On the stability of phase retrieval

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***Abstract*—In this paper we tend to study the stability of phase retrieval for functions $f \in L_p(\mathbb{R}) \cap L_2(\mathbb{R}), 1 \leq p < 2$ from their decomposition coefficients on continuous frames in $L_2(\mathbb{R})$. As a by-product, we prove a version of Hausdorff -Young inequality for continuous frames.**

***Index Terms*—Phase retrieval, continuous frame**

I. Introduction

Phase retrieval brings together a number of problems where an object is reconstructed from incomplete information. For example, an element being decomposed over a system, but the information about the phase of the coefficients is missing and only their modulus if known). This problem, first appeared in x-ray crystallography about a century ago, arises as well in other fields of physics, such as for example astronomy, optics, speech processing, diffraction imaging, computational biology and others.

From the most general point of view, the problem of phase retrieval consists in reconstructing a function f in a space X from the phaseless information of some transform of f. Let this transform be described by a linear operator T that maps a vector space X to another space Y of either real or complex functions and suppose that it has a bounded inverse.

The lack of phase information leads us to consider a mapping

$$A : f \mapsto |Tf|, \tag{1}$$

(which is now non-linear) and we aim to reconstitute f from Af data. In what follows, we start by an acquisition system $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda}$ indexed by a set Λ (which is not necessarily discrete); the mapping T represents the analysis operator of Φ : $(Tf)_\lambda = (f, \varphi_\lambda)$, and the corresponding operator with missing phase is denoted by A_Φ.

In order to provide the well-posedness of the inverse problem, one needs the *existence* of a solution, or surjectivity of A_Φ, *uniqueness*, or injectivity of A_Φ, and *stability*, i.e. continuity of A_Φ^{-1}. To fulfill the existence condition, it suffices to put $Y = Ran(T)$. However, two other requirements are far more tricky to fulfill :

A_Φ is not injective: there is an ambiguity coming from multiplying by a scalar of modulus 1:

$$A_\Phi f = A_\Phi(cf) \text{ for any } f \in X, |c| = 1. \tag{2}$$

This could be fixed by reformulating the problem with respect to the operator on the quotient space $\tilde{X} = X \setminus \{-1, 1\}(X \setminus S^1$ in complex case):

$$\tilde{A}_\Phi : \tilde{X} \to \mathbb{R}_+^\Lambda,\ x \mapsto \{|(x, \varphi_\lambda)|\}_{\lambda\in\Lambda}.$$

Then, the injectivity of the resulted operator $\tilde{A}_\Phi$ can be characterized via the complement property of Φ (introduced in [1] for the case of discrete frames in infinite dimensions and for the case of continuous frames in Banach spaces in [2]). This property provides a necessary condition of uniqueness, and this condition becomes also a sufficient one in real case :

Provided that the operator A_Φ with the chosen representation system Φ is injective, one wonders then whether the inverse operator A_Φ^{-1} is continuously invertible. One can distinguish between weak stability (when A_Φ^{-1} is continuously invertible on its range) and strong stability (when A_Φ^{-1} is uniformly continuously invertible on its range). While weak stability is usually attained (see [3] in the case of semi-discrete frame of Cauchy wavelets, and [2] in case of continuous frames in Banach spaces), it is not quite so rosy as for strong stability. In fact, it holds in finite dimension ([1], [4]) but no more in infinite dimensions neither for frames nor for continuous frames ([2]). This is proven via strong complement property :

Definition 1.1: The system $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset X$ verifies $\sigma-$strong complementary property in X if there exists $\sigma > 0$ such that for any subset $S \subset \Lambda$ we have

$$max(A_{\Phi_S}, A_{\Phi_{\Lambda\setminus S}}) \geq \sigma,$$

which appears to be a necessary condition for the stability (and also a sufficient one in the real-valued case) but unfortunately it never holds in infinite dimension. Moreover, as proven in [1], stability degrades while augmenting the dimension of the space.

In that light, the results concerning the case of Fourier transform ([5]) sound rather optimistic. The author proposes an estimate for function from $L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n), 1 \leq p < 2$ using the fact that if $|f|$ and $|g|$ are close in L_2, f and g are close in L_p then up to translations f and g are close in L_2. More generally, the following theorem holds:

Theorem 1.1: ([5]) Let $1 \leq p < 2$ and $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.Let us define $h_f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$h_f(x) = \left(8 \int_{|\hat{f}(\xi)| \leq 10x} |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2} + \begin{cases} x^2 & \text{if } p > 1; \\ 0 & \text{if } p = 1. \end{cases}$$

Then, for any $g \in L_p(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$

$$\|f-g\|^2_{L_2(\mathbb{R}^n)} \le 2\||\hat{f}|-|\hat{g}|\|^2_{L_2(\mathbb{R}^n)} + h_f(\|f-g\|_{L_p(\mathbb{R}^n)}) \\ +2\|\mathrm{Im}\ \overline{\hat{f}}|\hat{f}|^{-1}\hat{g}\|^2_{L_2(\mathbb{R}^n)}$$

In case when $p=1$ and f has a real-valued Fourier transform supported on a set of measure L, this result reads as follows: for any $g \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$

$$\|f-g\|_{L_2(\mathbb{R}^n)} \le 2\||\hat{f}|-|\hat{g}|\|_{L_2(\mathbb{R}^n)} + 30\sqrt{L}\|f-g\|_{L_1(\mathbb{R}^n)} \\ +2\|\mathrm{Im}\ \hat{g}\|_{L_2(\mathbb{R}^n)}.$$

Moreover, as it follows from the result shown in [13], the last imaginary term can be removed :

Theorem 1.2: ([13]) Let $1 \le p < 2$, $f \in L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$. Then for any $g \in L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$

$$\|f-g\|_2^2 \le \||\hat{f}|-|\hat{g}|\|_2^2 + c_{n,p}|supp(\hat{f}) \cap supp(\hat{g})|^{\frac{p}{2-p}}\|f-g\|_p^2.$$

(the result shown in [13] is more general and concerns the estimates in Bessel potential spaces).

A. Mathematical setup

Let Ω be a space with a positive $\sigma-$additive measure μ, X be a space with a positive measure ν, $H = L_2(X,\nu)$ be the (complex) Hilbert space of square integrable functions on X by measure ν with inner product $(\cdot,\cdot)$.

The following definition of continuous frames extends the one of discrete frames for the cases when indices belong to a measurable space, see [6], [7], [8]:

Definition 1.2: The mapping $\varphi : \Omega \to H$ is called a continuous frame with respect to (Ω, ν) if

- φ is weakly measurable (i.e. for any $f \in H$, $\omega \mapsto (f, \varphi_\omega) = \hat{f}(\omega)$ is a measurable function on Ω);
- there exist constants $0 < a \le b < \infty$ such that for any $f \in L_2(\Omega, \nu)$

$$a\|f\|^2 \le \int_\Omega |(f,\varphi_\omega)|^2\, d\nu \le b\|f\|^2.$$

The constants a and b are called frame bounds; if $a=b$ then the frame is called tight and if $a=b=1$ then it is called Parseval. For $\Omega = \mathbb{N}$ and ν a counting measure, we fall into the case of a discrete frame.

Besides apparent common properties, this extended definition features as well some differences with respect to the discrete framework; namely, continuous frames are not necessarily norm bounded (see the corresponding example in [9]).

B. Contributions.

In the next section, we will prove an estimate similar to the one in Thm.1.1. We will as well prove a version of Hausdorff-Young inequality for continuous frames, interesting in itself.

II. Stability

In what follows, let $\{\varphi_\omega\}_{\omega\in\Omega}$ be a continuous frame in $L_2(X,\nu)$ with constants a, b such that there exist $r : 2 < r \le \infty$ and a measurable finite function $M(\omega)$ such that

$$\|\varphi_\omega\|_{r,\nu} \le M(\omega) \text{ for a.e. } \omega \in \Omega, \tag{3}$$

with $\|\varphi_\omega\|_{r,\nu}$ being the norm in $L_2(X,\nu)$. Suppose that for any complex-valued function $b(\omega)$ such that $b(\omega)M(\omega) \in L_1(\Omega,\mu)$ the function $b(\omega)\varphi_\omega : \Omega \to L_r(X,\nu)$ is measurable. Let then s be defined by $1/r + 1/s = 1$.

Lemma 2.1: Let $f \in L_p(X,\nu)$, $s \le p \le 2$; then the coefficient function

$$\hat{f}(\omega) = \int\limits_X f(x)\overline{\varphi_\omega(x)}\, d\nu(x)$$

is a.e.finite and measurable on Ω.

Proof. For any $f \in L_p(X,\nu)$, let us decompose it as $f = f_1 + f_2$ where

$$f_2(x) = \begin{cases} f(x) & \text{if } |f(x)| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and $f_1(x) = f(x) - f_2(x)$. This implies $f_1 \in L_s(X,\nu), f_2 \in L_2(X,\nu)$. Since

$$|\hat{f}_1(\omega)| \le \|f_1\|_{s,\nu}\|\varphi_\omega\|_{r,\nu} < \infty,$$
$$|\hat{f}_2(\omega)| \le \|f_2\|_{2,\nu}\|\varphi_\omega\|_{2,\nu} < \infty,$$

the function $\hat{f}(\omega) = \hat{f}_1(\omega) + \hat{f}_2(\omega)$ is a.e.finite. Then, there exists a sequence $\{g_k\}_k$ of simple functions converging to f_1 in $L_s(\Omega,\nu)$; the functions g_k and f_2 belong to L_2 and thus $\hat{g}_k$ and $\hat{f}_2$ are measurable (by the definition of a continuous frame). The measurability of $\hat{f}$ follows then from the continuity of the scalar product.

The following theorem gives a version of Hausdorff-Young inequality in the case of continuous frames and generalizes the inequality from [10]. The proof is provided in the Appendix.

Theorem 2.1: Let $\{\varphi_\omega\}_{\omega\in\Omega}$ be a continuous frame verifying (3). Let s satisfy $\frac{1}{r} + \frac{1}{s} = 1$, p verify $s \le p \le 2$ and q being defined by $\frac{s}{p} + \frac{2-s}{q} = 1$. Then for any $f \in L_p(X,\nu)$

$$\int_\Omega |\hat{f}(\omega)|^q M^{2-q}(\omega) d\mu(\omega) \le b\|f\|^q_{p,\nu}. \tag{5}$$

Theorem 2.2: Let $s \le p < 2$ and $f \in L_p(\mathbb{R}) \cap L_2(\mathbb{R})$. Let $\{\varphi_\omega\}_{\omega\in\Omega}$ be a continuous frame verifying (3).

Then for all $g \in L_p(\mathbb{R}) \cap L_2(\mathbb{R})$

$$\|f-g\|_2^2 \le 2\||\hat{f}|-|\hat{g}|\|_2^2 + (h_f(\|f-g\|_p) \\ + b\|f-g\|_p^2 + 2\|\mathrm{Im}\overline{\hat{f}}|\hat{f}|^{-1}\hat{g}\|_2^2$$

(h_f being defined in the same way as in Thm 1.1).

Proof. First, since $\{\varphi_\omega\}_{\omega\in\Omega}$ is a continuous frame in $L_2(\mathbb{R}^n)$, for any $f, g \in \mathbb{R}^n$

$$\|f-g\|_2^2 \le \frac{1}{a}\int_{\mathbb{R}^n} |\hat{f}(\omega) - \hat{g}(\omega)|^2\, d\mu(\omega).$$

Let us fix $\varepsilon = \|f-g\|_p$ and split the integral into three parts:

$$\int_{\mathbb{R}^n} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, d\mu(\omega) =$$
$$\int\limits_{|\hat{f}(\omega)|\geq M(\omega)\varepsilon} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, d\mu(\omega)+$$
$$\int\limits_{|\hat{f}(\omega)|\leq M(\omega)\varepsilon} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, d\mu(\omega) =$$
$$\int\limits_{\substack{|\hat{f}(\omega)|\,\geq\, M(\omega)\varepsilon \\ |\hat{f}(\omega)-\hat{g}(\omega)|\,\leq\, M(\omega)\varepsilon}} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, d\mu(\omega)+$$
$$\int\limits_{\substack{|\hat{f}(\omega)|\,\geq\, M(\omega)\varepsilon \\ |\hat{f}(\omega)-\hat{g}(\omega)|\,\geq\, M(\omega)\varepsilon}} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, d\mu(\omega)+$$
$$\int\limits_{|\hat{f}(\omega)|\leq M(\omega)\varepsilon} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, d\mu(\omega) = I_1+I_2+I_3.$$

First term. The estimation of the first term is the same as for the case of Fourier transform thus we do not reproduce it here:

$$I_1 \leq \||\hat{f}|-|\hat{g}|\|_2^2 + \frac{6}{5}\|Im\hat{f}|\hat{f}|^{-1}\hat{g}\|_2^2.$$

Second term. Denoting by Ω_1 the integration domain :

$$\Omega_1 = \{\omega\in\mathbb{R}\ :\ |\hat{f}(\omega)|\geq M(\omega)\varepsilon, \left|\frac{\hat{f}(\omega)-\hat{g}(\omega)}{M(\omega)}\right| \geq \varepsilon\},$$

one has (by Holder inequality with $q/2, q/(q-2)$)

$$I_2 = \int_{\Omega_1} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, d\mu(\omega) =$$
$$\int_{\Omega_1} |\hat{f}(\omega)-\hat{g}(\omega)|^2 |M(\omega)|^2\, \frac{d\mu(\omega)}{|M(\omega)|^2} =$$
$$\int_{\Omega_1} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, \frac{d\alpha(\omega)}{|M(\omega)|^2} \leq$$
$$\left(\int_{\Omega_1}\left(\frac{|\hat{f}(\omega)-\hat{g}(\omega)|}{M(\omega)}\right)^q d\alpha(\omega)\right)^{\frac{2}{q}} (\alpha(\Omega_1))^{1-\frac{2}{q}}.$$

Using the theorem 2.1, we get

$$\int_{\Omega_1}\varepsilon^q M^2(\omega)\, d\mu(\omega) \leq \int_{\Omega_1}\left(\frac{|\hat{f}-\hat{g}|}{M(\omega)}\right)^q M^2(\omega)\, d\mu(\omega)$$
$$\leq b\|f-g\|_p^q$$

which results in

$$\alpha(\Omega_1) = \int_{\Omega_1} M^2(\omega)\, d\omega \leq \frac{1}{\varepsilon^q} b\|f-g\|_p^q = b.$$

Thus, using once more theorem 2.1, one gets

$$I_2 \leq \left(\int_{\Omega_1}\left(\frac{|\hat{f}(\omega)-\hat{g}(\omega)|}{M(\omega)}\right)^q d\alpha(\omega)\right)^{\frac{2}{q}} b^{1-\frac{2}{q}}$$
$$\leq \left(\int_{\mathbb{R}^n}\left(\frac{|\hat{f}(\omega)-\hat{g}(\omega)|}{M(\omega)}\right)^q d\alpha(\omega)\right)^{\frac{2}{q}} b^{1-\frac{2}{q}}$$
$$\leq b^{\frac{2}{q}}\|f-g\|_p^2 b^{1-\frac{2}{q}} = b\|f-g\|_p^2.$$

Third term.

$$I_3 = \int\limits_{|\hat{f}(\omega)|\leq M(\omega)\varepsilon} |\hat{f}(\omega)-\hat{g}(\omega)|^2\, d\mu(\omega) \leq$$
$$8\int\limits_{|\hat{f}(\omega)|\leq M(\omega)\varepsilon} |\hat{f}(\omega)|^2\, d\mu(\omega)+$$
$$2\int\limits_{|\hat{f}(\omega)|\leq M(\omega)\varepsilon} ||\hat{f}(\omega)|-|\hat{g}(\omega)||^2\, d\mu(\omega)$$
$$\leq 8\int\limits_{|\hat{f}(\omega)|\leq M(\omega)\varepsilon} |\hat{f}(\omega)|^2\, d\mu(\omega) + 2\||\hat{f}|-|\hat{g}|\|_2^2.$$

Appendix.

Proof of Theorem 2.1. One can define a measure α on X by taking $\alpha(A) = \int_A M^2(\omega)\, d\omega$ for any measurable subset A of X. This measure is $\sigma-$finite. Then for any $f\in L_2(X,\nu)$ we have

$$\int_X \left|\frac{\hat{f}(\omega)}{M(\omega)}\right|^2\, d\alpha(\omega) = \int_\Omega \left|\hat{f}(\omega)\right|^2\, d\mu(\omega) \leq b\|f\|_{2,\nu}^2,$$

and for any $f\in L_s(X,\nu)$

$$\left|\frac{\hat{f}(\omega)}{M(\omega)}\right| = \frac{1}{M(\omega)}\left|\int_X f(x)\overline{\varphi_\omega(x)}\, d\nu(x)\right|$$
$$\leq \frac{1}{M(\omega)}\|f\|_{s,\nu}\|\varphi(\omega)\|_{r,\nu} \leq \|f\|_{s,\nu} \text{ a.e. on } \Omega.$$

The operator

$$T_M : f\mapsto \frac{\hat{f}(\omega)}{M(\omega)}$$

is of strong type $(2,2)$ and of strong type $(s,+\infty)$. Since for $\theta = 1-\frac{2}{q}$ one has

$$\frac{1}{p} = \frac{1-\theta}{2}+\frac{\theta}{s},\ \frac{1}{q} = \frac{1-\theta}{2},$$

by Riesz-Thorin theorem T_M is of strong type (p,q) , i.e.

$$\|T_M f\|_{q,\alpha} \leq b^{1/q}\|f\|_{p,\nu} \text{ for any } f\in L_p(X,\nu).$$

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