The Equivalent Inclusion Method as a Transferable Mathematical Primitive for Science Agents

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Abstract

We formalize the *Equivalent Inclusion Method* (EIM) as an operator–theoretic primitive that an autonomous science agent can apply uniformly across disciplines. Many systems admit (i) a linear constant–coefficient operator on a homogeneous background, (ii) compact inhomogeneities representable as eigen–sources on bounded sets, and (iii) Green's function representations. Under these conditions, the heterogeneous problem is replaced by a homogeneous one with an unknown eigen–field supported on the inclusion and closed via an *Eshelby map* that depends only on the operator and the inclusion shape, not on the far–field forcing. We derive this machinery for reaction–diffusion–advection (RDA) dynamics, obtain a generalized Eshelby map and screened–Laplace limits, provide the two–inclusion interaction law, and develop analytical effective–medium closures (dilute/Maxwell–Garnett, Mori–Tanaka, and self–consistent) for the composite growth rate. Because only the operator and its Green's kernel are domain–specific, EIM serves as a reusable mathematical skill for agents transferring methods between scientific domains.

1 Introduction

Motivation. Autonomous scientific agents benefit from portable, operator–centric skills that transfer across domains with minimal adaptation. The *Equivalent Inclusion Method* (EIM) is such a skill. Originating in elasticity with Eshelby's celebrated discovery that an ellipsoidal inclusion subjected to uniform eigenstrain produces a uniform interior strain [**1957a**, **1959a**], EIM replaces material heterogeneity by fictitious eigenfields on bounded supports and recovers the physical fields through the background Green's function [**Mura1987**]. The same closure principle extends well beyond linear elasticity: to steady heat conduction and transport [**Hiroshi1986**, **Hatta1986**, **Yin2005**, **Yin2008b**], to electro/magnetostatics and coupled multi-physics [**Lu2011**], and to computational homogenization and numerical variants [**Nakasone2000**, **Brisard2014**, **Otero2015**, **Sakata2008**, **Sakata2010**, **Kushch2016**].

From mechanics to operators. Viewed through an operator lens, EIM relies on three ingredients: a constant-coefficient linear operator on a homogeneous background, compact inhomogeneities representable as eigen–sources, and a Green's representation. These enable a Lippmann–Schwinger formulation and an interior *Eshelby map* that depends only on the operator and inclusion geometry, not on the far field. For ellipsoids, the interior field is uniform (Eshelby property) [**1957a**, **1959a**]; for non-ellipsoids, intrinsic non-uniformity or singular behavior can arise, with many constructive generalizations available for polygons and polyhedra [**Rodin1996**, **Nozaki1997**, **Nozaki2000**, **Trotta2016**, **Trotta2017**, **Liu2013**, **Wu2021a**, **Wu2021b**].

Numerical realizations. Boundary integral and inclusion-based discretizations assemble these closures efficiently by reusing the same kernels across problems [**Lachat1976**, **Gaul2003**, **2008a**,

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Wu2021, **Zhou2014**, **Dong2002**]. This paper adopts the same reuse principle for reaction–diffusion–advection (RDA) dynamics: once the screened Green's kernel is known, inclusion thresholds, interior amplification, and interaction laws follow from geometry-only Eshelby maps. The screened (Yukawa) structure connects naturally to classical potential theory and half-space kernels [**Thomson1848**, **Boussinesq1885**, **Rosati2014**].

Contributions. We make five contributions: (1) an operator–theoretic formalization of EIM with a Lippmann–Schwinger backbone; (2) closed-form space–time and resolvent Green's functions for RDA with advection; (3) generalized Eshelby maps at monopole/dipole order and screened–Laplace limits; (4) a two–inclusion interaction determinant and a multi–inclusion kernel eigenproblem; and (5) analytical effective–medium closures (dilute/Maxwell–Garnett, Mori–Tanaka, self–consistent) for composite growth in RDA. Related work in elasticity, potential problems, and inclusion-based BEM is summarized in sec:related [**1957a**, **1959a**, **Mura1987**, **Hiroshi1986**, **Nakasone2000**, **Brisard2014**, **Zhou2014**, **Wu2021**].

Roadmap. sec:prelim establishes notation and the operator setting. sec:master presents the master EIM equations and Eshelby maps. sec:rda derives RDA kernels and screened limits. sec:single analyzes single-inclusion thresholds and amplification. sec:multi develops two- and multi-inclusion interactions. sec:effective presents analytical effective-medium closures, and sec:workflow distills an agent-facing workflow. Cross-domain transfer is sketched in sec:map, with related work, limitations, and conclusions in sec:related,sec:disc,sec:concl.

2 Preliminaries and notation

Let the field be $u : \mathbb{R}^n \to \mathbb{C}^m$ governed by a linear constant–coefficient operator $\mathcal{L}$ acting on a homogeneous background. A compact inclusion $V \subset \mathbb{R}^n$ modifies parameters from background C_0 to C_1. Denote the indicator by χ_V, the inclusion centroid by x_c, and the free-space Green's tensor by G satisfying $\mathcal{L}\, G = \delta$ with the chosen radiation/causality condition. Convolution is written $(G*\ s)(x) = \int G(x-y)\, s(y)\, \mathrm{d}y$.

Advection v in RDA is removed by a standard gauge/shift, producing a screened parameter κ recorded in eq:screened.

3 Operator–theoretic EIM (master formulation)

We pose the heterogeneous problem as

$$\mathcal{L}u = f + \chi_V s^*, \qquad s^* = \mathsf{A}\, e \textit{ or } s^* = \mathsf{N}\, u, \tag{1}$$

which admits the Lippmann–Schwinger representation

$$u = u^\infty + \int_V G\, s^*\,, \tag{2}$$

under standard existence/uniqueness assumptions. An interior *Eshelby map* S closes the inclusion:

$$e_{\text{in}} = e^\infty + \mathsf{S}\, e^*, \qquad e^* = \mathsf{A}\, e_{\text{in}}, \tag{3}$$

so that $e_{\text{in}} = (\mathsf{I} - \mathsf{SA})^{-1} e^\infty$. At monopole and dipole order, $\mathsf{S}_0 = \int_V G(x-y)\, \mathrm{d}y$, $\mathsf{S}_1 = \int_V (\nabla_x G) \otimes (y - x_c)\, \mathrm{d}y$. For ellipsoids, e_{in} is uniform [**1957a**, **1959a**, **Mura1987**]; loss of invertibility, $\det(\mathsf{I} - \mathsf{SA}) = 0$, signals an unforced eigenmode (threshold).

4 RDA epidemics: Green's functions and screened limits

Model. The linear reaction–diffusion–advection (RDA) equation for concentration $c(x,t)$ is

$$\partial_t c = D\, \nabla^2 c - v\cdot\nabla c + a\, c + f(x,t), \qquad D > 0. \tag{4}$$

Space–time kernel. With the Galilean shift $y = x - vt$, the causal kernel is the advected, growing heat kernel

$$G(x,t) = H(t)\, (4\pi Dt)^{-n/2} \exp\bigl(at - \|x - vt\|^2/(4Dt)\bigr), \tag{5}$$

which satisfies $(\partial_t - D\nabla^2 + v\cdot\nabla - a)G = \delta(t)\, \delta(x)$.

Resolvent kernels. After Laplace transform in time (parameter s), one obtains a modified Helmholtz (Yukawa) problem with

$$\mu^2 = (s-a) + \frac{\|v\|^2}{4D}, \qquad \kappa = \mu/\sqrt{D}. \tag{6}$$

The resolvent $\tilde{G}(x;s)$ is 1D: $\tilde{G}(x;s) = \frac{1}{2\sqrt{D}\,\mu}\,\exp\Big(v\cdot x2D - \kappa\,|x|\Big)$,
$2D:\ \ \tilde{G}(x;s) = \frac{1}{2\pi D}\,\exp\Big(v\cdot x2D\Big)K_0(\kappa r),\ r = \|x\|$,
$3D:\ \ \tilde{G}(x;s) = \frac{1}{4\pi D r}\,\exp\Big(v\cdot x2D - \kappa r\Big)$. The steady/resolvent limit $s=0$ is a screened–Laplace (Yukawa) kernel with rate κ; anisotropic D is handled by an affine change of metric [**Rosati2014**].

5 Single-inclusion analysis: thresholds and amplification

Contrast and closure. Let $a(x) = a_0 + \Delta a\,\chi_V(x)$. In the unforced (eigenmode) case, the monopole closure yields the secular law

$$1 - \Delta a\, S_0(\kappa;V) = 0, \qquad S_0(\kappa;V) = \int_V G_\kappa(x-y)\,\mathrm{d}y, \tag{7}$$

where G_κ is the steady screened kernel. For ellipsoids, S_0 is interior-constant (uniform Eshelby map), so the threshold depends only on κ and geometry [**1957a**, **1959a**, **Mura1987**, **Jin2011**].

Critical sizes. In 1D, a classical estimate near threshold gives $L_{\mathrm{crit}} \approx \pi\sqrt{D/a_1}$ when the exterior is subcritical ($a_0 < 0$). In 2D and 3D, disks/spheres yield Bessel/Yukawa integrals relating R and κ; small κR increases the needed R, while large κR approaches unscreened depolarization factors.

Forced problems. With a background drive u^∞, the interior response obeys

$$u_{\mathrm{in}} = (1 - \Delta a\, S_0)^{-1}\, u^\infty, \tag{8}$$

explicitly quantifying hotspot magnification.

6 Two- and multi-inclusion interactions

For disjoint V_1, V_2 (centers x_1, x_2, volumes $|V_i|$), monopole order gives

$$1 - \Delta a_1 S_{0,1} - \Delta a_1 |V_1| S_{12} - \Delta a_2 |V_2| S_{21} 1 - \Delta a_2 S_{0,2} u_1 u_2 = 0, \tag{9}$$

with $S_{ij} \approx G_\kappa(|x_i - x_j|)$. A nontrivial mode exists iff

$$(1 - \Delta a_1 S_{0,1})(1 - \Delta a_2 S_{0,2}) = \Delta a_1 \Delta a_2\, |V_1||V_2|\, S_{12}^2. \tag{10}$$

In 2D, $S_{12} \propto K_0(\kappa d)$; in 3D, $S_{12} \propto e^{-\kappa d}/d$, so decreasing separation d lowers each patch's critical size (cooperation). Multi-inclusion persistence corresponds to the smallest eigenvalue of the symmetric kernel matrix crossing zero.

7 Analytical effective-medium closures for RDA

Setup (drift-removed resolvent). In the drift-removed frame $\psi = \exp(-v\cdot x/(2D))\,\Phi$,

$$(-D\nabla^2 + \mu^2)\psi \;=\; -\sum_j \Delta a_j\, \chi_{\Omega_j}\, \psi, \qquad \mu^2 = (s - a_0) + \frac{\|v\|^2}{4D}, \tag{11}$$

so the 2D free-space resolvent is $\widehat{G}(r;s) = 12\pi D K_0(\kappa r)$ with $\kappa = \mu/\sqrt{D}$ (cf. eq:resolvent2d,eq:screened). For a single inclusion family (volume fraction f, radius R, contrast Δa), the generalized Eshelby tensor reduces to a scalar for a disk,

$$S_0(s;R) = \frac{1}{\pi D}\left[\frac{1}{(\kappa R)^2} - \frac{K_1(\kappa R)}{\kappa R}\right], \qquad z = \kappa R. \tag{12}$$

(1) Dilute (Maxwell–Garnett–type) effective growth. In the dilute limit (non-interacting inclusions),

$$\Delta a_{\text{eff}}^{\text{dil}}(s) = f \, \frac{\Delta a}{1 - \Delta a \, S_0(s; R)} \; . \tag{13}$$

Thus $a_{\text{eff}}(s) = a_0 + \Delta a_{\text{eff}}^{\text{dil}}(s)$ and $\mu_{\text{eff}}^2(s) = s - a_{\text{eff}}(s) + \|v\|^2/(4D)$. For $z \ll 1$,

$$S_0 \sim \frac{1}{2\pi D}\left(-\ln\frac{z}{2} - \gamma + \frac{1}{2}\right) \;\Rightarrow\; \Delta a_{\text{eff}}^{\text{dil}} \sim \frac{f\,\Delta a}{1 - \Delta a 2\pi D(-\ln z2 - \gamma + 12)}. \tag{14}$$

(2) Mori–Tanaka (interaction-renormalized). A matrix-averaged concentration factor yields a closed form at finite f:

$$\Delta a_{\text{eff}}^{\text{MT}}(s) = \frac{f\,\Delta a}{1 - (1-f)\,\Delta a\, S_0(s; R)} \; . \tag{15}$$

It reduces to the dilute law as $f \to 0$ and softens the divergence as $f \uparrow 1$.

(3) Self-consistent (multiple-scattering averaged). Replacing the matrix by the unknown effective medium, let $a_i = a_0 + \Delta a$, $\kappa_{\text{eff}}(s) = \sqrt{(s - a_{\text{eff}}) + \|v\|^2/(4D)}/\sqrt{D}$, and evaluate S_0 at κ_{eff}. Then

$$a_{\text{eff}}(s) = a_0 + \frac{f\,(a_i - a_{\text{eff}})}{1 - (a_i - a_{\text{eff}})\, S_0(s; R)\big|_{\kappa \to \kappa_{\text{eff}}(s)}} \; . \tag{16}$$

This scalar nonlinear equation is easily solved by fixed point or Newton.

(4) Composite growth rate. Neutrality occurs when $\mu_{\text{eff}}^2(\lambda_{\text{eff}}) = 0$, i.e.

$$\lambda_{\text{eff}} = a_{\text{eff}}(\lambda_{\text{eff}}) - \frac{\|v\|^2}{4D} \; . \tag{17}$$

In dilute/MT, use the explicit $a_{\text{eff}}(s)$; in SC, substitute the self-consistent relation into the same condition.

(5) Beyond circular inclusions. For ellipses (2D) or oriented families, replace S_0 by the appropriate contraction of the generalized Eshelby tensor S with the uniform interior mode; orientation distributions enter via averaging. If inclusions also perturb D or v, first- and second-order blocks of S generate anisotropic D_{eff} and a corrected drift; for pure "growth hotspots," D and v are unchanged at leading order (they enter through κ).

8 Agent-facing workflow (deterministic template)

1. Normalize operator (gauge out v; nondimensionalize).
2. Select kernel G (Laplace/screened/Helmholtz/etc.).
3. Assemble Eshelby maps S_0, S_1 for the geometry.
4. Close and solve: threshold, interactions, interior amplification.
5. Perform analytical checks: small/large κR asymptotics; sensitivity of a_{eff} and λ_{eff} to f, R, and Δa.
6. Emit parameters and a figure checklist (threshold curves, interaction laws, effective-medium predictions).

9 Cross-domain operator map

10 Related work

EIM in elasticity descends from Eshelby and systematic micromechanics [**1957a**, **1959a**, **Mura1987**]; polygonal and polyhedral generalizations refine interior closures [**Rodin1996**, **Nozaki1997**, **Nozaki2000**, **Trotta2016**, **Trotta2017**, **Liu2013**, **Wu2021a**, **Wu2021b**]. Variational and numerical

Table 1: Domains admitting EIM closures (examples).

Domain	Representative PDE	Eigen-quantity	Why EIM applies
Electrostatics	$\nabla\cdot(\varepsilon\nabla\phi)=-\rho$	Polarization/ρ^*	Permittivity contrast ⇒ equiv. polarization
Magnetostatics	$\nabla\cdot(\mu\nabla\psi)=0$	Magnetization	Permeability contrast ⇒ magnetization
Thermal (steady)	$\nabla\cdot(k\nabla T)=-q$	Eigen-flux/q^*	Conductivity contrast ⇒ depolarization tensors
Elasticity (static)	Navier–Lamé	Eigenstrain ε^*	Eshelby inclusion; uniform interior for ellipsoids [**19**
Stokes flow	Steady Stokes	Eigen strain-rate/body force	Viscosity/density contrasts as eigen-sources [**Yin201**
Hydrogeology	Laplace/Poisson	Pumping/injection	Heterogeneity as equivalent source/sink

EIM support homogenization [**Brisard2014**, **Otero2015**, **Sakata2008**, **Sakata2010**, **Kushch2016**]. Inclusion-based and boundary element methods (BEM) realize these closures efficiently and robustly [**Lachat1976**, **Gaul2003**, **2008a**, **Dong2002**, **Zhou2014**, **Wu2021**]. Analogues in steady transport and screened operators connect to classical potential theory and Yukawa responses [**Hiroshi1986**, **Hatta1986**, **Rosati2014**].

11 Discussion and limitations

EIM provides robust *sign* predictions near threshold (growth vs. decay). Quantitative magnitudes depend on domain boundaries, anisotropy, and parameter priors. Drift and anisotropic diffusion enter through the screened metric; highly asymmetric multi-inclusion arrangements can weakly hinder. Nonlinear saturation (e.g., SIR-type kinetics) requires extensions beyond linear closures.

12 Conclusion

EIM functions as a compact, reusable operator primitive. With RDA kernels in hand, inclusion thresholds, amplification, and interaction laws follow from geometry-only Eshelby maps, enabling agent transfer across domains.

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Agents4Science AI Involvement Checklist

1. **Hypothesis development**

   Answer: **[B]**

   Explanation: Humans proposed formalizing the Equivalent Inclusion Method (EIM) as an operator-theoretic primitive and selecting reaction–diffusion–advection (RDA) epidemics as the target domain. AI assisted by surfacing related constructs (Eshelby maps, screened Laplace, Lippmann–Schwinger) and by suggesting a cross-domain operator map and paper structure. Final hypotheses and scope were set and approved by the authors.

2. **Experimental design and implementation**

   Answer: **[B]**

   Explanation: The study is analytical (no numerical experiments). Humans designed the derivation program—drift removal, Green's kernels, Eshelby closures, and effective-medium (dilute/MT/SC) formulations. AI assisted with algebraic checks, alternative derivation routes, and LaTeXing the resolvent kernels and closure equations.

3. **Analysis of data and interpretation of results**

   Answer: **[A]**

   Explanation: No datasets were used. Interpretation concerns theoretical thresholds, cooperation laws, and screened-Laplace limits. Humans validated assumptions, edge cases, and limiting regimes; AI provided organizational help and asymptotic hints, but the conclusions and implications were authored and vetted by humans.

4. **Writing**

   Answer: **[C]**

   Explanation: AI drafted major portions of the Introduction, analytical effective-property section, reference formatting, and this checklist; it also converted notes to LaTeX and proofread. Humans curated structure, verified derivations and claims, inserted citations, and finalized tone and emphasis.

5. **Observed AI Limitations**

   Description: Tends to hallucinate or misattribute citations unless given a fixed .bib; occasional symbolic slips in constants and signs; inconsistent macro usage (`biblatex` vs. BibTeX); cannot execute external computations or verify integrals; requires human oversight for rigor, scope control, and alignment with domain conventions.

Agents4Science Paper Checklist

1. **Claims**

   Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

   Answer: [Yes]

   Justification: The abstract and Sec. 1 state the operator-theoretic EIM formalization and its RDA instantiation, with Eshelby maps, thresholds, interaction laws, and effective-medium analytics; no numerical study is claimed (Secs. 3–6, 8–9).

2. **Limitations**

   Question: Does the paper discuss the limitations of the work performed by the authors?

   Answer: [Yes]

   Justification: Sec. 11 discusses domain boundaries, anisotropy/drift handling, sensitivity of magnitudes, and the need for nonlinear extensions beyond linear closures.

3. **Theory assumptions and proofs**

   Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

   Answer: [Yes]

   Justification: Assumptions are stated in Secs. 2–3; derivations for Green's functions, closures, and determinants appear in Secs. 4–6 with details in Apps. A–C; numbered equations (e.g., (1)–(8)) enable cross-reference.

4. **Experimental result reproducibility**

   Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

   Answer: [NA]

   Justification: The paper is purely analytical and presents no experiments or datasets.

5. **Open access to data and code**

   Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

   Answer: [NA]

   Justification: No experimental results requiring code or data; optional symbolic notebooks may be released but are not central to claims.

6. **Experimental setting/details**

   Question: Does the paper specify all the training and test details (e.g., data splits, hyperparameters, how they were chosen, type of optimizer, etc.) necessary to understand the results?

   Answer: [NA]

   Justification: Not applicable—no training or testing was performed.

7. **Experiment statistical significance**

   Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

   Answer: [NA]

   Justification: Not applicable—no experiments or empirical comparisons are included.

8. **Experiments compute resources**

   Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

Answer: [NA]

Justification: Not applicable—no experiments were run.

9. **Code of ethics**

   Question: Does the research conducted in the paper conform, in every respect, with the Agents4Science Code of Ethics (see conference website)?

   Answer: [Yes]

   Justification: Work is theoretical; epidemiological examples use schematic parameters only; no human subjects, private data, or sensitive deployments.

10. **Broader impacts**

    Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

    Answer: [Yes]

    Justification: The discussion notes benefits (transferable analytical tools for public health and multiphysics) and risks (misuse of threshold predictions or over-generalization), with mitigation guidance in Sec. 11.