\dots,i_n)<\varepsilon^2$. It follows that $f_n\to 0$ almost everywhere (on some large measure space where all independent variables $i,i_1,i_2,\dots$ are defined). The sequence f_n is bounded by the integrable function $f_1(i,i_1)=\|r_i-r_{i_1}\|_2^2$.

It follows that for almost every item i and an infinite sequence $\{i_1, \ldots\}$ of items independently sampled from μ_I,

$$\lim_{k\to\infty} \left(\min_{i' \in \{0\} \cup \{i_1, \ldots, i_k\}} \|r_i - r_{i'}\|_2^2 \right) = 0.$$

The expression inside the lim is bounded by $\|r_i\|_2^2$. Hence, by the Lebesgue's dominated convergence theorem, its mean over $i \in I$ tends to zero.

Step 3 Take N large enough that

$$\mathrm{E}_i \left(\min_{i' \in \{0\} \cup \{i_1, \ldots, i_N\}} \|r_i - r_{i'}\|_2^2 \right) < \varepsilon.$$

Take $\{i_1, \ldots, i_N\}$ as our support items. Let $I_N = span(r_{i_1}, \ldots, r_{i_N})$ be a linear span of the first N random items in $L_2(Q)$. Then, for each choice of the regularization coefficient λ:

$$\mathrm{E}_i \min_{c_1, \ldots, c_N} \left(\lambda \sum_{k=1}^{N} c_k^2 + \|r_i - \sum_{k=1}^{N} c_k r_{i_k}\|_2^2 \right) < \lambda + \varepsilon.$$

For $v \in L_2(Q)$, let $proj_\lambda(v, I_N)$ be a linear combination $c_1 r_{i_1} + \ldots + c_N r_{i_N}$ that minimizes $\lambda \sum_{k=1}^N c_k^2 + \|v - \sum_{k=1}^N c_k r_{i_k}\|_2^2$. So, $\|r_i - proj_\lambda(r_i, I_N)\|_2^2 < \lambda + \varepsilon$. Take $\lambda = \varepsilon$.

Step 4 We know that $proj_\varepsilon(r_i, I_N)$ is close to r_i (the average l_2 distance is at most $\sqrt{2\varepsilon}$). Next, we will prove that the linear combination of the support items by which the regularized CUR approximates r_i is close to $proj_\varepsilon(r_i, I_N)$ (on average).

Let $\{q_1, q_2, \ldots\}$ be a sequence of random queries independently sampled from μ_Q. Let Q_m be a set of the first m queries. Then, for any pair of items $i, i' \in I$ let $\langle i, i' \rangle_m = \frac{1}{m} \sum_{k=1}^m R(i, q_k) \cdot R(i', q_k)$. In other words, $\langle i, i' \rangle_m$ is the Monte-Carlo estimate of the product of r_i and $r_{i'}$ in $L_2(Q)$. Consider $N \times m$ matrix $\hat{R}$ such that $\hat{R}_{a,b} = R(i_a, q_b)$. The matrix $\hat{G} = \hat{R}\hat{R}^T$ consists of pairwise estimates of products of support items $\hat{G}_{kl} = \langle i_k, i_l \rangle_m$. As m tends to infinity, the matrix $\hat{G}$ converges to the matrix G of the exact scalar products of support items in the space $L_2(Q)$: $G_{kl} = \mathrm{E}_q\left(r_{i_k}(q) r_{i_l}(q)\right)$.

For each item i, let $\langle i, I_N \rangle_m$ be a vector $(\langle i, i_1 \rangle_m, \ldots, \langle i, i_N \rangle_m)$ and $\langle i, I_N \rangle = (\langle i, i_1 \rangle, \ldots, \langle i, i_N \rangle) = \mathrm{E}_\mathrm{q} \langle i, I_N \rangle_m$. Each component of $\langle i, I_N \rangle_m$ is the average of m random variables of the form $r_i(q) r_{i_k}(q)$ which are conditionally mutually independent given i and I_N. Let us estimate the mean squared deviation of $\langle i, I_N \rangle_m$ from $\langle i, I_N \rangle$ (with the fixed set of support items I_N):

$$\begin{aligned}
\mathrm{E}_{i, q_1, \ldots, q_m} \|\langle i, I_N \rangle_m - \langle i, I_N \rangle_m\|_2^2 &= \frac{1}{m} \sum_{k=1}^{N} \mathrm{E}_{i,q} (\langle r_i, r_{i_k} \rangle - r_i(q) r_{i_k}(q))^2 \\
&\leqslant \frac{1}{m} \sum_{k=1}^{N} \mathrm{E}_{i,q} r_i^2(q) r_{i_k}^2(q)) \leqslant \frac{1}{m} \sum_{k=1}^{N} \mathrm{E}_i \|r_i^2\| \|r_{i_k}^2\| \\
&= \frac{1}{m} \mathrm{E}_i \|r_i^2\| \sum_{k=1}^{N} \|r_{i_k}^2\| \leqslant \frac{1}{m} \|R^2\| \sum_{k=1}^{N} \|r_{i_k}^2\|.
\end{aligned}$$

So, the mean squared deviation of $\langle i, I_N \rangle_m$ from the vector $\langle i, I_N \rangle$ tends to zero as $m \to \infty$.

Finally, let A be an operator that maps the coefficients $c_1, \ldots, c_N$ to linear combinations of support items $c_1 r_{i_1} + \ldots + c_N r_{i_N} \in L_2(Q)$. Note that the operator norm $\|A\|$ of A is finite as it is a finite rank operator (so $\|Av\| \leqslant \|v\| \|A\|$).

We have

$$proj_\varepsilon(r_i, I_N) = A(G + \varepsilon E_N)^{-1} \langle i, I_N \rangle,$$

where E_N is $N \times N$ identity matrix. While the regularized CUR approximation of r_i is:

$$CUR_\varepsilon(r_i, I_N) = A(\hat{G} + \varepsilon E_N)^{-1} \langle i, I_N \rangle_m.$$

Let δ_m be the vector $\langle i, I_N\rangle_m - \langle i, I_N\rangle$ and δG_m^{-1} be $(\hat{G} + \varepsilon E_N)^{-1} - (G + \varepsilon E_N)^{-1}$. Then, the norm of the difference between $CUR_\varepsilon(r_i, I_N)$ and $proj_\varepsilon(r_i, I_N)$ can be estimated as:

$$\begin{aligned}
&\|CUR_\varepsilon(r_i, I_N) - proj_\varepsilon(r_i, I_N)\| \\
&= \|A((G + \varepsilon E_N)^{-1} + \delta G_m^{-1})(\langle i, I_N\rangle + \delta_m) - A(G + \varepsilon E_N)^{-1}\langle i, I_N\rangle\| \\
&= \|A(\delta G_m^{-1}\langle i, I_N\rangle + \delta G_m^{-1}\delta_m + (G + \varepsilon E_N)^{-1}\delta_m)\| \\
&\leqslant \|A\|(\|\delta G_m^{-1}\|\|\langle i, I_N\rangle\| + \|\delta G_m^{-1}\|\|\delta_m\| + \|(G + \varepsilon E_N)^{-1}\|\|\delta_m\|) \\
&\leqslant \|A\| \cdot (\|\delta_m\|\frac{1}{\varepsilon} + \|\delta G_m^{-1}\|\|\langle i, I_N\rangle\| + \|\delta_m\|\|\delta G_m^{-1}\|),
\end{aligned}$$

where we use $\|(G + \varepsilon E_N)^{-1}\| \leqslant \frac{1}{\varepsilon}$.

We can take m large enough that the expectation of the square of that difference is arbitrarily small, say less than ε. Then,

$$\mathrm{E}_i\|r_i - CUR_\varepsilon(i)\| \leqslant \|r_i - proj_\varepsilon(r_i, I_n)\| + \|proj_\varepsilon(r_i, I_n) - CUR_\varepsilon(i)\| \leqslant \sqrt{2\varepsilon} + \sqrt{\varepsilon}.$$

B Greedy selection of support items

Let us denote $X := (I, S_Q)$, which is an $M \times n$ matrix (M is the total number of items). Then, our optimization problem can be formulated as follows.

We are given an $M \times n$ matrix X of real numbers and let x_i, $i = 1, \ldots, M$, be the rows of X. Choose m rows in such a way that the sum of squared distances from each row of the matrix to the space generated by the chosen rows would be minimal. In other words, find a subset of indices $S = \{i_1, \ldots, i_m\} \subset \{1, \ldots, M\}$ which minimizes following expression:

$$\sum_{i=1}^{M} \|x_i - \pi(x_i, span(x_{i_1}, \ldots, x_{i_m}))\|_2^2 = \sum_{i=1}^{M} \|x_i - X_S^T \mathrm{pinv}(X_S^T)x_i\|_2^2,$$

where $\pi(v, V)$ is orthogonal projection of vector v to subspace V, X_S is an $m \times n$ matrix consisting of rows with indices from S. This problem corresponds to the CUR-decomposition of X with m rows and all n columns.

A straightforward way is to choose items greedily. Suppose we have already chosen items $i_1, \ldots, i_t$. Then, we choose an item i_{t+1} so that

$$\sum_{i=1}^{M} \|x_i - \pi(x_i, span(x_{i_1}, \ldots, x_{i_{t+1}}))\|_2^2$$

is minimal.

Let us discuss how to choose $x_{i_{t+1}}$. Let Δ^t be the $M \times n$ matrix of our current approximation errors: $\Delta_i^t = x_i - \pi(x_i, span(x_{i_1}, \ldots, x_{i_t}))$ ($\Delta^0 = X$). Note that $span(x_{i_1}, \ldots, x_{i_t}, x_i) = span(x_{i_1}, \ldots, x_{i_t}, \Delta_i^t/\|\Delta_i^t\|_2)$, so for the purpose of evaluation our objective we can replace x_i with $o_i^t = \Delta_i^t/\|\Delta_i^t\|_2$. When we add x_i to the support set, the squared error on x_j reduces by $\langle x_j, o_i\rangle^2$ and Δ_j becomes $\Delta_j - \langle x_j, o_i\rangle o_i$. It can be seen by considering the orthonormal basis of $\mathbb{R}^m$, the first t elements of which generate $span(x_{i_1}, \ldots, x_{i_t})$ and $(t+1)$-th is o_i^t. Adding o_i^t to the support set will set to zero the $(t+1)$-th coordinate of the vector x_j^t (and Δ_j^t). And in the standard basis this coordinate may be calculated as $\langle x_j, o_i\rangle$. So we want to maximize over i:

$$\sum_{j=1}^{M} \langle x_j o_i^t\rangle^2 = \sum_{j=1}^{M} o_i^{tT} x_j x_j^T o_i^t = o_i^{tT} \left(\sum_{j=1}^{M} x_j x_j^T\right) o_i^t = o_i^{tT} X^T X o_i^t.$$

The procedure is summarized in Algorithm 1.

The choice of the next support item may be trivially implemented with $O(n^2 M)$ complexity. But it can be optimized: together with o_j^t we can keep the vectors $c_j^t = X^T X o_j^t$ that can be computed once initially in $O(n^2 M)$ and can be updated at each iteration synchronously with o_j^t. Updates

Algorithm 1 l_2-greedy support items selection

```
compute X^T X
compute normalized vectors o_i^0 = x_i/||x_i||_2
for t in [1, ..., m] do
    choose i_{t+1} which maximize o_i^{tT} X^T X o_i^t
    update all o_j
    for j in [1, ... N] do
        o_j^{t+1} ← o_j^t − o_{i_{t+1}}^t ⟨o_{i_{t+1}}^t, o_j^t⟩
        o_j^{t+1} ← o_j^{t+1} / ||o_j^{t+1}||_2
    end for
end for
```

of o_j^t at each iteration have the form $o_j^{t+1} = \alpha o_j^t + \beta o_{i_{t+1}}^t$, so c_j transforms analogously with the same coefficients: $c_j^{t+1} = \alpha c_j^t + \beta c_{i_{t+1}}^t$. So we can score all the items in $O(nM)$, calculating all the dot products $\langle o_j^t, c_j^t \rangle$ and update vectors o_j and c_j. The total complexity of the algorithm is $O(nM(n+m))$.

C Details on RBE implementation

In this section, we discuss our implementation of the relevance-based embeddings.

As a trainable mapping $f_Q(R(S_I, q), \theta_Q)$, we use the following variant:

$$f_Q(R(S_I, q), \theta_Q) := R(S_I, q) \,||\, F_Q^{mlp}(R(S_I, q), \theta_Q) \,||\, (1),$$

where F_Q^{mlp} is a 2-layer perceptron with the ELU activations, $||$ is the vector concatenation, and the last term is needed to represent the items offsets as a scalar product. The intuition here is that we split the representation into the prediction of AnnCUR and the trainable prediction of its error. In the experiments, such decomposition improves the convergence and training stability.

For the item mapping $f_I(R(i, S_Q), \theta_I)$, we use the following function:

$$f_I(R(i, S_Q), \theta_I) := t_I(R(i, S_Q), \theta_I) \,||\, F_I^{mlp}(t_I(R(i, S_Q), \tilde{\theta}_I)) \,||\, (c_i),$$

$$t_I(R(i, S_Q), \theta_I) := R(i, S_Q) \times P,$$

$$P = \mathrm{pinv}(R(S_I, Q_{train})),\ \theta_I := (P, c, \tilde{\theta}_I),$$

where c is a trainable bias vector, $\tilde{\theta}_I$ — perceptron trainable parameters. Although, as noted in Section 3.5, the transformation f_I acts in practice on a finite set I of elements and can be learned as an embedding matrix, the approach described above greatly accelerates the speed and stability of learning.

The mappings are trained using the Adam algorithm to optimize the following loss function inside the batches:

$$L := \frac{1}{|Q_{train}|} \sum_{q \in Q_{train}} \mathrm{softmax}(\tilde{R}(q, I)) \dot{(} 2 \cdot \mathbf{1}_{\mathrm{binRelevance(q)}} - 1),$$

$$binRelevance(q) := \tilde{R}(q, I) \geq q_{1-\frac{K}{|I|}}(R(q, I)),$$

where K is the desired top size and $q_x(v)$ calculates the x-th quantile of the vector v. We have experimented with various loss functions, but the one described above leads to consistently good results.

Our implementation of RBE has about 50K trainable parameters.

Table 7: Datasets sizes

	items	queries (used)
Yugioh	10031	3374
P.Wrest.	10133	1392
StarTrek	34430	4227
Dr.Who	40281	4000
Military	105K	2400
RecSys	16514	6958
RecSysLT	16514	6958
RecSys2	8950	10179
QA.Small	82326	9650
QA	0.8M	9650

Table 8: Dual encoder embeddings vs support relevances, different tops, RecSysLT

X	DUAL ENCODER	AXN_{DE}	RBE+K-MEANS	RBE+l_2-GREEDY
	HR(X+100, X)	HR(X+100, X)	HR(X, X)	HR(X, X)
100	0.7048	**0.7065**	0.6300	0.6682
200	0.6803	0.6769	0.6611	**0.6955**
300	0.6739	0.6740	0.6912	**0.7221**
400	0.6739	0.6769	0.7109	**0.7406**
500	0.6760	0.6820	0.7253	**0.7538**
600	0.6792	0.6883	0.7357	**0.7639**
700	0.6827	0.6958	0.7448	**0.7720**
800	0.6868	0.7054	0.7527	**0.7792**
900	0.6904	0.7161	0.7589	**0.7853**

D ADDITIONAL EXPERIMENTS

D.1 DUAL ENCODER EMBEDDINGS VS SUPPORT RELEVANCES (EXTENDED)

See Tables 8 and 9. For all datasets, DEs were trained on a significantly larger number of queries, the table refers to the data used for AnnCUR/RBE.

Table 9: Dual encoder embeddings vs support relevances, HitRate$(K, 100)$, RecSysLT

K	DUAL ENCODER	AXN_{DE}	RBE+K-MEANS	RBE+l_2-GREEDY
	HR(K+100, 100)	HR(K+100, 100)	HR(K, 100)	HR(K, 100)
100	0.7048	**0.7065**	0.6300	0.6682
200	0.7977	0.7970	0.8090	**0.8359**
300	0.8518	0.8538	0.8823	**0.9026**
400	0.8855	0.8902	0.9190	**0.9342**
500	0.9086	0.9153	0.9402	**0.9522**
600	0.9258	0.9331	0.9536	**0.9632**
700	0.9385	0.9465	0.9629	**0.9704**
800	0.9484	0.9572	0.9691	**0.9760**
900	0.9561	0.9660	0.9740	**0.9799**

D.2 CATEGORY PREDICTION FROM RELEVANCE-BASED EMBEDDING

In various practical applications, embeddings trained in one task are used to solve another. To check whether RBE has the ability to such transfer and how support items selection affects the quality of such a prediction, we trained a simple MLP category classifier on CUR-based item embeddings

$R(I, S_Q) \times \text{pinv}(R(S_I, S_Q))$ obtained with different S_I support items selection strategies and DE embeddings (sizes are the same). The categories of elements are marked by the authors of the content among 30 available options, multiple categories are allowed. Intersection over Union (IoU) is used as a metric:

$$IoU(C_{pred}, C_{true}) = \frac{C_{pred} \cap C_{true}}{C_{pred} \cup C_{true}}.$$

The results are presented in Table 10. Two conclusions can be made: first, in this setup, categories are predicted better by vectors derived from relevance than by vectors DE, and secondly, improving the algorithm for selecting support items also improves the quality of category prediction. Although these results are promising, we note that the possibility of such a transfer in other tasks and data requires a more detailed further study.

Table 10: Category prediction from CUR-Approx embeddings, RecSysLT dataset

S_I	IoU
(Dual Encoder)	0.6796
Random	0.7331
KMeans	0.7419
l_2-greedy	0.7516

E Scalability hints

Let us consider separately the selection of the support elements, training, and inference:

Support items selection Since different clusterization approaches have shown near-optimal quality, there are different options for scalable clustering:

- Clusterization on downsampled datasets: for extremely large (and dense) datasets it is natural to expect that cluster structure could be inferred from a significantly smaller subsample (according to the authors' experience, on the data of ads recommendation systems with billions of banners and downsampling to millions, this is so). Even for our research (small) datasets downsampling 75% of data reduces the quality of key selection in a discussable way (Table 11).
- Using data-driven clusterization: as shown in the third row of Table 1, choosing popular items from different categories/genres (in our case the global top of popular items is almost uniformly diversified) works extremely well.
- Using a distributed clustering algorithm.

Training As discussed in Sections 3.3 and 4.1.4, the training of such a model does not significantly differ from any dual-encoder-like models (which are commonly used in production recommendation

Table 11: Support items selection on downsampled data

	Yugioh	P.Wrest.	StarTrek	Dr.Who	Military
AnnCUR+Random	0.4724	0.4280	0.2287	0.1919	0.2455
AnnCUR+KMeans	0.5083	0.4850	0.3226	0.2517	0.3042
AnnCUR+1/4KMeans	0.5112	0.4685	0.3101	0.2514	0.2854
AnnCUR+1/8KMeans	0.5111	0.4694	0.3103	0.2450	0.2950
AnnCUR+l_2-greedy	0.5618	0.5119	0.3677	0.2960	0.3357
AnnCUR+1/4l_2-greedy	0.5529	0.4865	0.3582	0.2862	0.3270
AnnCUR+1/8l_2-greedy	0.5492	0.4712	0.3533	0.2822	0.3231

services) with relevance vectors as inputs. The only major difference is that relevances to fixed support items should be provided. Let us also note that efficient sampling of negative samples for the loss function should be used in order to train on such large datasets.

Inference It is also similar to dual-encoders: the item representations are precalculated and placed in the Approximate Nearest Neighbours index like HNSW, which accepts the embedding of the query as input.

F Statistical significance for AnnCUR (with random selection)

Since our baseline includes a random selection of support elements, the results can be noisy. In Table 12, we provide the average and standard deviation of HitRate(100) value (similar to Table 1), obtained by aggregating 15 runs of the algorithm with different initializations. As shown, even a random selection of support items gives fairly stable results. Moreover, it can be noted that the results of most non-random selection and RBE algorithms are several standard deviations higher.

Table 12: Average and standart deviation for AnnCUR over 15 launches

	Yugioh	**P.Wrest.**	**StarTrek**	**Dr.Who**	**Military**	**RecSys**	**RecSysLT**	**RecSys2**	**QA.Small**
avg	0.4599	0.4232	0.2447	0.1875	0.2557	0.6747	0.5908	0.1492	0.5803
std	0.0081	0.0062	0.0118	0.0060	0.0037	0.0048	0.0087	0.0066	0.0030

G Popular as a support key selection

The results for AnnCUR + popular are shown in the 3rd row of Table 13, the results for RBE+popular for ZeSHEL datasets are shown below in Table 13.

Table 13: Popular as a support key selection

	Yugioh	**P.Wrest.**	**StarTrek**	**Dr.Who**	**Military**
AnnCUR + Popular (Table 1)	0.2429	0.3001	0.1154	0.1197	0.1907
RBE + Popular	0.2637	0.3161	0.1330	0.1207	0.2038

H Information about licenses

- The Zero-Shot Entity Linking (ZESHEL) dataset (Logeswaran et al., 2019): CC-BY-SA;
- MsMarco (Nguyen et al., 2016) dataset: Custom (research-only, non-commercial);[8]
- RecSys, RecSysLT dataset: yet to be published;
- *all-mpnet-base-v2* from the SentenceTransformers (Reimers & Gurevych, 2019) library: Apache-2.0;
- CatBoost (Prokhorenkova et al., 2018) gradient boosting library: Apache-2.0.

[8] https://microsoft.github.io/msmarco/