# Improved Central Limit Theorem and Bootstrap Approximations for Linear Stochastic Approximation

Bogdan Butyrin [*], Eric Moulines [†], Alexey Naumov [‡], Sergey Samsonov [§], Qi-Man Shao [¶], Zhuo-Song Zhang [‖]

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**Abstract**

In this paper, we refine the Berry–Esseen bounds for the multivariate normal approximation of Polyak–Ruppert averaged iterates arising from the linear stochastic approximation (LSA) algorithm with decreasing step size. We consider the normal approximation by the Gaussian distribution with covariance matrix predicted by the Polyak-Juditsky central limit theorem and establish the rate up to order $n^{-1/3}$ in convex distance, where $n$ is the number of samples used in the algorithm. We also prove a non-asymptotic validity of the multiplier bootstrap procedure for approximating the distribution of the rescaled error of the averaged LSA estimator. We establish approximation rates of order up to $1/\sqrt{n}$ for the latter distribution, which significantly improves upon the previous results obtained by Samsonov et al. (2024).

## 1 Introduction

In this paper we consider the Linear Stochastic Approximation (LSA) algorithm, a simple yet foundational method with various applications in statistics and machine learning [16, 4, 21, 24]. The LSA procedure addresses the problem of approximating the unique solution $\theta^\star$ to a linear system of equations given by

$$\bar{\mathbf{A}}\theta^\star = \bar{\mathbf{b}} ,$$

where $\bar{\mathbf{A}} \in \mathbb{R}^{d\times d}$ is a non-degenerate matrix. This approximation is based on a sequence of observations $\{(\mathbf{A}(Z_k), \mathbf{b}(Z_k))\}_{k\in\mathbb{N}}$, where $\mathbf{A} : \mathsf{Z} \to \mathbb{R}^{d\times d}$ and $\mathbf{b} : \mathsf{Z} \to \mathbb{R}^d$ are measurable mappings. The sequence $(Z_k)_{k\in\mathbb{N}}$ consists of independent and identically distributed (i.i.d. ) random variables defined on a measurable space $(\mathsf{Z}, \mathcal{Z})$ with distribution $\pi$, satisfying $\mathbb{E}[\mathbf{A}(Z_k)] = \bar{\mathbf{A}}$ and $\mathbb{E}[\mathbf{b}(Z_k)] = \bar{\mathbf{b}}$. Often in the applications $(Z_k)_{k\in\mathbb{N}}$ are not independent and instead form a Markov chain, see [12, 30, 55]. In this paper, we do not consider this setting and postpone it as a direction for a future

[*] HSE University, Moscow, Russia, bbutyrin@hse.ru.

[†] Ecole Polytechnique, Paris, France, and Mohamed bin Zayed University of Artificial Intelligence (MBZUAI), UAE, eric.moulines@polytechnique.edu.

[‡] HSE University, Moscow, Russia, anaumov@hse.ru.

[§] HSE University, Moscow, Russia, svsamsonov@hse.ru.

[¶] Shenzhen International Center of Mathematics, Southern University of Science and Technology, Xueyuan Blvd., 518000, Shenzhen, P.R. China, shaoqm@sustech.edu.cn.

[‖] Shenzhen International Center of Mathematics, Southern University of Science and Technology, Xueyuan Blvd., 518000, Shenzhen, P.R. China, zhangzs3@sustech.edu.cn.

work. Given a sequence of decreasing step sizes $(\alpha_k)_{k\in\mathbb{N}}$ and an initialization $\theta_0 \in \mathbb{R}^d$, we define the iterative estimates $(\theta_k)_{k\in\mathbb{N}}$ and their Polyak–Ruppert averaged counterparts $(\bar{\theta}_n)_{n\in\mathbb{N}}$ by

$$\theta_k = \theta_{k-1} - \alpha_k \left(\mathbf{A}(Z_k)\theta_{k-1} - \mathbf{b}(Z_k)\right), \quad k \geq 1, \quad \bar{\theta}_n = n^{-1}\sum_{k=0}^{n-1} \theta_k, \quad n \geq 1. \tag{1.1}$$

The idea of using averaged estimates $\bar{\theta}_n$ was proposed in the works of Ruppert [40] and Polyak and Juditsky [36, 37]. Using the averaged iterates $\bar{\theta}_n$ instead of the last iterate $\theta_n$ has been shown to stabilize stochastic approximation procedures and accelerate their convergence. Moreover, it is known (see [37]) that the estimator $\bar{\theta}_n$ is asymptotically normal under appropriate regularity conditions on the step sizes $(\alpha_k)_{k\in\mathbb{N}}$ and the noise observations $(\mathbf{A}(Z_k))_{k\in\mathbb{N}}$, that is,

$$\sqrt{n}(\bar{\theta}_n - \theta^\star) \xrightarrow{d} \mathcal{N}(0, \Sigma_\infty)\,. \tag{1.2}$$

The expression for $\Sigma_\infty$ is given below in Section 3.2 and corresponds to the preconditioned version of the sequence $\theta_k$, which uses the optimal preconditioner $\bar{\mathbf{A}}^{-1}$, see [18, 37].

Both asymptotic [37, 6] and non-asymptotic [25, 48, 29, 12] properties of the averaged LSA errors $\bar{\theta}_n - \theta^\star$ attained lot of research interest. Many of the mentioned works primarily focus on providing the moment bounds and concentration inequalities for the scaled estimation error $\sqrt{n}(\bar{\theta}_n - \theta^\star)$. The primary aim of these concentration bounds is to obtain results with explicit dependence on the number of samples $n$, the problem dimension $d$, and other problem-specific quantities related to $\bar{\mathbf{A}}$ and the noise observations $(\mathbf{A}(Z_k))_{k\in\mathbb{N}}$. It is also important to study the rate of convergence in (1.2) in a sense of appropriate distance between the probability distributions. Recent papers consider approximation either in Wasserstein distance [47], class of smooth test functions [2], or in convex distance [44, 41, 54]. The latter type of results can be directly applied when ensuring the non-asymptotic validity of the confidence sets for $\theta^\star$, and we follow the same direction in our paper.

The primary aim of the analysis of the approximation rate in (1.2) is the need to construct confidence intervals for $\theta^\star$. The principal difficulty is the fact that $\Sigma_\infty$ is unknown in practice, hence, (1.2) can not be applied directly. Classical approaches suggest to approximate $\Sigma_\infty$ directly based on either plug-in estimates [8, 54], or various modifications of batch-mean approach [8, 53, 26]. Typically these methods constructs an estimator $\hat{\Sigma}_n$ of $\Sigma_\infty$, and often provide non-asymptotic on the closeness between $\hat{\Sigma}_n$ and $\Sigma_\infty$. Yet the there are only asymptotic guarantees on coverage probabilities of $\theta^\star$ with constructed confidence sets. The notable exceptions are recent works [41] and [54], where the authors provide non-asymptotic error bounds for coverage probabilities. The paper [41] considers general LSA setting and multiplier bootstrap procedure adopted from [17], while the authors of [54] considered a plug-in based approach for estimating $\Sigma_\infty$ and focused on the particular setting of the temporal difference (TD) learning algorithm. In this paper we revisit the analysis of [41], derive the error rates in coverage probabilities of order up to $1/\sqrt{n}$. Our contributions can be summarized as follows:

- We refine the high-order moment bounds for $\sqrt{n}(\bar{\theta}_n - \theta^\star)$, improving the previous results of [29] and [12]. Namely, our results yield, for $p \geq 2$, the bound

$$\mathbb{E}^{1/p}\left[\|\bar{\theta}_n - \theta^\star\|^p\right] \lesssim \frac{\sqrt{p}\sqrt{\operatorname{Tr}\Sigma_\infty}}{\sqrt{n}} + \frac{p^{3/2}}{n^{5/6}}\,,$$

  provided that the step sizes $\alpha_k$ are appropriately chosen. Note that the leading term of this bound aligns with the moment bound for the Gaussian vector $\mathcal{N}(0, \Sigma_\infty)$.

- We establish a Berry–Esseen bound characterizing the rate of normal approximation in (1.2) in a sense of convex distance (see Section 3.2) between distributions. We show the approximation rate in (1.2) of order up to $n^{-1/3}$, up to logarithmic factors in $n$. This convergence rate improves the previous rate of order $n^{-1/4}$ obtained in [41] for the general LSA procedure, and aligns with the rate achieved in [54] for the particular setting of the temporal difference (TD) learning algorithm. Similar to [41] and [54], our proof approach builds upon the techniques developed for nonlinear statistics in [44].

- We derive an approximation of the distribution of the scaled Polyak–Ruppert estimator $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ based on a multiplier bootstrap procedure. In particular, we show that the coverage probabilities of the true value $\theta^\star$ under the true distribution $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ can be approximated by its bootstrap-based counterpart with a rate approaching $n^{-1/2}$ up to logarithmic factors in $n$. This rate is achieved for step sizes of the form $\alpha_k = c_0/(k+k_0)^\gamma$ when $\gamma \to 1$. Our results provide an improvement over the existing non-asymptotic bounds obtained in [41] for similar procedure. The main reason for this improvement is the observation that the distribution of $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ can be effectively approximated by a normal distribution $\mathcal{N}(0, \Sigma_n)$ with a suitably chosen covariance matrix $\Sigma_n$, bypassing the direct approximation with $\mathcal{N}(0, \Sigma_\infty)$. The obtained rate is in sharp contrast with [54] and other related works based on direct approximating of the limiting covariance $\Sigma_\infty$.

**Notations.** For matrix $A \in \mathbb{R}^{d \times d}$ we denote by $\|A\|$ its operator norm. Given a sequence of matrices $\{A_\ell\}_{\ell \in \mathbb{N}}$, $A_\ell \in \mathbb{R}^{d \times d}$, we use the following convention for matrix products: $\prod_{\ell=m}^{k} A_\ell = A_k A_{k-1} \ldots A_m$, where $m \le k$. For symmetric and positive-definite matrix $Q = Q^\top \succ 0$ , $Q \in \mathbb{R}^{d \times d}$, and $x \in \mathbb{R}^d$ we define the corresponding norm $\|x\|_Q = \sqrt{x^\top Q x}$, and define the respective matrix $Q$-norm of the matrix $B \in \mathbb{R}^{d \times d}$ by $\|B\|_Q = \sup_{x \neq 0} \|Bx\|_Q / \|x\|_Q$. For sequences $a_n$ and $b_n$, we write $a_n \lesssim b_n$ if there exist a constant $c > 0$ such that $a_n \le c b_n$ for any $n \in \mathbb{N}$. In the present text, the following abbreviations are used: "w.r.t." stands for "with respect to", "i.i.d. " - for "independent and identically distributed".

## 2 Related works

Asymptotic properties of Linear Stochastic Approximation (LSA) algorithms were studied in [37, 24, 6, 4]. These works established asymptotic normality and almost sure convergence under both i.i.d. and Markovian noise. Non-asymptotic analyses of LSA (and of the non-linear setting, corresponding to the SGD algorithm) have been carried out in [38, 32, 5, 25, 30], where mean squared error (MSE) bounds for LSA iterates and their Polyak-Ruppert averaged versions were obtained. Further works [29, 13, 12] establish high-probability bounds (moment bounds or Bernstein-type bounds) for the estimation error $\bar{\theta}_n - \theta^\star$. However, the concentration bounds for the LSA error given in [29, 13, 12, 30] do not yield convergence rates of the rescaled error $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ to the normal distribution in Wasserstein or Kolmogorov distance.

Non-asymptotic convergence rates towards normality were investigated in [2] using Stein's method and measured in terms of the integral probability metric associated with smooth test functions (smoothed Wasserstein distance). Recent advances include [47], which studied convergence rates in Wasserstein distance for LSA with Markov observations. The bounds derived in these works exhibit less favorable dependence on the trajectory length $n$ than those presented here. Further, [41] analyzed normal approximation rates for $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ and obtained convex distance bounds of order $n^{-1/4}$ for general LSA. This result was later improved by [54] for the specific setting of the temporal difference (TD) learning algorithm. In this paper, we show that the actual rate of normal approximation for LSA is also $n^{-1/3}$ up to logarithmic factors, matching the result of [54]. A detailed comparison with these works is provided in the discussion following Theorem 3.

The bootstrap approach [15] is one of the widely used methods for constructing confidence intervals in parametric models. This method has been extensively studied theoretically; see [9, 10, 46, 20]. In these works, the validity of the bootstrap relies on Gaussian comparison techniques and anticoncentration results, tailored to particular subclasses of convex sets (spherical or rectangular). Bootstrap validity has also been analyzed in the context of spectral projectors of covariance matrices [31, 23]. At the same time, extending classical bootstrap methods to online learning algorithms poses considerable theoretical and practical challenges. In particular, the iterates $\{\theta_k\}_{k\in\mathbb{N}}$ generated by the iterative scheme (1.1) are typically not stored in memory, making standard bootstrap methods inapplicable. Instead, one can employ the multiplier bootstrap technique introduced in [17], designed specifically for the iterates of Stochastic Gradient Descent (SGD). A non-asymptotic analysis of this procedure was carried out in [41], which established approximation rates for the distribution of $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ of order up to $n^{-1/4}$ in convex distance. In this paper, we show that the actual approximation rate can be significantly faster, up to $n^{-1/2}$. However, the attempt in [39] to generalize this procedure to the case of Markovian noise leads to an inconsistent method, as demonstrated in [28, Proposition 1]. Thus, the question of appropriate generalizations of the multiplier bootstrap approach to stochastic approximation algorithms with Markov data remains, to our knowledge, open.

Other methods for constructing confidence intervals, not based on the bootstrap approach, rely on the direct estimation of the asymptotic covariance matrix $\Sigma_\infty$; see, e.g., [8, 27, 53]. In this approach, the authors typically construct an estimator $\widehat{\Sigma}_n$ of $\Sigma_\infty$ and provide bounds on $\mathbb{E}[\|\widehat{\Sigma}_n - \Sigma_\infty\|]$ with explicit dependence on $n$. To our knowledge, within this approach there are no error bounds for the coverage probabilities of $\theta^\star$ or error rates for approximating, for example, the distribution of the true statistic $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ with $\mathcal{N}(0, \widehat{\Sigma}_n)$.

# 3 Main results

We begin this section by specifying the set of assumptions that will be used for the non-asymptotic central limit theorem for LSA iterates. To simplify notation and whenever clarity permits, we write simply $\mathbf{A}_k = \mathbf{A}(Z_k)$ and $\mathbf{b}_k = \mathbf{b}(Z_k)$. Starting from equation (1.1), algebraic manipulations yield the recurrence

$$\theta_k - \theta^\star = (\mathrm{I} - \alpha_k \mathbf{A}_k)(\theta_{k-1} - \theta^\star) - \alpha_k \varepsilon_k, \tag{3.1}$$

where we have introduced the noise term $\varepsilon_k = \varepsilon(Z_k)$, defined by

$$\varepsilon(z) = \tilde{\mathbf{A}}(z)\theta^\star - \tilde{\mathbf{b}}(z), \quad \tilde{\mathbf{A}}(z) = \mathbf{A}(z) - \bar{\mathbf{A}}, \quad \tilde{\mathbf{b}}(z) = \mathbf{b}(z) - \bar{\mathbf{b}}.$$

The random variable $\varepsilon(Z_k)$ corresponds to the noise measured at the solution $\theta^\star$. We introduce the following assumptions on $\{Z_k\}$ and mappings $\mathbf{A}(\cdot), \mathbf{b}(\cdot)$:

**A 1.** *The sequence $\{Z_k\}_{k\in\mathbb{N}}$ consists of independent and identically distributed (i.i.d.) random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution $\pi$.*

**A 2.** *$\int_{\mathsf{Z}} \mathbf{A}(z)\mathrm{d}\pi(z) = \bar{\mathbf{A}}$ and $\int_{\mathsf{Z}} \mathbf{b}(z)\mathrm{d}\pi(z) = \bar{\mathbf{b}}$, with the matrix $-\bar{\mathbf{A}}$ being Hurwitz. Moreover, $\|\varepsilon\|_\infty = \sup_{z\in\mathsf{Z}} \|\varepsilon(z)\| < +\infty$, and the mapping $z \to \mathbf{A}(z)$ is bounded, that is,*

$$\mathrm{C}_{\mathbf{A}} = \sup_{z\in\mathsf{Z}} \|\mathbf{A}(z)\| \vee \sup_{z\in\mathsf{Z}} \|\tilde{\mathbf{A}}(z)\| < \infty\ .$$

*Moreover, the smallest eigenvalue of the noise covariance matrix* $\Sigma_\varepsilon = \int_{\mathsf{Z}} \varepsilon(z)\varepsilon(z)^\top \mathrm{d}\pi(z)$ *is bounded away from* 0*, that is,*

$$\lambda_{\min} := \lambda_{\min}(\Sigma_\varepsilon) > 0 \, .$$

The fact that the matrix $-\bar{\mathbf{A}}$ is Hurwitz implies that the linear system $\bar{\mathbf{A}}\theta = \bar{\mathbf{b}}$ has a unique solution $\theta^\star$. Moreover, this fact is sufficient to show that $\|\mathrm{I} - \alpha\bar{\mathbf{A}}\|_Q^2 \leq 1 - \alpha a$ for appropriately chosen matrix $Q = Q^\top > 0$ and $a > 0$, provided that $\alpha > 0$ is small enough. Precisely, the following proposition holds:

**Proposition 1** (Proposition 1 in [41])**.** *Let* $-\bar{\mathbf{A}}$ *be a Hurwitz matrix. Then for any* $P = P^\top \succ 0$*, there exists a unique matrix* $Q = Q^\top \succ 0$*, satisfying the Lyapunov equation* $\bar{\mathbf{A}}^\top Q + Q\bar{\mathbf{A}} = P$*. Moreover, setting*

$$a = \tfrac{\lambda_{\min}(P)}{2\|Q\|} \, , \quad \text{and} \quad \alpha_\infty = \tfrac{\lambda_{\min}(P)}{2\kappa_Q \|\bar{\mathbf{A}}\|_Q^2} \wedge \tfrac{\|Q\|}{\lambda_{\min}(P)} \, , \tag{3.2}$$

*where* $\kappa_Q = \lambda_{\max}(Q)/\lambda_{\min}(Q)$*, it holds for any* $\alpha \in [0, \alpha_\infty]$ *that* $\alpha a \leq 1/2$*, and*

$$\|\mathrm{I} - \alpha\bar{\mathbf{A}}\|_Q^2 \leq 1 - \alpha a \, . \tag{3.3}$$

**Remark 1.** *One of the important particular examples of the LSA procedure is the setting of the temporal difference (TD) learning algorithm [49, 50]. In the TD algorithm, we consider a discounted MDP (Markov Decision Process) given by a tuple* $(\mathcal{S}, \mathcal{A}, \mathrm{P}, r, \gamma)$*. Where* $\mathcal{S}$ *and* $\mathcal{A}$ *stand for state and action spaces, and* $\gamma \in (0,1)$ *is a discount factor, and we want to evaluate the value function of a* policy $\nu(\cdot|s)$*, which is the distribution over the action space* $\mathcal{A}$ *at a fixed state* $s \in \mathcal{S}$*. Many recent contributions to the analysis of TD learning deal with the linear function approximation when* $V^\nu(s) \approx \varphi^\top(s)\theta$*, where* $\theta \in \mathbb{R}^d$ *and* $\varphi(s) : \mathcal{S} \to \mathbb{R}^d$ *is a feature mapping. Under these conditions, the problem of finding optimal approximation parameters* $\theta^\star$ *is reduced to an instance of a linear stochastic approximation problem by the projected Bellman equation [52]. All the results given below in Sections 3.1, 3.2 and 4 apply directly to the TD learning with linear function approximation under the generative model assumptions studied in [41] and [54]. Namely, the assumptions A 1 and A 2 hold, and Proposition 1 holds with* $Q = \mathrm{I}$ *and* $P = \bar{\mathbf{A}} + \bar{\mathbf{A}}^\top$*, where* $\bar{\mathbf{A}}$ *is a system matrix corresponding to the projected TD learning equations, see [41, Section 5].*

We also consider the family of assumptions on the step sizes $\alpha_k$. Namely, for $p \geq 2$ consider the following assumptions A 3($p$):

**A 3** ($p$)**.** *The step sizes* $\{\alpha_k\}_{k \in \mathbb{N}}$ *have a form* $\alpha_k = \frac{c_0}{(k+k_0)^\gamma}$*, where* $\gamma \in (1/2; 1)$ *and* $c_0 \in (0; \alpha_\infty]$*. Assume additionally that*

$$k_0 \geq \left(\frac{16}{a c_0}\right)^{1/(1-\gamma)} \vee \left(\frac{2p\kappa_Q \, \mathrm{C}_{\mathbf{A}}^2}{a c_0}\right)^{1/\gamma} \, .$$

In our main results we often apply A 3($p$) with $p = \log d$. This particular choice of $p$ imposes a logarithmic dependence of $k_0$ upon the problem dimension $d$. This relaxes the polynomial bounds on $d$, which were previously considered in [30]. At this stage we assume that $k_0$ is a fixed constant that does not depend on time horizon $n$ used in (1.1).

### 3.1 Moment bounds for Polyak-Ruppert averaged LSA iterates.

We first present results for the $p$-th norm of the averaged LSA error, that is, $\mathbb{E}^{1/p}[\|\bar{\theta}_n - \theta^\star\|^p]$, where $\bar{\theta}_n$ is given in (1.1). We first define the product of random matrices

$$\Gamma_{m:k} = \prod_{\ell=m}^{k} (\mathrm{I} - \alpha_\ell \mathbf{A}_\ell) \;, \; m \le k \;, \quad \text{and } \Gamma_{m:k} = \mathrm{I} \;, \quad m > k \;. \tag{3.4}$$

Using the recurrence relation (3.1), we obtain the following decomposition of the LSA error:

$$\theta_k - \theta^\star = \tilde{\theta}_k^{(\mathsf{tr})} + \tilde{\theta}_k^{(\mathsf{fl})} \;, \quad \tilde{\theta}_k^{(\mathsf{tr})} = \Gamma_{1:k}(\theta_0 - \theta^\star) \;, \quad \tilde{\theta}_k^{(\mathsf{fl})} = -\sum_{\ell=1}^{k} \alpha_\ell \Gamma_{\ell+1:k} \varepsilon_\ell \;. \tag{3.5}$$

The term $\tilde{\theta}_k^{(\mathsf{tr})}$ above is a transient term, which reflects the forgetting of the initial error $\theta_0 - \theta^\star$, while $\tilde{\theta}_k^{(\mathsf{fl})}$ is a fluctuation term. Controlling the $p$-th order moments of the transient component $\tilde{\theta}_k^{(\mathsf{tr})}$ is essentially equivalent to bounding the $p$-th moment of the product of random matrices $\Gamma_{m:k}$. For this purpose, we use techniques for proving the stability of products of random matrices from [22] and [14]. We establish the following bound, which is referred to as the *exponential stability* of the product of random matrices:

**Lemma 1.** *Let $p \ge 2$ and assume A 1, A 2 , A 3($p \vee \log d$). Then for any $k \le n$, $1 \le m \le k$, it holds that*

$$\mathbb{E}^{1/p}\left[\|\Gamma_{m:k}\|^p\right] \le \sqrt{\kappa_Q}\mathrm{e} \prod_{\ell=m}^{k} \big(1 - \frac{a\alpha_\ell}{2}\big) \le \sqrt{\kappa_Q}\mathrm{e} \exp\big\{-\frac{a}{2}\sum_{\ell=m}^{k} \alpha_\ell\big\} \;.$$

The proof of Lemma 1 is given in Section B. We further decompose $\tilde{\theta}_k^{(\mathsf{fl})}$ based on the perturbation-expansion approach of [1], see also [12]. Namely, we notice that $\tilde{\theta}_k^{(\mathsf{fl})}$ satisfies the recurrence $\tilde{\theta}_k^{(\mathsf{fl})} = (\mathrm{I} - \alpha_k \mathbf{A}_k)\tilde{\theta}_{k-1}^{(\mathsf{fl})} - \alpha_k \varepsilon_k$, with $\tilde{\theta}_0^{(\mathsf{fl})} = 0$. Extracting its linear part, we represent $\tilde{\theta}_k^{(\mathsf{fl})}$ as

$$\tilde{\theta}_k^{(\mathsf{fl})} = J_k^{(0)} + H_k^{(0)} \;, \tag{3.6}$$

where the latter terms are defined by the following pair of recursions

$$J_k^{(0)} = \left(\mathrm{I} - \alpha_k \bar{\mathbf{A}}\right) J_{k-1}^{(0)} - \alpha_k \varepsilon_k \;, \qquad J_0^{(0)} = 0 \;, \tag{3.7}$$

$$H_k^{(0)} = (\mathrm{I} - \alpha_k \mathbf{A}_k)\, H_{k-1}^{(0)} - \alpha_k \tilde{\mathbf{A}}_k J_{k-1}^{(0)} \;, \qquad H_0^{(0)} = 0 \;. \tag{3.8}$$

Here the term $J_k^{(0)}$ represents the leading (w.r.t. $\alpha_k$) part of the error $\tilde{\theta}_k^{(\mathsf{fl})}$. Informally, one can show that $\mathbb{E}^{1/2}[\|J_k^{(0)}\|^2] \lesssim \alpha_k^{1/2}$, and similarly $\mathbb{E}^{1/2}[\|H_k^{(0)}\|^2] \lesssim \alpha_k$. Thus, $J_k^{(0)}$ is a leading term of $\tilde{\theta}_k^{(\mathsf{fl})}$ in terms of its moments, and $H_k^{(0)}$ is a remainder one, a phenomenon, that is referred to as a *separation of scales*.

The linear part $J_k^{(0)}$ plays an important role in our further analysis. In particular, we note that the outlined representation of the last iterate error (3.5) implies that

$$\sqrt{n}(\bar{\theta}_n - \theta^\star) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_k^{(0)} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} H_k^{(0)} + \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} \Gamma_{1:k}(\theta_0 - \theta^\star) \;. \tag{3.9}$$

The representation (3.9) plays a key role in our subsequent analysis of both the moment bounds and Gaussian approximation for $\sqrt{n}(\bar{\theta}_n - \theta^\star)$. Indeed, this representation allows us to represent the statistic $\sqrt{n}(\bar{\theta}_n - \theta^\star)$, which is non-linear as a function of $Z_1, \ldots, Z_{n-1}$, as a sum of a linear statistic $\frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} J_k^{(0)}$ and a remainder non-linear part, which is of smaller scale. We further denote

$$\Sigma_n = \frac{1}{n} \operatorname{Var}\big[\sum_{k=1}^{n-1} J_k^{(0)}\big] = \frac{1}{n} \sum_{k=1}^{n-1} Q_k \Sigma_\varepsilon Q_k^\top \;, \quad Q_\ell = \alpha_\ell \sum_{j=\ell}^{n-1} G_{\ell+1:j} \;, \quad G_{m:k} = \prod_{\ell=m}^{k} (\mathrm{I} - \alpha_\ell \bar{\mathbf{A}}) \;. \tag{3.10}$$

We also define the sequence $\varphi_n, n \in \mathbb{N}$, as follows:

$$\varphi_n = \begin{cases} \frac{2c_0^{3/2}}{(1-3\gamma/2)n^{3\gamma/2-1/2}} \;, & 1/2 < \gamma < 2/3; \\ \frac{c_0^{3/2} \log n}{n^{1/2}} \;, & \gamma = 2/3; \\ \frac{c_0^{3/2}}{(3\gamma/2-1)n^{1/2}} \;, & 2/3 < \gamma < 1 \;. \end{cases} \tag{3.11}$$

As a first main result of this section, we obtain the following $p$-th moment bound with the leading term given by the trace of the covariance matrix $\Sigma_n$. Precisely, the following bound holds:

**Theorem 1.** *Let $p \geq 2$ and assume A 1, A 2, and A 3($p \vee \log d$). Then, it holds that*

$$\mathbb{E}^{1/p}\big[\|\bar{\theta}_n - \theta^\star\|^p\big] \leq \frac{\mathsf{C}_{1,1}\sqrt{p}\sqrt{\operatorname{Tr}\Sigma_n}}{\sqrt{n}} + \Delta^{(\mathrm{fl})}(n,p,\gamma) + \frac{\mathsf{C}_{1,5}\|\theta_0 - \theta^\star\|}{n} \;, \tag{3.12}$$

*where we set*

$$\Delta^{(\mathrm{fl})}(n,p,\gamma) = \frac{\mathsf{C}_{1,2}p^{3/2}}{n^{1/2+\gamma/2}} + \frac{\mathsf{C}_{1,3}p^{5/2}\varphi_n}{n^{1/2}} + \frac{\mathsf{C}_{1,4}p}{n} \;,$$

*and the constants $\{\mathsf{C}_{1,i}\}_{i=1}^5$, depending on $\gamma, \kappa_Q, a, \mathrm{C}_{\mathbf{A}}, c_0, k_0$ and $\|\varepsilon\|_\infty$, are given in Section 5.1, see (5.5).*

**Remark 2.** *In order to study the scaling of the bound (3.12) with the problem dimension $d$, we assume the natural scaling $\|\varepsilon\|_\infty \leq \sqrt{d}\mathsf{C}_\varepsilon$, where $\mathsf{C}_\varepsilon$ is dimension-free. Then Theorem 1 implies that*

$$\mathbb{E}^{1/p}\big[\|\bar{\theta}_n - \theta^\star\|^p\big] \lesssim \frac{\sqrt{p}\sqrt{\operatorname{Tr}\Sigma_n}}{\sqrt{n}} + \frac{p^{3/2}\sqrt{d}}{n^{1/2+\gamma/2}} + \frac{p^{5/2}\sqrt{d}\varphi_n}{n^{1/2}} + \frac{p\sqrt{d}}{n} \;,$$

*where $\lesssim$ stands for constant not depending upon $p, n$, and $d$.*

The proof of Theorem 1 is provided in Section 5.1. Note that the leading in $n$ term of the above bound appears with the coefficient $\sqrt{\operatorname{Tr}\Sigma_n}$, where $\Sigma_n$ is the variance of the linear statistic extracted in the representation (3.9). It is possible to switch from the bound provided by Theorem 1 to the moment bound with the leading term matching the CLT covariance given by

$$\Sigma_\infty = \bar{\mathbf{A}}^{-1}\Sigma_\varepsilon\bar{\mathbf{A}}^{-\top}, \tag{3.13}$$

and $\Sigma_\varepsilon$ is defined in A 2. Precisely, the following bound holds:

**Corollary 1.** *Assume A 1, A 2, A 3($p \vee \log d$). Then, it holds that*

$$\mathbb{E}^{1/p}\big[\|\bar{\theta}_n - \theta^\star\|^p\big] \leq \frac{\mathsf{C}_{1,1}\sqrt{p}\sqrt{\operatorname{Tr}\Sigma_\infty}}{\sqrt{n}} + \frac{\mathsf{C}_1 d\sqrt{p}}{n^{3/2-\gamma}} + \Delta^{(\mathrm{fl})}(n,p,\gamma) + \frac{\mathsf{C}_{1,5}\|\theta_0 - \theta^\star\|}{n} \;. \tag{3.14}$$

*where the constant $\mathsf{C}_1$ is defined in (5.7).*

The proof of Corollary 1 is provided in Section 5.1. Optimizing the r.h.s. of (3.14) over $\gamma$, we obtain that the optimal value is $\gamma = 2/3$. This choice implies the moment bound

$$\mathbb{E}^{1/p}\big[\|\bar{\theta}_n - \theta^\star\|^p\big] \lesssim \frac{\sqrt{p}\sqrt{\operatorname{Tr}\Sigma_\infty}}{\sqrt{n}} + \frac{p^{3/2}}{n^{5/6}} \,, \tag{3.15}$$

where $\lesssim$ stands for constant not depending upon $n$ and $p$. The bound (3.15) improves upon previous bounds of this type obtained in [29] and [12]. Both of these papers considered constant step-size LSA. [12, Proposition 5] showed a bound of the form (3.15) with a residual term of order $\mathcal{O}(p^2/n^{3/4})$. The improvement in the dependence on $n$, compared to the latter paper, arises from the fact that the authors used a summation by parts formula applied to $\bar{\theta}_n - \theta^\star$, which yields a counterpart of (3.9) with a different linear statistic identified as the leading term. [29, Theorem 2] obtained a counterpart of (3.15) for one-dimensional projections of the error. Unlike typical results in linear stochastic approximation, where stepsizes often decay as $n^{-\gamma}$ for $\gamma \in (1/2, 1)$, [29] requires a slower rate of $n^{-1/3}$, leading to a second-order term of order $\mathcal{O}(n^{-5/6})$, similar to (3.15).

## 3.2 Gaussian approximation for Polyak-Ruppert averaged LSA iterates.

In this section, we analyze the rate of Gaussian approximation for the statistic $\sqrt{n}(\bar{\theta}_n - \theta^\star)$. The result of Polyak and Juditsky [37] states that, under assumptions A 1-A 3, it holds that

$$\sqrt{n}(\bar{\theta}_n - \theta^\star) \xrightarrow{d} \mathcal{N}(0, \Sigma_\infty) \,, \tag{3.16}$$

where the asymptotic covariance matrix $\Sigma_\infty$ is defined in (3.13). We are interested to quantify the rate of convergence in (3.16) w.r.t. the available sample size $n$ and other problem parameters, such as dimension $d$. To measure the approximation quality, we use the convex distance, defined for a pair of probability measures $\mu, \nu$ on $\mathbb{R}^d$ as

$$\rho^{\mathrm{Conv}}(\mu, \nu) = \sup_{B \in \mathrm{Conv}(\mathbb{R}^d)} |\mu(B) - \nu(B)| \,,$$

where $\mathrm{Conv}(\mathbb{R}^d)$ denotes the collection of all convex sets in $\mathbb{R}^d$. With a slight abuse of notation, we write $\rho^{\mathrm{Conv}}(X, Y)$ for random vectors $X$ and $Y$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ instead of their distributions under $\mathbb{P}$ whenever there is no risk of confusion.

**Gaussian approximation with randomized concentration inequalities.** To establish the Gaussian approximation for $\sqrt{n}(\bar{\theta}_n - \theta^\star)$, we consider it as a non-linear statistic of independent random variables $Z_k$ outlined in (1.1). Then we consider this statistic as a sum of a linear term and a remainder term of smaller order in $n$. This framework is presented in [44], and we summarize below the key results that will be later used to establish our findings. In this paragraph, we present all results for statistics defined in terms of the random variables $X_1, \ldots, X_n$, rather than $Z_1, \ldots, Z_n$ as used in the remainder of the paper.

Let $X_1, \ldots, X_n$ be independent random variables taking values in a measurable space $\mathsf{X}$, and consider a $d$-dimensional statistic $T = T(X_1, \ldots, X_n)$, which admits the decomposition $T = W + D$, where

$$W = \sum_{\ell=1}^{n} \xi_\ell, \quad D := D(X_1, \ldots, X_n) = T - W \,. \tag{3.17}$$

Here $\xi_\ell = h_\ell(X_\ell)$, and $h_\ell : \mathsf{X} \to \mathbb{R}^d$ are measurable functions. The term $D$ represents a potentially nonlinear component of the statistic $T$, which is assumed to be small compared to $W$ in an appropriate sense. Assume that $\mathbb{E}[\xi_\ell] = 0$ and $\sum_{\ell=1}^n \mathbb{E}[\xi_\ell \xi_\ell^\top] = \mathrm{I}_d$. Define $\Upsilon_n = \sum_{\ell=1}^n \mathbb{E}[\|\xi_\ell\|^3]$. Then, letting $\eta \sim \mathcal{N}(0, \mathrm{I}_d)$, [44, Theorem 2.1] yields that

$$\rho^{\mathrm{Conv}}(T, \eta) \leq 259 d^{1/2} \Upsilon_n + 2\mathbb{E}[\|W\| \|D\|] + 2 \sum_{\ell=1}^n \mathbb{E}[\|\xi_\ell\| \|D - D^{(\ell)}\|], \tag{3.18}$$

where $D^{(\ell)} = D(X_1, \ldots, X_{\ell-1}, X_\ell', X_{\ell+1}, \ldots, X_n)$, and $(X_1', \ldots, X_n')$ is an independent copy of $(X_1, \ldots, X_n)$. One can modify the bound (3.18) for the setting when $\sum_{\ell=1}^n \mathbb{E}[\xi_\ell \xi_\ell^\top] = \Sigma \succ 0$, see [44, Corollary 2.3].

**Gaussian approximation for the LSA algorithm**. In the setting of linear stochastic approximation we use the decomposition (3.17), identify $T = T(Z_1, \ldots, Z_{n-1}) = \sqrt{n} \Sigma_n^{-1/2} (\bar{\theta}_n - \theta^\star)$, and write

$$W = \frac{1}{\sqrt{n}} \Sigma_n^{-1/2} \sum_{k=1}^{n-1} J_k^{(0)}, \quad D = \frac{1}{\sqrt{n}} \Sigma_n^{-1/2} \sum_{k=1}^{n-1} H_k^{(0)} + \frac{1}{\sqrt{n}} \Sigma_n^{-1/2} \sum_{k=0}^{n-1} \Gamma_{1:k} (\theta_0 - \theta^\star) \,. \tag{3.19}$$

Changing the order of summation, we get with $Q_\ell$ defined in (3.10), that

$$W = -\frac{1}{\sqrt{n}} \sum_{\ell=1}^{n-1} \Sigma_n^{-1/2} Q_\ell \varepsilon_\ell \ , \tag{3.20}$$

i.e. $W$ is a weighted sum of i.i.d. random vectors with mean zero and $\mathbb{E}[W W^\top] = \mathrm{I}_d$. The decomposition (3.19) and (3.20) allows to apply the general Gaussian approximation result of (3.18). Application of the above result requires that the matrix $\Sigma_n$ is non-degenerate, which is guaranteed by the following lemma:

**Lemma 2.** *Let $p \geq 2$ and assume A 1, A 2, A 3(p). Let also $n \geq k_0 + 1$. Then it holds that*

$$\|\Sigma_n - \Sigma_\infty\| \leq \mathsf{C}_2 n^{\gamma - 1} \ , \tag{3.21}$$

*where the constant $\mathsf{C}_2$ is given in (5.34).*

The proof of Lemma 2 is given in Section 5.5. With Lidskii's inequality, we obtain that

$$\lambda_{\min}(\Sigma_n) \geq \lambda_{\min}(\Sigma_\infty) - \|\Sigma_\infty - \Sigma_n\| \ .$$

Therefore, using Lemma 2, we can lower bound $\lambda_{\min}(\Sigma_n)$, provided that $n$ is large enough. This is formalized in the following assumption:

**A 4.** *The sample size $n$ satisfies the conditions $n \geq k_0 + 1$ and $n^{1-\gamma} \geq 2\mathsf{C}_2 / \lambda_{\min}(\Sigma_\infty)$.*

With the assumptions above, we obtain the following Gaussian approximation result.

**Theorem 2.** *Assume A 1, A 2, A 3($2 \vee \log d$), A 4. Then, with $\eta \sim \mathcal{N}(0, \mathrm{I})$,*

$$\rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n - \theta^\star), \Sigma_n^{1/2} \eta) \leq \frac{\mathsf{C}_{2,1}}{\sqrt{n}} + \frac{\mathsf{C}_{2,2}}{n^{\gamma/2}} + \mathsf{C}_{2,3} \varphi_n + \frac{\mathsf{C}_{2,4} \|\theta_0 - \theta^\star\|}{n} \ ,$$

*where $\varphi_n$ is defined in (3.11) and $\mathsf{C}_{2,1}$, $\mathsf{C}_{2,2}$, $\mathsf{C}_{2,3}$, $\mathsf{C}_{2,4}$ are constants defined in (5.8).*

The constants $\mathsf{C}_{2,2} - \mathsf{C}_{2,4}$ contain factors that scale as $1/(1-\gamma)$, and the result in the stated form is not valid when setting $\gamma = 1$. At the same time, following the technique of Shao and Zhang [44, Theorem 3.4], it is possible to show that a counterpart of Theorem 2 holds when $\gamma = 1$, at the cost of additional $\log n$ factors arising in the r.h.s. of the bound and under additional constraints on the constant $c_0$, which cannot be chosen too small in this case.

**Remark 3.** *Under a natural scaling $\|\varepsilon\|_\infty \leq \sqrt{d}\mathsf{C}_\varepsilon$, where $\mathsf{C}_\varepsilon$ is dimension-free, Theorem 2 implies that*

$$\rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n - \theta^\star), \Sigma_n^{1/2}\eta) \lesssim \frac{d^2}{\sqrt{n}} + \frac{d^{3/2}}{n^{\gamma/2}} + d\varphi_n + \frac{d\log(d)\|\theta_0 - \theta^\star\|}{n} ,$$

*where $\lesssim$ stands for inequality up to a constant not depending upon $n$ and $d$.*

The proof of Theorem 2 is provided in Section 5.2. Note that the term $\frac{\mathsf{C}_{2,1}}{\sqrt{n}}$ above corresponds to the summand $\Upsilon_n$ from (3.18), which is related with the sum of third moments of random vectors forming the linear statistic $W$. The result of Theorem 2 shows, that the rate of normal approximation of $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ by $\mathcal{N}(0, \Sigma_n)$ improves when the step sizes $\alpha_k$ are chosen to be less aggressive, that is, when the power $\gamma$ approaches 1 in A 3. As already mentioned, constants $\mathsf{C}_{2,2} - \mathsf{C}_{2,4}$ scales with $1/(1-\gamma)$, so the latter conclusion applies when the available number of observations $n$ is large. This aligns with the phenomenon, previously observed for the SGD algorithm [44], [45] and TD learning [54].

Given the result of Theorem 2 and Lemma 2, it is possible to quantify the rate of convergence in (3.16). Precisely, the following result holds.

**Theorem 3.** *Assume A 1, A 2, A 3($2 \vee \log d$), A 4. Then, with $\eta \sim \mathcal{N}(0, \mathrm{I})$,*

$$\rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n - \theta^\star), \Sigma_\infty^{1/2}\eta) \leq \frac{\mathsf{C}_{2,1}}{\sqrt{n}} + \frac{\mathsf{C}_{2,2}}{n^{\gamma/2}} + \mathsf{C}_{2,3}\varphi_n + \frac{\mathsf{C}_{2,4}\|\theta_0 - \theta^\star\|}{n} + \frac{\mathsf{C}_3}{n^{1-\gamma}} , \qquad (3.22)$$

*where the constant $\mathsf{C}_3$ is given in (5.13).*

The proof of Theorem 3 is provided in Section 5.2.

**Discussion.** The bound established in Theorem 3 achieves the optimal normal approximation error rate of $n^{-1/3}$ for Polyak-Ruppert averaged estimates. This optimal rate is attained using step sizes $\alpha_k = c_0/(k+k_0)^{2/3}$, corresponding to the decay exponent $\gamma = 2/3$ in (3.22).

This $n^{-1/3}$ rate aligns with recent results for policy evaluation in reinforcement learning. Wu et al. [54] established the same convergence rate for the temporal difference (TD) learning algorithm. Their analysis employs step sizes scaling as $c_0/k^{2/3}$, which is consistent with the optimal choice predicted by Theorem 3. Related work Wu et al. [55] studies TD learning under Markov noise, achieving a slightly slower rate of $n^{-1/4}$ (up to logarithmic factors) in convex distance. Another relevant contribution is provided by Srikant [47], who analyzed temporal-difference learning with Markov noise and established a convergence rate of $n^{-1/6}$ in Wasserstein distance for the step sizes $\alpha_k = c_0/k^{2/3}$. Applying the relation between convex distance and Wasserstein distance [33, Eq. (3)], this bound translates to a convergence rate of order $n^{-1/12}$ in convex distance.

The fastest known rate for $\rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n - \theta^\star), \Sigma_\infty^{1/2}\eta)$ in the general LSA problem is $n^{-1/4}$ and is due to [41]. Our rate improvement compared to this work is achieved through a tighter analysis of the normal approximation with $\mathcal{N}(0, \Sigma_n)$, which is carried out in Theorem 2. We then estimate $\rho^{\mathrm{Conv}}(\mathcal{N}(0, \Sigma_n), \mathcal{N}(0, \Sigma_\infty))$ using the Gaussian comparison inequality [3, 11]. The authors of [41] used a different error decomposition for the statistic $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ based on the summation by parts representation [29, 12], with a linear statistic with covariance matrix $\Sigma_\infty$. This approach avoids the Gaussian comparison step but induces a slower approximation rate compared to Theorem 3.

Several related studies [2, 44, 45] have investigated the normal approximation problem (3.16) for stochastic gradient descent (SGD) algorithms targeting strongly convex objective functions. We provide a comparative analysis of these results relative to our LSA framework. Anastasiou et al. [2] studied both SGD setting and LSA with symmetric positive-definite system matrix $\bar{A} = \bar{A}^\top \succ 0$, achieving normal approximation rates of order $n^{-1/2}$ for integral probability metrics $\mathsf{d}_{[2]}$ induced by twice-differentiable test functions. Precise definition of $\mathsf{d}_{[2]}$ is given in Section A. This result has two important limitations. First, the relation between the Kolmogorov distance and $\mathsf{d}_{[2]}$ metric (see e.g. [19, Proposition 2.1]) suggests that the rate $n^{-1/2}$ translates to the one of $n^{-1/6}$ when considering the Kolmogorov distance. Hence, the implied rate in convex distance is not faster than $n^{-1/6}$, which is substantially slower than the $n^{-1/3}$ rate achieved in (3.22). Second, a detailed examination of [2, Theorem 4] reveals that their bound depends on a quantity $\rho(\eta, t)$ which scales, in notations of the current paper, with the sample size $n$ and the step-size exponent $\gamma$. It is not clear that this term can be uniformly bounded independently of $n$, suggesting that the convergence rate in a sense of $\mathsf{d}_{[2]}$ is actually slower than $n^{-1/2}$, depending upon $\gamma$.

Shao and Zhang [44] developed the SGD counterpart of our result of Theorem 2. Their analysis focused on Gaussian approximation with the normal distribution $\mathcal{N}(0, \Sigma_n)$ from (3.10), rather than $\mathcal{N}(0, \Sigma_\infty)$. These results were further developed in Sheshukova et al. [45], where the authors shown a counterpart of Theorem 3 with a convergence rate of order $n^{-1/4}$ when setting $\gamma = 3/4$. This rate is slower than the one corresponding to the LSA setting. This gap arises from the nonlinearity of SGD recursions, which introduces an additional error term in the r.h.s. of (3.18).

### 3.3 Lower bounds for the LSA algorithm.

Lower bounds for the convex distance $\rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n - \theta^\star), \Sigma_\infty^{1/2}\eta)$ were studied in [45] for the setting of SGD algorithm. The particular instance of this algorithm, which covers also to the LSA setting, can be written as follows. Consider the simplest 1-dimensional LSA problem with $\bar{\mathbf{A}} = 1$, $\bar{\mathbf{b}} = 0$, that is, simply the equation

$$\theta = 0\;.$$

and $\mathbf{A}_k = 1$, $\mathbf{b}_k \sim \mathcal{N}(0, 1)$ for any $k \in \mathbb{N}$. Here $\theta^\star = 0$. The corresponding sequence of LSA updates can be written as follows:

$$\theta_{k+1} = \theta_k - \alpha_k(\theta_k + \xi_{k+1}),\ k \geq 0\;,$$

where $\theta_0 \in \mathbb{R}$, $\alpha_k = c_0(1+k)^{-\gamma}$, $1/2 < \gamma < 1$, and $\xi_k = -\mathbf{b}_k$ are i.i.d. standard gaussian random variables. Then [45, Proposition 1] shows that for large enough $n$ it holds that

$$\rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n - \theta^\star), \mathcal{N}(0, 1)) > \frac{C(\gamma, c_0)}{n^{1-\gamma}}\;, \tag{3.23}$$

where $C(\gamma, c_0)$ is a constant that depends upon $\gamma, c_0$. This result implies that the rate of convergence in Theorem 3 is optimal for $\gamma \in [2/3, 1)$, since the term $\frac{\mathsf{C}_3}{n^{1-\gamma}}$ dominates the r.h.s. in this regime. Similar result for TD learning was shown in [54]. However, to the best of our knowledge, there is no matching lower bound for the setting when $\gamma \in (1/2, 2/3)$.

## 4 Multiplier bootstrap for LSA

To perform statistical inference with the Polyak-Ruppert estimator $\bar{\theta}_n$, we propose an online bootstrap procedure that recursively updates the LSA estimate and a set of randomly perturbed LSA trajectories using the same set of noise variables $Z_k$. The proposed method follows the procedure outlined in [17]. This approach does not rely on the asymptotic distribution of the error $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ and does not require approximation of the covariance matrix $\Sigma_\infty$, which is known to be computationally expensive [8].

We describe the suggested procedure as follows. We assume that on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the sequence $\{Z_k\}_{k\in\mathbb{N}}$ is defined, we can construct $M \in \mathbb{N}$ sequences of i.i.d. random variables $\{w_k^\ell\}$, $1 \le k \le n$ and $1 \le \ell \le M$, which are independent of $\{Z_k\}_{k\in\mathbb{N}}$. We assume that $\mathbb{E}[w_k^\ell] = 1$, $\mathrm{Var}[w_k^\ell] = 1$, and $\mathbb{E}[|w_k^\ell - 1|^3] = m_3 < \infty$. Using these weight sequences, we recursively update $M$ randomly perturbed LSA estimates according to:

$$
\begin{aligned}
&\theta_k^{\mathsf{b},\ell} = \theta_{k-1}^{\mathsf{b},\ell} - \alpha_k w_k^\ell \{\mathbf{A}(Z_k)\theta_{k-1}^{\mathsf{b},\ell} - \mathbf{b}(Z_k)\} \;, \quad k \ge 1 \;, \quad \theta_0^{\mathsf{b},\ell} = \theta_0 \;, \\
&\bar{\theta}_n^{\mathsf{b},\ell} = n^{-1} \textstyle\sum_{k=0}^{n-1} \theta_k^{\mathsf{b},\ell} \;, \quad n \ge 1 \;.
\end{aligned} \tag{4.1}
$$

These weights add additional random perturbations to the LSA process (1.1). We set $\mathcal{Z}^{n-1} = \{Z_\ell\}_{1\le\ell\le n-1}$ and use the notation $\mathbb{P}^{\mathsf{b}} = \mathbb{P}(\cdot|\mathcal{Z}^{n-1})$ and $\mathbb{E}^{\mathsf{b}} = \mathbb{E}(\cdot|\mathcal{Z}^{n-1})$ for the corresponding conditional probability and expectation. We refer to them as the "bootstrap world" probability and expectation, respectively. We adopt the shorthand notation $\bar{\theta}_n^{\mathsf{b}}$ for $\bar{\theta}_n^{\mathsf{b},1}$.

The fundamental principle behind (4.1) is that the conditional distribution of the perturbed bootstrap samples $\sqrt{n}(\bar{\theta}_n^{\mathsf{b}} - \bar{\theta}_n)$ given the observed data $\mathcal{Z}^{n-1}$ (the "bootstrap world" distribution) approximates the distribution of the target quantity $\sqrt{n}(\bar{\theta}_n - \theta^\star)$. Specifically, [17] established that

$$
\sup_{B \in \mathrm{Conv}(\mathbb{R}^d)} |\mathbb{P}^{\mathsf{b}}(\sqrt{n}(\bar{\theta}_n^{\mathsf{b}} - \bar{\theta}_n) \in B) - \mathbb{P}(\sqrt{n}(\bar{\theta}_n - \theta^\star) \in B)| \to 0 \tag{4.2}
$$

in $\mathbb{P}$-probability as $n \to \infty$. We refer to this result as the asymptotic validity of the procedure (4.1) and aim to quantify the rate in (4.2). While no closed-form expression exists for $\mathbb{P}^{\mathsf{b}}(\sqrt{n}(\bar{\theta}_n^{\mathsf{b}} - \bar{\theta}_n) \in B)$, this probability can be approximated numerically via (4.1) by simulating a sufficiently large number $M$ of perturbed trajectories. Standard Monte Carlo theory (see, e.g., [43, Section 5.1]) indicates that this approximation achieves accuracy of order $M^{-1/2}$. Consider the following assumption:

**A 5.** *The step size offset $k_0$ satisfies*

$$
k_0^\gamma \ge \max\left\{2h(n)\, \mathrm{C}_{\mathbf{A}} \sqrt{\kappa_Q},\ \tfrac{c_0 h(n)}{\min\{1,\alpha_\infty\}},\ \tfrac{8\,\mathrm{C}_{\mathbf{A}}^2\, c_0 \sqrt{\kappa_Q} e h(n)}{a(2-2^\gamma)},\ \tfrac{c_0 \log^2(5n)}{\min\{1,\, a\}}\right\} \,,
$$

*where $h(n)$ is defined as*

$$
h(n) := \left\lceil \left( \tfrac{8\,\mathrm{C}_{\mathbf{A}} \sqrt{\kappa_Q}(1+2\log(10n^3 d))}{a(2-2^\gamma)} \right)^2 \right\rceil \,. \tag{4.3}
$$

*Additionally, the sample size $n$ must be sufficiently large such that*

$$
\lambda_{\min}(\Sigma_\infty) \ge \frac{8\sqrt{2}\|\varepsilon\|_\infty^2 \mathsf{C}_4^2 \sqrt{\log(10dn)}}{\sqrt{n}} + \frac{8\|\varepsilon\|_\infty^2 \mathsf{C}_4^2 \log(10dn)}{3n} \,.
$$

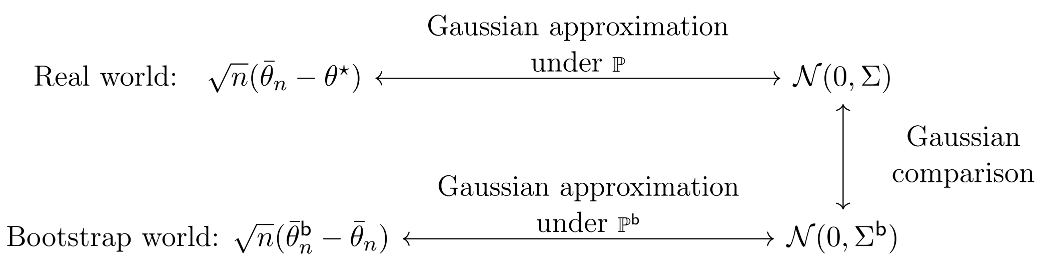
The condition A 5 ensures that the initial step sizes are not too large, which is crucial for the bootstrap approximation to be valid. We now present the main theoretical result of this section. Our analysis focuses on polynomially decaying step sizes $\gamma_n = c_0/(k_0+n)^\gamma$ with decay exponent $\gamma \in (1/2, 1)$.

**Theorem 4.** *Assume A 1, A 2, A 3*($\log(5n^3) \vee \log d$)*, A 4, A 5. Then with $\mathbb{P}$ – probability at least $1 - 1/n$ it holds that*

$$\sup_{B \in \mathrm{Conv}(\mathbb{R}^d)} |\mathbb{P}^{\mathsf{b}}(\sqrt{n}(\bar{\theta}_n^{\mathsf{b}} - \bar{\theta}_n) \in B) - \mathbb{P}(\sqrt{n}(\bar{\theta}_n - \theta^\star) \in B)| \leq \frac{\mathsf{C}_4 \|\theta_0 - \theta^\star\| + \Delta_{4,1}}{\sqrt{n}} + \frac{\Delta_{4,2}}{n^{\gamma/2}} + \Delta_{4,3}\varphi_n + \frac{\Delta_{4,4}}{n} \,,$$

*where $\mathsf{C}_4$ is a constant and $\{\Delta_{4,i}\}_{i=1}^4$ are polynomials in $\log(n)$ that are defined in Section 5.3, see* (5.28).

*Proof.* We provide here a high-level overview of the proof and refer the reader to Section 5.3 for a detailed exposition. The main ingredient of the proof is a Gaussian approximation via the randomized concentration inequalities approach [44]. The latter is carried out both for $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ under $\mathbb{P}$ and for $\sqrt{n}(\bar{\theta}_n^{\mathsf{b}} - \bar{\theta}_n)$ under $\mathbb{P}^{\mathsf{b}}$. These two results are then combined using a suitable Gaussian comparison inequality. The main steps of the proof are outlined in the diagram presented below: The principal




question that arises here is related with the choice of the approximating normal distribution $\Sigma$ and its bootstrap counterpart $\Sigma^{\mathsf{b}}$. In the earlier work [41], the authors used $\Sigma = \Sigma_\infty$. As indicated by Theorem 2 and Theorem 3, this does not appear to be an optimal choice, as it fails to provide an approximation rate faster than $n^{-1/3}$—at least when $\gamma \in (2/3; 1)$—due to the lower bound (3.23). At the same time, $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ can be approximated by $\mathcal{N}(0, \Sigma_n)$ at a rate approaching $1/\sqrt{n}$. This is the reason why we use $\Sigma = \Sigma_n$ in the present paper. The second principal difficulty in the proof is more technical and is related to the fact that applying a randomized concentration approach under $\mathbb{P}^{\mathsf{b}}$ requires a representation

$$\sqrt{n}(\bar{\theta}_n^{\mathsf{b}} - \bar{\theta}_n) = W^{\mathsf{b}} + D^{\mathsf{b}} \,, \tag{4.4}$$

where $W^{\mathsf{b}}$ is a linear statistic with $\mathbb{E}^{\mathsf{b}}[W^{\mathsf{b}}\{W^{\mathsf{b}}\}^\top] =: \Sigma_n^{\mathsf{b}}$. Since we aim to prove Gaussian approximation under $\mathbb{P}^{\mathsf{b}}$, by "linear statistic" we mean linearity in the bootstrap weights $w_\ell$. In addition to (4.4), we need to ensure that $\Sigma_n^{\mathsf{b}}$ is "close" to $\Sigma_n$ in an appropriate sense. We provide a detailed exposition, together with the definition of the statistics $W^{\mathsf{b}}$, $D^{\mathsf{b}}$, and $\Sigma_n^{\mathsf{b}}$, in Section 5.3.1. □

The terms $\{\Delta_{4,i}\}_{i=1}^4$ exhibit similar behavior as the constants from Theorem 2 and scale with the factor $1/(1-\gamma)$. Thus the special setting of $\gamma = 1$ requires separate treatment and is not covered by Theorem 4 in the present form. Similarly to Theorem 2, we expect that in the particular setting of $\gamma = 1$, the conclusion of Theorem 4 is still valid, probably with additional $\log n$ factors appearing in the r.h.s. and given additional constraints on $c_0$.

**Remark 4.** *Assuming a natural scaling $\|\varepsilon\|_\infty \leq \sqrt{d}\mathsf{C}_\varepsilon$, where $\mathsf{C}_\varepsilon$ is dimension-free, Theorem 4* *writes as*

$$\sup_{B\in\mathrm{Conv}(\mathbb{R}^d)} |\mathbb{P}^{\mathsf{b}}(\sqrt{n}(\bar{\theta}_n^{\mathsf{b}}-\bar{\theta}_n)\in B)-\mathbb{P}(\sqrt{n}(\bar{\theta}_n-\theta^\star)\in B)| \lesssim \frac{d^2+d^{3/2}\sqrt{\log(dn)}}{\sqrt{n}} + \frac{d^{3/2}\log n+\sqrt{d}\log^{2\gamma} n}{n^{\gamma/2}} + d\varphi_n + \frac{d^{3/2}\log(dn)}{n} \;.$$

**Discussion.** The direct counterpart of Theorem 4 with the slower approximation rate (with order up to order $n^{-1/4}$ up to logarithmic factors in $n$) was obtained in [41]. The main reason for improvement in the current paper is the choice of the approximating matrix $\Sigma$ in Section 4. The authors in [41] used $\Sigma=\Sigma_n$ contrary to the choice $\Sigma=\Sigma_\infty$ employed in Theorem 4. To our knowledge, the closest result to ours is the one of [54]. In this paper within the plug-in methods the authors obtain an estimator $\widehat{\Sigma}_n$ of the asymptotic covariance $\Sigma_\infty$ and provide high-probability error bounds

$$\rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n-\theta^\star),\mathcal{N}(0,\widehat{\Sigma}_n)) \lesssim \frac{1}{n^{1/3}} \;,$$

which is attained when the step size exponent $\gamma=2/3$. Theorem 4 provides approximation of order up to $1/\sqrt{n}$ when $\gamma\to 1$.

# 5 Proofs

## 5.1 Proofs of Section 3.1

In this section we provide additional details on the perturbation-expansion technique [1, 12]. Recall that we can represent the fluctuation component of the error $\tilde{\theta}_k^{(\mathsf{fl})}$ defined in (3.5) as $\tilde{\theta}_k^{(\mathsf{fl})}=J_k^{(0)}+H_k^{(0)}$ where the terms $J_k^{(0)}$ and $H_k^{(0)}$ are given in (3.7) and (3.8), respectively. The term $H_k^{(0)}$ can be further expanded. One can check with simple algebra that for any $L\in\mathbb{N}$ the term $H_k^{(0)}$ can be decomposed as

$$H_k^{(0)}=\sum_{\ell=1}^{L} J_k^{(\ell)}+H_k^{(L)} \;, \tag{5.1}$$

where the terms $J_k^{(\ell)}$ and $H_k^{(\ell)}$ are given by the following recurrences:

$$\begin{aligned} J_k^{(\ell)} &= \left(\mathrm{I}-\alpha_k\bar{\mathbf{A}}\right) J_{k-1}^{(\ell)}-\alpha_k\tilde{\mathbf{A}}(Z_k)J_{k-1}^{(\ell-1)} \;, \qquad J_0^{(\ell)}=0 \;, \\ H_k^{(\ell)} &= \left(\mathrm{I}-\alpha_k\mathbf{A}(Z_k)\right) H_{k-1}^{(\ell)}-\alpha_k\tilde{\mathbf{A}}(Z_k)J_{k-1}^{(\ell)} \;, \quad H_0^{(\ell)}=0 \;. \end{aligned} \tag{5.2}$$

It is possible to show that, under assumptions A1, A2, and A3, it holds that $\mathbb{E}^{1/p}[\|J_k^{(\ell)}\|^p]\le c_\ell\alpha_k^{(\ell+1)/2}$, and similarly $\mathbb{E}^{1/p}[\|H_k^{(\ell)}\|^p]\le c_\ell\alpha_k^{(\ell+1)/2}$, where the constant $c_\ell$ can depend upon $d,p$, and problem-related quantities, but not upon $k$. Thus the expansion depth $L$ in (5.1) controls the desired approximation accuracy. Our analysis of the $p$-th moment of the last iterate error $\theta_k-\theta^\star$ will not require the expansion (3.6). At the same time, more delicate bounds for $\mathbb{E}^{1/p}[\|\bar{\theta}_n-\theta^\star\|^p]$ will require to use (3.6) and (5.1) with $L=2$. We recall the $p$-th moment bound of last iterate, adapted from [41, Proposition 4].

**Proposition 2.** *Let $p \geq 2$ and assume A 1, A 2, and A 3($p \vee \log d$). Then for any $1 \leq k \leq n-1$, it holds that*

$$\mathbb{E}^{1/p}[\|\theta_k - \theta^\star\|^p] \leq \sqrt{\kappa_Q}\mathrm{e}\|\theta_0 - \theta^\star\| \prod_{\ell=1}^{k}(1 - \frac{a}{2}\alpha_\ell) + \mathsf{C}_2 p\sqrt{\alpha_k} \;, \text{ where } \mathsf{C}_2 = \sqrt{6}\mathrm{e}\|\varepsilon\|_\infty\sqrt{\kappa_Q/a} \;.$$

Now we provide moment bounds for the terms $J_k^{(\ell)}, H_k^{(\ell)}$, $\ell \in \{0, \ldots, L\}$.

**Lemma 3.** *Let $p \geq 2$. Assume A 1, A 2, and A 3($p \vee \log d$). Then for any $\ell \in \{0, 1, 2\}$ it holds that*

$$\mathbb{E}^{1/p}[\|J_k^{(\ell)}\|^p] \leq \mathsf{C}_3^{(\mathrm{J},\ell)} p^{\ell+1/2}\alpha_k^{(\ell+1)/2} \;,$$

$$\mathbb{E}^{1/p}[\|H_k^{(\ell)}\|^p] \leq \mathsf{C}_3^{(\mathrm{H},\ell)} p^{\ell+1/2}\alpha_k^{(\ell+1)/2} \;,$$

*where the constants $\mathsf{C}_3^{(\mathrm{J},\ell)}$, $\mathsf{C}_3^{(\mathrm{H},\ell)}$ satisfy the recurrence*

$$\mathsf{C}_3^{(\mathrm{J},0)} = \frac{4\sqrt{3}\kappa_Q^{1/2}\|\varepsilon\|_\infty}{a^{1/2}} \;, \; \mathsf{C}_3^{(\mathrm{J},\ell)} = \frac{2\sqrt{6}\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}}{a^{1/2}}\mathsf{C}_3^{(\mathrm{J},\ell-1)} \;, \; \mathsf{C}_3^{(\mathrm{H},\ell)} = \frac{12\kappa_Q^{1/2}\mathrm{e}}{a}\mathsf{C}_3^{(\mathrm{J},\ell)} \;.$$

We also state here the lemma, which is instrumental for our further results and bounds $\|Q_\ell\|$ for matrices $Q_\ell$ defined in (3.10).

**Lemma 4.** *Assume A 1, A 2, and A 3($2 \vee \log(d)$). Then, for any $\ell \in \{1, \ldots, n-1\}$,*

$$\|Q_\ell\| \leq \alpha_\ell \sum_{j=\ell}^{n-1}\|G_{\ell+1:j}\| \leq \mathsf{C}_4 \;, \text{ where } \mathsf{C}_4 = \kappa_Q^{1/2}\big(c_0 + \frac{2}{a(1-\gamma)}\big) \;. \tag{5.4}$$

*Moreover, $\sum_{j=1}^{n-1}\|G_{1:j}\| \leq (1+k_0)^\gamma \mathsf{C}_4/c_0$.*

*Proof of Theorem 1.* We first define the constants outlined in the statement of the theorem:

$$\mathsf{C}_{1,1} = 60\mathrm{e} \;, \; \mathsf{C}_{1,2} = \frac{\sqrt{c_0}\mathsf{C}_3^{(\mathrm{J},0)}\mathsf{C}_4\,\mathrm{C}_{\mathbf{A}}}{\sqrt{1-\gamma}} \;, \; \mathsf{C}_{1,3} = \mathsf{C}_3^{(\mathrm{J},2)} + \mathsf{C}_3^{(\mathrm{H},2)} \;, \tag{5.5}$$
$$\mathsf{C}_{1,4} = 60\mathsf{C}_4\|\varepsilon\|_\infty \;, \; \mathsf{C}_{1,5} = 1 + \frac{\sqrt{\kappa_Q}\mathrm{e}(1+k_0)^\gamma}{c_0}\big(c_0 + \frac{2}{a(1-\gamma)}\big) \;.$$

Combining the representations (3.5) and (3.6), we get

$$\bar{\theta}_n - \theta^\star = n^{-1}\sum_{k=1}^{n-1}\Gamma_{1:k}(\theta_0 - \theta^\star) + n^{-1}\sum_{k=1}^{n-1}J_k^{(0)} + n^{-1}\sum_{k=1}^{n-1}H_k^{(0)} \;. \tag{5.6}$$

Now we proceed with different terms in (5.6) separately. Applying Lemma 25, we obtain that

$$\mathbb{E}^{1/p}\big[\|n^{-1}\sum_{k=0}^{n-1}\Gamma_{1:k}(\theta_0 - \theta^\star)\|^p\big] \leq \frac{\|\theta_0 - \theta^\star\|\mathsf{C}_{1,5}}{n} \;.$$

Now we proceed with the term $n^{-1}\sum_{k=1}^{n-1} J_k^{(0)} = -n^{-1}\sum_{\ell=1}^{n-1} Q_\ell \varepsilon_\ell$, where $Q_\ell$ is defined in (3.10). Applying the version of Rosenthal inequality due to Pinelis [35, Theorem 4.1], we get

$$\mathbb{E}^{1/p}\|n^{-1}\sum_{\ell=1}^{n-1} Q_\ell \varepsilon_\ell\|^p \leq \frac{\mathrm{C}_{\mathrm{Ros},1}\, p^{1/2}\{\operatorname{Tr}\Sigma_n\}^{1/2}}{n^{1/2}} + \frac{\mathrm{C}_{\mathrm{Ros},2}\, p \mathbb{E}^{1/p}[\max_{1\leq \ell \leq n}\|Q_\ell \varepsilon_\ell\|^p]}{n} ,$$

where $\mathrm{C}_{\mathrm{Ros},1} = 60\mathrm{e}$ and $\mathrm{C}_{\mathrm{Ros},2} = 60$ are constants from [35]. Applying Lemma 4, we get that $\|Q_\ell\| \leq \mathsf{C}_4$, where $\mathsf{C}_4$ is defined in (5.4). Hence,

$$\mathbb{E}^{1/p}\|n^{-1}\sum_{\ell=1}^{n-1} Q_\ell \varepsilon_\ell\|^p \leq \frac{\mathrm{C}_{\mathrm{Ros},1}\, p^{1/2}\{\operatorname{Tr}\Sigma_n\}^{1/2}}{n^{1/2}} + \frac{\mathrm{C}_{\mathrm{Ros},2}\, p \mathsf{C}_4 \|\varepsilon\|_\infty}{n} .$$

Now we proceed with the next-order terms in $n$ corresponding to $n^{-1}\sum_{k=1}^{n-1} H_k^{(0)}$. Note that $H_k^{(0)} = J_k^{(1)} + H_k^{(1)}$ where $J_k^{(1)}$ and $H_k^{(1)}$ are given by $J_k^{(1)} = -\sum_{\ell=1}^{k} \alpha_\ell G_{\ell+1:k} \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)}$ and $H_k^{(1)} = -\sum_{m=1}^{k} \alpha_m \Gamma_{m+1:k} J_m^{(1)}$. Applying Lemma 3 and Minkowski's inequality, we obtain that

$$\mathbb{E}^{1/p}[\|H_k^{(1)}\|^p] \leq \mathbb{E}^{1/p}[\|J_k^{(2)}\|^p] + \mathbb{E}^{1/p}[\|H_k^{(2)}\|^p] \leq (\mathsf{C}_3^{(\mathrm{J},2)} + \mathsf{C}_3^{(\mathrm{H},2)}) p^{5/2} \alpha_k^{3/2} .$$

which implies that

$$n^{-1}\mathbb{E}^{1/p}[\|\sum_{k=1}^{n-1} H_k^{(1)}\|^p] \leq n^{-1}(\mathsf{C}_3^{(\mathrm{J},2)} + \mathsf{C}_3^{(\mathrm{H},2)}) p^{5/2} \sum_{k=1}^{n-1} \alpha_k^{3/2} \leq \frac{\varphi_n}{n^{1/2}}(\mathsf{C}_3^{(\mathrm{J},2)} + \mathsf{C}_3^{(\mathrm{H},2)}) p^{5/2} ,$$

where the sequence $\varphi_n$ is defined in (3.11). It remains to proceed with $\sum_{k=1}^{n-1} J_k^{(1)}$. Note that

$$\sum_{k=1}^{n-1} J_k^{(1)} = -\sum_{k=1}^{n-1}\sum_{\ell=1}^{k} \alpha_\ell G_{\ell+1:k} \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)} = -\sum_{\ell=1}^{n-1} Q_\ell \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)} .$$

Using the fact that $Q_\ell \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)}$ is a martingale-difference w.r.t. $\mathcal{F}_{\ell-1} = \sigma(Z_s : s \leq \ell - 1)$, we obtain, applying Burkholder's inequality [34, Theorem 8.6] and Lemma 3,

$$\begin{aligned}\frac{1}{n}\mathbb{E}^{1/p}[\|\sum_{k=1}^{n} J_k^{(1)}\|^p] \leq \frac{p}{n}\Big(\sum_{\ell=1}^{n-1} \mathbb{E}^{2/p}\|Q_\ell \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)}\|^p\Big)^{1/2} &\leq \frac{\mathsf{C}_3^{(\mathrm{J},0)}\mathsf{C}_4\,\mathrm{C}_{\mathbf{A}}\, p^{3/2}\{\sum_{\ell=1}^{n-1}\alpha_\ell\}^{1/2}}{n} \\ &\leq \frac{\sqrt{c_0}\mathsf{C}_3^{(\mathrm{J},0)}\mathsf{C}_4\,\mathrm{C}_{\mathbf{A}}}{\sqrt{1-\gamma}}\frac{p^{3/2}}{n^{(1+\gamma)/2}} .\end{aligned}$$

It remains to combine the above bounds in (5.6). □

*Proof of Corollary 1.* First, we define the constant $\mathsf{C}_1$ outlined in the statement:

$$\mathsf{C}_1 = \frac{\mathsf{C}_{1,1}\mathsf{C}_2}{2\sqrt{\operatorname{Tr}(\Sigma_\infty)}} , \tag{5.7}$$

where the constant $\mathsf{C}_2$ is introduced in Lemma 2. Using that for $a > 0$, $b \geq 0$, $\sqrt{a+b} = \sqrt{a(1+b/a)} \leq \sqrt{a} + b/(2\sqrt{a})$ and Lemma 2, we get

$$\sqrt{\operatorname{Tr}(\Sigma_n)} \leq \sqrt{\operatorname{Tr}(\Sigma_\infty)} + \frac{|\operatorname{Tr}(\Sigma_n - \Sigma_\infty)|}{2\sqrt{\operatorname{Tr}(\Sigma_\infty)}} \leq \sqrt{\operatorname{Tr}(\Sigma_\infty)} + \frac{d\mathsf{C}_2 n^{\gamma-1}}{2\sqrt{\operatorname{Tr}(\Sigma_\infty)}} .$$

The proof is concluded using Theorem 1. □

## 5.2 Proofs of Section 3.2

We first define the constants outlined in the statement of the theorem:

$$
\begin{aligned}
\mathsf{C}_{2,1} &= 259 d^{3/2} \mathsf{C}_6^{-1/2} \|\varepsilon\|_\infty \mathsf{C}_4 \;, \\
\mathsf{C}_{2,2} &= d^{1/2} \frac{2^{11/2}\, \mathrm{C}_{\mathbf{A}}\, \mathsf{C}_4 \mathsf{C}_3^{(\mathrm{J},0)} \sqrt{c_0} \|\varepsilon\|_\infty}{\mathsf{C}_6^{1/2} \sqrt{1-\gamma}} + 2 d^{1/2} \mathsf{C}_6^{-1} \|\varepsilon\|_\infty \mathsf{C}_4 \frac{\sqrt{c_0}}{1-\gamma} \mathsf{C}_8 \;, \\
\mathsf{C}_{2,3} &= \sqrt{d} (2^{7/2} \mathsf{C}_6^{-1/2} \mathsf{C}_3^{(\mathrm{J},2)} + 2^{7/2} \mathsf{C}_6^{-1/2} \mathsf{C}_3^{(\mathrm{H},2)}) \;, \\
\mathsf{C}_{2,4} &= d^{1/2} \mathsf{C}_6^{-1/2} \mathsf{C}_{1,5} + 2 d^{1/2} \frac{2(1+k_0)^\gamma}{\mathsf{C}_6} \|\varepsilon\|_\infty \mathsf{C}_7 (1 + \frac{4}{a c_0 (1-\gamma)}) \;,
\end{aligned}
\tag{5.8}
$$

*Proof of Theorem 2.* Recall that we use the representation $\sqrt{n}\Sigma_n^{-1/2}(\bar{\theta}_n - \theta^\star) = W + D$ , where

$$
W = \frac{1}{\sqrt{n}} \Sigma_n^{-1/2} \sum_{k=1}^{n-1} J_k^{(0)}, \; D = \frac{1}{\sqrt{n}} \Sigma_n^{-1/2} \sum_{k=1}^{n-1} H_k^{(0)} + \frac{1}{\sqrt{n}} \Sigma_n^{-1/2} \sum_{k=0}^{n-1} \Gamma_{1:k}(\theta_0 - \theta^\star) \,.
$$

Alternative representation for $W$ is given in (3.20). Recall that we write $\eta$ for a random vector with standard normal distribution $\eta \sim \mathcal{N}(0, \mathrm{I}_d)$ under $\mathbb{P}$. Then, setting

$$
\xi_k = \frac{1}{\sqrt{n}} (\Sigma_n)^{-1/2} Q_k \varepsilon_k \;, \quad \Upsilon_n = \sum_{k=1}^{n-1} \mathbb{E}[\|\xi_k\|^3] \;,
$$

we obtain from [44, Theorem 2.1]:

$$
\rho^{\mathrm{Conv}}(\sqrt{n}\Sigma_n^{-1/2}(\bar{\theta}_n - \theta^\star), \eta) \le 259 d^{1/2} \Upsilon_n + 2\mathbb{E}[\|W\|\|D\|] + 2 \sum_{\ell=1}^{n-1} \mathbb{E}[\|\xi_\ell\|\|D - D^{(\ell)}\|]. \tag{5.9}
$$

Note that Lemma 6 and Lemma 4 imply $\|\xi_k\| \le \frac{1}{\sqrt{n}} \mathsf{C}_6^{-1/2} \|\varepsilon\|_\infty \mathsf{C}_4$. Hence,

$$
\Upsilon_n \le \frac{1}{\sqrt{n}} \mathsf{C}_6^{-1/2} \|\varepsilon\|_\infty \mathsf{C}_4 \sum_{k=1}^{n-1} \mathbb{E}[\|\xi_k\|^2] = \frac{d}{\sqrt{n}} \mathsf{C}_6^{-1/2} \|\varepsilon\|_\infty \mathsf{C}_4 \,. \tag{5.10}
$$

To proceed with the second term in (5.9), we use the representation for the statistic $D$ from (3.19):

$$
D = \frac{1}{\sqrt{n}} \Sigma_n^{-1/2} \sum_{k=1}^{n-1} H_k^{(0)} + \frac{1}{\sqrt{n}} \Sigma_n^{-1/2} \sum_{k=0}^{n-1} \Gamma_{1:k}(\theta_0 - \theta^\star) \,.
$$

Lemma 5 implies that

$$
\frac{\mathsf{C}_6^{-1/2}}{\sqrt{n}} \mathbb{E}\big[\|\sum_{k=0}^{n-1} \Gamma_{1:k}(\theta_0 - \theta^\star)\|^2\big] \le \frac{\mathsf{C}_6^{-1/2} \|\theta_0 - \theta^\star\| \mathsf{C}_{1,5}}{\sqrt{n}} \,.
$$

Using the representation $H_k^{(0)} = J_k^{(1)} + J_k^{(2)} + H_k^{(2)}$ and Minkowski's inequality,

$$
\mathbb{E}^{1/2}\big[\|\sum_{k=1}^{n-1} H_k^{(0)}\|^2\big] \le \mathbb{E}\big[\|\sum_{k=1}^{n-1} J_k^{(1)}\|^2\big] + \mathbb{E}\big[\|\sum_{k=1}^{n-1} J_k^{(2)}\|^2\big] + \mathbb{E}\big[\|\sum_{k=1}^{n-1} H_k^{(2)}\|^2\big] \,.
$$

Applying Lemma 3 with $p = 2$ and Minkowski's inequality, we get

$$n^{-1/2}\mathbb{E}^{1/2}\big[\|\sum_{k=1}^{n-1} J_k^{(2)}\|^2\big] \leq n^{-1/2}\sum_{k=1}^{n-1} \mathsf{C}_3^{(\mathrm{J},2)} 2^{5/2}\alpha_k^{3/2} \leq 2^{5/2}\mathsf{C}_3^{(\mathrm{J},2)}\varphi_n \;,$$

where the function $\varphi_n$ is defined in (3.11). Similarly, it holds that

$$n^{-1/2}\mathbb{E}^{1/2}\big[\|\sum_{k=1}^{n-1} H_k^{(2)}\|^2\big] \leq n^{-1/2}\sum_{k=1}^{n-1} 2^{5/2}\mathsf{C}_3^{(\mathrm{H},2)}\alpha_k^{3/2} \leq 2^{5/2}\mathsf{C}_3^{(\mathrm{H},2)}\varphi_n \;.$$

Rewrite the sum of $J_k^{(1)}$:

$$n^{-1/2}\mathbb{E}^{1/2}\big[\|\sum_{k=1}^{n-1} J_k^{(1)}\|^2\big] = n^{-1/2}\mathbb{E}^{1/2}\big[\|\sum_{k=1}^{n-1}\sum_{\ell=1}^{k} \alpha_\ell G_{\ell+1:k}\tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)}\|^2\big] = n^{-1/2}\mathbb{E}^{1/2}\big[\|\sum_{\ell=1}^{n-1} Q_\ell\tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)}\|^2\big] \;.$$

Since $Q_\ell\tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)}$ is a martingale-difference sequence, Lemma 4 and Burkholder's inequality [34, Theorem 9.1] imply

$$n^{-1/2}\mathbb{E}^{1/2}\big[\|\sum_{\ell=1}^{n-1} Q_\ell\tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)}\|^2\big] \leq 2n^{-1/2}\mathsf{C}_4\big(\sum_{\ell=1}^{n-1}\mathbb{E}[\|\tilde{\mathbf{A}}_\ell J_{\ell-1}^{(0)}\|^2]\big)^{1/2} \leq 2n^{-1/2}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_4\big(\sum_{\ell=1}^{n-1}\mathbb{E}[\|J_{\ell-1}^{(0)}\|^2]\big)^{1/2} \;.$$

Now we use Lemma 3 and get:

$$n^{-1/2}\|\mathbb{E}^{1/2}\big[\sum_{k=1}^{n-1} J_k^{(1)}\|^2\big] \leq 2^{9/2}n^{-1/2}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_4\mathsf{C}_3^{(\mathrm{J},0)}\sqrt{c_0\sum_{\ell=1}^{n-2}\ell^{-\gamma}} \leq \frac{2^{9/2}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_4\mathsf{C}_3^{(\mathrm{J},0)}\sqrt{c_0}}{\sqrt{1-\gamma}}n^{-\gamma/2} \;.$$

Combining the above bounds with $\mathbb{E}[\|W\|^2] = d$ we obtain:

$$\mathbb{E}\big[\|W\|\|D\|\big] \leq \frac{2^{9/2}d^{1/2}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_4\mathsf{C}_3^{(\mathrm{J},0)}\sqrt{c_0}\|\varepsilon\|_\infty}{n^{\gamma/2}\mathsf{C}_6^{1/2}\sqrt{1-\gamma}} + d^{1/2}2^{5/2}\mathsf{C}_6^{-1/2}\varphi_n(\mathsf{C}_3^{(\mathrm{J},2)} + \mathsf{C}_3^{(\mathrm{H},2)}) + \frac{d^{1/2}\|\theta_0-\theta^\star\|}{\mathsf{C}_6^{1/2}n^{1/2}}\mathsf{C}_{1,5} \;, \tag{5.11}$$

To derive a bound for the third term in (5.9), we introduce the following notations:

$$D_1^{(i)} = n^{-1/2}\sum_{k=1}^{n-1}(H_k^{(0)} - H_k^{(0,i)}) \;, \quad D_2^{(i)} = n^{-1/2}\sum_{k=1}^{n-1}(\Gamma_{1:k} - \Gamma_{1:k}^{(i)})(\theta_0-\theta^\star) \;.$$

Hence, one can check that

$$D - D^{(i)} = \Sigma_n^{-1/2}(D_1^{(i)} + D_2^{(i)}) \;.$$

Thus, combining Lemma 4, Lemma 7, Lemma 8 with Minkowski's inequality, we get

$$\begin{aligned}\mathbb{E}\big[\|\xi_i\|\|D - D^{(i)}\|\big] &\leq n^{-1/2}\mathsf{C}_6^{-1}\|\varepsilon\|_\infty\mathsf{C}_4\left(\mathbb{E}^{1/2}\big[\|D_1^{(i)}\|^2\big] + \mathbb{E}^{1/2}\big[\|D_2^{(i)}\|^2\big]\right)\\ &\leq \mathsf{C}_6^{-1}\|\varepsilon\|_\infty\left(\frac{\|\theta_0-\theta^\star\|}{n}\mathsf{C}_7\prod_{m=1}^{i-1}(1-\frac{a\alpha_m}{2}) + \frac{1}{n}\sqrt{\alpha_i}\mathsf{C}_8\right) \;.\end{aligned}$$

Hence, using Lemma 25, we finish the proof:

$$\sum_{i=1}^{n-1} \mathbb{E}[\|\xi_i\| \|D - D^{(i)}\|] \leq \frac{\|\varepsilon\|_\infty \mathsf{C}_4 \frac{\sqrt{c_0}}{1-\gamma} \mathsf{C}_8}{\mathsf{C}_6 n^{\gamma/2}} + \frac{\|\theta_0 - \theta^\star\|}{n} \frac{\|\varepsilon\|_\infty \mathsf{C}_7}{\mathsf{C}_6} (c_0 + \frac{2}{a(1-\gamma)}) \frac{(1+k_0)^\gamma}{c_0} \ . \quad (5.12)$$

The proof follows from (5.10), (5.11), (5.12) by rearranging the terms. □

We now state the technical lemmas that we use in the proof of Theorem 3.

**Lemma 5.** *Let $p \geq 2$. Assume A 1, A 2, A 3($p \vee \log d$), A 4. Then, it holds that*

$$\mathbb{E}^{1/p}\big[\| \sum\nolimits_{k=0}^{n-1} \Gamma_{1:k}(\theta_0 - \theta^\star)\|^p\big] \leq \|\theta_0 - \theta^\star\| \mathsf{C}_{1,5} \ ,$$

*where the constant $\mathsf{C}_{1,5}$ is given in (5.5).*

**Lemma 6.** *Let $p \geq 2$. Assume A 1, A 2, A 3($p \vee \log d$), A 4. Then it holds that*

$$\lambda_{\min}(\Sigma_n) \geq \mathsf{C}_6 \ , \ \textit{where } \mathsf{C}_6 = \frac{\lambda_{\min}(\Sigma_\infty)}{2} \ .$$

Now we introduce the vector $(Z_1', \ldots, Z_{n-1}')$ an independent copy of $(Z_1, \ldots, Z_{n-1})$, and introduce the following notation for $\ell \leq m$:

$$\Gamma_{\ell:m}^{(i)} = \begin{cases} \Gamma_{\ell:m}, & \text{if } i \notin [\ell, m], \\ \Gamma_{\ell:m}(Z_\ell, \ldots, Z_{i-1}, Z_i', Z_{i+1}, \ldots, Z_m), & \text{if } i \in [\ell, m] \ ; \end{cases}$$

$$J_k^{(\ell,i)} = \begin{cases} J_k^{(\ell)}, & \text{if } k < i, \\ J_k^{(\ell)}(Z_1, \ldots, Z_{i-1}, Z_i', Z_{i+1}, \ldots, Z_k), & \text{if } k \geq i \ ; \end{cases}$$

$$H_k^{(\ell,i)} = \begin{cases} H_k^{(\ell)}, & \text{if } k < i, \\ H_k^{(\ell)}(Z_1, \ldots, Z_{i-1}, Z_i', Z_{i+1}, \ldots, Z_k), & \text{if } k \geq i \ ; \end{cases}$$

$$D^{(i)} = n^{-1/2}\Sigma_n^{-1/2} \sum_{k=1}^{n-1} H_k^{(0,i)} + n^{-1/2}\Sigma_n^{-1/2} \sum_{k=1}^{n-1} \Gamma_{1:k}^{(i)}(\theta_0 - \theta^\star) \ .$$

Here $D^{(i)}$, $1 \leq i \leq n-1$, is a counterpart of $D$ with $Z_i$ substituted with $Z_i'$. In order to control $\mathbb{E}[\|D - D^{(i)}\|^2]$, we use the following two auxiliary lemmas.

**Lemma 7.** *Assume A 1, A 2, A 3($2 \vee \log d$) and A 4. Then it holds that*

$$\mathbb{E}^{1/2}\big[\|\frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} (\Gamma_{1:k} - \Gamma_{1:k}^{(i)})(\theta_0 - \theta^\star)\|^2\big] \leq \frac{\|\theta_0 - \theta^\star\|}{\sqrt{n}} \mathsf{C}_7 \prod_{m=1}^{i-1} (1 - \frac{a\alpha_m}{2}) \ ,$$

*where $\mathsf{C}_7 = \mathrm{C}_{\mathbf{A}}\, \kappa_Q \mathrm{e}^2 (c_0 + 2/((1-\gamma)a))$.*

**Lemma 8.** *Assume A 1, A 2, A 3($2 \vee \log d$) and A 4. Then it holds that*

$$n^{-1/2}\mathbb{E}^{1/2}\big[\| \sum_{k=1}^{n-1} (H_k^{(0)} - H_k^{(0,i)})\|^2\big] \leq \frac{\mathsf{C}_8}{\sqrt{n}} \sqrt{\alpha_i} \ ,$$

*where $\mathsf{C}_8$ is given by*

$$\mathsf{C}_8 = \kappa_Q \mathrm{e}^2 \|\varepsilon\|_\infty \mathrm{C}_{\mathbf{A}} \bigg(c_0 + \frac{2}{a(1-\gamma)}\bigg)^{3/2} + 2\, \mathrm{C}_{\mathbf{A}} \sqrt{\kappa_Q} \mathrm{e}(\mathsf{C}_3^{(\mathrm{J},0)} + \mathsf{C}_3^{(\mathrm{H},0)}) \bigg(c_0 + \frac{2}{a(1-\gamma)}\bigg) \ .$$

**Proof of Theorem 3** We first introduce the constant

$$\mathsf{C}_3 = \frac{3\sqrt{d}\mathsf{C}_2}{2\lambda_{\min}(\Sigma_\infty)} \, . \tag{5.13}$$

Applying the triangle inequality,

$$\rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n - \theta^\star), \Sigma_\infty^{1/2}\eta) \le \rho^{\mathrm{Conv}}(\sqrt{n}(\bar{\theta}_n - \theta^\star), \Sigma_n^{1/2}\zeta) + \rho^{\mathrm{Conv}}(\Sigma_n^{1/2}\zeta, \Sigma_\infty^{1/2}\eta) \, , \tag{5.14}$$

where $\eta, \zeta \sim \mathcal{N}(0, \mathrm{I}_d)$. Then the first term in r.h.s. is controlled with Theorem 2, and it remains to upper bound $\rho^{\mathrm{Conv}}(\Sigma_\infty^{1/2}\eta, \Sigma_n^{1/2}\zeta)$. Towards this aim, we apply the Gaussian comparison inequality of [11, Theorem 1.1], see also [3]. Assumption A 4 and Lemma 2 imply that

$$\|\Sigma_\infty^{-1/2}\Sigma_n\Sigma_\infty^{-1/2} - \mathrm{I}_d\| \le \|\Sigma_\infty^{-1}\|\|\Sigma_n - \Sigma_\infty\| \le \frac{\mathsf{C}_2}{\lambda_{\min}(\Sigma_\infty)n^{1-\gamma}} \le \frac{1}{2} \, .$$

On the other hand, the following bound holds:

$$\mathrm{Tr}\,(\Sigma_\infty^{-1/2}\Sigma_n\Sigma_\infty^{-1/2} - \mathrm{I}_d)^2 \le \frac{\mathsf{C}_2^2 d}{\lambda_{\min}^2(\Sigma_\infty)n^{2(1-\gamma)}} \, .$$

Hence, applying [11, Theorem 1.1], we get

$$\rho^{\mathrm{Conv}}(\Sigma_n^{1/2}\zeta, \Sigma_\infty^{1/2}\eta) \le \frac{3\mathsf{C}_2}{2\lambda_{\min}(\Sigma_\infty)} \frac{\sqrt{d}}{n^{1-\gamma}} \, ,$$

and it remains to substitute this bound into (5.14).

## 5.3 Proofs of Section 4

### 5.3.1 Preliminary steps for Gaussian approximation under $\mathbb{P}^{\mathsf{b}}$

We first identify the linear ($W^{\mathsf{b}}$) and non-linear ($D^{\mathsf{b}}$) parts of the error decomposition (4.4). We start from the decomposition

$$\theta_k^{\mathsf{b}} - \theta_k = (\mathrm{I} - \alpha_k w_k \mathbf{A}_k)(\theta_{k-1}^{\mathsf{b}} - \theta_{k-1}) - \alpha_k(w_k - 1)\tilde{\varepsilon}_k, \tag{5.15}$$

where we have set

$$\tilde{\varepsilon}_k = \mathbf{A}_k(\theta_{k-1} - \theta^\star) + \varepsilon_k \, . \tag{5.16}$$

To simplify the notation, we omit the bootstrap replication index, which is implicit in the sequel. Expanding the recurrence above till $k = 0$, and using the fact that $\theta_0^{\mathsf{b}} = \theta_0$, we obtain from (5.15) that

$$\theta_k^{\mathsf{b}} - \theta_k = -\sum_{\ell=1}^{k} \alpha_\ell (w_\ell - 1)\Gamma_{\ell+1:k}^{\mathsf{b}}\tilde{\varepsilon}_\ell \, .$$

where we have defined, similarly to (3.4), the product of random matrices

$$\Gamma_{m:k}^{\mathsf{b}} = \prod_{\ell=m}^{k} (\mathrm{I} - \alpha_\ell w_\ell \mathbf{A}_\ell) \, , \; m \le k \, , \quad \text{and } \Gamma_{m:k}^{\mathsf{b}} = \mathrm{I} \, , \quad m > k \, . \tag{5.17}$$

Proceeding as in (3.7), we consider the decomposition $\theta_k^{\mathsf{b}} - \theta_k = J_k^{(\mathsf{b},0)} + H_k^{(\mathsf{b},0)}$, where we have set

$$J_k^{(\mathsf{b},0)} = (\mathrm{I} - \alpha_k \mathbf{A}_k)\, J_{k-1}^{(\mathsf{b},0)} - \alpha_k (w_k - 1)\tilde{\varepsilon_k} \;, \qquad J_0^{(\mathsf{b},0)} = 0 \;, \tag{5.18}$$

$$H_k^{(\mathsf{b},0)} = (\mathrm{I} - \alpha_k w_k \mathbf{A}_k)\, H_{k-1}^{(\mathsf{b},0)} - \alpha_k (w_k - 1)\mathbf{A}_k J_{k-1}^{(\mathsf{b},0)} \;, \qquad H_0^{(\mathsf{b},0)} = 0 \;. \tag{5.19}$$

The idea of the decomposition (D.2)-(D.3) is similar to the one outlined before in (3.7)-(3.8), since the statistic $J_k^{(\mathsf{b},0)}$ is linear when considered under $\mathbb{P}^{\mathsf{b}}$ (that is, when we consider only the randomness due to the bootstrap weights $(w_k)$). With the decomposition (D.2)-(D.3), we get by averaging the iterates

$$\sqrt{n}(\bar{\theta}_n^{\mathsf{b}} - \bar{\theta}_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} J_k^{(\mathsf{b},0)} + \frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} H_k^{(\mathsf{b},0)} \;. \tag{5.20}$$

Unfortunately, the representation (5.20) does not exactly match the one for $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ outlined in (3.19). Indeed, the latter one shows that $\sqrt{n}(\bar{\theta}_n - \theta^\star) = \Sigma_n^{1/2} W + \Sigma_n^{1/2} D$, and $\mathbb{E}[\Sigma_n^{1/2} W W^\top \{\Sigma_n^{1/2}\}^\top] = \Sigma_n$. At the same time, simple calculations show that

$$\mathbb{E}\big[\mathrm{Var}^{\mathsf{b}}\big[\frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} J_{k,0}^{(\mathsf{b},0)}\big]\big] \neq \Sigma_n \;.$$

This issue is due to additional term $\mathbf{A}_\ell(\theta_{\ell-1} - \theta^\star)$ arising in the definition of $\tilde{\varepsilon_k}$ in (5.16). In order to overcome this problem, we further represent $J_k^{(\mathsf{b},0)} = \sum_{i=0}^{2} J_{k,i}^{(\mathsf{b},0)}$, where

$$\begin{aligned} J_{k,0}^{(\mathsf{b},0)} &= -\sum_{\ell=1}^{k} \alpha_\ell (w_\ell - 1) G_{\ell+1:k} \varepsilon_\ell, \quad J_{k,1}^{(\mathsf{b},0)} = -\sum_{\ell=1}^{k} \alpha_\ell (w_\ell - 1)(\Gamma_{\ell+1:k} - G_{\ell+1:k})\varepsilon_\ell, \\ J_{k,2}^{(\mathsf{b},0)} &= -\sum_{\ell=1}^{k} \alpha_\ell (w_\ell - 1)\Gamma_{\ell+1:k} \mathbf{A}_\ell (\theta_{\ell-1} - \theta^\star), \end{aligned} \tag{5.21}$$

It is easily seen that $\sum_{k=1}^{n-1} J_{k,0}^{(\mathsf{b},0)} = \sum_{\ell=1}^{n-1} (w_\ell - 1) Q_\ell \varepsilon_\ell$, where $(Q_\ell)$ is defined in (3.10), moreover,

$$\Sigma_n^{\mathsf{b}} := \mathrm{Var}^{\mathsf{b}}\big[\frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} J_{k,0}^{(\mathsf{b},0)}\big] = \frac{1}{n} \sum_{\ell=1}^{n-1} Q_\ell \varepsilon_\ell \varepsilon_\ell^\top Q_\ell^\top \;, \text{ and } \mathbb{E}[\Sigma_n^{\mathsf{b}}] = \Sigma_n \;.$$

Later we show that $J_{k,i}^{(\mathsf{b},0)}$, $i = 1, 2$ are negligible relative to the leading term $J_{k,0}^{(\mathsf{b},0)}$. Now we rely on the "bootstrap-world" decomposition

$$\sqrt{n}(\bar{\theta}_n^{\mathsf{b}} - \bar{\theta}_n) = (\Sigma_n^{\mathsf{b}})^{1/2} W^{\mathsf{b}} + (\Sigma_n^{\mathsf{b}})^{1/2} D^{\mathsf{b}} \;,$$

where we have set

$$W^{\mathsf{b}} = n^{-1/2} (\Sigma_n^{\mathsf{b}})^{-1/2} \sum_{k=1}^{n-1} J_{k,0}^{(\mathsf{b},0)} =: \sum_{k=1}^{n-1} \xi_k^{\mathsf{b}}, \quad \text{where } \xi_k^{\mathsf{b}} = n^{-1/2} (\Sigma_n^{\mathsf{b}})^{-1/2} (w_k - 1) Q_k \varepsilon_k, \tag{5.22}$$

$$D^{\mathsf{b}} = (\Sigma_n^{\mathsf{b}})^{-1/2} \bigg( \frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} J_{k,1}^{(\mathsf{b},0)} + \frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} J_{k,2}^{(\mathsf{b},0)} + \frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} H_k^{(\mathsf{b},0)} \bigg) \;. \tag{5.23}$$

In this decomposition, $W^{\mathsf{b}}$ is the linear part whereas $D^{\mathsf{b}}$ is the nonlinear part. With these notations and preliminary results, we are in a position to provide the proof of Theorem 4. Some proofs of technical lemmas are postponed to the appendix.

The following decomposition allows us to formalize the structure outlined in the sketch of proof given in Section 4:

$$\sup_{B\in\mathrm{Conv}(\mathbb{R}^d)} |\mathbb{P}^{\mathsf{b}}(\sqrt{n}(\bar{\theta}_n^{\mathsf{b}}-\bar{\theta}_n)\in B)-\mathbb{P}(\sqrt{n}(\bar{\theta}_n-\theta^\star)\in B)| \le T_1+T_2+T_3\ ,$$

where

$$\begin{aligned}
T_1 &:= \sup_{B\in\mathrm{Conv}(\mathbb{R}^d)} \left|\mathbb{P}\big(\sqrt{n}(\bar{\theta}_n-\theta^\star)\in B\big)-\mathbb{P}(\Sigma_n^{1/2}\eta\in B)\right|\ ,\\
T_2 &:= \sup_{B\in\mathrm{Conv}(\mathbb{R}^d)} \left|\mathbb{P}(\Sigma_n^{1/2}\eta\in B)-\mathbb{P}^{\mathsf{b}}((\Sigma_n^{\mathsf{b}})^{1/2}\eta\in B)\right|\ ,\\
T_3 &:= \sup_{B\in\mathrm{Conv}(\mathbb{R}^d)} \left|\mathbb{P}^{\mathsf{b}}(\sqrt{n}(\bar{\theta}_n^{\mathsf{b}}-\bar{\theta}_n)\in B)-\mathbb{P}^{\mathsf{b}}((\Sigma_n^{\mathsf{b}})^{1/2}\eta\in B)\right|\ ,
\end{aligned}$$

and $\eta\sim\mathcal{N}(0,\mathrm{I})$ under $\mathbb{P}$ and $\mathbb{P}^{\mathsf{b}}$. Our next objective is to obtain bounds on these three terms. For the term $T_1$, it suffices to apply Theorem 2. Consider now $T_2$. In this case, we are comparing two centered Gaussian distributions that differ in their covariance matrices. We begin by applying Pinsker's inequality to bound the total variation distance, which itself serves as an upper bound for the convex distance, using [11, Theorem 1.1]:

$$\|\mathcal{N}(0,\Sigma_1)-\mathcal{N}(0,\Sigma_2)\|_{\mathsf{TV}} \le \frac{3\sqrt{d}}{2}\|\Sigma_1^{-1/2}\Sigma_2\Sigma_1^{-1/2}-\mathrm{I}\|\ .$$

Applying the inequality above yields

$$T_2 \le \frac{3\sqrt{d}}{2\,\lambda_{\min}(\Sigma_n)}\,\|\Sigma_n^{\mathsf{b}}-\Sigma_n\|.$$

Bounding $T_2$ therefore boils down to obtain a high-probability bound for $\|\Sigma_n^{\mathsf{b}}-\Sigma_n\|$. Such bound follow from the matrix Bernstein inequalities, developed in [51]. Detailed argument is given below.

The main technical challenge arises in controlling the term $T_3$, which requires decomposing the quantity $H_k^{(\mathsf{b},0)}$ in a manner analogous to the decomposition in (5.1). It is worth noting, however, that once again, the quantities introduced and the method used to derive the bounds are markedly different than the ones used in Section 5.1. Along the lines of (5.2), we expand $H_k^{(\mathsf{b},0)}$ as follows:

$$\theta_k^{\mathsf{b}}-\theta_k = J_k^{\mathsf{b},0}+\sum_{j=1}^{L} J_k^{\mathsf{b},j}+H_k^{\mathsf{b},L}, \tag{5.24}$$

where

$$\begin{aligned}
J_k^{(\mathsf{b},0)} &= -\sum_{\ell=1}^{k}\alpha_\ell(w_\ell-1)\Gamma_{\ell+1:k}\tilde{\varepsilon}_\ell, \quad J_k^{(\mathsf{b},j)} = -\sum_{\ell=1}^{k}\alpha_\ell(w_\ell-1)\Gamma_{\ell+1:k}A_\ell J_{\ell-1}^{(\mathsf{b},j-1)}, \quad j\in[1,L]\\
H_k^{(\mathsf{b},L)} &= -\sum_{\ell=n+1}^{k}\alpha_\ell(w_\ell-1)\Gamma_{\ell+1:k}^{\mathsf{b}}A_\ell J_{\ell-1}^{(\mathsf{b},L)}\ ,
\end{aligned} \tag{5.25}$$

Similar to Section 5.1, we will establish upper bounds on the $p$-th moments under $\mathbb{P}^{\mathsf{b}}$ of $J_k^{(\mathsf{b},j)}$, $H_k^{(\mathsf{b},j)}$, $j \in [0; L]$ and $\tilde{\varepsilon}_\ell$. However, the proofs differ significantly from the previous case, which relied heavily on the exponential stability of products of random matrices $\Gamma_{m:k}$ (see Lemma 1). The proof strategy consists of two steps. First, we define certain events in the 'original world' under which the various quantities of interest can be controlled. Second, we show that these events occur with high probability—specifically, of order $1 - \iota/n$, for an appropriately chosen $\iota > 0$. We define first:

$$
\Omega_1 = \bigcap_{k=1}^{n-1} \big\{ \|\theta_k - \theta^\star\| \le g(k, \|\theta_0 - \theta^\star\|, n) \ \big\} \ ,
$$

where we have set

$$
g(k,t,n) = \sqrt{\kappa_Q} \mathrm{e}^2 t \prod_{\ell=1}^{k} (1 - \frac{a}{2}\alpha_\ell) + 2\mathrm{e} \log(5n) \mathsf{C}_2 \sqrt{\alpha_k} \ .
$$

Applying Proposition 2 with for $2 \le p \le \log(5n^2)$ and then Lemma 24, we get that for every fixed $k \in [1; n-1]$,

$$
\mathbb{P}\left(\|\theta_k - \theta^\star\| \ge g(k, \|\theta_0 - \theta^\star\|, n)\right) \le \frac{1}{5n^2} \ .
$$

By the union bound, we obtain that $\mathbb{P}(\Omega_1) \ge 1 - 1/(5n)$. We may show that

**Lemma 9.** *Under the assumptions of Theorem 4, on the event $\Omega_1$, it holds for any $\ell \ge 1$ that*

$$
\|\tilde{\varepsilon}_\ell\| \le \mathsf{C}_9, \ \textit{where } \mathsf{C}_9 = \|\varepsilon\|_\infty + 2\mathrm{e}\, \mathrm{C}_{\mathbf{A}}\, \mathsf{C}_2 + \sqrt{\kappa_Q} \mathrm{e}^3\, \mathrm{C}_{\mathbf{A}}\, \|\theta_0 - \theta^\star\| \ .
$$

Similarly, we introduce the following event

$$
\Omega_2 = \bigcap_{1 \le m \le k \le n} \big\{ \|\Gamma_{m:k}\| \le \mathsf{C}_1 \prod_{j=m}^{k} \big(1 - \frac{a\alpha_j}{2}\big) \big\} \ .
$$

Using the exponential stability of $\Gamma_{m:k}$ (see Lemma 1) with $p = \log(5n^3)$ and Lemma 24, we get that with probability at least $1 - 1/(5n^3)$,

$$
\|\Gamma_{m:k}\| \le \mathsf{C}_1 \prod_{\ell=m}^{k} \big(1 - \frac{a\alpha_\ell}{2}\big) \le \mathsf{C}_1 \exp\big\{-\frac{a}{2} \sum_{\ell=m}^{k} \alpha_\ell\big\} \quad \text{where } \mathsf{C}_1 = \sqrt{\kappa_Q} \mathrm{e}^2 \ .
$$

By the union bound, we get $\mathbb{P}(\Omega_2) \ge 1 - 1/(5n)$. It is also required to consider

$$
\Omega_3 = \bigcap_{\ell=1}^{n} \big\{ \|\alpha_\ell \sum_{k=\ell}^{n-1} (\Gamma_{\ell+1:k} - G_{\ell+1:k}) \varepsilon_\ell\| \le \mathsf{C}_{17} \sqrt{\alpha_\ell} \log(5n) \big\} \ ,
$$

Here again, we may show that $\mathbb{P}(\Omega_3) \ge 1 - 1/(5n)$. A detailed proof is given in Lemma 13. On the event $\bigcap_{i=1}^{3} \Omega_i$, we derive below bounds for $J_{k,i}^{\mathsf{b},0}$, $i = 1, 2$, defined in (5.21).

**Lemma 10.** *Under the assumptions of Theorem 4, on the event $\Omega_3$, it holds that*

$$
\big\{\mathbb{E}^{\mathsf{b}}\big[\|n^{-1/2} \textstyle\sum_{k=1}^{n-1} J_{k,1}^{\mathsf{b},0}\|^2\big]\big\}^{1/2} \le \mathsf{C}_{10} \frac{\log(5n)}{n^{\gamma/2}} \ , \ \textit{where } \mathsf{C}_{10} = \frac{\|\varepsilon\|_\infty \sqrt{c_0} \mathsf{C}_{17}}{\sqrt{1-\gamma}} \ .
$$

**Lemma 11.** *Under the assumptions of Theorem 4, on the event $\Omega_1 \cap \Omega_2$, it holds*

$$\{\mathbb{E}^{\mathsf{b}}\big[\|n^{-1/2}\sum_{k=1}^{n-1} J_{k,2}^{\mathsf{b},0}\|^2\big]\}^{1/2} \le \frac{\mathsf{C}_{11,1}\log(5n)}{n^{\gamma/2}} + \frac{\mathsf{C}_{11,2}(1+k_0)^{\gamma/2}\|\theta_0-\theta^\star\|}{\sqrt{n}} ,$$

*where we have defined*

$$\mathsf{C}_{11,1} = \sqrt{2}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_1 \mathrm{e}\big(c_0 + \frac{2}{a(1-\gamma)}\big)\mathsf{C}_2 \frac{\sqrt{c_0}}{\sqrt{1-\gamma}} , \quad \mathsf{C}_{11,2} = 2\sqrt{2}c_0^{-1/2}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_1 \kappa_Q^{1/2}\mathrm{e}^2\big(c_0 + \frac{2}{a(1-\gamma)}\big)^{3/2} .$$

We introduce an additional event, which is essential for establishing exponential stability of the *bootstrap world* random matrix product $\Gamma^{\mathsf{b}}_{m:k}$ defined in (5.17).

$$\Omega_4 = \bigcap_{h=1}^{n}\bigcap_{m=0}^{n-h-1}\{\|\sum_{\ell=m+1}^{m+h}\alpha_\ell(\mathbf{A}_\ell - \bar{\mathbf{A}})\| \le 2\,\mathrm{C}_{\mathbf{A}}\{\sum_{\ell=m+1}^{m+h}\alpha_\ell^2\}^{1/2}\log\big(10n^3 d\big)\} , \tag{5.26}$$

Here again, we may show that $\mathbb{P}(\Omega_4) \ge 1 - 1/(5n)$. The proof is a straightforward application of matrix Bernstein inequality; details are given in Lemma 14.

Under the event $\Omega_1 \cap \Omega_2 \cap \Omega_4$, we can provide bounds to the terms appearing in the expansion of $H_j^{(\mathsf{b},0)}$, given in (5.25).

**Lemma 12.** *Under the assumptions of Theorem 4, on the event $\Omega_1 \cap \Omega_2 \cap \Omega_4$, for $j, L \in \{0,1,2\}$ it holds that*

$$\{\mathbb{E}^{\mathsf{b}}[\|J_k^{\mathsf{b},j}\|^2]\}^{1/2} \le \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},j)}\alpha_k^{(j+1)/2} , \quad \{\mathbb{E}^{\mathsf{b}}[\|H_k^{\mathsf{b},L}\|^2]\}^{1/2} \le \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},L)}\alpha_k^{(L+1)/2} .$$

*where $\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},j)}$, $\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},L)}$ satisfy the recurrence*

$$\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},0)} = \frac{2\sqrt{3}\mathsf{C}_9\mathsf{C}_1}{\sqrt{a}} , \; \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},j)} = \frac{2\sqrt{3}}{\sqrt{a}}\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},j-1)}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_1 , \; \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},L)} = \frac{4\sqrt{3}}{\sqrt{a}}\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},L)}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_3 .$$

We may now proceed to the proof of the theorem.

*Proof of Theorem 4.* We start with $T_2$. We first show that the bootstrap word covariance $\Sigma_n^{\mathsf{b}}$ approximates $\Sigma_n$. More precisely, set $\mathsf{C}_{15} = 2\|\varepsilon\|_\infty^2\mathsf{C}_4^2$ and consider the event

$$\Omega_5 = \{\|\Sigma_n^{\mathsf{b}} - \Sigma_n\| \le \frac{\sqrt{2}\mathsf{C}_{15}\sqrt{\log(10dn)}}{\sqrt{n}} + \frac{\mathsf{C}_{15}\log(10dn)}{3n}\} ,$$

It is shown in Lemma 15 that $\mathbb{P}(\Omega_5) \ge 1 - 1/5n$ and in Lemma 16, that $\lambda_{\min}(\Sigma_n) \ge \lambda_{\min}(\Sigma_\infty)/2$. Combining these two results, we then get that on the event $\Omega_5$,

$$T_2 \le \frac{3\sqrt{d}}{2\lambda_{\min}(\Sigma_n)}\left(\frac{\sqrt{2}\mathsf{C}_{15}\sqrt{\log(10dn)}}{\sqrt{n}} + \frac{\mathsf{C}_{15}\log(10dn)}{3n}\right)$$

Finally, we consider $T_3$. As emphasized in the preliminaries of the proof, we again invoke the approach of [44], where $W^b$ (defined in (5.22)) plays the role of the linear term and $D^b$ (defined in (5.23)) that of the nonlinear remainder. [44, Theorem 2.1] shows that

$$T_3 \le 259 d^{1/2}\Upsilon^{\mathsf{b}} + 2\mathbb{E}^{\mathsf{b}}[\|W^{\mathsf{b}}\|\|D^{\mathsf{b}}\|] + 2\sum_{\ell=1}^{n}\mathbb{E}^{\mathsf{b}}[\|\xi_\ell^{\mathsf{b}}\|\|D^{\mathsf{b}} - D^{\mathsf{b},\ell}\|], \tag{5.27}$$

where $\Upsilon^{\mathsf{b}} = \sum_{i=1}^{n-1} \mathbb{E}[\|\xi_i^{\mathsf{b}}\|^3]$, and $\xi_\ell^{\mathsf{b}}$ is defined in (5.22). It follows Lemma 4 and Lemma 16 that

$$\|\xi_\ell^{\mathsf{b}}\| \leq n^{-1/2}\{\lambda_{\min}(\Sigma_n^{\mathsf{b}})\}^{-1/2}|w_\ell - 1|\|Q_\ell\|\|\varepsilon\|_\infty \leq |w_\ell - 1|\|\varepsilon\|_\infty \mathsf{C}_4/(\sqrt{\mathsf{C}_{16}}\sqrt{n}).$$

Since by construction $\mathbb{E}^{\mathsf{b}}[\|W^{\mathsf{b}}\|^2] = \sum_{\ell=1}^{n} \mathbb{E}^{\mathsf{b}}[\|\xi_\ell^{\mathsf{b}}\|^2] = d$ and $\mathbb{E}^{\mathsf{b}}[|w_\ell - 1|^3] = m_3$ for all $\ell$, we get:

$$\Upsilon^{\mathsf{b}} \leq \frac{m_3\|\varepsilon\|_\infty \mathsf{C}_4}{\sqrt{n}\sqrt{\mathsf{C}_{16}}} \sum_{i=1}^{n-1} \mathbb{E}^{\mathsf{b}}[\|\xi_\ell^{\mathsf{b}}\|^2] \leq \frac{d}{\sqrt{n}} \frac{m_3\|\varepsilon\|_\infty \mathsf{C}_4}{\sqrt{\mathsf{C}_{16}}} \; .$$

To proceed with the second term in (5.27), first note that

$$\mathbb{E}^{\mathsf{b}}\big[\|W^{\mathsf{b}}\|\|D^{\mathsf{b}}\|\big] \leq \{\mathbb{E}^{\mathsf{b}}\big[\|W^{\mathsf{b}}\|^2\big]\}^{1/2}\{\mathbb{E}^{\mathsf{b}}\big[\|D^{\mathsf{b}}\|^2\big]\}^{1/2} = d^{1/2}\{\mathbb{E}^{\mathsf{b}}\big[\|D^{\mathsf{b}}\|^2\big]\}^{1/2} \; .$$

We use the expression of $D^{\mathsf{b}}$ given in (5.23) and further expand $H_k^{(\mathsf{b},0)} = \sum_{j=1}^{2} J_k^{(\mathsf{b},j)} + H_k^{(\mathsf{b},2)}$, using (5.24) with $L = 2$. Lemmas 10 and 11 provide bounds for $\big\{\mathbb{E}^{\mathsf{b}}\big[\|n^{-1/2}\sum_{k=1}^{n-1} J_{k,j}^{\mathsf{b},0}\|^2\big]\big\}^{1/2}$, $j = 1, 2$. Lemma 18 give the bound for $\big\{\mathbb{E}^{\mathsf{b}}\big[\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_k^{\mathsf{b},1}\|^2\big]\big\}^{1/2}$. Finally, Lemma 12 show that

$$\big\{\mathbb{E}^{\mathsf{b}}\big[\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_k^{\mathsf{b},2}\|^2\big]\big\}^{1/2} \leq \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},2)}\alpha_k^{3/2} \leq \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},2)}\varphi_n \; ,$$
$$\big\{\mathbb{E}^{\mathsf{b}}\big[\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} H_k^{\mathsf{b},2}\|^2\big]\big\}^{1/2} \leq \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},2)}\alpha_k^{3/2} \leq \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},2)}\varphi_n \; .$$

By combining the inequalities above, we obtain

$$\mathbb{E}^{\mathsf{b}}\big[\|W^{\mathsf{b}}\|\|D^{\mathsf{b}}\|\big] \leq \frac{d^{1/2}\log(5n)}{n^{\gamma/2}\mathsf{C}_{16}^{1/2}}(\mathsf{C}_{10} + \mathsf{C}_{11,1} + \mathsf{C}_{18}) + \frac{d^{1/2}(1+k_0)^{\gamma/2}\|\theta_0 - \theta^\star\|\mathsf{C}_{11,2}}{n^{1/2}\mathsf{C}_{16}^{1/2}} + \frac{d^{1/2}}{\mathsf{C}_{16}^{1/2}}(\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},2)} + \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},2)})\varphi_n \; .$$

Cauchy-Schwarz inequality and Lemma 20 imply the bound for the third term in (5.27):

$$\mathbb{E}^{\mathsf{b}}[\|\xi_i^{\mathsf{b}}\|\|D^{\mathsf{b}} - D^{\mathsf{b},i}\|] \leq \{\mathbb{E}^{\mathsf{b}}[\|\xi_i^{\mathsf{b}}\|^2]\}^{1/2}\{\mathbb{E}^{\mathsf{b}}[\|D^{\mathsf{b}} - D^{\mathsf{b},i}\|^2]\}^{1/2} \leq \frac{1}{n}\frac{1}{\sqrt{\mathsf{C}_{16}}}\mathsf{C}_4\|\varepsilon\|_\infty\sqrt{\alpha_i}\log(5n)\mathsf{C}_{20} \; .$$

Thus, it holds that

$$\sum_{i=1}^{n-1}\mathbb{E}^{\mathsf{b}}[\|\xi_i^{\mathsf{b}}\|\|D^{\mathsf{b}} - D^{\mathsf{b},i}\|] \leq n^{-\gamma/2}\log(n)\frac{1}{\sqrt{\mathsf{C}_{16}}}\mathsf{C}_4\|\varepsilon\|_\infty\frac{\sqrt{c_0}}{1-\gamma/2}\mathsf{C}_{20,1} + n^{-\gamma}\frac{\|\theta_0 - \theta^\star\|}{\sqrt{\mathsf{C}_{16}}}\mathsf{C}_4\|\varepsilon\|_\infty\frac{c_0}{1-\gamma/2}\mathsf{C}_{20,2} \; .$$

By collecting the inequalities derived above, we ultimately obtain the final bound.

$$T_3 \leq \mathsf{C}_{4,1}\frac{\log(5n)}{n^{\gamma/2}} + \mathsf{C}_{4,2}\frac{(1+k_0)^{\gamma/2}}{n^{\gamma/2}} + \mathsf{C}_{4,3}\varphi_n + \mathsf{C}_{4,4}\frac{1}{\sqrt{n}} + \mathsf{C}_{4,5}\frac{1}{n^\gamma} \; ,$$

where the constants $\mathsf{C}_{4,1}$, $\mathsf{C}_{4,2}$, $\mathsf{C}_{4,3}$, are given by:

$$\mathsf{C}_{4,1} = 2d^{1/2}\mathsf{C}_{16}^{-1/2}(\mathsf{C}_{10} + \mathsf{C}_{11,1} + \mathsf{C}_{18}) + 2\mathsf{C}_{16}^{-1/2}\mathsf{C}_{4}\|\varepsilon\|_\infty \frac{\sqrt{c_0}}{1-\gamma/2}\mathsf{C}_{20} \;, \quad \mathsf{C}_{4,2} = 2d^{1/2}\mathsf{C}_{16}^{-1/2}\mathsf{C}_{11,2} \;,$$

$$\mathsf{C}_{4,3} = 2d^{1/2}\mathsf{C}_{16}^{-1/2}(\mathsf{C}_{12,1}^{(\mathrm{b},\mathrm{J},2)} + \mathsf{C}_{12,1}^{(\mathrm{b},\mathrm{H},2)}) \;, \; \mathsf{C}_{4,4} = 259 m_3 \mathsf{C}_{16}^{-1/2} d^{3/2}\|\varepsilon\|_\infty \mathsf{C}_{4} \;, \; \mathsf{C}_{4,5} = \frac{2}{\sqrt{\mathsf{C}_{16}}}\mathsf{C}_{4}\|\varepsilon\|_\infty \frac{c_0}{1-\gamma/2}\mathsf{C}_{20,2} \;.$$

Rearranging the terms above yields the statement with the expressions $\Delta_{4,1}$ to $\Delta_{4,4}$ and $\mathsf{C}_4$ given by

$$\begin{aligned}
\mathsf{C}_{4} &= \mathsf{C}_{2,4} + \frac{2}{\sqrt{\mathsf{C}_{16}}}\mathsf{C}_{4}\|\varepsilon\|_\infty \frac{c_0}{1-\gamma/2}\mathsf{C}_{20,2} \;, \\
\Delta_{4,1} &= \mathsf{C}_{2,1} + \frac{3\sqrt{2}\sqrt{d}\mathsf{C}_{15}\sqrt{\log(10dn)}}{2\mathsf{C}_{6}} + \mathsf{C}_{4,4} \;, \\
\Delta_{4,2} &= \mathsf{C}_{2,2} + \mathsf{C}_{4,1}\log(5n) + \mathsf{C}_{4,2}(1+k_0)^{\gamma/2} \;, \\
\Delta_{4,3} &= \mathsf{C}_{2,3} + \mathsf{C}_{4,3} \;, \\
\Delta_{4,4} &= \frac{3\sqrt{d}\mathsf{C}_{15}\log(10dn)}{6\mathsf{C}_{6}} \;.
\end{aligned} \tag{5.28}$$

□

**Lemma 13.** *Under the assumptions of Theorem 4,* $\mathbb{P}(\Omega_3) \geq 1 - 1/(5n)$

*Proof.* The proof follows from Lemma 17 and union bound. □

**Lemma 14.** *Under the assumptions of Theorem 4,* $\mathbb{P}(\Omega_4) \geq 1 - 1/(5n)$

*Proof.* To show that $\mathbb{P}(\Omega_4) \geq 1 - 1/(5n)$, we fix $h \in [1, n]$ and $m \in [0, n-h-1]$, and define the random variable

$$T_n = \|\sum_{\ell=m+1}^{m+h} \alpha_\ell(\mathbf{A}_\ell - \bar{\mathbf{A}})\|.$$

We first control its variance. By standard matrix inequalities, we have:

$$\max\left( \|\sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \,\mathbb{E}\big[(\mathbf{A}_\ell - \bar{\mathbf{A}})(\mathbf{A}_\ell - \bar{\mathbf{A}})^\top\big]\|, \; \|\sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \,\mathbb{E}\big[(\mathbf{A}_\ell - \bar{\mathbf{A}})^\top(\mathbf{A}_\ell - \bar{\mathbf{A}})\big]\| \right) \leq \mathrm{C}_{\mathbf{A}}^2 \sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \;,$$

and note that for each $\ell$, the operator norm satisfies $\|(\mathbf{A}_\ell - \bar{\mathbf{A}})(\mathbf{A}_\ell - \bar{\mathbf{A}})^\top\| \leq \mathrm{C}_{\mathbf{A}}^2$. Applying the matrix Bernstein inequality [51], we obtain that with probability at least $1 - \delta/n^2$,

$$T_n \leq \mathrm{C}_{\mathbf{A}} \sqrt{2\sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \log\left(\frac{2n^2 d}{\delta}\right)} + \frac{\alpha_{m+1}\mathrm{C}_{\mathbf{A}}}{3}\log\left(\frac{2n^2 d}{\delta}\right) \;.$$

The remainder of the proof follows by setting $\delta = 1/(5n)$ and applying a union bound over all valid pairs $(h, m)$, along with the inequality $\|B\|_Q^2 \leq \kappa_Q \|B\|^2$, which holds for any matrix $B \in \mathbb{R}^{d\times d}$. □

**Lemma 15.** *Under the assumption of Theorem 4,* $\mathbb{P}(\Omega_5) \geq 1 - 1/(5n)$*, where* $\mathsf{C}_{15} = 2\|\varepsilon\|_\infty^2 \mathsf{C}_4^2$.

*Proof.* Introduce a random matrix $U_\ell = Q_\ell(\varepsilon_\ell \varepsilon_\ell^\top - \Sigma_\varepsilon)Q_\ell^\top$. Note that $\mathbb{E}[U_\ell] = 0$ and $\Sigma_n^{\mathsf{b}} - \Sigma_n = \frac{1}{n}\sum_{\ell=1}^{n-1} U_\ell$. Moreover, Lemma 4, A 1, and A 2 imply that

$$\|U_\ell\| \le (\|\varepsilon\|_\infty^2 + \|\Sigma_\varepsilon\|)\mathsf{C}_4^2 \le 2\|\varepsilon\|_\infty^2 \mathsf{C}_4^2 = \mathsf{C}_{15} \;.$$

Hence, the matrix Bernstein inequality [51, Theorem 6.1.1] implies

$$\mathbb{P}\left(\|\Sigma_n^{\mathsf{b}} - \Sigma_n\| \ge t\right) = \mathbb{P}(\|\sum_{\ell=1}^{n-1} U_\ell\| \ge nt) \le 2d \exp\left(-\frac{nt^2}{2\mathsf{C}_{15}^2 + 2\mathsf{C}_{15} t/3}\right) \;.$$

Equivalently (see e.g. [7, Theorem 2.10]), with probability at least $1-\delta$, it holds that

$$\|\Sigma_n^{\mathsf{b}} - \Sigma_n\| \le \frac{\sqrt{2}\mathsf{C}_{15}\sqrt{\log(2d/\delta)}}{\sqrt{n}} + \frac{\mathsf{C}_{15}\log(2d/\delta)}{3n} \;.$$

To complete the proof it remains to take $\delta = 1/(5n)$. $\square$

**Lemma 16.** *Under the assumption of Theorem 4, on the event $\Omega_5$ it holds that*

$$\lambda_{\min}(\Sigma_n^{\mathsf{b}}) \ge \mathsf{C}_{16} \;, \quad \textit{where } \mathsf{C}_{16} = \frac{\lambda_{\min}(\Sigma_\infty)}{4} \;.$$

*Proof.* Using Lidskii's inequality for Hermitian matrices, we get that

$$\lambda_{\min}(\Sigma_n^{\mathsf{b}}) \ge \lambda_{\min}(\Sigma_n) + \lambda_{\min}(\Sigma_n^{\mathsf{b}} - \Sigma_n) \ge \lambda_{\min}(\Sigma_n) - \|\Sigma_n^{\mathsf{b}} - \Sigma_n\|.$$

Hence, on the event $\Omega_5$, we get that

$$\lambda_{\min}(\Sigma_n^{\mathsf{b}}) \ge \lambda_{\min}(\Sigma_n) - \left(\frac{\sqrt{2}\mathsf{C}_{15}\sqrt{\log(10dn)}}{\sqrt{n}} + \frac{\mathsf{C}_{15}\log(10dn)}{3n}\right).$$

Under A 5, the sample size $n$ is chosen large enough so that

$$\lambda_{\min}(\Sigma_n^{\mathsf{b}}) \ge \lambda_{\min}(\Sigma_n) - \lambda_{\min}(\Sigma_\infty)/4.$$

From (3.21), using again Lidskii's inequality this time with $\Sigma_n$ and $\Sigma_\infty$, we know that under A 3, $\lambda_{\min}(\Sigma_n) \ge \lambda_{\min}(\Sigma_\infty)/2$. The proof follows. $\square$

**Lemma 17.** *Under the assumptions of Theorem 4, For each $\ell \in \{1, \ldots, n-1\}$, it holds that*

$$\mathbb{P}\left(\|\alpha_\ell \sum_{k=\ell}^{n-1}(\Gamma_{\ell+1:k} - G_{\ell+1:k})\varepsilon_\ell\| \ge \log(5n)\mathsf{C}_{17}\sqrt{\alpha_\ell}\right) \le \frac{1}{5n^2} \;.$$

*where we have defined*

$$\mathsf{C}_{17} = 2(\sqrt{8}/\sqrt{7})\mathrm{e}^2\, \mathrm{C}_{\mathbf{A}}\, \|\varepsilon\|_\infty \mathsf{C}_4 \kappa_Q^{1/2}\left(c_0 + \frac{2}{a(1-\gamma)}\right)^{1/2} \;.$$

**Lemma 18.** *Under the assumptions of Theorem 4, conditionally on the event $\Omega_0$, it holds*

$$\big\{\mathbb{E}^{\mathsf{b}}\big[\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_k^{\mathsf{b},1}\|^2\big]\big\}^{1/2} \leq \frac{\mathsf{C}_{18}}{n^{\gamma/2}} ,$$

*where the constant $\mathsf{C}_{18}$ is defined as follows*

$$\mathsf{C}_{18} = \frac{2\sqrt{3}}{\sqrt{a}}\,\mathrm{C}_{\mathbf{A}}\big(c_0 + \frac{2}{a(1-\gamma)}\big)\sqrt{\frac{c_0}{1-\gamma}}\mathsf{C}_1\mathsf{C}_9 .$$

Let $w_i'$ be a copy of $w_i$ independent from $w_1, \ldots, w_{n-1}$. Introduce the following notation for $\ell \leq m$:

$$\begin{aligned}
\Gamma_{\ell:m}^{\mathsf{b},i} &= \begin{cases} \Gamma_{\ell:m}^{\mathsf{b}}, & \text{if } i \notin [\ell, m], \\ \Gamma_{\ell:m}^{\mathsf{b}}(w_\ell, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_m), & \text{if } i \in [\ell, m] \end{cases} \\
J_{k,m}^{(\mathsf{b},\ell,i)} &= \begin{cases} J_{k,m}^{(\mathsf{b},\ell)}, & \text{if } k < i, \\ J_{k,m}^{(\mathsf{b},\ell)}(w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k), & \text{if } k \geq i \end{cases} \\
H_k^{(\mathsf{b},\ell,i)} &= \begin{cases} H_k^{(\mathsf{b},\ell)}, & \text{if } k < i, \\ H_k^{(\mathsf{b},\ell)}(w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k), & \text{if } k \geq i \end{cases} \\
D^{\mathsf{b},i} &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_{k,1}^{(\mathsf{b},0,i)} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_{k,2}^{(\mathsf{b},0,i)} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} H_k^{(\mathsf{b},0,i)}
\end{aligned}$$

For simplicity we introduce the following constants:

$$\mathsf{C}_{5.29}^{(1)} = c_0 + \frac{16}{a(1-\gamma)} , \ \mathsf{C}_{5.29}^{(2)} = \frac{1}{1-ac_0}\big(1 + \frac{16}{ac_0(1-\gamma)}\big) . \tag{5.29}$$

**Lemma 19.** *Under the assumptions of Theorem 4, let $w_i'$ be a copy of $w_i$ independent from $w_1, w_2, \ldots, w_{n-1}$. Then on the event $\Omega_0$ it holds that*

$$\big\{\mathbb{E}^{\mathsf{b}}\big[\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} H_k^{(\mathsf{b},0)} - \frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} H_k^{(\mathsf{b},0,i)}\|^2\big]\big\}^{1/2} \leq n^{-1/2}\sqrt{\alpha_i}\mathsf{C}_{19}$$

*where the constant $\mathsf{C}_{19}$ is given by*

$$\mathsf{C}_{19} = \mathrm{C}_{\mathbf{A}}\,\mathsf{C}_3(\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},0)} + \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},0)})\mathsf{C}_{5.29}^{(1)} + \mathrm{C}_{\mathbf{A}}(\mathsf{C}_{5.29}^{(1)})^{3/2}\mathsf{C}_3\mathsf{C}_1\mathsf{C}_9 .$$

**Lemma 20.** *Under the assumptions of Theorem 4, conditionally on $\Omega_0$, it holds that*

$$\{\mathbb{E}^{\mathsf{b}}[\|D^{\mathsf{b}} - D^{\mathsf{b},i}\|^2]\}^{1/2} \leq \frac{1}{\sqrt{n}}\sqrt{\alpha_i}\log(5n)\mathsf{C}_{20,1} + \mathsf{C}_{20,2}\frac{\alpha_i\|\theta_0 - \theta^\star\|}{\sqrt{n}} ,$$

*where the constant $\mathsf{C}_{20}$ is given by*

$$\mathsf{C}_{20,1} = \frac{2\sqrt{2}\mathsf{C}_{17}}{\sqrt{\mathsf{C}_{16}}} + \frac{\mathsf{C}_{19}}{\sqrt{\mathsf{C}_{16}}} + \frac{2\sqrt{2}\mathrm{e}\mathsf{C}_1\,\mathrm{C}_{\mathbf{A}}\,\kappa_Q^{1/2}\|\varepsilon\|_\infty\mathsf{C}_{5.29}^{(1)}}{\sqrt{\mathsf{C}_{16}}} , \ \mathsf{C}_{20,2} = \frac{\sqrt{2}\kappa_Q^{1/2}\mathrm{e}^2\mathsf{C}_1\,\mathrm{C}_{\mathbf{A}}\,\mathsf{C}_{5.29}^{(2)}}{\sqrt{\mathsf{C}_{16}}(1-a/2)^2} .$$

### 5.4 Proofs on products of random matrices

We first introduce some notations and definitions. For a matrix $B \in \mathbb{R}^{d\times d}$ we denote by $(\sigma_\ell(B))_{\ell=1}^d$ its singular values. For $q \geq 1$, the Shatten $q$-norm of $B$ is denoted by $\|B\|_q = \{\sum_{\ell=1}^d \sigma_\ell^q(B)\}^{1/q}$. For $q,p \geq 1$ and a random matrix $\mathbf{X}$ we write $\|\mathbf{X}\|_{q,p} = \{\mathbb{E}[\|\mathbf{X}\|_q^p]\}^{1/p}$. We use a result of [22], sharpened in [12].

**Lemma 21** (Proposition 15 in [12])**.** *Let $\{\mathbf{Y}_\ell\}_{\ell\in\mathbb{N}}$ be an independent sequence and $P$ be a positive definite matrix. Assume that for each $\ell \in \mathbb{N}$ there exist $m_\ell \in (0,1)$ and $\sigma_\ell > 0$ such that $\|\mathbb{E}[\mathbf{Y}_\ell]\|_P^2 \leq 1 - m_\ell$ and $\|\mathbf{Y}_\ell - \mathbb{E}[\mathbf{Y}_\ell]\|_P \leq \sigma_\ell$ almost surely. Define $\mathbf{Z}_k = \prod_{\ell=0}^k \mathbf{Y}_\ell = \mathbf{Y}_k\mathbf{Z}_{k-1}$, for $k \geq 1$ and starting from $\mathbf{Z}_0$. Then, for any $2 \leq q \leq p$ and $k \geq 1$,*

$$\|\mathbf{Z}_k\|_{p,q}^2 \leq \kappa_P \prod_{\ell=1}^k (1 - m_\ell + (p-1)\sigma_\ell^2)\|P^{1/2}\mathbf{Z}_0 P^{-1/2}\|_{p,q}^2 \;, \tag{5.30}$$

*where $\kappa_P = \lambda_{\mathsf{min}}^{-1}(P)\lambda_{\mathsf{max}}(P)$.*

Now we aim to bound $\Gamma_{m:k}$ defined in (3.4) using Lemma 21. Set $\mathbf{Y}_\ell = \mathrm{I} - \alpha_\ell \mathbf{A}_\ell, \ell \geq 1$, and $\mathbf{Y}_0 = \mathrm{I}$. Applying the bound (3.3), we get $\|\mathbb{E}[\mathbf{Y}_\ell]\|_Q^2 = \|\mathrm{I} - \alpha_\ell \bar{\mathbf{A}}\|_Q^2 \leq 1 - a\alpha_\ell$. Further, assumption A 2 implies that almost surely,

$$\|\mathbf{Y}_\ell - \mathbb{E}[\mathbf{Y}_\ell]\|_Q = \alpha_\ell \|\mathbf{A}_\ell - \bar{\mathbf{A}}\|_Q \leq \alpha_\ell \sqrt{\kappa_Q}\,\mathrm{C}_{\mathbf{A}} = b_Q \alpha_\ell \;.$$

Therefore, (5.30) holds with $m_\ell = a\alpha_\ell$ and $\sigma_\ell = b_Q\alpha_\ell$. As $\|\mathrm{I}\|_p = d^{1/p}$, we obtain the following corollary.

**Corollary 2.** *Assume A 1 and A 2. Then, for any $\alpha_\ell \in [0, \alpha_\infty]$, $2 \leq q \leq p$, and $1 \leq m \leq k$, it holds*

$$\mathbb{E}^{1/q}\left[\|\Gamma_{m:k}\|^q\right] \leq \|\Gamma_{m:k}\|_{p,q} \leq \sqrt{\kappa_Q} d^{1/p} \prod_{\ell=m}^k (1 - a\alpha_\ell + (p-1)b_Q^2\alpha_\ell^2) \;,$$

*where $\alpha_\infty$ was defined in (3.2), and $b_Q = \sqrt{\kappa_Q}\,\mathrm{C}_{\mathbf{A}}$.*

**Proposition 3.** *Assume A 1, A 2, A 3($\log(5n^2) \vee \log d$), A 4, A 5. Then on the set $\Omega_4$ defined in (5.26), it holds for any $0 \leq m \leq k \leq n$, that*

$$\big\{\mathbb{E}^{\mathsf{b}}[\|\Gamma_{m+1:k}^{\mathsf{b}}\|^2]\big\}^{1/2} \leq \mathsf{C}_3 \exp\big\{-\frac{a}{4}\sum_{\ell=m+1}^k \alpha_\ell\big\} \;,\; \mathsf{C}_3 = \kappa_Q^{3/2}\mathrm{e}^{9/8} \;.$$

**Lemma 22.** *Assume A 1, A 2, A 3($\log(5n^2) \vee \log d$), A 4, A 5. Then on the event $\Omega_4$, defined in (5.26), with $h = h(n)$ defined in (4.3), it holds for any $m \in [0; n-h-1]$, that*

$$\big\{\mathbb{E}^{\mathsf{b}}[\|\Gamma_{m+1:m+h}^{\mathsf{b}}\|_Q^2]\big\}^{1/2} \leq \exp\big\{-\frac{a}{4}\sum_{\ell=m+1}^{m+h} \alpha_\ell\big\} \;,$$

*where $\Gamma_{m+1:m+h}^{\mathsf{b}}$ is defined in (5.17).*

### 5.5 Proof of Lemma 2

Before proceeding with the actual proof, we introduce a decomposition of $\Sigma_n - \Sigma_\infty$ that forms the backbone of the argument. This decomposition is built upon non-trivial identities involving both $Q_t - \bar{\mathbf{A}}^{-1}$ and the cumulative sum $\sum_{t=1}^{n-1}(Q_t - \bar{\mathbf{A}}^{-1})$, as outlined and established in [54, pp. 26–30].

$$Q_t - \bar{\mathbf{A}}^{-1} = S_t - \bar{\mathbf{A}}^{-1} G_{t:n},\ S_t = \sum_{j=t+1}^{n-1} (\alpha_t - \alpha_j) G_{t+1:j-1} \tag{5.31}$$

$$\sum_{t=1}^{n-1} (Q_t - \bar{\mathbf{A}}^{-1}) = -\bar{\mathbf{A}}^{-1} \sum_{j=1}^{n-1} G_{1:j} \tag{5.32}$$

In the following, we will require a bound on the operator norm of the matrix $S_t$, which is provided below:

**Lemma 23.** *Assume A 1 to 3 with $p = 2 \vee \log(d)$. Let $c_0 \in (0, \alpha_\infty]$ and $t \in \mathbb{N}$. Then*

$$\|S_t\| \leq \sqrt{\kappa_Q} \mathsf{C}_{23} (t + k_0)^{\gamma - 1} ,$$

*where $\mathsf{C}_{23}$ is given by*

$$\mathsf{C}_{23} = \frac{c_0}{1-\gamma} \exp\left(a c_0 + \frac{a c_0}{2(1-\gamma)}\right) \left(\frac{a c_0}{2(1-\gamma)}\right)^{-\frac{1}{1-\gamma}} \left( \max\{1, \phi(x_\gamma)\} \left(\frac{a c_0}{2} + x_\gamma\right) + \int_{x_\gamma}^{\infty} \phi(x) \right) ,$$

*and $x_\gamma = \frac{\gamma}{1-\gamma}$, $\phi(x) = x^{\frac{\gamma}{1-\gamma}} \exp(-x)$.*

Since $\Sigma_\infty = \bar{\mathbf{A}}^{-1} \Sigma_\varepsilon \bar{\mathbf{A}}^{-\top}$, elementary manipulations with $\Sigma_n - \Sigma_\infty$ imply the following equality:

$$\Sigma_n - \Sigma_\infty = \underbrace{\frac{1}{n} \sum_{t=1}^{n-1} (Q_t - \bar{\mathbf{A}}^{-1}) \Sigma_\varepsilon \bar{\mathbf{A}}^{-\top} + \frac{1}{n} \sum_{t=1}^{n-1} \bar{\mathbf{A}}^{-1} \Sigma_\varepsilon (Q_t - \bar{\mathbf{A}}^{-1})^\top}_{D_1}$$
$$+ \underbrace{\frac{1}{n} \sum_{t=1}^{n-1} (Q_t - \bar{\mathbf{A}}^{-1}) \Sigma_\varepsilon (Q_t - \bar{\mathbf{A}}^{-1})^\top}_{D_2} - \frac{1}{n} \Sigma_\infty \tag{5.33}$$

The decomposition (5.33) is crucial to obtain the convergence rate of $\Sigma_n - \Sigma_\infty$. The proof would follow from estimating $D_1$ and $D_2$ separately by expressions of order $n^{\gamma-1}$. For simplicity, we introduce the following notation:

$$g_{k:m}^{(\gamma)} = \sum_{\ell=k}^{m} (\ell + k_0)^{-\gamma},\ k \leq m .$$

*Proof of Lemma 2.* We first provide an expression for the constant $\mathsf{C}_2$:

$$\mathsf{C}_2 = \|\Sigma_\infty\| + \frac{2^{1+\gamma} \|\Sigma_\infty\| \sqrt{\kappa_Q} \mathsf{C}_4}{c_0} + \frac{\|\Sigma_\varepsilon\| \kappa_Q (\mathsf{C}_{23})^2}{2\gamma - 1} + \frac{2^\gamma \kappa_Q \|\Sigma_\infty\|}{a c_0 - (a c_0/2)^2} + \frac{4 \kappa_Q \|\bar{\mathbf{A}} \Sigma_\infty\| \mathsf{C}_{23}}{a c_0} . \tag{5.34}$$

Using (5.33), we get

$$\|\Sigma_n - \Sigma_\infty\| \le \frac{1}{n}\|\Sigma_\infty\| + \|D_1\| + \|D_2\| \,.$$

We first bound $D_1$. The operator norms of both terms are equal because one is a transposed version of another, so it is sufficient to bound only one of them. Note that $G_{n:m}$, $Q_t$, $\bar{\mathbf{A}}$, $\bar{\mathbf{A}}^{-1}$ commute as polynomials in $\bar{\mathbf{A}}$. Now we use (5.32) and obtain

$$\|\frac{1}{n}\sum_{t=1}^{n-1}(Q_t - \bar{\mathbf{A}}^{-1})\Sigma_\varepsilon \bar{\mathbf{A}}^{-\top}\| = \| -\frac{1}{n}\bar{\mathbf{A}}^{-1}\sum_{j=1}^{n-1} G_{1:j}\Sigma_\varepsilon \bar{\mathbf{A}}^{-\top}\| = \|n^{-1}\Sigma_\infty \sum_{j=1}^{n-1} G_{1:j}\| \le n^{-1}\|\Sigma_\infty\|\|\sum_{j=1}^{n-1} G_{1:j}\| \,.$$

Lemma 4 directly implies the bound for $D_1$:

$$\frac{1}{n}\|\sum_{t=1}^{n-1}(Q_t - \bar{\mathbf{A}}^{-1})\Sigma_\varepsilon \bar{\mathbf{A}}^{-\top}\| \le \frac{\|\Sigma_\infty\|\|\sum_{j=1}^{n-1} G_{1:j}\|}{n} \le \frac{\|\Sigma_\infty\|\sqrt{\kappa_Q}\mathsf{C}_4(1+k_0)^\gamma}{nc_0} \le \frac{2^\gamma n^{\gamma-1}\|\Sigma_\infty\|\sqrt{\kappa_Q}\mathsf{C}_4}{c_0} \,.$$

Hence,

$$\|D_1\| \le \frac{2^{1+\gamma} n^{\gamma-1}\|\Sigma_\infty\|\sqrt{\kappa_Q}\mathsf{C}_4}{c_0} \,. \tag{5.35}$$

We now consider $D_2$. Using (5.31), we get

$$n^{-1}\sum_{t=1}^{n-1}(Q_t - \bar{\mathbf{A}}^{-1})\Sigma_\varepsilon(Q_t - \bar{\mathbf{A}}^{-1})^\top = \underbrace{n^{-1}\sum_{t=1}^{n-1} S_t \Sigma_\varepsilon S_t^\top}_{D_{21}} + \underbrace{n^{-1}\sum_{t=1}^{n-1}\bar{\mathbf{A}}^{-1}\prod_{k=t}^{n-1}(\mathrm{I} - \alpha_k \bar{\mathbf{A}})\Sigma_\varepsilon \bar{\mathbf{A}}^{-\top}\prod_{k=t}^{n-1}(\mathrm{I} - \alpha_k\bar{\mathbf{A}})^\top}_{D_{22}}$$
$$\underbrace{- n^{-1}\sum_{t=1}^{n-1}\bar{\mathbf{A}}^{-1}\prod_{k=t}^{n-1}(\mathrm{I} - \alpha_k\bar{\mathbf{A}})\Sigma_\varepsilon S_t^\top}_{D_{23}} \underbrace{- n^{-1}\sum_{t=1}^{n-1} S_t\Sigma_\varepsilon \bar{\mathbf{A}}^{-\top}\prod_{k=t}^{n-1}(\mathrm{I} - \alpha_k\bar{\mathbf{A}})^\top}_{D_{24}} \,.$$

Lemma 23 reveal an evident bound for $\|D_{21}\|$:

$$\|D_{21}\| = \|n^{-1}\sum_{t=1}^{n-1} S_t\Sigma_\varepsilon S_t^\top\| \le n^{-1}\|\Sigma_\varepsilon\|\sum_{t=1}^{n-1}\kappa_Q(\mathsf{C}_{23})^2 t^{2(\gamma-1)} \le n^{2(\gamma-1)}\frac{\|\Sigma_\varepsilon\|\kappa_Q(\mathsf{C}_{23})^2}{2\gamma - 1} \,. \tag{5.36}$$

Note that

$$\sum_{t=1}^{n-1}\|G_{t:n-1}\|^2 \le \kappa_Q \sum_{t=1}^{n-1}\prod_{k=t}^{n-1}(1 - \frac{ac_0}{2}(k+k_0)^{-\gamma})^2 \le \kappa_Q\sum_{t=1}^{n-1}(1 - \frac{ac_0}{2}(n-1+k_0)^{-\gamma})^{n-t} \,,$$

The bound for $\|D_{22}\|$ follows from the above inequality:

$$\|D_{22}\| \le n^{-1}\|\Sigma_\infty\|\sum_{t=1}^{n-1}\|G_{t:n-1}\|^2 \le n^{-1}\frac{\kappa_Q\|\Sigma_\infty\|}{ac_0(n+k_0)^{-\gamma} - (ac_0/2)^2(n+k_0)^{-2\gamma}} \le n^{\gamma-1}\frac{2^\gamma\kappa_Q\|\Sigma_\infty\|}{ac_0 - (ac_0/2)^2} \,. \tag{5.37}$$

Since $D_{23} = D_{24}^{\top}$, we concentrate on $\|D_{24}\|$. Lemma 23 and Lemma 25-i imply the following bound:

$$\|D_{24}\| = \|n^{-1}\sum_{t=1}^{n-1} S_t \Sigma_\varepsilon \bar{\mathbf{A}}^{-\top} \prod_{k=t}^{n-1}(\mathrm{I} - \alpha_k \bar{\mathbf{A}})^{\top}\| \leq n^{-1}\|\Sigma_\varepsilon \bar{\mathbf{A}}^{-\top}\| \sum_{t=1}^{n-1}\|S_t\| \| \prod_{k=t}^{n-1}(\mathrm{I} - \alpha_k \bar{\mathbf{A}})^{\top}\|$$

$$\leq n^{-1}\|\bar{\mathbf{A}}\Sigma_\infty\| \sum_{t=1}^{n-1} \sqrt{\kappa_Q}(t+k_0)^{\gamma-1}\mathsf{C}_{23}\sqrt{\kappa_Q} \prod_{k=t+1}^{n-1}(1 - \frac{ac_0}{2}(k+k_0)^{-\gamma}) \leq n^{2(\gamma-1)}\frac{2\kappa_Q\|\bar{\mathbf{A}}\Sigma_\infty\|\mathsf{C}_{23}}{ac_0} .$$

Hence,

$$\|D_{23}\| + \|D_{24}\| \leq n^{2(\gamma-1)}\frac{4\kappa_Q\|\bar{\mathbf{A}}\Sigma_\infty\|\mathsf{C}_{23}}{ac_0} . \tag{5.38}$$

The needed result follows from (5.35), (5.36), (5.37), (5.38). □

## 5.6 Technical Lemmas

**Lemma 24** (Lemma 1 in [41])**.** *Fix $\delta \in (0, 1/\mathrm{e}^2)$ and let $Y$ be a positive random variable, such that $\mathbb{E}^{1/p}[Y^p] \leq C_1 + C_2 p$ for any $2 \leq p \leq \log(1/\delta)$. Then it holds with probability at least $1-\delta$, that*

$$Y \leq \mathrm{e}C_1 + \mathrm{e}C_2 \log(1/\delta) .$$

**Lemma 25.** *The following statement holds:*

*(i)* *Let $b > 0$ and $(\alpha_k)_{k\geq 0}$ be a non-increasing sequence such that $\alpha_0 \leq 1/b$. Then*

$$\sum_{j=1}^{k} \alpha_j \prod_{l=j+1}^{k}(1-\alpha_l b) = \frac{1}{b}\left\{1 - \prod_{l=1}^{k}(1-\alpha_l b)\right\} .$$

*(ii)* *Let $b > 0$ and $\alpha_k = \frac{c_0}{(k+k_0)^\gamma}$, $\gamma \in (0,1)$, such that $c_0 \leq 1/b$ and $k_0^{1-\gamma} \geq \frac{8\gamma}{bc_0}$. Then for any $q \in (1,4]$ it holds that*

$$\sum_{j=1}^{k} \alpha_j^q \prod_{\ell=j+1}^{k}(1-\alpha_\ell b) \leq \frac{6}{b}\alpha_k^{q-1} .$$

*(iii)* *Let $b, c_0, k_0 > 0$ and $\alpha_\ell = c_0(\ell+k_0)^{-\gamma}$ for $\gamma \in (1/2,1)$ and $\ell \in \mathbb{N}$. Assume that $bc_0 < 1$ and $k_0^{1-\gamma} \geq \frac{1}{bc_0}$. Then, for any $\ell, n \in \mathbb{N}$, $\ell \leq n$, it holds that*

$$\sum_{k=\ell}^{n-1} \alpha_\ell \prod_{j=\ell+1}^{k}(1-b\alpha_j) \leq c_0 + \frac{1}{b(1-\gamma)} .$$

*Proof.* Lemma 25-i follows from Lemma 24 in [14]. Lemma 25-ii follows from Lemma 33 in [42]. (iii) is elementary. □

**Lemma 26** (Lemma 36 in [42])**.** *For any $A > 0$, any $1 \leq i \leq n-1$, and $\gamma \in (1/2,1)$ it holds*

$$\sum_{j=i}^{n-1}\exp\left\{-A(j^{1-\gamma} - i^{1-\gamma})\right\} \leq \begin{cases} 1 + \exp\{\frac{1}{1-\gamma}\}\frac{1}{A^{1/(1-\gamma)}(1-\gamma)}\Gamma(\frac{1}{1-\gamma}) , & \text{if } Ai^{1-\gamma} \leq \frac{1}{1-\gamma} ; \\ 1 + \frac{1}{A(1-\gamma)^2} i^\gamma , & \text{if } Ai^{1-\gamma} > \frac{1}{1-\gamma} . \end{cases}$$

# 6 Conclusion

In this paper, we have obtained a novel bound for the Gaussian approximation of the distribution of the Polyak–Ruppert averaged LSA iterates in the sense of convex distance. Compared to the previous analysis established in [41], the fastest achievable rate of normal approximation has been improved from $n^{-1/4}$ to $n^{-1/3}$. We also derived a bootstrap-based approximation for the distribution $\sqrt{n}(\bar{\theta}_n - \theta^\star)$ with an error of order up to $1/\sqrt{n}$. Importantly, this result does not rely on the Gaussian approximation with the limiting covariance matrix $\Sigma\infty$. Among further directions, we list the generalization of the randomized concentration approach of [44] to the Markov setting, which enables the analysis of stochastic approximation problems with Markov noise. Current approaches [47, 55, 42] rely on versions of the Berry–Esseen inequalities for martingales, which require additional step size constraints and introduce extra $\log n$ factors. Another research direction would be to tighten the lower bound (3.23) in the regime where the step size exponent $\gamma \in (1/2, 2/3)$. Establishing a counterpart of (3.23) with the term $n^{-\gamma/2}$ would imply the optimality of the rate $n^{-1/3}$ in Theorem 3.

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## A Definitions of integral probability metrics

In this section we closely follow the exposition outlined in [19]. Consider two $\mathbb{R}^d$-valued random variables $X$ and $Y$. The integral probability metric [56], associated with a class of test functions $\mathcal{H} = \{h : \mathbb{R}^d \to \mathbb{R} : \mathbb{E}[|h(X)|] < \infty, \mathbb{E}[|h(Y)|] < \infty\}$, is defined as

$$\mathsf{d}_{\mathcal{H}}(X,Y) = \sup_{h\in\mathcal{H}} \big|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]\big| \,.$$

Different choices of the function class $\mathcal{H}$ induce different probability metrics. We consider the following important examples:

$$\begin{aligned}
\mathcal{H}_K &= \{\mathbf{1}_{(-\infty,u_1]\times\cdots\times(-\infty,u_d]}, \quad u = (u_1,\dots,u_d) \in \mathbb{R}^d\} \\
\mathcal{H}_{Conv} &= \{\mathbf{1}_B, \quad B \in \mathrm{Conv}(\mathbb{R}^d)\} \\
\mathcal{H}_W &= \{h : \mathbb{R}^d \to \mathbb{R}, \quad \|h\|_{\mathrm{Lip}} \le 1\} \\
\mathcal{H}_{[m]} &= \{h : \mathbb{R}^d \to \mathbb{R}, \quad h \in C^{m-1}(\mathbb{R}^d) \text{ with } |h|_j \le 1 \text{ for } 1 \le j \le m\} \,,
\end{aligned}$$

where $\mathrm{Conv}(\mathbb{R}^d)$ denotes the collection of all convex subsets of $\mathbb{R}^d$, $\|h\|_{\mathrm{Lip}} = \sup_{x\neq y} \frac{\|h(x)-h(y)\|}{\|x-y\|}$ is the Lipschitz constant, $C^{m-1}(\mathbb{R}^d)$ represents the space of $(m-1)$-times continuously differentiable functions, and the seminorm $|h|_j$ is defined as

$$|h|_j = \max_{i_1,\dots,i_j\in\{1,\dots,d\}} \|\frac{\partial^j h}{\partial u_{i_1}\cdots\partial u_{i_j}}\|_\infty \,.$$

Thus, for each $m \in \mathbb{N}$, the function class $\mathcal{H}_{[m]}$ consists of functions whose partial derivatives up to order $m$ are uniformly bounded.

These function classes generate well-established probability metrics in the literature. The class $\mathcal{H}_K$ induces the classical Kolmogorov metric between distributions [56], while $\mathcal{H}_{Conv}$ generates the convex distance $\rho^{\mathrm{Conv}}$ defined for a pair of probability measures $\mu, \nu$ on $\mathbb{R}^d$ as

$$\rho^{\mathrm{Conv}}(\mu,\nu) = \sup_{B\in\mathrm{Conv}(\mathbb{R}^d)} |\mu(B) - \nu(B)| \,,$$

which is the primary focus of this paper. The class $\mathcal{H}_W$ yields the Wasserstein-1 distance, and the classes $\mathcal{H}_{[m]}$ define the smoothed Wasserstein metrics of order $m$. We denote the corresponding metrics as $\mathsf{d}_K$, $\rho^{\mathrm{Conv}}$, $\mathsf{d}_W$, and $\mathsf{d}_{[m]}$, respectively.

An important hierarchy exists among these metrics: for any pair of random vectors $X$ and $Y$, we have

$$\mathsf{d}_K(X,Y) \le \rho^{\mathrm{Conv}}(X,Y),$$

since every rectangular set is convex, implying $\mathcal{H}_K \subset \mathcal{H}_{Conv}$.Other relations among these metrics are substantially more intricate. For instance, when $Y$ is a multivariate normal vector, it is well-established (see, e.g., [33]) that

$$\rho^{\mathrm{Conv}}(X,Y) \le C\sqrt{\mathsf{d}_W(X,Y)} \,,$$

where the constant $C$ depends explicitly on the covariance matrix of the vector $Y$. This inequality serves as the theoretical basis for comparing the bounds provided in Theorem 3 with the results obtained in [47].

## B Proofs of Section 3.1

*Proof of Lemma 1.* Note that A 3$(p \vee \log d)$ implies for all $\ell$ that

$$1 - a\alpha_\ell + (p-1)b_Q^2\alpha_\ell^2 \le 1 - \frac{a\alpha_\ell}{2} \,.$$

To finish the proof we combine the latter inequality with Corollary 2. □

*Proof of Proposition 2.* Using the decomposition (3.5), we obtain that, with $p \ge 2$, it holds

$$\mathbb{E}^{1/p}[\|\theta_k - \theta^\star\|^p] \le \mathbb{E}^{1/p}[\|\Gamma_{1:k}\{\theta_0 - \theta^\star\}\|^p] + \mathbb{E}^{1/p}[\|\sum_{j=1}^{k}\alpha_j\Gamma_{j+1:k}\varepsilon_j\|^p] \,, \tag{B.1}$$

and we bound both terms separately. Applying Lemma 1, we get for $2 \le p \le \log(5n^2)$:

$$\mathbb{E}^{1/p}[\|\Gamma_{1:k}\{\theta_0 - \theta^\star\}\|^p] \le \sqrt{\kappa_Q}\mathrm{e}\|\theta_0 - \theta^\star\|\prod_{\ell=1}^{k}(1 - \frac{a}{2}\alpha_\ell) \,.$$

Now we proceed with the second term in (B.1). Applying Burholder's inequality [34, Theorem 8.6], we obtain that

$$\begin{aligned}\mathbb{E}^{1/p}[\|\sum\nolimits_{j=1}^{k}\alpha_j\Gamma_{j+1:k}\varepsilon_j\|^p] &\le p\left(\mathbb{E}^{2/p}\left[\left(\sum\nolimits_{j=1}^{k}\alpha_j^2\|\Gamma_{j+1:k}\varepsilon_j\|^2\right)^{p/2}\right]\right)^{1/2} \le p\left(\sum\nolimits_{j=1}^{k}\alpha_j^2\mathbb{E}^{2/p}\left[\|\Gamma_{j+1:k}\varepsilon_j\|^p\right]\right)^{1/2} \\ &\le p\sqrt{\kappa_Q}\mathrm{e}\|\varepsilon\|_\infty\left(\sum\nolimits_{j=1}^{k}\alpha_j^2\prod_{\ell=j+1}^{k}(1 - \frac{a\alpha_\ell}{2})\right)^{1/2} \overset{(a)}{\le} \mathsf{C}_2 p\sqrt{\alpha_k} \,,\end{aligned}$$

where in *(a)* we additionally applied Lemma 25-ii. □

*Proof of Lemma 3.* First, we derive a bound for $J_k^{(0)} = -\sum_{\ell=1}^{k}\alpha_\ell G_{\ell+1:k}\varepsilon_\ell$ which is a sum of independent random vectors, satisfying $\|\alpha_\ell G_{\ell+1:k}\varepsilon_\ell\| \le \alpha_\ell\kappa_Q^{1/2}\prod_{j=\ell+1}^{k}(1-\alpha_j a)^{1/2}\|\varepsilon\|_\infty$. Hence, applying the Pinelis inequality [35, Theorem 3.5], we obtain that, for any $t \ge 0$,

$$\mathbb{P}(\|J_k^{(0)}\| \ge t) \le 2\exp\left(-\frac{t^2}{2\sigma_k^2}\right) \,, \qquad \text{where } \sigma_k^2 = \kappa_Q\|\varepsilon\|_\infty^2\sum_{\ell=1}^{k}\alpha_\ell^2\prod_{j=\ell+1}^{k}(1-\alpha_j a) \le \alpha_k c_1 \,,$$

and $c_1 = 24\kappa_Q\|\varepsilon\|_\infty^2/a$. Thus, applying [12, Lemma 7], we obtain that, for $p \ge 2$, it holds

$$\mathbb{E}^{1/p}\left[\|J_k^{(0)}\|^p\right] \le 2^{1/p}\sqrt{p}\sqrt{\alpha_k}\sqrt{c_1} \,,$$

and the bound for $J_k^{(0)}$ follows. Now we bound $J_k^{(\ell)}$ by induction. Using the equation (5.2), $J_k^{(\ell)}$, $\ell \ge 1$, can be represented as

$$J_k^{(\ell)} = -\sum_{m=1}^{k}\alpha_m G_{m+1:k}\tilde{\mathbf{A}}_m J_{m-1}^{(\ell-1)} \,.$$

Note that $\alpha_m G_{m+1:k} J_{m-1}^{(\ell-1)}$ is a martingale-difference sequence w.r.t. the filtration $\mathcal{F}_m = \sigma(Z_s : 1 \leq s \leq m)$. Hence, Burkholder's inequality [34, Theorem 8.6] implies that

$$\mathbb{E}^{1/p}[\|J_k^{(\ell)}\|^p] \leq p\Big(\sum_{m=1}^{k} \mathbb{E}^{2/p}[\|\alpha_m G_{m+1:k}\tilde{\mathbf{A}}_m J_{m-1}^{(\ell-1)}\|^p]\Big)^{1/2} \leq \mathrm{C}_{\mathbf{A}}\, p^{\ell+1/2}\mathsf{C}_3^{(\mathrm{J},\ell-1)}\big(\sum_{m=1}^{k} \alpha_m^{\ell+2}\|G_{m+1:k}\|^2\big)^{1/2}$$
$$\overset{(a)}{\leq} \mathrm{C}_{\mathbf{A}}\, \mathsf{C}_3^{(\mathrm{J},\ell-1)} \cdot \frac{2\sqrt{6}\sqrt{\kappa_Q}}{\sqrt{a}} p^{\ell+1/2}\alpha_k^{(\ell+1)/2} ,$$

where in (a) we used Lemma 25. Now we prove the bound for $H_k^{(\ell)}$. Recall that $H_k^{(\ell)} = -\sum_{m=1}^{k}\alpha_m\Gamma_{m+1:k}J_m^{(\ell)}$. Since $\Gamma_{m+1:k}$ and $J_m^{(\ell)}$ are independent for all $m$, the desired result follows from Lemma 3, Lemma 25, and Minkowski's inequality:

$$\mathbb{E}^{1/2}[\|H_k^{(\ell)}\|^p] \leq \sum_{m=1}^{k}\alpha_m\mathbb{E}^{1/p}[\|\Gamma_{m+1:k}\|^p]\mathbb{E}^{1/p}[\|J_m^{(\ell)}\|^p] \overset{(a)}{\leq} \sqrt{\kappa_Q}\mathrm{e}\mathsf{C}_3^{(\mathrm{J},\ell)}p^{\ell+1/2}\sum_{m=1}^{k}\alpha_m^{(\ell+3)/2}\prod_{\ell=m}^{k}(1-\frac{a\alpha_\ell}{2})$$
$$\overset{(b)}{\leq} \mathsf{C}_3^{(\mathrm{H},\ell)}p^{\ell+1/2}\alpha_k^{(\ell+1)/2} ,$$

where in $(a)$ we used the moment bound for $J_m^{(\ell)}$, and in $(b)$ we used Lemma 25. □

*Proof of Lemma 4.* Using the triangle inequality we get:

$$\|Q_\ell\| \leq \alpha_\ell\sum_{k=\ell}^{n-1}\|G_{\ell+1:k}\| \leq \sqrt{\kappa_Q}\sum_{k=\ell}^{n-1}\alpha_\ell\prod_{j=\ell+1}^{k}(1-\frac{ac_0}{2}j^{-\gamma})$$

The rest of the proof follows from Lemma 25. □

## C Proofs of Section 3.2

*Proof of Lemma 5.* Applying Minkowski's inequality and Corollary 2,

$$\mathbb{E}^{1/p}[\|\sum_{k=0}^{n-1}\Gamma_{1:k}\|^p] \leq \sum_{k=0}^{n-1}\mathbb{E}^{1/p}[\|\Gamma_{1:k}\|^p] \leq 1+\sqrt{\kappa_Q}\mathrm{e}\sum_{k=1}^{n-1}\prod_{\ell=1}^{k}(1-\frac{a\alpha_\ell}{2}) .$$

Thus, applying Lemma 25, $\sum_{k=1}^{n-1}\prod_{\ell=1}^{k}(1-\frac{a\alpha_\ell}{2}) \leq \frac{(1+k_0)^\gamma}{c_0}(c_0+\frac{2}{a(1-\gamma)})$, and the statement follows. □

*Proof of Lemma 6.* Decomposing $\Sigma_n = \Sigma_\infty + (\Sigma_n - \Sigma_\infty)$ and then applying Lidskii's inequality, we obtain $\lambda_{\min}(\Sigma_n) \geq \lambda_{\min}(\Sigma_\infty) - \|\Sigma_n - \Sigma_\infty\|$. The conclusion follows from Lemma 2 and A 3, which imply $\|\Sigma_n - \Sigma_\infty\| \leq \mathsf{C}_2 n^{\gamma-1} \leq \frac{\lambda_{\min}(\Sigma_\infty)}{2}$. □

*Proof of Lemma 7.* First, we rewrite the sum:

$$\sum_{k=1}^{n-1}(\Gamma_{1:k}-\Gamma_{1:k}^{(i)}) = \sum_{k=i}^{n-1}\alpha_i\Gamma_{1:i-1}(\mathbf{A}(Z_i)-\mathbf{A}(Z_i'))\Gamma_{i+1:k} = \Gamma_{1:i-1}(\mathbf{A}(Z_i)-\mathbf{A}(Z_i'))\sum_{k=i}^{n-1}\alpha_i\Gamma_{i+1:k} .$$

Hence, it holds that

$$\|\sum_{k=1}^{n-1}(\Gamma_{1:k}-\Gamma_{1:k}^{(i)})\| \le \mathrm{C}_{\mathbf{A}}\,\|\Gamma_{1:i-1}\|\,\|\sum_{k=i}^{n-1}\alpha_i\Gamma_{i+1:k}\|\;.$$

Lemma 1 implies that $\mathbb{E}^{1/2}\big[\|\Gamma_{1:i-1}\|^2\big] \le \sqrt{\kappa_Q}\mathrm{e}\prod_{m=1}^{i-1}(1-a\alpha_m/2)$. On the other hand, combining Minkowski's inequality with Lemma 25, we obtain

$$\mathbb{E}^{1/2}\big[\|\sum_{k=i}^{n-1}\alpha_i\Gamma_{i+1:k}\|^2\big] \le \sqrt{\kappa_Q}\mathrm{e}\sum_{k=i}^{n-1}\alpha_i\prod_{m=i+1}^{k}(1-a\alpha_j/2) \le \sqrt{\kappa_Q}\mathrm{e}(c_0+\frac{2}{a(1-\gamma)})\;.$$

To finish the proof, it remains to notice that $\Gamma_{1:i-1}$ is independent from $\Gamma_{i+1:k}$. □

*Proof of Lemma 8.* First, note that $H_k^{(0)}-H_k^{(0,i)}=0$, if $k<i$. On the other hand, for $k\ge i$ we get

$$H_k^{(0)}-H_k^{(0,i)} = \underbrace{\Gamma_{i+1:k}(H_i^{(0)}-H_i^{(0,i)})}_{T_1^{(k)}} - \underbrace{\sum_{j=i+1}^{k}\alpha_j\Gamma_{j+1:k}\tilde{\mathbf{A}}_j(J_{j-1}^{(0)}-J_{j-1}^{(0,i)})}_{T_2^{(k)}}\;.$$

Introduce $\varepsilon_i'=\varepsilon(Z_i')$ and $\mathbf{A}_i'=\mathbf{A}(Z_i')$. Then, for $\ell\ge i+1$: $J_{\ell-1}^{(0)}-J_{\ell-1}^{(0,i)}=\alpha_iG_{i+1:\ell-1}(\varepsilon_i'-\varepsilon_i)$. Thus, since $T_2^{(i)}=0$, we obtain that

$$\sum_{k=i}^{n-1}T_2^{(k)} = \sum_{k=i+1}^{n-1}\sum_{j=i+1}^{k}\alpha_j\Gamma_{j+1:k}\tilde{\mathbf{A}}_k\alpha_iG_{i+1:k-1}(\varepsilon_i'-\varepsilon_i) = \sum_{j=i+1}^{n-1}\underbrace{\alpha_i(\sum_{k=j}^{n-1}\alpha_j\Gamma_{j+1:k})\tilde{\mathbf{A}}_jG_{i+1:j-1}(\varepsilon_i'-\varepsilon_i)}_{U_j}\;.$$

Note that $U_j$ is a reverse martingale-difference sequence with respect to the filtration $\mathcal{F}_{j,i}=\sigma(Z_i,Z_i',Z_j,Z_{j+1},\ldots,Z_{n-1})$. Hence,

$$\mathbb{E}^{1/2}\big[\|n^{-1/2}\sum_{\ell=i+1}^{n-1}U_j\|^2\big] = n^{-1/2}\big(\sum_{j=i+1}^{n-1}\mathbb{E}\left[\|U_j\|^2\right]\big)^{1/2}\;.$$

For simplicity we set $u_{\ell:m}=\prod_{t=\ell}^{m}(1-a\alpha_t/2)$. Applying Corollary 2 and Lemma 25, we obtain

$$\mathbb{E}\left[\|U_j\|^2\right] \le \alpha_i^2\Big(\kappa_Q\mathrm{e}^2\|\varepsilon\|_\infty\,\mathrm{C}_{\mathbf{A}}\Big)^2\Bigg(\sum_{k=j}^{n-1}\alpha_ju_{j+1:k}\Bigg)^2u_{i+1:j-1}^2 \le \alpha_i^2\underbrace{\Big(\kappa_Q\mathrm{e}^2\|\varepsilon\|_\infty\,\mathrm{C}_{\mathbf{A}}(c_0+\frac{2}{a(1-\gamma)})\Big)^2}_{R_U^2}u_{i+1:j-1}\;.$$

Thus, it holds that

$$\mathbb{E}^{1/2}\big[\|n^{-1/2}\sum_{\ell=i+1}^{n-1}U_j\|^2\big] \le \frac{1}{n^{1/2}}\sqrt{\alpha_i}R_U\big(\alpha_i\sum_{j=i}^{n-2}u_{i+1:j}\big)^{1/2} \le \frac{R_U\sqrt{c_0+\frac{2}{a(1-\gamma)}}}{n^{1/2}}\sqrt{\alpha_i}\;.$$

The recurrent rule (3.8) implies the following representation for $T_1^{(k)}$:

$$T_1^{(k)} = \Gamma_{i+1:k}(-\alpha_i(\mathbf{A}_i - \mathbf{A}_i')H_{i-1}^{(0)} - \alpha_i(\mathbf{A}_i - \mathbf{A}_i')J_{i-1}^{(0)}) \; .$$

Therefore, Lemma 3 together with $\alpha_{i-1} \leq 2\alpha_i$ implies that

$$\mathbb{E}^{1/2}\big[\|T_1^{(k)}\|^2\big] \leq 2\,\mathrm{C}_{\mathbf{A}}\,\sqrt{\kappa_Q}\mathrm{e}u_{i+1:k}\alpha_i^{3/2}(\mathsf{C}_3^{(\mathrm{J},0)} + \mathsf{C}_3^{(\mathrm{H},0)}) \; .$$

Thus, using Lemma 25, we get

$$\begin{aligned}\mathbb{E}^{1/2}\big[\|\sum_{k=i}^{n-1} T_1^{(k)}\|^2\big] &\leq 2\,\mathrm{C}_{\mathbf{A}}\,\sqrt{\kappa_Q}\mathrm{e}u_{i+1:k}\alpha_i^{1/2}(\mathsf{C}_3^{(\mathrm{J},0)} + \mathsf{C}_3^{(\mathrm{H},0)})\sum_{k=i}^{n-1}\alpha_i u_{i+1:k} \\ &\leq \alpha_i^{1/2} 2\,\mathrm{C}_{\mathbf{A}}\,\sqrt{\kappa_Q}\mathrm{e}(\mathsf{C}_3^{(\mathrm{J},0)} + \mathsf{C}_3^{(\mathrm{H},0)})(c_0 + \frac{2}{a(1-\gamma)}) \; .\end{aligned}$$

It remains to note that

$$\sum_{k=1}^{n-1}(H_k^{(0)} - H_k^{(0,i)}) = \sum_{k=1}^{n-1}(T_1^{(k)} + T_2^{(k)}) = \sum_{k=i}^{n-1}(T_1^{(k)} + T_2^{(k)}) \; ,$$

and use Minkowski's inequality. □

# D Proofs of Section 4

*Proof of Lemma 9.* Writing, $\tilde{\varepsilon}_\ell = \varepsilon_\ell + \mathbf{A}_\ell(\theta_{\ell-1} - \theta^\star)$, the proof follows from the definition of $\Omega_1$ and A 5. □

*Proof of Lemma 10.* Applying Lemma 17, we get

$$\begin{aligned}\mathbb{E}^{\mathsf{b}}\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_{k,1}^{\mathsf{b},0}\|^2 &= n^{-1}\mathbb{E}^{\mathsf{b}}\|\sum_{\ell=1}^{n-1}\alpha_\ell(w_\ell - 1)\sum_{k=\ell}^{n-1}(\Gamma_{\ell+1:k} - G_{\ell+1:k})\varepsilon_\ell\|^2 = n^{-1}\sum_{\ell=1}^{n-1}\|\alpha_\ell\sum_{k=\ell}^{n-1}(\Gamma_{\ell+1:k} - G_{\ell+1:k})\varepsilon_\ell\|^2 \\ &\leq n^{-1}\log^2(5n)\mathsf{C}_{17}^2\|\varepsilon\|_\infty^2\sum_{\ell=1}^{n-1}\alpha_\ell \leq \log^2(5n)\frac{c_0\mathsf{C}_{17}^2\|\varepsilon\|_\infty^2}{(1-\gamma)n^\gamma} \; .\end{aligned}$$

□

*Proof of Lemma 11.* First we rewrite the sum and obtain

$$\begin{aligned}\mathbb{E}^{\mathsf{b}}\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_{k,2}^{\mathsf{b},0}\|^2 &= \frac{1}{n}\mathbb{E}^{\mathsf{b}}\|\sum_{k=1}^{n-1}\sum_{\ell=1}^{k}\alpha_\ell(w_\ell - 1)\Gamma_{\ell+1:k}\mathbf{A}_\ell(\theta_{\ell-1} - \theta^\star)\|^2 \\ &= \frac{1}{n}\sum_{\ell=1}^{n-1}\alpha_\ell^2\|\sum_{k=\ell}^{n-1}\Gamma_{\ell+1:k}\mathbf{A}_\ell(\theta_{\ell-1} - \theta^\star)\|^2 \leq \frac{1}{n}\,\mathrm{C}_{\mathbf{A}}^2\sum_{\ell=1}^{n-1}\alpha_\ell^2\|\sum_{k=\ell}^{n-1}\Gamma_{\ell+1:k}\|^2\|\theta_{\ell-1} - \theta^\star\|^2 \; .\end{aligned}$$

Hence, using Lemma 1 and Lemma 25 with $b = a/2$, we get that on the event $\Omega_2$, it holds that

$$\begin{aligned}\mathbb{E}^{\mathsf{b}}\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_{k,2}^{\mathsf{b},0}\|^2 &\leq n^{-1}\,\mathrm{C}_{\mathbf{A}}^2\,\mathsf{C}_1^2\sum_{\ell=1}^{n-1}\big(\alpha_\ell\sum_{k=\ell}^{n-1}\prod_{j=\ell+1}^{k}(1-\frac{a\alpha_j}{2})\big)^2\|\theta_{\ell-1}-\theta^\star\|^2 \\ &\leq n^{-1}\,\mathrm{C}_{\mathbf{A}}^2\,\mathsf{C}_1^2\big(c_0+\frac{2}{a(1-\gamma)}\big)^2\sum_{\ell=1}^{n-1}\|\theta_{\ell-1}-\theta^\star\|^2 .\end{aligned}$$

Using an elementary inequality $(a+b)^2 \leq 2a^2+2b^2$, we get that on the event $\Omega_1$, it holds

$$\begin{aligned}\sum_{\ell=1}^{n-1}\|\theta_{\ell-1}-\theta^\star\|^2 &\leq \sum_{\ell=1}^{n-1}\big\{2\kappa_Q\mathrm{e}^4\|\theta_0-\theta^\star\|^2\prod_{j=1}^{\ell-1}(1-\frac{a\alpha_j}{2})+8\mathrm{e}^2\log^2(5n)\mathsf{C}_2^2\alpha_{\ell-1}\big\} \\ &\leq \frac{2\kappa_Q\mathrm{e}^4(1+k_0)^\gamma\big(c_0+\frac{2}{a(1-\gamma)}\big)}{c_0}\|\theta_0-\theta^\star\|^2+8\mathrm{e}^2\log^2(5n)\mathsf{C}_2^2 c_0\frac{n^{1-\gamma}}{1-\gamma} .\end{aligned}$$

It remains to combine the above bounds. □

*Proof of Lemma 12.* We start from the decomposition

$$\theta_k^{\mathsf{b}}-\theta_k = (\mathrm{I}-\alpha_k w_k\mathbf{A}_k)(\theta_{k-1}^{\mathsf{b}}-\theta_{k-1})-\alpha_k(w_k-1)\tilde{\varepsilon}_k. \tag{D.1}$$

Expanding the recurrence above till $k=0$, and using the fact that $\theta_0^{\mathsf{b}}=\theta_0$, we get running the recurrence (D.1), that

$$\theta_k^{\mathsf{b}}-\theta_k = -\sum_{\ell=n+1}^{k}\alpha_\ell(w_\ell-1)\Gamma_{\ell+1:k}^{\mathsf{b}}\tilde{\varepsilon}_\ell .$$

Hence, proceeding as in (3.7), we obtain the representation

$$J_k^{(\mathsf{b},0)} = (\mathrm{I}-\alpha_k\mathbf{A}_k)\,J_{k-1}^{(\mathsf{b},0)}-\alpha_k(w_k-1)\tilde{\varepsilon_k}\ , \qquad J_0^{(\mathsf{b},0)}=0\ , \tag{D.2}$$

$$H_k^{(\mathsf{b},0)} = (\mathrm{I}-\alpha_k w_k\mathbf{A}_k)\,H_{k-1}^{(\mathsf{b},0)}-\alpha_k(w_k-1)\mathbf{A}_k J_{k-1}^{(\mathsf{b},0)}\ , \qquad H_0^{(\mathsf{b},0)}=0\ . \tag{D.3}$$

Hence, using Lemma 9 together with the definition of $J_k^{\mathsf{b},0}$, we obtain that

$$\mathbb{E}^{\mathsf{b}}[\|J_k^{\mathsf{b},0}\|^2] = \sum_{\ell=1}^{k}\alpha_\ell^2\|\Gamma_{\ell+1:k}\tilde{\varepsilon}_\ell\|^2 \leq \alpha_k\mathsf{C}_9^2\mathsf{C}_1^2\sum_{\ell=1}^{k}\alpha_\ell^2\prod_{j=\ell+1}^{k}(1-a\alpha_\ell/2) \leq \alpha_k\underbrace{\frac{12\mathsf{C}_9^2\mathsf{C}_1^2}{a}}_{(\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},0)})^2} .$$

Assume now that the bound on $J_k^{\mathsf{b},j-1}$ has a form $\{\mathbb{E}^{\mathsf{b}}[\|J_k^{\mathsf{b},j-1}\|^2]\}^{1/2} \leq \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},j-1)}\alpha_k^{j/2}$. Then, using the martingale property of $J_k^{\mathsf{b},j}$, we write that

$$\mathbb{E}^{\mathsf{b}}[\|J_k^{\mathsf{b},j}\|^2] = \sum_{\ell=1}^{k}\alpha_\ell^2\mathbb{E}^{\mathsf{b}}[\|\Gamma_{\ell+1:k}\mathbf{A}_\ell J_{\ell-1}^{\mathsf{b},j-1}\|^2] \leq (\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},j-1)})^2\sum_{\ell=1}^{k}\alpha_\ell^{j+2}\,\mathrm{C}_{\mathbf{A}}^2\,\mathsf{C}_1^2\prod_{t=\ell+1}^{k}(1-a\alpha_t/2)^2 .$$

Hence, applying Lemma 25 we get

$$\mathbb{E}^{\mathsf{b}}[\|J_k^{\mathsf{b},j}\|^2] \le \alpha_k^{j+1} \underbrace{(\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},j-1)})^2 \mathsf{C}_1^2 \,\mathrm{C}_{\mathbf{A}}^2 \,\frac{12}{a}}_{(\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},j)})^2} \, .$$

and, thus, the moment bound for $J_k^{\mathsf{b},j}$ is proved. Moreover, using the definition of $H_k^{\mathsf{b},L}$ and Minkowski's inequality, we obtain that

$$\begin{aligned}(\mathbb{E}^{\mathsf{b}}[\|H_k^{\mathsf{b},L}\|^2])^{1/2} \le \mathrm{C}_{\mathbf{A}} \sum_{\ell=1}^{k} \alpha_\ell (\mathbb{E}^{\mathsf{b}}[\|\Gamma_{\ell+1:k}^{\mathsf{b}}\|^2])^{1/2} (\mathbb{E}^{\mathsf{b}}[\|J_{\ell-1}^{\mathsf{b},L}\|^2])^{1/2} &\le \mathrm{C}_{\mathbf{A}} \,\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},L)} \mathsf{C}_3 \sum_{\ell=1}^{k} \alpha_\ell^{\frac{L+3}{2}} \prod_{t=\ell+1}^{k} (1 - \frac{a\alpha_t}{8}) \\ &\le \alpha_\ell^{(L+1)/2} \underbrace{\mathrm{C}_{\mathbf{A}} \,\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},L)} \mathsf{C}_3 \frac{48}{a}}_{\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},L)}} \, .\end{aligned}$$

and the moment bound for $H_k^{\mathsf{b},L}$ follows. ☐

*Proof of Lemma 17.* For any matrix-valued sequences $(U_n)_{n\in\mathbb{N}}$, $(V_n)_{n\in\mathbb{N}}$ and any $M \in \mathbb{N}$, it holds that:

$$\prod_{k=1}^{M} U_k - \prod_{k=1}^{M} V_k = \sum_{k=1}^{M} \Big( \prod_{j=k+1}^{M} V_j \Big) (U_k - V_k) \Big( \prod_{j=1}^{k-1} U_j \Big) \, . \tag{D.4}$$

Using (D.4) and changing the order of summation, we get

$$\alpha_\ell \sum_{k=\ell}^{n-1} (\Gamma_{\ell+1:k} - G_{\ell+1:k}) \varepsilon_\ell = \alpha_\ell \sum_{j=\ell+1}^{n-1} \underbrace{(\alpha_j \sum_{k=j}^{n-1} G_{j+1:k}) (\mathbf{A}_j - \bar{\mathbf{A}}) \Gamma_{\ell+1:j-1} \varepsilon_\ell}_{U_j} \, . \tag{D.5}$$

Applying Lemma 4, we get $\|\alpha_j \sum_{k=j}^{n-1} G_{j+1:k}\| = \|Q_j\| \le \mathsf{C}_4$, hence, $\|U_j\| \le 2\mathsf{C}_4 \,\mathrm{C}_{\mathbf{A}} \,\|\Gamma_{\ell+1:j-1}\|$. Consider the sigma-algebras

$$\mathcal{F}_{m:k} = \begin{cases} \sigma(Z_s : m \le s \le k), & \text{if } m \le k \, , \\ \{\emptyset, \mathsf{Z}\}, & \text{otherwise.} \end{cases}$$

Note that $U_j$ is a martingale-difference sequence w.r.t. the filtration $\mathcal{F}_{\ell+1:\ell+1} \subseteq \mathcal{F}_{\ell+1:\ell+2} \subseteq \ldots \subseteq \mathcal{F}_{\ell+1:2n}$, thus Burkholder's inequality [34, Theorem 8.6] implies

$$\mathbb{E}^{1/p}\big[\| \sum_{j=\ell+1}^{n-1} U_j \|^p\big] \le p \big( \sum_{j=\ell+1}^{n-1} \mathbb{E}^{2/p} \|U_j\|^p \big)^{1/2} \le 2p\sqrt{d} \mathsf{C}_\varepsilon \,\mathrm{C}_{\mathbf{A}} \,\mathsf{C}_4 \big( \sum_{j=\ell+1}^{n-1} \mathbb{E}^{2/p} \|\Gamma_{\ell+1:j-1}\|^p \big)^{1/2} \, .$$

Applying now Lemma 1 together with the fact $\alpha_\infty a \le 1/2$, we get

$$\begin{aligned}\alpha_\ell \sum_{j=\ell+1}^{n-1} \mathbb{E}^{2/p} \|\Gamma_{\ell+1:j-1}\|^p \le \sum_{j=\ell+1}^{n-1} \alpha_\ell \kappa_Q \mathrm{e}^2 \prod_{t=\ell+1}^{j-1} (1 - \frac{a\alpha_t}{2}) &\le (8/7) \kappa_Q \mathrm{e}^2 \sum_{j=\ell}^{n-1} \alpha_\ell \prod_{t=\ell+1}^{j} (1 - \frac{a\alpha_m}{2}) \\ &\overset{(a)}{\le} (8/7) \kappa_Q \mathrm{e}^2 \left( c_0 + \frac{2}{a(1-\gamma)} \right) \, .\end{aligned}$$

In (a) we additionally used Lemma 25 with $b = a/2$. It remains to combine the above bounds in (D.5). To conclude the proof, we need to apply Lemma 24 with $p = \log(5n^2)$. ☐

*Proof of Lemma 18.* First we rewrite the expression using the recurrent formula for $J_k^{\mathsf{b},1}$ proven in Lemma 12 and swapping the order of summation:

$$\mathbb{E}^{\mathsf{b}}\big[\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1} J_k^{\mathsf{b},1}\|^2\big] = \frac{1}{n}\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=1}^{n-1}\sum_{\ell=1}^{k}\alpha_\ell(w_\ell-1)\Gamma_{\ell+1:k}\mathbf{A}_\ell J_{\ell-1}^{\mathsf{b},0}\|^2\big] = \frac{1}{n}\sum_{\ell=1}^{n-1}\alpha_\ell^2\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=\ell}^{n-1}\Gamma_{\ell+1:k}\mathbf{A}_\ell J_{\ell-1}^{\mathsf{b},0}\|^2\big]$$
$$\le \frac{1}{n}\,\mathrm{C}_{\mathbf{A}}^2\sum_{\ell=1}^{n-1}\alpha_\ell^2\|\sum_{k=\ell}^{n-1}\Gamma_{\ell+1:k}\|^2\mathbb{E}^{\mathsf{b}}\big[\|J_{\ell-1}^{\mathsf{b},0}\|^2\big] \overset{(a)}{\le} n^{-1}\,\mathrm{C}_{\mathbf{A}}^2\,\mathsf{C}_1^2\big(c_0+\frac{2}{a(1-\gamma)}\big)^2\sum_{\ell=1}^{n-1}\mathbb{E}^{\mathsf{b}}\big[\|J_{\ell-1}^{\mathsf{b},0}\|^2\big]\,.$$

Here in (a) we applied Lemma 1 and Lemma 25 with $b=a/2$. Now we will provide a bound for $\mathbb{E}^{\mathsf{b}}\big[\|J_{\ell-1}^{\mathsf{b},0}\|^2\big]$ using a technique similar to the written above. Lemma 12 and Lemma 9 imply that

$$\mathbb{E}^{\mathsf{b}}\big[\|J_{\ell-1}^{\mathsf{b},0}\|^2\big] = \mathbb{E}^{\mathsf{b}}\big[\|\sum_{j=1}^{\ell-1}\alpha_j(w_j-1)\Gamma_{j+1:\ell-1}\tilde{\varepsilon}_j\|^2\big] \le \sum_{j=1}^{\ell-1}\alpha_j^2\|\Gamma_{j+1:\ell-1}\|^2\|\tilde{\varepsilon}_j\|^2 \le \mathsf{C}_1^2\mathsf{C}_9^2\sum_{j=1}^{\ell-1}\alpha_j^2\prod_{t=j+1}^{\ell-1}(1-\frac{a\alpha_t}{2})\,.$$

Therefore, we obtain using Lemma 25:

$$\mathbb{E}^{\mathsf{b}}\big[\|J_{\ell-1}^{\mathsf{b},0}\|^2\big] \le \mathsf{C}_1^2\mathsf{C}_9^2\alpha_\ell\frac{12}{a}\,.$$

Introduce the constant

$$\mathsf{C}_{18}^2 = \mathrm{C}_{\mathbf{A}}^2\,\mathsf{C}_1^2\big(c_0+\frac{2}{a(1-\gamma)}\big)^2\frac{c_0}{1-\gamma}\mathsf{C}_9^2\frac{12}{a}\,.$$

Now we obtain that

$$\mathbb{E}^{\mathsf{b}}\big[\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n-1}J_k^{\mathsf{b},1}\|^2\big] \le \frac{\mathsf{C}_{18}^2}{n}(1-\gamma)\sum_{k=1}^{n-1}k^{-\gamma} \le \mathsf{C}_{18}^2 n^{-\gamma}$$

which concludes the proof. □

*Proof of Lemma 19.* First, note that $H_k^{(\mathsf{b},0)} - H_k^{(\mathsf{b},0,i)} = 0$ if $k<i$. On the other hand, for $k\ge i$ we get

$$H_k^{(\mathsf{b},0)} - H_k^{(\mathsf{b},0,i)} = \underbrace{\Gamma_{i+1:k}^{\mathsf{b}}(H_i^{(\mathsf{b},0)} - H_i^{(\mathsf{b},0,i)})}_{T_1^{(k)}} - \underbrace{\sum_{\ell=i+1}^{k}\alpha_\ell(w_\ell-1)\Gamma_{\ell+1:k}^{\mathsf{b}}\mathbf{A}_\ell(J_{\ell-1}^{(\mathsf{b},0)} - J_{\ell-1}^{(\mathsf{b},0,i)})}_{T_2^{(k)}}$$

For simplicity we introduce $v_{m:k} = \prod_{j=m}^{k}(1-a\alpha_j/8)$. Consider $T_1^{(k)}$. Note that the following decomposition holds:

$$T_1^{(k)} = \Gamma_{i+1:k}^{\mathsf{b}}(-\alpha_i\mathbf{A}_i(w_i-w_i')H_{i-1}^{(\mathsf{b},0)} - \alpha_i\mathbf{A}_i(w_i-w_i')J_{i-1}^{(\mathsf{b},0)})\,.$$

Hence, we get

$$\mathbb{E}^{\mathsf{b}}\big[n^{-1/2}\|\sum_{k=i}^{n-1}T_1^{(k)}\|\big] \le \mathsf{C}_3\,\mathrm{C}_{\mathbf{A}}\,n^{-1/2}\sqrt{\alpha_i}(\mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{J},0)} + \mathsf{C}_{12,1}^{(\mathsf{b},\mathrm{H},0)})\sum_{k=i}^{n-1}\alpha_i v_{i+1:k}\,.$$

Recall the definition of $\mathsf{C}^{(2)}_{5.29}$, $\mathsf{C}^{(1)}_{5.29}$ (5.29). Therefore, Lemma 25 and Lemma 4 imply that

$$\mathbb{E}^{\mathsf{b}}\big[n^{-1/2}\|\sum_{k=i}^{n-1} T_1^{(k)}\|\big] \leq \mathsf{C}_3\, \mathrm{C}_{\mathbf{A}}\, n^{-1/2}\sqrt{\alpha_i}(\mathsf{C}^{(\mathsf{b},\mathrm{J},0)}_{12,1} + \mathsf{C}^{(\mathsf{b},\mathrm{H},0)}_{12,1})\mathsf{C}^{(1)}_{5.29} \,.$$

Consider $T_2^{(k)}$. First, we note that

$$J^{(\mathsf{b},0)}_{\ell-1} - J^{(\mathsf{b},0,i)}_{\ell-1} = \alpha_i(w_i' - w_i)\Gamma_{i+1:\ell-1}\tilde{\varepsilon}_i \,.$$

Thus, we rewrite the sum and get

$$\sum_{k=i}^{n-1} T_2^{(k)} = \sum_{\ell=i+1}^{n-1} \underbrace{\alpha_i(w_\ell - 1)(w_i' - w_i)\sum_{k=\ell}^{n-1}\alpha_\ell \Gamma^{\mathsf{b}}_{\ell+1:k}\mathbf{A}_\ell\Gamma_{i+1:\ell-1}\tilde{\varepsilon}_i}_{U_\ell} \,.$$

Note that $U_\ell$ is a martingale-difference sequence w.r.t. the filtration $\mathcal{F}_\ell = \sigma(W_i, W_i', W_\ell, \ldots, W_{n-1})$. Minkowski's inequality and Proposition 3 reveal that

$$\{\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=\ell}^{n-1}\alpha_\ell\Gamma^{\mathsf{b}}_{\ell+1:k}\|^2\big]\}^{1/2} \leq \mathsf{C}^{(1)}_{5.29}\mathsf{C}_3 \,.$$

Therefore, we obtain

$$\mathbb{E}^{\mathsf{b}}\big[\|U_\ell\|^2\big] \leq \alpha_i^2(\mathsf{C}^{(1)}_{5.29}\mathsf{C}_3)^2\,\mathrm{C}^2_{\mathbf{A}}\,\mathsf{C}_1^2\mathsf{C}_9^2\prod_{j=i+1}^{\ell-1}(1 - a\alpha_j/2) \,.$$

Hence, we get using Lemma 4:

$$\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=i}^{n-1} T_2^{(k)}\|^2\big] \leq \alpha_i(\mathsf{C}^{(1)}_{5.29}\mathsf{C}_3)^2\,\mathrm{C}^2_{\mathbf{A}}\,\mathsf{C}_1^2\mathsf{C}_9^2\sum_{\ell=i}^{n-1}\alpha_i\prod_{j=i+1}^{\ell}(1 - a\alpha_j/2) \leq \alpha_i(\mathsf{C}^{(1)}_{5.29}\mathsf{C}_3)^2\,\mathrm{C}^2_{\mathbf{A}}\,\mathsf{C}_1^2\mathsf{C}_9^2\mathsf{C}^{(1)}_{5.29} \,.$$

Now we combine the obtained bounds with Minkowski's inequality and finish the proof:

$$\{\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=1}^{n-1} H_k^{(\mathsf{b},0)} - \sum_{k=1}^{n-1} H_k^{(\mathsf{b},0,i)}\|^2\big]\}^{1/2} \leq \sum_{j=1}^{2}\{\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=i}^{n-1} T_j^{(k)}\|^2\big]\}^{1/2} \leq \sqrt{\alpha_i}\mathsf{C}_{19} \,,$$

where we have set

$$\mathsf{C}_{19} = \mathrm{C}_{\mathbf{A}}\,\mathsf{C}_3(\mathsf{C}^{(\mathsf{b},\mathrm{J},0)}_{12,1} + \mathsf{C}^{(\mathsf{b},\mathrm{H},0)}_{12,1})\mathsf{C}^{(1)}_{5.29} + \mathrm{C}_{\mathbf{A}}(\mathsf{C}^{(1)}_{5.29})^{3/2}\mathsf{C}_3\mathsf{C}_1\mathsf{C}_9 \,.$$

□

*Proof of Lemma 20.* Applying Minkowski's inequality we obtain that

$$\{\mathbb{E}^{\mathsf{b}}[\|D^{\mathsf{b}} - D^{\mathsf{b},i}\|^2]\}^{1/2} \leq \frac{1}{\sqrt{n}\sqrt{\mathsf{C}_{16}}}\{\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=i}^{n-1}(J^{(\mathsf{b},0)}_{k,1} - J^{(\mathsf{b},0,i)}_{k,1})\|^2\big]\}^{1/2}$$

$$+ \frac{1}{\sqrt{n}\sqrt{\mathsf{C}_{16}}}\{\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=i}^{n-1}(J^{(\mathsf{b},0)}_{k,1} - J^{(\mathsf{b},0,i)}_{k,2})\|^2\big]\}^{1/2} + \frac{1}{\sqrt{n}\sqrt{\mathsf{C}_{16}}}\{\mathbb{E}^{\mathsf{b}}\big[\|\sum_{k=i}^{n-1}(H^{(\mathsf{b},0)}_{k,1} - H^{(\mathsf{b},0,i)}_{k,1})\|^2\big]\}^{1/2} \,.$$

Consider the first term. Note that

$$J_{k,1}^{(\mathsf{b},0)} - J_{k,1}^{(\mathsf{b},0,i)} = \alpha_i (w_i' - w_i)(\Gamma_{i+1:k} - G_{i+1:k})\varepsilon_i \; .$$

Thus, using the definition of $\Omega_5$ we get

$$\| \sum_{k=i}^{n-1} (J_{k,1}^{(\mathsf{b},0)} - J_{k,1}^{(\mathsf{b},0,i)}) \| \leq |w_i' - w_i| \mathsf{C}_{17} \sqrt{\alpha_i} \log(5n) \; .$$

Hence, it holds that

$$\{ \mathbb{E}^{\mathsf{b}} \big[ \| \sum_{k=i}^{n-1} (J_{k,1}^{(\mathsf{b},0)} - J_{k,1}^{(\mathsf{b},0,i)}) \|^2 \big] \}^{1/2} \leq \sqrt{2} \mathsf{C}_{17} \sqrt{\alpha_i} \log(5n) \; .$$

On the other hand, we obtain the following representation for $J_{k,2}^{(\mathsf{b},0)} - J_{k,2}^{(\mathsf{b},0,i)}$:

$$J_{k,2}^{(\mathsf{b},0)} - J_{k,2}^{(\mathsf{b},0,i)} = \alpha_i (w_i' - w_i) \Gamma_{i+1:k} \mathbf{A}_i (\theta_{i-1} - \theta^\star) \; .$$

Introduce the following notation

$$u_{m:k} = \prod_{j=m}^{k} (1 - a\alpha_j/2) \; .$$

Therefore, applying Minkowski's inequality, Lemma 4, Lemma 25 and using the definition of $\Omega_1$, we obtain the following:

$$\begin{aligned}
\{ \mathbb{E}^{\mathsf{b}} \big[ \| \sum_{k=i}^{n-1} (J_{k,2}^{(\mathsf{b},0)} - J_{k,2}^{(\mathsf{b},0,i)}) \|^2 \big] \}^{1/2} &\leq \sqrt{2} \mathsf{C}_1 \, \mathrm{C}_{\mathbf{A}} \sum_{k=i}^{n-1} \alpha_i u_{i+1:k} (\kappa_Q^{1/2} \mathrm{e}^2 u_{1:i-2} \|\theta_0 - \theta^\star\| + 2\mathrm{e} \log(5n) \kappa_Q^{1/2} \|\varepsilon\|_\infty \sqrt{\alpha_i}) \\
&\leq \alpha_i \frac{\sqrt{2} \kappa_Q^{1/2} \mathrm{e}^2 \mathsf{C}_1 \, \mathrm{C}_{\mathbf{A}} \, \|\theta_0 - \theta^\star\|}{(1 - a/2)^2} \mathsf{C}_{5.29}^{(2)} + 2\sqrt{2} \mathrm{e} \mathsf{C}_1 \, \mathrm{C}_{\mathbf{A}} \, \kappa_Q^{1/2} \|\varepsilon\|_\infty \mathsf{C}_{5.29}^{(1)} \log(5n) \sqrt{\alpha_i} \; .
\end{aligned}$$

Hence, applying Lemma 19 and gathering similar terms the proof follows. □

# E Proofs of products of random matrices

*Proof of Proposition 3.* Our proof relies on the auxiliary result of Lemma 22 below together with the blocking technique. Indeed, let us represent

$$k - m = Nh + r \; ,$$

where $r < h$ and $h = h(n)$ is a block size defined in (4.3). Then we obtain, using the independence of bootstrap weights $w_{m+1}, \ldots, w_k$, that

$$\begin{aligned}
\{ \mathbb{E}^{\mathsf{b}} [ \| \Gamma_{m+1:k}^{\mathsf{b}} \|^2 ] \}^{1/2} &\leq \sqrt{\kappa_Q} \{ \mathbb{E}^{\mathsf{b}} [ \| \Gamma_{m+1:k}^{\mathsf{b}} \|_Q^2 ] \}^{1/2} \\
&= \sqrt{\kappa_Q} \prod_{j=1}^{N} \big\{ \mathbb{E}^{\mathsf{b}} [ \| \Gamma_{m+1+(j-1)h:m+1+jh}^{\mathsf{b}} \|_Q^2 ] \big\}^{1/2} \big\{ \mathbb{E}^{\mathsf{b}} [ \| \Gamma_{m+1+Nh:k}^{\mathsf{b}} \|_Q^2 ] \big\}^{1/2} \\
&\leq \sqrt{\kappa_Q} \exp\big\{ -\frac{a}{4} \sum_{\ell=m+1}^{k} \alpha_\ell \big\} \big\{ \mathbb{E}^{\mathsf{b}} [ \| \Gamma_{m+1+Nh:k}^{\mathsf{b}} \|_Q^2 ] \big\}^{1/2} \exp\big\{ \frac{a}{4} \sum_{\ell=m+1+Nh:k}^{k} \alpha_\ell \big\} \; .
\end{aligned}$$

In the last inequality we applied Lemma 22 to each of the blocks of length $h$ in the first bound. It remains to upper bound the residual terms. Since the remainder block has length less then $h$, we have due to (E.5) (which holds according to A 5), that

$$\exp\Big\{\frac{a}{4}\sum_{\ell=m+1+Nh:k}^{k}\alpha_\ell\Big\} \le \exp\Big\{\frac{\alpha_\infty a}{4}\Big\} \le \mathrm{e}^{1/8}\;,$$

where the last inequality is due to Proposition 1. Next,

$$\begin{aligned}\big\{\mathbb{E}^{\mathsf{b}}[\|\Gamma^{\mathsf{b}}_{m+1+Nh:k}\|^2]\big\}^{1/2} &\le \kappa_Q \prod_{\ell=m+1+Nh:k}^{k}\{\mathbb{E}^{\mathsf{b}}[\|(\mathrm{I}-\alpha_\ell w_\ell \mathbf{A}_\ell)\|^2]\}^{1/2} \\ &\le \kappa_Q \prod_{\ell=m+1+Nh:k}^{k}\{\mathbb{E}^{\mathsf{b}}[1+2\alpha_\ell|w_\ell|\,\mathrm{C}_{\mathbf{A}}+\alpha_\ell^2 w_\ell^2\,\mathrm{C}_{\mathbf{A}}^2]\}^{1/2}\;.\end{aligned}$$

Since $\mathbb{E}[|w_\ell|] \le \mathbb{E}^{1/2}[w_\ell^2] = \big\{(\mathbb{E}[w_\ell])^2 + \mathrm{Var}[w_\ell]\big\}^{1/2} = \sqrt{2}$ , we get from previous bound

$$\begin{aligned}\big\{\mathbb{E}^{\mathsf{b}}[\|\Gamma^{\mathsf{b}}_{m+1+Nh:k}\|^2]\big\}^{1/2} &\le \kappa_Q \prod_{\ell=m+1+Nh:k}^{k}(1+2\sqrt{2}\alpha_\ell\,\mathrm{C}_{\mathbf{A}}+2\alpha_\ell^2\,\mathrm{C}_{\mathbf{A}}^2)^{1/2} \\ &\le \kappa_Q \exp\big\{\sqrt{2}\,\mathrm{C}_{\mathbf{A}}\sum_{\ell=m+1+Nh:k}^{k}\alpha_\ell\big\} \le \kappa_Q\exp\{\sqrt{2}\,\mathrm{C}_{\mathbf{A}}\frac{c_0 h}{k_0^\gamma}\} \le \kappa_Q \mathrm{e}\;,\end{aligned}$$

where in the last line we additionally used A 5 and the inequality

$$\sum_{\ell=m+1+Nh}^{k}\alpha_\ell \le \sum_{\ell=1}^{h}\alpha_\ell \le c_0\int_{k_0}^{k_0+h} x^{-\gamma}\,dx = c_0\frac{(h+k_0)^{1-\gamma}-k_0^{1-\gamma}}{1-\gamma} \le c_0\frac{h}{k_0^\gamma}\;. \tag{E.1}$$

□

*Proof of Lemma 22.* Let $h \in \mathbb{N}$ be a block length given in (4.3). Then the product $\Gamma^{\mathsf{b}}_{m+1:m+h}$ writes as

$$\Gamma^{\mathsf{b}}_{m+1:m+h} = \mathrm{I} - \sum_{\ell=m+1}^{m+h}\alpha_\ell\mathbf{A}_\ell - \mathbf{S} + \mathbf{R} = \mathrm{I} - \sum_{\ell=m+1}^{m+h}\alpha_\ell\bar{\mathbf{A}} - \sum_{\ell=m+1}^{m+h}\alpha_\ell(\mathbf{A}_\ell-\bar{\mathbf{A}}) - \mathbf{S} + \mathbf{R}\;, \tag{E.2}$$

where $\mathbf{S} = \sum_{\ell=m+1}^{m+h}\alpha_\ell(w_\ell-1)\mathbf{A}_\ell$ is a linear statistics in $\{w_\ell\}_{\ell=m+1}^{m+h}$, and the remainder $\mathbf{R}$ collects the higher-order terms:

$$\mathbf{R} = \sum_{r=2}^{h}(-1)^r\sum_{(i_1,\ldots,i_r)\in\mathsf{I}_r}\prod_{u=1}^{r}\alpha_{i_u}w_{i_u}\mathbf{A}_{i_u}\;.$$

with $\mathsf{I}_r = \{(i_1,\ldots,i_r)\in\{m+1,\ldots,m+h\}^r : i_1<\cdots<i_r\}$. We first consider the contracting part in matrix $Q$-norm. Indeed, applying (3.3), we obtain that

$$\|\mathrm{I} - \textstyle\sum_{\ell=m+1}^{m+h}\alpha_\ell\bar{\mathbf{A}}\|_Q^2 \le 1 - a\textstyle\sum_{\ell=m+1}^{m+h}\alpha_\ell\;, \tag{E.3}$$

The definition of block size $h$ combined with an integral bound (E.1) guarantees that that $\sum_{\ell=m+1}^{m+h} \alpha_\ell \leq \alpha_\infty$, where $\alpha_\infty$ is defined in (3.2). Thus, we get from (E.3) that the following bound holds

$$\|\mathrm{I} - \textstyle\sum_{\ell=m+1}^{m+h} \alpha_\ell \bar{\mathbf{A}}\|_Q \leq 1 - (a/2) \textstyle\sum_{\ell=m+1}^{m+h} \alpha_\ell \;.$$

Now we need to estimate the remainders in the representation (E.2). On the set $\Omega_4$, it holds that

$$\| \textstyle\sum_{\ell=m+1}^{m+h} \alpha_\ell (\mathbf{A}_\ell - \bar{\mathbf{A}})\|_Q \leq 2\,\mathrm{C}_{\mathbf{A}} \sqrt{\kappa_Q} \Big\{ \sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \Big\}^{1/2} \log(10n^3 d) \;.$$

Moreover, it is straightforward to check that

$$\mathbb{E}^{\mathsf{b}}[\|\mathbf{S}\|_Q^2] \leq \mathrm{C}_{\mathbf{A}}^2\, \kappa_Q \sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \;.$$

In order to bound the remainder term $\mathbf{R}$, we note that for any $i \in \{1,\ldots,n\}$, $\mathbb{E}^{\mathsf{b}}[|w_{i_u}|] \leq \sqrt{2}$, and

$$\begin{aligned} \mathbb{E}^{\mathsf{b}}[\|\mathbf{R}\|_Q] &\leq \sqrt{\kappa_Q} \sum_{r=2}^{h} \binom{h}{r} \alpha_{m+1}^r 2^{r/2}\, \mathrm{C}_{\mathbf{A}}^r \leq 2\alpha_{m+1}^2\, \mathrm{C}_{\mathbf{A}}^2 \sqrt{\kappa_Q} \sum_{r=0}^{h-2} \binom{h}{r+2} \alpha_{m+1}^r 2^{r/2}\, \mathrm{C}_{\mathbf{A}}^r \\ &\leq \alpha_{m+1}^2 h^2\, \mathrm{C}_{\mathbf{A}}^2 \sqrt{\kappa_Q} \exp\big\{\sqrt{2} h \alpha_{m+1}\, \mathrm{C}_{\mathbf{A}}\big\} \leq \alpha_{m+1}^2 h^2\, \mathrm{C}_{\mathbf{A}}^2 \sqrt{\kappa_Q} \mathrm{e} \;. \end{aligned}$$

To complete the proof it remains to set the parameter $h$ in such a way that we can guarantee the following:

$$\mathrm{C}_{\mathbf{A}} \sqrt{\kappa_Q} \Big\{ \sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \Big\}^{1/2} \big(1 + 2\log(10n^3 d)\big) + \alpha_{m+1}^2 h^2\, \mathrm{C}_{\mathbf{A}}^2 \sqrt{\kappa_Q} \mathrm{e} \leq \frac{a}{4} \sum_{\ell=m+1}^{m+h} \alpha_\ell \;. \tag{E.4}$$

Now it remains to ensure that our choice of $h$ satisfies

$$\begin{cases} \mathrm{C}_{\mathbf{A}} \sqrt{\kappa_Q} \big\{ \sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \big\}^{1/2} \big(1 + 2\log(10n^3 d)\big) \leq \frac{a}{8} \sum_{\ell=m+1}^{m+h} \alpha_\ell \\ \alpha_{m+1}^2 h^2\, \mathrm{C}_{\mathbf{A}}^2 \sqrt{\kappa_Q} \mathrm{e} \leq \frac{a}{8} \sum_{\ell=m+1}^{m+h} \alpha_\ell \;. \end{cases}$$

Using an integral bound, we get

$$\sum_{\ell=m+1}^{m+h} \alpha_\ell \geq c_0 \frac{(m+k_0+h+1)^{1-\gamma} - (m+k_0+1)^{1-\gamma}}{1-\gamma} \geq c_0 (m+k_0+1)^{1-\gamma} \frac{(2^{1-\gamma}-1)h}{(m+k_0+1)} \geq \frac{c_0(2-2^\gamma)h}{(m+k_0)^\gamma} \;, \tag{E.5}$$

Similarly, one can check that

$$\sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \leq c_0^2 \frac{(m+k_0)^{1-2\gamma} - (m+k_0+h)^{1-2\gamma}}{2\gamma - 1} \leq c_0^2 \frac{h}{(m+k_0)(m+k_0+h)^{2\gamma-1}} \leq c_0^2 \frac{h}{(m+k_0)^{2\gamma}} \;. \tag{E.6}$$

Hence, taking into account (E.5) and (E.6), the inequality (E.4) would follow from the bounds

$$\begin{cases} \mathrm{C}_{\mathbf{A}} \sqrt{\kappa_Q} \big\{ \sum_{\ell=m+1}^{m+h} \alpha_\ell^2 \big\}^{1/2} \big(1 + 2\log(10n^3 d)\big) \leq \frac{ac_0(2-2^\gamma)}{8} \frac{h}{(m+k_0)^\gamma} \\ \alpha_{m+1}^2 h^2\, \mathrm{C}_{\mathbf{A}}^2 \sqrt{\kappa_Q} \mathrm{e} \leq \frac{ac_0(2-2^\gamma)}{8} \frac{h}{(m+k_0)^\gamma} \;. \end{cases}$$

Note that the latter inequalities follow from A 5. Thus, all previous inequalities will be fulfilled. Hence, the following holds:

$$\big\{\mathbb{E}^{\mathsf{b}}[\|\Gamma^{\mathsf{b}}_{m+1:m+h}\|^2_Q]\big\}^{1/2} \leq 1-(a/4)\sum_{\ell=m+1}^{m+h}\alpha_\ell \;,$$

and the statement follows from an elementary inequality $1+x \leq \mathrm{e}^x$. □

# F Proof of Lemma 2

*Proof of Lemma 23.* We follow the approach of [54, pp. 25–26]. By the definition of $S_t$

$$S_t = \sum_{j=t+1}^{n-1} (\alpha_t - \alpha_j)\, G_{t+1:j-1},$$

we have

$$\|S_t\| \;\leq\; \sum_{j=t+1}^{n-1} (\alpha_t - \alpha_j)\, \|G_{t+1:j-1}\| \;\leq\; \sqrt{\kappa_Q}\, c_0 \sum_{j=t+1}^{n-1} \Big[(j+k_0)^{-\gamma} - (j+1+k_0)^{-\gamma}\Big] e^{-\frac{ac_0}{2}\, g_{t+1:j}},$$

where

$$g_{t+1:j} \;=\; \sum_{\ell=t+1}^{j} (\ell+k_0)^{-\gamma} \;\geq\; \int_{t+k_0}^{j+1+k_0} x^{-\gamma}\, dx = \frac{(j+1+k_0)^{1-\gamma} - (t+k_0)^{1-\gamma}}{1-\gamma}.$$

Set

$$a_{t:j} = 1 + (1-\gamma)\, g_{t:j}, \qquad s_{t:j} = \frac{ac_0}{2(1-\gamma)}\, a_{t:j}.$$

Then one checks

$$(j+k_0)^{-\gamma} - (j+1+k_0)^{-\gamma} \;\leq\; \frac{a_{t:j+1} - a_{t:j}}{1-\gamma}\, (t+k_0)^{-\gamma}, \qquad e^{-\frac{ac_0}{2}\, g_{t+1:j}} = e^{-s_{t:j}}\, e^{\frac{ac_0}{2(t+k_0)^{\gamma}}}.$$

Hence

$$\|S_t\| \;\leq\; \frac{\sqrt{\kappa_Q}\, c_0}{1-\gamma}\, e^{\frac{ac_0}{2} + \frac{ac_0}{2(1-\gamma)}}\, (t+k_0)^{\gamma-1} \sum_{j=t}^{n-2} \big(a_{t:j+1} - a_{t:j}\big)\, a_{t:j}^{\frac{\gamma}{1-\gamma}}\, e^{-s_{t:j}}.$$

Since $\phi(x) = x^{\frac{\gamma}{1-\gamma}} e^{-x}$ attains its maximum at $x_\gamma = \frac{\gamma}{1-\gamma}$, a discrete summation-by-parts gives

$$\sum_{j=t}^{n-2} \big(a_{t:j+1} - a_{t:j}\big)\, a_{t:j}^{\frac{\gamma}{1-\gamma}}\, e^{-s_{t:j}} \;\leq\; \max\{1, \phi(x_\gamma)\}\Big(\tfrac{ac_0}{2} + x_\gamma\Big) + \int_{x_\gamma}^{\infty} \phi(x)\, dx.$$

Collecting constants yields

$$\|S_t\| \;\leq\; \sqrt{\kappa_Q}\, (t+k_0)^{\gamma-1}\, \mathsf{C}_{23},$$

as claimed. □