Quadrature formulas on graphs

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***Abstract*—We consider a disjoint cover (partition) of an undirected weighted finite graph G by $|J|$ connected subgraphs (clusters) $S_{j_{j\in J}}$ and select a function $\psi_j \geq 0$ on each of the clusters. For a given signal f on G its weighted average samples are defined via inner products $\{\langle \psi_j, f\rangle\}_{j\in J}$. The goal of the paper is to describe subspaces of bandlimited functions for which there exist quadrature formulas with positive coefficients based on weighted average samples.**

***Index Terms*—combinatorial graphs, combinatorial Laplace operators, cubature formulas on graphs, frames, Poincaré and Plancherel-Polya-type inequalities**

I. INTRODUCTION

During last years in connection with a variety of important applications, the problems about sampling, interpolation and quadrature formulas in the setting of combinatorial graphs attracted attention of many mathematicians and engineers. Here are some of the relevant papers [1]- [3], [6], [8], [10]-[20].

In the present paper we consider an undirected weighted finite graph G which is equipped with positive, symmetric weights $w(v,u)$ for every edge (v,u). Let $\{S_j\}_{j\in J}$ be a family of connected subgraphs (clusters) which form a disjoint cover of the set of vertices $V(G)$. We select a set of functions $\psi_j \geq 0$ where ψ_j has non-trivial support in S_j. Throughout the paper the notation $\Xi(G) = (\{S_j\}_{j\in J}, \{\psi_j\}_{j\in J})$ will be used. For a given signal f on G its weighted average samples are defined via inner products $\{\langle \psi_j, f\rangle\}_{j\in J}$. Let $\ell^2(G)$ denote the space of all complex-valued functions with the inner product

$$\langle f, g\rangle = \sum_{v\in V(G)} f(v)\overline{g(v)},$$

and let $L : \ell^2(G) \mapsto \ell^2(G)$ be the combinatorial Laplace operator associated with the graph G, i.e.

$$(Lf)(v) = \sum_{u\in V(G)} (f(v) - f(u))w(v,u). \qquad (1)$$

A subspace of bandlimited functions $\mathbf{E}_\omega(G) \subset \ell^2(G)$, $\omega > 0$ is defined as the span of all eigenfunctions of combinatorial Laplace operator L whose corresponding eigenvalues which are not greater than $\omega > 0$. Our main result is the following.

Theorem 1.1: For a given G and $\Xi = \Xi(G) = (\{S_j\}_{j\in J}, \{\psi_j\}_{j\in J})$, there exist non-negative weights $\widetilde{\beta}_j$ and a positive constant $\widetilde{C(G,\Xi)} > 0$ such that the following formula holds for all $f \in \mathbf{E}_\omega(G)$

$$\sum_{v\in V(G)} f(v) = \sum_j \widetilde{\beta}_j \langle \psi_j, f\rangle,$$

as long as

$$0 \leq \sqrt{\omega} < \widetilde{C(G,\Xi)}, \qquad (2)$$

In connection with this statement one has to verify if there exist graphs for which the corresponding interval $[0, \widetilde{C(G,\Xi)})$ contains non-trivial eigenvalues of the Laplacian L. We answer this question in section III where we prove in Lemma 4.4 that for a community graph $G = \cup_{j\in J} S_j$ (see [7]) and any $\delta > 0$ the interval $(0,\delta)$ contains at least $|J| - 1$ nontrivial frequencies (counted with multiplicity) of the corresponding Laplace operator L if the weights of edges between communities are small enough.

II. MARCINKIEWICZ-ZYGMUND-TYPE INEQUALITIES IN $\ell^2(G)$.

For a finite G we consider the following assumptions and notations. We assume that $\mathcal{S} = \{S_j\}_{j\in J}$ form a disjoint cover of $V(G)$

$$\bigcup_{j\in J} S_j = V(G). \qquad (3)$$

Let L_j be the Laplacian for the **induced subgraph** S_j. We assume that every $S_j \subset V(G)$, $j \in J$, is a *finite and connected graph with more than one vertex.* The spectrum of the operator L_j will be denoted as $0 = \lambda_{0,j} < \lambda_{1,j} \leq ... \leq \lambda_{|S_j|-1,j}$ and we use $\{\varphi_{k,j}\}_{k=0}^{|S_j|}$ for an corresponding o.n.b. of eigenfunctions. In particular, the first non-zero eigenvalue for a subgraph S_j is $\lambda_{1,j}$, and $\varphi_{0,j} = \mathbf{1}/\sqrt{|S_j|}$.

Let $\|\nabla_j g\|_{\ell^2(S_j)}$, $g \in \ell^2(S_j)$, be the weighted gradient for the induced subgraph S_j, i.e.

$$\|\nabla_j g\|_{\ell^2(S_j)} = \left(\sum_{u,v\in V(S_j)} \frac{1}{2}|g(u) - g(v)|^2 w(u,v)\right)^{1/2}. \qquad (4)$$

With every S_j, $j \in J$, we associate a function $\psi_j \in \ell^2(S_j)$ whose support is in S_j and introduce the functionals Ψ_j on $\ell^2(S_j)$ defined by these functions $\langle \psi_j, f\rangle$, i.e.

$$\Psi_j(g) = \sum_{v\in V(S_j)} \psi_j(v)\overline{g(v)}, \quad g \in \ell^2(S_j). \qquad (5)$$

Notation χ_j will be used for the characteristic function of S_j and we use f_j for $f\chi_j,\ f \in \ell^2(G)$.

We assume that $\langle \psi_j, \varphi_{0,j}\rangle \neq 0$ for every $j \in J$ and introduce the following notations

$$A_j = \frac{\|\psi\|^2}{\lambda_{1,j}|\langle\psi_j,\varphi_{0,j}\rangle|^2}, \quad A_\Xi = \max_{j\in J} A_j,$$
$$\Xi = \left(\{S_j\}_{j\in J},\ \{\psi_j\}_{j\in J}\right), \tag{6}$$

and define the functions

$$\zeta_j = \frac{\psi_j}{|S_j|^{1/2}\langle\psi_j,\varphi_{0,j}\rangle}, \ \ \langle\psi_j,\varphi_{0,j}\rangle \neq 0.$$

The following Poincare-type inequality can be proved.

Theorem 2.1: Let G be a connected finite graph and $\mathcal{S} = \{S_j\}$ is its disjoint cover by finite sets. Let L_j be the Laplace operator of the induced subgraph S_j whose first nonzero eigenvalue is $\lambda_{1,j}$ and $\varphi_{0,j} = \mathbf{1}/\sqrt{|S_j|}$ is its normalized eigenfunction with eigenvalue zero. Assume that for every j function $\psi_j \in \ell^2(G)$ has support in S_j, and $\langle\psi_j,\varphi_{0,j}\rangle \neq 0$. Then the following inequality holds true for every $f \in \ell^2(G)$

$$\sum_{j\in J}\sum_{v\in V(S_j)} |f_j(v) - \langle\zeta_j, f\rangle\chi_j(v)|^2 \leq A_\Xi \|\nabla f\|^2_{\ell^2(G)}, \tag{7}$$

where $f_j = f\chi_j$.
We set

$$a_\Xi = \max_{j\in J}\frac{1}{|\langle\psi_j,\varphi_{0,j}\rangle|^2}, \quad c_\Xi = \max_{j\in J}\|\psi_j\|^2,$$

and then pick any $\omega > 0$, $\tau > 0$ which satisfy the inequality

$$\omega(1+\tau)A_\Xi < 1. \tag{8}$$

By using (7) we prove the following Marcinkiewicz-Zygmund-type (Plancherel-Polya-type, or frame) inequalities in the norm of $\ell^2(G)$.

Theorem 2.2: In the previous notations for every $f \in \mathbf{E}_\omega(G)$ with ω satisfying (8) the Marcinkiewicz-Zygmund-type inequality holds true

$$\frac{(1-\mu)\tau}{(1+\tau)a_\Xi}\|f\|^2 \leq \sum_{j\in J}|\langle\zeta_j, f\rangle|^2 \leq c_\Xi\|f\|^2, \tag{9}$$

where $\mu = \omega(1+\tau)A_\Xi < 1$.

III. Marcinkiewicz-Zygmund-type inequality in $\ell^1(G)$

Consider a finite dimensional space $\mathbb{R}^{|J|}$ of all sequences $\{\alpha_j\}$, $1 \leq j \leq |J|$, equipped with the norm

$$|||\{\alpha_j\}_{j\in J}||| = \sum_{j\in J}|\alpha_j|\,|S_j|, \tag{10}$$

where $|S_j|$ is cardinality of a subgraph S_j, and define the following *sampling operator* Q as

$$Q: f \in \mathbf{E}_\omega(G) \mapsto \{f_j\}_{j\in J} = \{\langle\zeta_j, f\rangle\}_{j\in J} \in \mathbb{R}^{|J|}. \tag{11}$$

The Marcinkiewicz-Zygmund-type inequality (9) already shows that Q is continues and injective. However, we will need a more accurate version of (9) in the norm $\ell^1(G)$.

We set

$$C_\Xi = \sqrt{|J|\max_{j\in J}(A_j|S_j|)}. \tag{12}$$

Theorem 3.1: In the same notations as above, if

$$\sqrt{\omega}\,C_\Xi = \gamma \in (0,\ 1/2), \tag{13}$$

then the following double inequality holds

$$(1-\gamma)\sum_{v\in V(G)}|f(v)| \leq \sum_j |\langle\zeta_j, f\rangle||S_j| \leq$$
$$(1+\gamma)\sum_{v\in V(G)}|f(v)|, \tag{14}$$

for every $f \in \mathbf{E}_\omega(G)$.

Indeed, we have

$$\sum_{v\in V(S_j)}|f_j(v) - \langle\zeta_j, f\rangle\chi_j(v)|^2 \leq A_j\|\nabla_j f_j\|^2_{\ell^2(S_j)},$$

and together with the Schwartz inequality it gives

$$\sum_{v\in V(S_j)}|f_j(v) - \langle\zeta_j, f\rangle\chi_j(v)| \leq$$
$$|S_j|^{1/2}\left(\sum_{v\in V(S_j)}|f_j(v) - \langle\zeta_j, f\rangle\chi_j(v)|^2\right)^{1/2} \leq$$
$$\sqrt{|S_j|A_j}\,\|\nabla_j f_j\|_{\ell^2(S_j)}.$$

We also have the inequality

$$\sum_{j\in J}\|\nabla_j f_j\|_{\ell^2(S_j)} \leq |J|^{1/2}\left(\sum_{j\in J}\|\nabla_j f_j\|^2_{\ell^2(S_j)}\right)^{1/2} \leq$$
$$\sqrt{|J|}\|\nabla f\|_{\ell^2(G)},$$

which implies

$$\sum_{j\in J}\sum_{v\in V(S_j)}|f_j(v) - \langle\zeta_j, f\rangle\chi_j(v)| \leq$$
$$\sum_{j\in J}\sqrt{|S_j|A_j}\|\nabla_j f_j\|_{\ell^2(S_j)} \leq$$
$$\sqrt{|J|\max_{j\in J}(|S_j|A_j)}\|\nabla f\|_{\ell^2(G)}.$$

After we obtain

$$\sum_{j\in J}\sum_{v\in V(S_j)}|f_j(v) - \langle\zeta_j, f\rangle\chi_j(v)| \leq$$
$$\sqrt{\omega}\,C_\Xi\sum_{v\in G}|f(v)|, \quad f \in \mathcal{X}_\omega(G), \quad f_j = f|_{S_j},$$

here $C_\Xi = \sqrt{|J|\max_{j\in J}(|S_j|A_j)}$. Theorem is proven.

IV. A POSITIVE QUADRATURE FORMULA

Suppose $\mathcal{E}$ is a linear normed space, $\mathcal{F} \subset \mathcal{E}$ is a subspace of $\mathcal{E}$, and $\mathcal{B}$ is a convex cone in $\mathcal{E}$, which determines an order on $\mathcal{E}$. The following theorem can be found in [4], [9] and it will be used in proving our cubature formulas.

Theorem 4.1: (Bauer-Namioka). Let Θ be a linear form on a subspace $\mathcal{F}$ of $\mathcal{E}$. There exists a continuous positive linear extension of Θ to $\mathcal{E}$ if and only if there exists a neighborhood $\mathcal{U}$ of 0 such that the set $\Theta(\mathcal{F} \cap (\mathcal{U} + \mathcal{B}))$ is bounded from below.

We consider the sampling operator Q which is continuous and injective. It is worth to emphasize that in general one can expect only an inclusion $\mathcal{R}(Q) \subset \mathbb{R}^{|J|}$ and not an equality $\mathcal{R}(Q) = \mathbb{R}^{|J|}$. The continues inverse operator Q^{-1} is defined on $\mathcal{R}(Q)$:

$$Q^{-1}\left(\{f_j\}_{j\in J}\right) = f \in \mathbf{E}_\omega(G), \tag{15}$$

and it means that every sequence $\{f_j\}_{j\in J} \in \mathcal{R}(Q)$ is mapped to a unique function in $\mathbf{E}_\omega(G)$ for which $f_j = \langle f, \zeta_j\rangle$, $1 \le j \le |J|$. We pick a number $0 < \gamma < 1/2$ and introduce the functional Θ on $\mathcal{R}(Q)$ by using the formula

$$\Theta\left(\{f_j\}\right) = \sum_{v\in V(G)} Q^{-1}\left(\{f_j\}\right)(v) - \frac{1-2\gamma}{1-\gamma}|||\{f_j\}||| =$$

$$\sum_{v\in V(G)} f(v) - \frac{1-2\gamma}{1-\gamma}\sum_j \langle\zeta_j, f\rangle|S_j|, \quad \{f_j\} \in \mathcal{R}(Q), \tag{16}$$

where $f = Q^{-1}(\{f_j\})$ is a unique function in $\mathbf{E}_\omega(G)$ for which $\langle f, \zeta_j\rangle = f_j$ for all j.

To apply Theorem 4.1 we treat $\left(\mathbb{R}^{|J|},\, |||\cdot|||\right)$ as the space $\mathcal{E}$, the $(\mathcal{R}(Q),\, |||\cdot|||)$ as the subspace $\mathcal{F}$, and also

$$\mathcal{B} = \left\{\{s_j\}_{j\in J} \in \mathcal{E} : s_j \ge 0,\ 1 \le j \le |J|\right\},$$

$$\mathcal{U} = \left\{\{s_j\}_{j\in J} \in \mathcal{E} : |||\{s_j\}_{j\in J}||| \le 1\right\}.$$

Since by assumption the functions ψ_j are not negative, the functionals $\langle\zeta_j, \cdot\rangle$ are also non-negative.

Lemma 4.2: If the functionals $\langle\zeta_j, \cdot\rangle$, $j \in J$, are positive and the positive γ in (13) is less than $1/2$ then the functional Θ is bounded from below on the set $\mathcal{F} \cap (\mathcal{U} + \mathcal{B})$. Namely, the following estimate holds

$$\Theta(\mathcal{F} \cap (\mathcal{U} + \mathcal{B})) > \frac{2\gamma}{\gamma - 1}, \quad 0 < \gamma < 1/2.$$

We are ready to formulate our statement about existence of quadrature formulas with positive weights.

Theorem 4.3: If the constant

$$\gamma = \sqrt{\omega}\, C_\Xi$$

satisfies

$$0 < \gamma < 1/2,$$

then there exist weights β_j

$$\frac{1-2\gamma}{1-\gamma}|S_j| \le \beta_j \le \frac{3-2\gamma}{1-\gamma}|S_j|, \tag{17}$$

such that the following formula holds for all $f \in \mathbf{E}_\omega(G)$

$$\sum_{v\in V(G)} f(v) = \sum_j \beta_j \langle\zeta_j, f\rangle,$$

where

$$C_\Xi = \sqrt{|J| \max_{j\in J}\left(|S_j| A_j\right)}, \tag{18}$$

$$A_j = \frac{\|\psi_j\|^2}{\lambda_{1,j}|\langle\psi_j, \varphi_{0,j}\rangle|^2}, \quad \Xi = \left(\{S_j\}_{j\in J},\ \{\psi_j\}_{j\in J}\right).$$

The condition that the constant $\gamma = \sqrt{\omega}\, C_\Xi$ is less than $1/2$ implies the following interval for frequencies

$$0 \le \omega < \frac{1}{4C_\Xi^2}. \tag{19}$$

In this connection the important question arises: *for what graphs does the interval* $\left[0,\ \frac{1}{4C_\Xi^2}\right)$ *contain non-trivial eigenvalues of the Laplacian* L_G?

Lemma 4.4: For every natural N and any $\rho > 0$ there exist community graphs with N communities such that the interval $[0,\, \rho)$ contains at least $N-1$ non-trivial eigenvalues (counting with multiplicities) of the corresponding Laplace operator.

The proof of this lemma shows that if in a community graph the weights of edges between communities are sufficiently small compared to weights of edges inside communities, then there are ”many” eigenvalues of a Laplacian which are close to zero.

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