A Gaussian Regularization for Higher-Order Derivative Sampling Expansion in the Linear Canonical Transform Domain

Rashad M. Asharabi
Department of Mathematics, College of Arts and Sciences,
Najran University, Najran, Saudi Arabia
E-mail: rashad1974@hotmail.com

***Abstract*—The derivative sampling theorem for bandlimited functions within the fractional Fourier transform (FrFT) domain, which involves samples from the function and its r-derivatives, was introduced by Jing and his collaborators in 2019. In this paper, we extend this type of sampling expansion to the linear canonical transform (LCT) domain and propose an alternative representation of this sampling using a contour integral. The convergence rate of the reformulated sampling expansion remains slow, particularly at the order of $O(\ln(N)/N)$. To address this slow convergence, we develop a Gaussian regularization method for higher-order derivative sampling in the LCT domain. This regularization sampling method applies to a broader range of functions within the Paley-Wiener space in the LCT domain, including entire functions that may not necessarily belong to the space $L^2(\mathbb{R})$ when their domain is restricted to $\mathbb{R}$. Notably, this regularized sampling still achieves an exponential convergence rate and requires only a finite number of samples from the original function and its first r derivatives. The results of Asharabi (2016) within the classical Fourier transform domain and Annaby et al. (2023) within the LCT domain will be special cases of the findings presented in this paper. Furthermore, we present a numerical example that demonstrates excellent consistency with our theoretical analysis.**

***Index Terms*—Derivative sampling theorem; Generalized Hermite interpolation, Linear canonical transform, Contour integral, Gaussian regularization.**

I. Introduction

The Fractional Fourier transform (FrFT) of order α for a function f, denoted as $\mathcal{F}_\alpha$, is formally defined as, cf. eg. [15],

$$\mathcal{F}_\alpha[f](\omega) = \int_{\mathbb{R}} f(t) K_\alpha(t,\omega)dt, \tag{1}$$

where

$$K_\alpha(t,\omega) = \begin{cases} c(\alpha)\mathrm{e}^{-\mathrm{i}c(\alpha)(t^2+\omega^2)-b(\alpha)wt}, & \alpha \neq k\pi, \\ \delta(t-\omega), & \alpha = 2k\pi. \\ \delta(t+\omega), & \alpha = (2k-1)\pi, \end{cases} \tag{2}$$

is the transformation kernel with

$$a(\alpha) = \cot(\alpha)/2, \quad b(\alpha) = \csc(\alpha), \quad c(\alpha) = \sqrt{\frac{1-\mathrm{i}\cot(\alpha)}{2\pi}}, \tag{3}$$

where $\mathrm{i} = \sqrt{-1}$, $\alpha \in \mathbb{R}$, k is an integer, and δ is the Dirac delta function. The Paley-Wiener space in the FrFT domain, denoted by $B^2_\alpha(\mathbb{R})$, consists of all bandlimited $L^2(\mathbb{R})$-functions with bandwidth Ω in the FrFT domain of order α, as defined in [11], [24]. This means that

$$B^2_{\alpha,\Omega}(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \quad \mathcal{F}_\alpha[f](\omega) = 0 \quad \text{for} \quad |\omega| > \Omega \right\}. \tag{4}$$

If $f \in B^2_\alpha(\mathbb{R})$, then the original function f can be reconstructed using its sampling points along with those of its derivatives through the following FrFT derivative sampling formula, as detailed in [11, Theorem 1]

$$\begin{aligned} f(t) &= \mathrm{e}^{-\mathrm{i}a(\alpha)t^2} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{r} \left(\mathrm{e}^{\mathrm{i}a(\alpha)(nT_{r,\alpha})^2} f\left(nT_{r,\alpha}\right)\right)^{(l)} \\ &\quad \times s_l\left(\pi T_{r,\alpha}^{-1} - n\pi\right), \end{aligned} \tag{5}$$

where

$$s_l(t) = \sum_{k=l}^{r} a_{kl}\overline{s_k}(t), \quad l = 0,1,\ldots,r, \tag{6}$$

$$\overline{s_k}(t) = \frac{1}{k!} t^k \operatorname{sinc}^{r+1}(t), \tag{7}$$

and the coefficients a_{kl} are the solutions of

$$s_l^{(l')}(0) = \sum_{k=l}^{r} \overline{s_k}^{(l')}(0)a_{kl} = \delta_{ll'}, \quad l' = l,\ldots,r, \; l = 0,\ldots,r. \tag{8}$$

Here $T_{r,\alpha} = (r+1)\pi\sin(\alpha)/\Omega$, Ω is the bandwidth depends on α and the sinc function that appears in Eq. (7) is defined as

$$\operatorname{sinc}(x) := \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

In this paper, we extend the type of sampling expansion presented in (5) to the linear canonical transform (LCT) domain and propose an alternative representation of this sampling using a contour integral. The convergence rate of the reformulated sampling expansion remains slow, particularly on the order of $O(\ln(N)/N)$. To address this slow convergence, we develop a Gaussian regularization method for the reformulated sampling expansion in (5). This regularization method applies to a broader range of functions within the Paley-Wiener space in the LCT domain, including entire functions that may not necessarily belong to the space $L^2(\mathbb{R})$ when their domain is restricted to $\mathbb{R}$. Additionally, we discuss the uniform convergence on both the real and complex domains for the reformulated sampling expansion in (5) within the LCT domain.

II. Extension of the expansion in (5) to LCT domain

In this section, we extend the higher-order derivative sampling expansion in (5) to the LCT domain. For $f \in L^2(\mathbb{R})$, we define the LCT with the parameter matrix $A = (a, b, c, d)$ to be, cf. [13], [20], [27],

$$\mathcal{L}_A[f](u) = \begin{cases} \int_{\mathbb{R}} f(z)\mathcal{K}_A(x,u)dx, & b \neq 0, \\ \sqrt{d}\, \mathrm{e}^{\mathrm{i}(1/2)cdu^2} f(du), & b = 0, \end{cases} \tag{9}$$

where a, b, c, d are real numbers satisfying $ad - cb = 1$, and the kernel $\mathcal{K}_A(x,u)$ is defined as

$$\mathcal{K}_A(x,u) = \frac{1}{\sqrt{2\mathrm{i}\pi b}}\, \mathrm{e}^{-\frac{\mathrm{i}}{2b}(ax^2 - 2ux + du^2)}. \tag{10}$$

The Paley-Wiener space in the LCT domain, denoted by $\mathcal{B}^2_A(\mathbb{R})$, consists of all bandlimited $L^2(\mathbb{R})$-functions with bandwidth Ω in the LCT sense. This means that, cf. e.g. [13], [30],

$$\mathcal{B}^2_A(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}), \quad \mathcal{L}_A[f](u) = 0 \quad \text{for} \quad |u| > \Omega \right\}. \tag{11}$$

According to the Paley-Wiener theorem in the LCT setting, as discussed in [12, Corollary 3], if $f \in \mathcal{B}^2_A(\mathbb{R})$, then f can be extended to an entire function in the complex plane, which satisfies the inequality

$$|f(z)| \leq C\, \mathrm{e}^{\frac{a}{b}\Re z \Im z}\, \mathrm{e}^{\frac{\Omega}{b}|\Im z|}, \quad z \in \mathbb{C}. \tag{12}$$

In the following theorem, we extend the higher-order derivative sampling expansion in (5) to the LCT domain.

Theorem 2.1: Consider $f \in \mathcal{B}^2_A(\mathbb{R})$. Then, f can be represented as the following higher-order derivative sampling series in the LCT domian

$$\begin{aligned} f(t) &= \mathrm{e}^{-\mathrm{i}(\frac{a}{2b})t^2} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{r} \left(\mathrm{e}^{\mathrm{i}(\frac{a}{2b})(nh_r)^2} f(nh_r) \right)^{(l)} \\ &\quad \times s_l\left(\pi h_r^{-1} t - n\pi\right), \end{aligned} \tag{13}$$

where $h_r = (r+1)\pi b/\Omega$, and the function s_l is previously given in (6) with (7) and (8).

A. Equivalent representation for the expansion in (13)

In this section, we introduce an alternative representation for the higher-order derivative sampling series in (13) using a contour integral approach. This new formulation is more straightforward and practical than the one in (13). It expands the series' applicability to a broader range of functions, including entire functions that may not necessarily belong to the space $L^2(\mathbb{R})$ when restricted to $\mathbb{R}$. Let $\mathcal{E}_\Omega$ denote the class of entire functions that satisfy one of the following growth conditions:

$$|f(\zeta)| \leq \frac{A_f\, \mathrm{e}^{\frac{a}{b}\Re\zeta\Im\zeta}\, \mathrm{e}^{\frac{\Omega}{b}|\Im\zeta|}}{1 + |\Re\zeta|}, \quad |f(\zeta)| \leq \frac{A_f\, \mathrm{e}^{\frac{a}{b}\Re\zeta\Im\zeta}\, \mathrm{e}^{\frac{\Omega}{b}|\Im\zeta|}}{1 + |\Im\zeta|}, \tag{14}$$

where A_f is a positive real number.

Theorem 2.2: Consider $f \in \mathcal{E}_\Omega$. Then, f can be represented as the following higher-order derivative sampling series in the LCT domian

$$\begin{aligned} f(z) &= \mathrm{e}^{-\mathrm{i}(\frac{a}{2b})z^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\frac{a}{2b})(nh_r)^2} \sum_{i+j+k+l=r} f^{(i)}(nh_r) \\ &\quad \times P_{n,i,j,k,l}(z)\, \mathrm{sinc}^{\,r+1}\left(\pi h_r^{-1} z - n\pi\right), \end{aligned} \tag{15}$$

where h_r is previously given and $P_{n,i,j,k,l}$ is a polynomial of degree $r + j - l$ defined as

$$P_{n,i,j,k,l}(z) = \frac{\delta_{r,k}\mathcal{P}_j(nh)}{i!j!k!}(z - nh_r)^{r-l}, \tag{16}$$

and the constant $\delta_{r,k}$ is defined as

$$\delta_{r,k} = \left[\frac{d^k}{d\zeta^k}\left(\frac{1}{\mathrm{sinc}^{\,r+1}\left(\pi h_r^{-1}\zeta - n\pi\right)} \right) \right]_{\zeta = nh_r}. \tag{17}$$

Here $\mathcal{P}_j(\zeta)$ is the j-th degree Hermite polynomial defined as

$$\mathcal{P}_j(\zeta) = \mathrm{e}^{-\mathrm{i}(\frac{a}{2b})\zeta^2} \frac{d^j}{d\zeta^j}\left\{ \mathrm{e}^{\mathrm{i}(\frac{a}{2b})\zeta^2} \right\}. \tag{18}$$

The series in (15) converges uniformly on any compact subset of $\mathbb{C}$ for every $r \in \mathbb{N}_0$.

In the following corollaries, we present four well-known series that are specific cases of the series in (15). The first case is the generalized Hermite interpolation involving derivatives in the classical Fourier transform (FT) domain, as established by Shin in [18]. The second case is the sampling series in the FrFT domain, as introduced by Xia and Zayed in [26], [29]. The third case, where f is a bandlimited function in the LCT sense, was presented by Li et al. in [13, Theorem 1]. The forth case is the derivative sampling series within the LCT domain, which was also established by Li et al. in [13, Theorem 2] (see also [1], [14]). The space $\mathcal{E}_\Omega$ introduced in this paper includes a broader range of functions compared to those in [1], [13], [18], [26], [29], as it includes entire functions that may not belong to $L^2(\mathbb{R})$ when their domain is restricted to the real line.

Corollary 2.3: Let $f \in \mathcal{E}_\Omega$. Then, f can be represented as the following generalized Hermite series

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} \sum_{i+k+l=r} f^{(i)}(nh) \frac{\sin^{r+1}\left(\pi h_r h^{-1} z\right)}{i!\,k!\,\left(z - nh_r\right)^{l+1}} \\ &\quad \times \left[\frac{d^k}{dz^k}\left(\frac{z - nh_r}{\sin\left(\pi h_r^{-1} z\right)} \right)^{r+1} \right]_{z = nh_r}, \end{aligned} \tag{19}$$

where $h_r = (r+1)\pi/\Omega$. The series in (19) converges uniformly on any compact subset of $\mathbb{C}$.

Corollary 2.4: Let $f \in \mathcal{E}_\Omega$. Then, f can be expressed using the following FrFT sampling series

$$\begin{aligned} f(z) &= \mathrm{e}^{-\mathrm{i}a(\alpha)z^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}a(\alpha)\left(\frac{n\pi\sin(\alpha)}{\Omega}\right)^2} f\left(\frac{n\pi\sin(\alpha)}{\Omega}\right) \\ &\quad \times \mathrm{sinc}\left(\frac{\Omega}{\sin(\alpha)}z - n\pi\right), \end{aligned} \tag{20}$$

where $\alpha \neq k\pi$, $k \in \mathbb{Z}$. Series (20) converges uniformly on any compact subset of $\mathbb{C}$.

Corollary 2.5: Consider $f \in \mathcal{E}_\Omega$. Then, f can be represented by the following LCT sampling series

$$f(z) = \mathrm{e}^{-\mathrm{i}(\frac{a}{2b})z^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\frac{a}{2b})(nh_0)^2} f(nh_0) \times \operatorname{sinc}\left(\frac{\pi z}{h_0} - n\pi\right), \quad (21)$$

where h_0 in this situation is fixed within the interval $(0, \pi b/\Omega]$. Series (21) converges uniformly on any compact subset of $\mathbb{C}$.

Corollary 2.6: Let $f \in \mathcal{E}_\Omega$. Then, f can be represented as the following first-order derivative sampling series in the LCT domain

$$f(z) = \mathrm{e}^{-\mathrm{i}(\frac{a}{2b})z^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\frac{a}{2b})(nh_1)^2} \left\{\left(1 + \frac{\mathrm{i}anh_1}{b}(z - nh_1)\right) \times f(nh_1) + (z - nh_1) f'(nh_1)\right\} \operatorname{sinc}^2\left(\frac{\pi z}{h_1} - n\pi\right), \quad (22)$$

where $h_1 \in (0, 2\pi b/\Omega]$. Series (21) converges uniformly on any compact subset of $\mathbb{C}$.

As discussed earlier, the convergence rate of the expansion in (15) is $O(\ln(N)/N)$ as $N \to \infty$. This rate is insufficient to achieve uniform convergence on $\mathbb{R}$. However, if $f \in \mathcal{B}_A^2(\mathbb{R})$ and exhibits a faster decay than typical $\mathcal{B}_A^2(\mathbb{R})$-functions, a uniform convergence on $\mathbb{R}$ can be demonstrated, as shown in the following theorem.

Theorem 2.7: If $f \in \mathcal{B}_A^2(\mathbb{R})$ and $z^k f(z) \in L^2(\mathbb{R})$ for all $0 \leq k \leq r$, then the expansion in (15) converges uniformly on $\mathbb{R}$ for every $r \in \mathbb{N}_0$.

III. Gaussian Regularization sampling

This section focuses on introducing the Gaussian regularization for the higher-order derivative LCT sampling series (15), denoted by $\mathcal{G}_{r,N}[f](z)$. It also provides an estimate for the error bound $|f(z) - \mathcal{G}_{r,N}[f](z)|$ for functions f that belong to a broader class of functions, including entire functions that are not necessarily belong to the space $L^2(\mathbb{R})$. Specifically, we consider the class $E_{\Omega/b}(\varphi)$, defined as follows.

$$E_{\Omega/b}(\varphi) := \{f : \mathbb{C} \to \mathbb{C} \mid \text{ is entire such that } |f(z)| \leq \varphi(|\Re z|)\, \mathrm{e}^{\frac{a}{b}\Re z . \Im z}\, \mathrm{e}^{\frac{\Omega}{b}|\Im z|}\}, \quad (23)$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing function. The Gaussian regularization for classical sampling series in the Fourier transform (FT) domain was first introduced in [16], while the Gaussian regularization for higher-order derivative sampling series in the FT domain was introduced in [7]. The space $E_{\Omega/b}(\phi)$ was first introduced in [3] and subsequently utilized in [2]. For $z \in \mathbb{C}$, $N \in \mathbb{N}$, let the integer-type interval

$$\mathbb{Z}_N(z) = \left\{n \in \mathbb{Z} : \left|n - \lfloor h^{-1}\Re z + 1/2 \rfloor\right| \leq N\right\}, \quad (24)$$

where $\lfloor \cdot \rfloor$ is the floor function. The Gaussian regularization for the higher-order derivative LCT sampling operator $\mathcal{G}_{r,N} : E_{\Omega/b}(\varphi) \to E_{\Omega/b}(\varphi)$ as follows:

$$\mathcal{G}_{r,N}[f](z) = \sum_{n \in \mathbb{Z}_N(z)} \mathrm{e}^{-\mathrm{i}(\frac{a}{2b})(z^2 - n^2 h_r^2)} \sum_{i+j+s+k+l=r} f^{(i)}(nh_r) \times \mathcal{B}_{n,i,j,k,l,s}(z) \operatorname{sinc}^{r+1}\left(\frac{\pi z}{h_r} - n\pi\right), \times \exp\left(-\frac{\alpha_r (z - nh_r)^2}{N h_r^2}\right) \quad (25)$$

where $h_r = (r+1)\pi b/\Omega$ and $\mathcal{B}_{n,i,j,k,l,s}$ is the polynomial of degree $r + k - j$ defined as

$$\mathcal{B}_{n,i,j,k,l,s}(z) := \frac{\delta_{r,l} \mathcal{P}_s(nh_r)(z - n\pi)^{r-j}}{i!k!l!s!} H_k\left(\frac{\sqrt{\alpha_r}(z - nh_r)}{{}^N h_r}\right). \quad (26)$$

Here, $H_k(z)$ represents the Hermite polynomial of degree k, defined by

$$H_k(z) := (-1)^k \exp(z^2) \frac{d^k}{dz^k} \exp(-z^2) \quad (27)$$

where the constant $\delta_{r,k}$ is previously defined in (17), and $\mathcal{P}_s$ corresponds to the Hermite polynomial of degree s, defined in (27).

In order to establish the operator $\mathcal{G}_{N,r}$, it is essential to introduce a kernel function, $\mathcal{K}_{r,z}$, which is defined as follows

$$\mathcal{K}_{r,z}(\zeta) = \frac{\sin^{r+1}\left(\pi h_r^{-1} z\right)\, \mathrm{e}^{-\mathrm{i}(\frac{a}{2b})(z^2 - \zeta^2) - \frac{\alpha_r}{N h_r^2}(z - \zeta)^2}}{2\pi\mathrm{i}(\zeta - z)\sin^{r+1}\left(\pi h_r^{-1}\zeta\right)}, \quad (28)$$

where α_r and h_r are previously defined. For $z \in \mathbb{C}$ and $z \neq nh_r$, the function $\mathcal{K}_{r,z}(\zeta)$, when considered as a function of ζ, is holomorphic. It has a simple pole at $\zeta = z$ and poles of order $r + 1$ at $\zeta = nh_r$, where $n \in \mathbb{N}$.

Lemma 3.8: For $f \in E_{\Omega/b}(\varphi)$, we can express

$$f(z) - \mathcal{G}_{r,N}[f](z) = \begin{cases} \dfrac{1}{2\pi\mathrm{i}} \displaystyle\oint_{\mathcal{C}} \mathcal{K}_{r,z}(\zeta) f(\zeta) d\zeta, & z \in \mathbb{C} \backslash \{nh_r\}, \\ 0, & z = nh_r, \end{cases} \quad (29)$$

where $\mathcal{C}$ is a positively oriented simple closed curve encloses the poles $\zeta = z$ and $\zeta = nh_r$, with $n \in \mathbb{Z}_N(z)$.

Below, we present two well-known operators that are specific cases of the sampling operator (30). The first operator corresponds to the Gaussian regularization for sampling reconstruction of functions in the LCT domain, obtained by setting $r = 0$ in (30). This operator was introduced by Annaby et al. in [3] and is expressed as:

$$\mathcal{G}_{0,N}[f](z) = \sum_{n \in \mathbb{Z}_N(z)} \mathrm{e}^{-\mathrm{i}(\frac{a}{2b})(z^2 - n^2 h_0^2)} f(nh_0) \times \operatorname{sinc}\left(\frac{\pi z}{h_0} - n\pi\right) \exp\left(-\frac{\alpha_0 (z - nh_0)^2}{N h_0^2}\right), \quad (30)$$

The second operator represents the Gaussian regularization for derivative sampling interpolation of functions in the LCT domain, obtained by setting $r=1$ in (30). This operator was also introduced by Annaby et al. in [2] and is given by the following expression:

$$\begin{aligned}\mathcal{G}_{1,N}[f](z) &= \sum_{n\in\mathbb{Z}_N(z)} \mathrm{e}^{-\mathrm{i}\left(\frac{a}{2b}\right)\left(z^2-n^2h_1^2\right)}\left\{\left(1+\frac{\mathrm{i}anh_1}{b}(z-nh_1)\right)\right.\\ &+ \left.\frac{2\alpha_1}{Nh_1^2}(z-nh_1)f(nh_1)+(z-nh_1)f'(nh_1)\right\}\\ &\times \operatorname{sinc}^2\left(\frac{\pi z}{h_1}-n\pi\right)\exp\left(-\frac{\alpha_1(z-nh_1)^2}{Nh_1^2}\right), \quad (31)\end{aligned}$$

The following theorem provides an estimate for the error bound $|f(z)-\mathcal{G}_{r,N}[f](z)|$.

Theorem 3.9: Suppose $f\in E_{\Omega/b}(\varphi)$ with $\sigma>0$. For all $z\in\mathbb{C}$ with $|\Im z|<hN$, the following inequality holds:

$$\begin{aligned}|f(z)-\mathcal{G}_{r,N}[f](z)| &\le 2^r\mathrm{e}^{\frac{a}{b}\Re z\Im z}\,\varphi\left(|\Re z|+h_r(N+1)\right)\\ &\times\left|\sin^{r+1}(\pi h^{-1}z)\right|\chi_{r,N}(\Im z)\frac{\mathrm{e}^{-\alpha_r N}}{\sqrt{\pi\alpha_r N}}, \quad (32)\end{aligned}$$

where the function φ was defined earlier, and $\chi_{r,N}$ is given as

$$\chi_{r,N}(t)=2\cosh(2\alpha_r t)+O(N^{-1/2}), \quad \text{as} \quad N\to\infty. \quad (33)$$

IV. Numerical Example

In this section, we present a numerical example that demonstrates excellent consistency with our theoretical analysis Consider the function $f(z)=\mathrm{e}^{-\mathrm{i}\frac{z^2}{2}}\sin\left(\sqrt{2}z\right)$, where $z\in\mathbb{C}$. It is evident that $\left|\mathrm{e}^{-\mathrm{i}\frac{z^2}{2}}\sin\left(\sqrt{2}z\right)\right|\le \mathrm{e}^{\Re z\Im z}\,\mathrm{e}^{\sqrt{2}|\Im z|}$ for $z\in\mathbb{C}$. Thus, this function belongs to the space $E_{\Omega/b}(\varphi)$ with $\varphi=1$, $\sigma=1$, and $a=b=1/\sqrt{2}$. As a result, Theorem 3.9 is applicable. In this example, the bound provided in Equation (3.9) is uniform and can be expressed as

$$\mathcal{B}_{h,N,r}(x)=2^r\left|\sin^{r+1}(\pi h^{-1}x)\right|\chi_{r,N}(0)\frac{\mathrm{e}^{-\alpha_r N}}{\sqrt{\pi\alpha_r N}}, \quad (34)$$

where the associated functions have been defined previously. The numerical results are presented in Table I, showing the exact error $|f(x)-\mathcal{G}_{r,N}[f](x)|$ alongside the bound $\mathcal{B}_{r,N}(z)$ at the points $x_{j,h_r} := (j-1/2)h_r$, with $N=8$ and $r=0,1,2$. Additionally, the errors $\Re\left(f(x)-\mathcal{G}_{r,N}[f](x)\right)$ and $\Im\left(f(x)-\mathcal{G}_{r,N}[f](x)\right)$ are illustrated in Figures 1 and 2.

TABLE I
Absolute error and the pointwise bound associated with approximating a function f at the points x_j for $N=8$, $h=1$, and $r=0,1,2$

r	j	$\lvert f(x)-\mathcal{G}_{r,N}[f](x)\rvert$	$\mathcal{B}_{r,N}[f](x)$
0	1	1.4353×10^{-4}	6.12464×10^{-4}
	2	1.76192×10^{-4}	
	3	8.85781×10^{-5}	
	4	2.03818×10^{-4}	
	5	2.50098×10^{-5}	
1	1	5.82920×10^{-10}	2.23536×10^{-9}
	2	7.44461×10^{-10}	
	3	3.50732×10^{-10}	
	4	8.53850×10^{-10}	
	5	8.44278×10^{-11}	
2	1	3.12642×10^{-15}	1.16097×10^{-14}
	2	3.94754×10^{-15}	
	3	1.72174×10^{-15}	
	4	4.48654×10^{-15}	
	5	9.31467×10^{-16}	

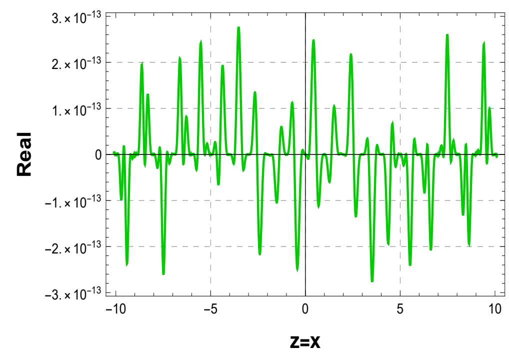

Fig. 1. The figure illustrates the error $\Re\left(f(x)-\mathcal{G}_{r,N}[f](x)\right)$ with parameters $h=1$, $r=2$, and $N=7$.

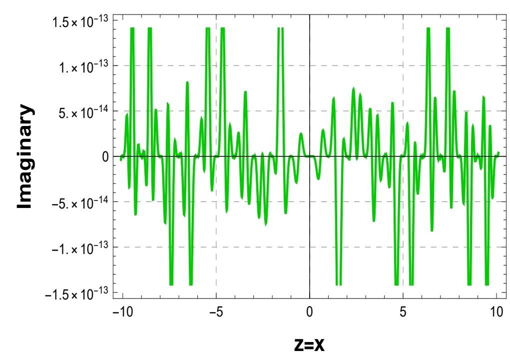

Fig. 2. The figure shows the error$\Im\left(f(x)-\mathcal{G}_{r,N}[f](x)\right)$ with parameters $h=1$, $r=2$, and $N=7$.

References

[1] M.H. Annaby, I.A. Al-Abdi, A.F. Ghaleb and M.S. Abou-Dina, Hermite interpolation theorems for band-limited functions of the linear canonical transforms with equidistant samples, *Numer. Algor.*, **94** (2023), 1281–1308

[2] M.H. Annaby and I.A. Al-Abdi, A Gaussian regularization for derivative sampling interpolation of signals in the linear canonical transform representations, *Signal, Image and Video Processing*, **17** (2023), 2157–2165

[3] M.H. Annaby, I.A. Al-Abdi, M.S. Abou-Dina and A.F. Ghaleb, Regularized sampling reconstruction of signals in the linear canonical transform domain, *Signal Process.* **198** (2022), 108569.
[4] M.H Annaby and R.M. Asharabi, Derivative sampling expansions for the linear canonical transform: convergence and error analysis, *J. Comput. Math.* **37** (2019), 431-446.
[5] M.H Annaby and R.M. Asharabi, Error estimates associated with sampling series of the linear canonical transforms, *IMA J. Numer. Anal.* **35** (2015), 931–946.
[6] R. M. Asharabi, Periodic Nonuniform Sinc-Gauss Sampling, *Filomat*, **37** (2023), 279-–292.
[7] R.M. Asharabi, Generalized sinc-Gaussian sampling involving derivatives, *Numer. Algor.* **73** (2016), 1055–1072.
[8] B. Deng, R. Tao and Y. Wang, Convolution theorems for the linear canonical transform and their applications, *Science in China (Ser. F, Information Science)* **49** (2006), 592–603.
[9] J. R. Higgins, *Sampling Theory in Fourier and Signal Analysis Foundations*, Oxford University Press, Oxford, 1996.
[10] R-M Jing and B-Z Li, Higher order derivatives sampling of random signals related to the fractional Fourier transform, *IAENG Int. J. Appl. Math.* **48** (2018), 1–7.
[11] R-M Jing, Q. Feng and B-Z Li, Higher-order derivative sampling associated with fractional Fourier transform, *Circuits, Systems, and Signal Processing* **38** (2019), 1751–1774.
[12] K-I Kou, R-H Xu, and Y-H Zhang, Paley-Wiener theorems and uncertainty principles for the windowed linear canonical transform, *Math. Methods Appl. Sci.* **35** (2012), 2122–2132.
[13] B.Z. Li, R. Tao and Y. Wang, New sampling formulae related to linear canonical transform, *Signal Process.* **87** (2007), 983–990.
[14] Y. Liu, K. Kou and I. Ho, New sampling formulae for nonbandlimited signals associated with linear canonical transform and non linear Fourier atoms, *Signal Process.* **90** (2010), 933–945.
[15] H. Ozaktas, Z. Zalevsky and M. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, J. Wiley, New York, 2001.
[16] G. Schmeisser and F. Stenger, Sinc approximation with a Gaussian multiplier, *Sampl. Theory Signal Image Process.* **6** (2007), 199–221.
[17] A. Stern, Sampling of linear canonical transformed signals, *Signal Process.*, **86** (2006), 1421–1425.
[18] C.-E. Shin, Generalized Hermite interpolation and sampling theorem involving derivatives, *Commun. Korean Math. Soc.* **17** (2002), 731–740.
[19] X. Shuiqing, H. Lei, C. Yi and H. Yigang, Nonuniform Sampling Theorems for Bandlimited Signals in the Offset Linear Canonical Transform, *Circuits Syst. Signal Process* **37** (2018), 3227–3244.
[20] M. Moshinsky and C. Quesne, Linear canonical transformations and their unitary representations, *J. Math. Phys.* **12** (1971), 1772–1783.
[21] A. Nathan, On sampling a function and its derivatives, *Information and Control* **22** (1973), 172–182.
[22] R. Tao, B-z Li, Y. Wang and G. K. Aggrey, On sampling of bandlimited signals associated with the linear canonical transform, *IEEE Trans. Signal Process.* **56** (2008), 5454–5464.
[23] J. Wang, S. Ren, Z. Chen and W. Wang, Periodically nonuniform sampling and reconstruction of signals in function spaces associated with the linear canonical transform, *IEEE Commun. Lett.* , **22** , (2018), 756–759
[24] D. Wei, Q. Ran, and Y. Li, Generalized sampling expansion for bandlimited signals associated with the fractional Fourier transform, *IEEE Signal Process. Lett.* , **17**, (2010), 595—598.
[25] D. Wei and Y. Li, Reconstruction of multidimensional bandlimited signals from multichannel samples in linear canonical transform domain, *IET Signal Process. Lett.* , **8** (2014), 647–657.
[26] X-G. Xia, On bandlimited signals with fractional Fourier transform, *IEEE Signal Process. Lett.* , **3** , (1996), 72–74.
[27] T.Z. Xu and B.Z. Li, *Linear Canonical Transform and Its Application*, Science Press, Beijing, China, 2013.
[28] P. Ye and Z. Song, Truncation and aliasing errors for Whittaker-Kotelnikov-Shannon sampling expansion, Applied Mathematics-A Journal of Chinese Universities 27 (2012), 412–418.
[29] A.I. Zayed, On the relationship between the Fourier and fractional Fourier transforms, *IEEE Signal Process. Lett.* , **3** , (1996), 310—311.
[30] H. Zhao, Q.-W. Ran, J. Ma and L.Y. Tan, On bandlimited signals associated with linear cannonical transform, *IEEE Signal Process. Lett.*, **16** , (2009), 343–345.