Towards a conjecture concerning minors of Fourier matrices

Andrei Caragea
MIDS, MGF
Katholische Universität Eichstätt-Ingolstadt
Ingolstadt, Germany
andrei.caragea@ku.de

Götz Pfander
MIDS, MGF
Katholische Universität Eichstätt-Ingolstadt
Ingolstadt, Germany
pfander@ku.de

Dae Gwan Lee
Department of Applied Mathematics
Pukyong National University
Busan, Republic of Korea
daegwan@pknu.ac.kr

Romanos-Diogenes Malikiosis
Department of Mathematics
Aristotle University
Thessaloniki, Greece
romanos@math.auth.gr

***Abstract*—A result by Chebotarëv states that all minors of a prime-sized Fourier matrix are non-zero. The authors have conjectured that for every Fourier matrix of composite size there exists a permutation of the columns that ensures that all *principal* minors of the resulting matrix are non-zero. After correcting numerical issues, we now know that this conjecture is false for $N = 16$. We present some partial results and numerics relating to this conjecture.**

***Index Terms*—Fourier matrix, minors, Chebotarëv's theorem, exponential basis, permutation.**

I. Introduction

The study of submatrices of the Fourier matrix is a classic topic in harmonic analysis with Chebotarëv's theorem (see, e.g., [5], [10] for recent proofs) establishing that *all* submatrices of a Fourier matrix of prime size are invertible. Classical applications range from compressed sensing (e.g., [2]) to uncertainty principles relating to the Fourier transform (e.g., [8], [10]).

More recently, the invertibility of Fourier submatrices has been connected in [6] with the existence of Riesz bases of exponentials for finite unions of intervals. This idea has been extended in [3], [9] to so-called hierarchical Riesz bases of exponentials for unions of intervals (i.e., bases where taking a partial union of the involved frequencies gives rise to a Riesz exponential basis for the corresponding partial union of intervals). In [3], the authors showed that such a hierarchical Riesz basis of exponentials exists for a finite union of intervals as long as the endpoints of the intervals are rationally independent. The authors realised that the same methodology can be used to prove the existence of hierarchical Riesz bases of exponentials in the case that all the intervals involved have *rational* endpoints, as long as the following conjecture is true.

Conjecture 1: For every $N \in \mathbb{N}$ there exists a permutation of the columns of the $N \times N$ Fourier matrix such that all principal minors of the permuted matrix are non-zero.

Supported by DFG grants PF 450/11-1 and CA 3683/1-1

In fact, we conjecture (along with the authors of [1]) that in the case when N is square-free (i.e., not divisible by any p^2 with p prime), then the identity permutation suffices.

The remainder of this paper is structured as follows. Section II contains definitions, notation and preliminaries. Section III contains some partial results towards Conjecture 1 obtained by the authors. Section IV contains technical results involving determinants of Fourier submatrices and permutations. Section V discusses numerical experiments performed concerning Conjecture 1 and details about the $N = 16$ case.

II. Preliminaries

Let $N \in \mathbb{N}$. Denote $\underline{N} := \{0, 1, \ldots, N-1\}$ and $\underline{N}^* := \{n \in \underline{N} : n \text{ is coprime with } N\}$. Note that $\#\underline{N}^* = \varphi(N)$, where φ is the Euler totient function. Let Σ_N denote the set of permutations acting on $\underline{N}$ and let $\omega := e^{2\pi i/N}$ be a principal N-th root of unity. The $N \times N$ Fourier matrix is defined as

$$\mathcal{F}_N := \left(\omega^{k\cdot\ell}\right)_{k,\ell\in\underline{N}}.$$

Given a permutation $\sigma \in \Sigma_N$, we define the permuted Fourier matrix as

$$\mathcal{F}_N^\sigma := \left(\omega^{k\cdot\sigma(\ell)}\right)_{k,\ell\in\underline{N}}.$$

If the permutation is the identity, we will simply write $\mathcal{F}_N$. If $K, L \subset \underline{N}$ satisfy $\#K = \#L$ (i.e., they have the same cardinality), we define the associated submatrix and, respectively, minor of the permuted Fourier matrix by

$$\mathcal{F}_N^\sigma[K, L] := \left(\omega^{k\cdot\sigma(\ell)}\right)_{k\in K,\ell\in L}$$

and

$$\Delta_N^\sigma[K, L] := \det \mathcal{F}_N^\sigma[K, L].$$

When dealing with principal minors, i.e., when $K = L$, we will simply write $\Delta_N^\sigma[K]$. With this notation, the conjectured statements in Section I become:

For every N, there exists a permutation σ (depending on N) such that $\Delta_N^\sigma[K] \neq 0$ for all $K \subset \underline{N}$. If N is square-free, the identity permutation suffices.

III. Partial results

In [4], we proved the following two results.

Theorem 1: If $N \geq 4$ is square-free, then all principal minors of $\mathcal{F}_N$ of size $r \times r$ are non-zero for $r \in \{2, 3, N-3, N-2\}$.

The result is based on elementary matrix manipulations (as described in Section IV) and properties of complex numbers. It is important to point out that the 2×2 case is independently discussed in [1].

Theorem 2: If $N \geq 4$ is not square-free, then for any $2 \leq r \leq N-2$, there exists a zero $r \times r$ principal minor of $\mathcal{F}_N$.

The main idea of this result is to construct a minor with index set containing multiples of pm if $N = p^2 m$. The corresponding rows can be shown to be linearly dependent.

Theorems 1 and 2 make use of a result concerning complementary principal minors [7].

In a manuscript currently under development, we prove, based on Galois theory and a refinement on Zhang's result from [11]

Theorem 3: Let $N = pq$ with p, q distinct primes. If $q \in \{2, 3, 5, 7\}$, then all principal minors of $\mathcal{F}_N$ are non-zero.

and

Theorem 4: Let $N = p_1 p_2 \dots p_k$, where $p_1 < p_2 < \cdots < p_k$ are primes. For $j \geq 2$, set $P_j := p_1 p_2 \dots p_{j-1}$. If for all $2 \leq j \leq k$ we have $p_j > \left(\frac{P_j}{2}\right)^{P_j \varphi(P_j)/4}$, where φ denotes the Euler totient function, then all principal minors of $\mathcal{F}_N$ are non-zero.

IV. Technicalities concerning permutations and cyclotomic polynomials

Observe that the determinant of any submatrix of $\mathcal{F}_N$ can be viewed as an integer coefficient polynomial evaluated at the root of unity ω. This is an immediate consequence of the Laplace expansion of the determinant. We will write

$$\mathcal{P}_{K,L}(\omega) = \Delta_N^\sigma[K, L]$$

or simply $\mathcal{P}_K(\omega)$ if we are dealing with principal minors, i.e., when $K = L$.

A fundamental fact of Galois theory is that if ψ_N is the N-th cyclotomic polynomial (i.e., the minimal integer coefficient polynomial that has ω as a root), then the roots of ψ_N are precisely ω^r with $r \in \underline{N}^*$. Therefore, if an integer coefficient polynomial has ω as a root, it must have all N-th principal roots of unity as roots. In particular, this implies

$$\mathcal{P}_{K,L}(\omega) = 0 \Leftrightarrow \mathcal{P}_{rK,sL}(\omega) = 0 \text{ for all } r, s \in \underline{N}^*.$$

This observation, together with elementary row and column operation considerations, motivate the introduction of two special classes of permutations:

$$\begin{aligned} a_k(x) &= x + k, \quad && \text{for } k \in \underline{N} \\ m_r(x) &= r \cdot x, \quad && \text{for } r \in \underline{N}^*. \end{aligned}$$

Note that m_r is a permutation if and only if r is a unit in $\mathbb{Z}_N$, i.e., $r \in \underline{N}^*$. The operations are performed modulo N.

The following lemma is elementary.

Lemma 1:

a. m_r always fixes $0 \in \underline{N}$.
b. a_k and m_r almost commute, $m_r \circ a_k \equiv a_{rk} \circ m_r$.
c. Let $\sigma \in \Sigma_N$ be arbitrary. If $k, k' \in \underline{N}$ and $r, r' \in \underline{N}^*$ are such that $a_k \circ m_r \circ \sigma \equiv a_{k'} \circ m_{r'} \circ \sigma$ or $\sigma \circ m_r \circ a_k \equiv \sigma \circ m_{r'} \circ a_{k'}$, then we necessarily have that $k = k'$ and $r = r'$.

Let us denote $\sigma_{s,\ell}^{k,r} := a_k \circ m_r \circ \sigma \circ m_s \circ a_\ell$ for $\sigma \in \Sigma_N$, $k, \ell \in \underline{N}$ and $r, s \in \underline{N}^*$. If $k = 0$ and $r = 1$, we will simply write $\sigma_{s,\ell}$. Similarly, if $s = 1$ and $\ell = 0$, we will simply write $\sigma^{k,r}$ for the compositions. Additionally, we write $\mathcal{O}(\sigma)$ for the orbit of σ under post-compositions, i.e.,

$$\mathcal{O}(\sigma) := \{\sigma^{k,r} : k \in \underline{N},\, r \in \underline{N}^*\}.$$

As a consequence of Lemma 1, the relation $\tau \sim \sigma$ if $\tau = \sigma^{k,r}$ is an equivalence relation and each equivalence class $\mathcal{O}(\sigma)$ has the same cardinality, namely $N\varphi(N)$.

We call σ a *good permutation* if all principal minors of the σ-permuted Fourier matrix are non-zero, i.e., if $\Delta_N^\sigma[K] \neq 0$ for all $K \subset \underline{N}$.

Proposition 1: σ is a good permutation if and only if all $\sigma_{s,\ell}^{k,r}$ are good permutations.

Proof: In view of Lemma 1, it suffices to show the separate implications σ good $\Rightarrow \sigma^{k,r}$ good and σ good $\Rightarrow \sigma_{s,\ell}$ good.

$\sigma \Rightarrow \sigma^{k,r}$: Let $K = \{x_1, x_2, \dots, x_m\} \subset \underline{N}$ be arbitrary. Since we are assuming that σ is a good permutation, we have that $\Delta_N^\sigma[K] \neq 0$. Let us denote by $\mathcal{P}_K$ the integer coefficient polynomial that satisfies $\mathcal{P}_K(\omega) = \Delta_N^\sigma[K]$. Using row operations on the determinant, we get

$$\Delta_N^{\sigma^{k,r}}[K] = \omega^{k(x_1 + \cdots + x_m)} \mathcal{P}_K(\omega^r),$$

from which the conclusion follows.

$\sigma \Rightarrow \sigma_{s,\ell}$: Again, let $K = \{x_1, x_2, \dots, x_m\} \subset \underline{N}$ be arbitrary. Define the subset

$$L := s(K + \ell) = \{s x_j + s\ell : 1 \leq j \leq m\}.$$

Since σ is assumed to be a good permutation, it follows that $\mathcal{P}_L(\omega) \neq 0$. This time using elementary column operations, we see that

$$\Delta_N^{\sigma_{s,\ell}}[K] = \omega^{-\ell s(x_1 + \cdots + x_m + m\ell)} P_L(\omega^{s^{-1}}).$$

■

Another line of investigation concerns the restrictions imposed on a permutation σ by ensuring that small principal minors of $\mathcal{F}_N^\sigma$ are non-zero.

Proposition 2: Let $N = n \cdot m$ for some naturals $n, m \geq 2$. Let σ be a permutation on $\underline{N}$ such that there exist $a, b \in \underline{N}$ satisfying $a \equiv b \bmod n$ and $\sigma(a) \equiv \sigma(b) \bmod m$. Then σ is *not* a good permutation.

Proof: Let us consider $K := \{a, b\}$ and let us write $a = a'n + c$ and $b = b'n + c$ for some $c \in \underline{n}$. For simplicity, let $\xi := \omega^n = e^{2\pi i n/N}$. Note that this implies $\xi^m = 1$.

$$\Delta_N^\sigma[K] = \begin{vmatrix} \omega^{a\sigma(a)} & \omega^{a\sigma(b)} \\ \omega^{b\sigma(a)} & \omega^{b\sigma(b)} \end{vmatrix} = \begin{vmatrix} \xi^{a'\sigma(a)}\omega^{c\sigma(a)} & \xi^{a'\sigma(b)}\omega^{c\sigma(b)} \\ \xi^{b'\sigma(a)}\omega^{c\sigma(a)} & \xi^{b'\sigma(b)}\omega^{c\sigma(b)} \end{vmatrix}$$
$$= \omega^{c(\sigma(a)+\sigma(b))} \begin{vmatrix} (\xi^{\sigma(a) \bmod m})^{a'} & (\xi^{\sigma(b) \bmod m})^{a'} \\ (\xi^{\sigma(a) \bmod m})^{b'} & (\xi^{\sigma(b) \bmod m})^{b'} \end{vmatrix}$$
$$= 0 \quad \text{since } \sigma(a) \bmod m = \sigma(b) \bmod m.$$

■

***Corollary** 1:* Let $N = n \cdot m$ as above and assume that σ is a good permutation. Then there must exist permutations $\tau_0, \tau_1, \ldots, \tau_{n-1} \in \Sigma_m$ and $\rho_0, \rho_1, \ldots, \rho_{m-1} \in \Sigma_n$ such that

$$\sigma(kn+\ell) = \tau_\ell(k)+m\cdot\rho_{\tau_\ell(k)}(\ell) \quad \text{for all } k \in \underline{m}, \ell \in \underline{n}. \quad (1)$$

***Theorem** 5:* If the permutation σ is good, then each of the permutations τ_ℓ and ρ_k from (1) must be good permutations in Σ_m and Σ_n respectively.

Proof: As usual, set $\omega := e^{2\pi i/N}$.

First, let us fix $\ell \in \underline{n}$ arbitrarily. Consider the set $K := \{\ell, n+\ell, \ldots, (m-1)n+\ell\}$. Then

$$\begin{aligned}\Delta_N^\sigma[K] &= \det\left(\omega^{(kn+\ell)\cdot\sigma(k'n+\ell)}\right)_{k,k'\in\underline{m}} \\ &= \det\left(\omega^{(kn+\ell)\cdot(\tau_\ell(k')+m\cdot\rho_{\tau_\ell(k')}(\ell))}\right)_{k,k'\in\underline{m}} \\ &= \det\left(\omega^{nk\tau_\ell(k')}\cdot\omega^{Nk\rho_{\tau_\ell(k')}(\ell)}\cdot\omega^{\ell\tau_\ell(k')}\cdot\omega^{m\ell\rho_{\tau_\ell(k')}(\ell)}\right) \\ &= \omega^\delta\cdot\det\left(\left(\omega^n\right)^{k\tau_\ell(k')}\right)_{k,k'\in\underline{m}} = \omega^\delta\cdot\det\mathcal{F}_m^{\tau_\ell},\end{aligned}$$

where, due to column operations,

$$\delta = \ell\sum_{k'=0}^{m-1}\tau_\ell(k')+m\ell\sum_{k'=0}^{m-1}\rho_{\tau_\ell(k')}(\ell) = \ell\Big(\sum_{j=0}^{m-1}j+m\sum_{j=0}^{m-1}\rho_j(\ell)\Big)$$

so, in particular, $|\omega^\delta| = 1$. It follows that any zero principal minor of $\mathcal{F}_m^{\tau_\ell}$ will give rise to a zero principal minor of $\mathcal{F}_N^\sigma$.

Now, let us fix $j \in \underline{m}$ arbitrarily. Note that for each $\ell \in \underline{n}$ there exists a unique $k_\ell \in \underline{m}$ such that $\tau_\ell(k_\ell) = j$. Set $L := \{k_0 n, k_1 n+1, \ldots, k_{n-1}n+n-1\}$ and compute

$$\begin{aligned}\Delta_N^\sigma[L] &= \det\left(\omega^{(k_\ell n+\ell)\cdot\sigma(k_{\ell'}n+\ell')}\right)_{\ell,\ell'\in\underline{n}} \\ &= \det\left(\omega^{(k_\ell n+\ell)\cdot(j+m\cdot\rho_j(\ell'))}\right)_{\ell,\ell'\in\underline{n}} \\ &= \det\left(\omega^{nk_\ell j}\cdot\omega^{Nk_\ell\rho_j(\ell')}\cdot\omega^{j\ell}\cdot\omega^{m\ell\rho_j(\ell')}\right)_{\ell,\ell'\in\underline{n}} \\ &= \omega^\Delta\cdot\det\left(\left(\omega^m\right)^{\ell\rho_j(\ell')}\right)_{\ell,\ell'\in\underline{n}} = \omega^\Delta\cdot\det\mathcal{F}_n^{\rho_j},\end{aligned}$$

where, due to row operations,

$$\Delta = j\sum_{\ell=0}^{n-1}\ell + nj\sum_{\ell=0}^{n-1}k_\ell.$$

Again, this implies that $|\omega^\Delta| = 1$, so any zero principal minor of $\mathcal{F}_n^{\rho_j}$ is a zero principal minor of $\mathcal{F}_N^\sigma$. ■

***Corollary** 2:* Assume that there are no good permutations for N. Then there are no good permutations for any multiple of N.

V. Numerical results and future work

The numerical results were obtained in $Python$ and C++ on personal hardware and the NHR@FAU cluster based on singular value decompositions.

Proposition 1 has two significant implications concerning the run time of numerical testing:

- The freedom to choose s, ℓ in $\sigma_{s,\ell}^{k,r}$ means that one only needs to check principal minors associated to $K = \{0, 1, x_1, x_2, \ldots, x_m\}$ where $x_1, \ldots, x_m \in \{2, 3, \ldots, N-1\}$ or $K = \{0, x_1, x_2, \ldots, x_m\}$ with $x_1, x_2, \ldots, x_m \in \underline{N} \setminus \underline{N}^*$.
- The choice of k, r allows one to only check one representative from each equivalence class $\mathcal{O}(\sigma)$ when attempting an exhaustive search for good permutations. In order to avoid memory issues with storing checked permutations, the code simply checks if the currently investigated permutation is first with respect to the lexicographic order in its equivalence class.

Furthermore, as a consequence of Jacobi's complementary determinants identity (see [7]), only minors of size at most $\lfloor\frac{N}{2}\rfloor \times \lfloor\frac{N}{2}\rfloor$ need to be checked to determine if a permutation is good.

The first numerical result obtained was generating exhaustive lists of good permutations for $N \leq 16$. After correcting a numerical precision issue in our earlier implementations, we realized that there are no good permutations for $N = 16$ and that this is the smallest number with this property. Note that in view of Corollary 2, this implies that there are infinitely many naturals for which Conjecture 1 fails, at least all multiples of 16. The most efficient way we've found of exhaustively generating the good permutations is based on a tree search type algorithm proposed by Florian Lange:

- Start with $K = \{0, 1\}$ and run through all possible choices $L = \{0, \ell\}$ for $\ell \in \{1, 2, \ldots, N-1\}$. The choice of the first element of L being 0 is justified by Proposition 1 and the above discussion.
- For each such pair (K, L), check numerically if $\Delta_N[K, L]$ is zero. This is achieved by computing the smallest singular value of $\mathcal{F}_N[K, L]$ with `numpy.linalg.svd` and assesing if it is smaller than a numerical threshold (usually chosen to be 10^{-10}). If $\Delta_N[K, L]$ is deemed to be zero, discard L.
- If L survived the previous step, append 2 to K and run through all possible extensions of L, i.e., $L = \{0, \ell, j\}$ for $1 \leq j \leq N-1$, $j \neq \ell$, such that L can be viewed as a truncation to the first 3 elements of a permutation that is a class representative with respect to the equivalence class discussed after Lemma 1 (e.g., by ensuring that the truncation comes from a 'lexicographically first' permutation).
- Check numerically all 2×2 and 3×3 submatrices of $\mathcal{F}_N[K, L]$ that involve the last row and column of $\mathcal{F}_N[K, L]$. Again, discard the L that yield singular such submatrices.

- Iterate this process by always appending the next consecutive index to K and any possible index to L that respects the uniqueness of truncations with respect to the equivalence class described before. For the sake of run time efficiency, only test submatrices that involve the last appended row and column and, in view of the complementarity result in [7], only check submatrices of size at most $\lfloor \frac{N}{2} \rfloor \times \lfloor \frac{N}{2} \rfloor$.
- When the check for $K = \underline{N}$ is complete, the collection of all remaining L gives the images of all the good permutations up to the equivalence relation described above. From this the full list of good permutations can easily be obtained by applying m_r and a_k permutations as described in Lemma 1.

For $2 \leq N \leq 16$, this yields the following table

N	# good	N	# good	N	# good
2	2	7	5040	12	38 880
3	6	8	2304	13	13!
4	16	9	46 656	14	51 685 200
5	120	10	43 400	15	23 079 600
6	144	11	11!	16	0

It is not clear at the moment whether the issue with $N = 16$ is strictly due to 16 being a power of 2. A natural question to ask is whether $N = 81$ also has no good permutations. However, this is definitely outside the scope of a numerical implementation. For this reason and in order to avoid the rather delicate issue of picking the correct numerical threshold in the above algorithm, we hope to be able to prove directly (i.e., 'with pen and paper') based on an extension of the ideas from Theorem 5 that $N = 16$ admits no good permutations and see how this generalizes. This is in the scope of a manuscript in preparation.

Additional results were obtained for $N \leq 36$:

- For $N \in 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$, the identity permutation suffices by Chebotarëv's theorem.
- For $N \in \{6, 10, 14, 15, 21, 22, 26, 30, 33, 34, 35\}$, the identity permutation was verified to be good numerically, consistent with the fact that in this case N is always square-free.
- For $N \in \{4, 9, 25\}$, the bit reversal permutation was verified to be good. For $N = p^2$ with p prime, the bit reversal permutation is defined as $ap + b \mapsto bp + a$ for all $a, b \in 0, 1, \ldots, p - 1$.
- For $N \in \{12, 18, 20, 28\}$, which are all of the form $p^2 q$ with p, q distinct primes, the permutation obtained by tensoring the identity permutation on $\mathbb{Z}_q$ with the bit reversal permutation on $\mathbb{Z}_{p^2}$ was verified to be good.
- For $N \in \{8, 24\}$ good permutations were found by random or exhaustive search based on the algorithm described above, but no purely algebraic way of generating good permutations has been identified yet.
- For $N \in \{16, 32\}$ there are no good permutations. As described before, the case $N = 16$ was exhaustively checked numerically and the case $N = 32$ follows from Corollary 2.
- For $N \in \{27, 36\}$ quick randomized searches have not yet yielded any good permutations, but we hope to have a clear answer for these cases soon, based on longer numerical searches.

It should be pointed out that whenever a good permutation was found numerically, it was also then, whenever feasible given the much longer run times, verified symbolically using `sympy` by checking the remainder of the determinants, written as polynomials evaluated at the root of unity, after dividing by the N-th cyclotomic polynomial.

Acknowledgment

The authors would like to thank Florian Lange and Aditi Bhushan for their invaluable help with the numerical experiments and the use of the NHR@FAU cluster. AC was supported by DFG Grant CA 3683/1-1.

References

[1] C. Cabrelli, U. Molter and F. Negreira, "Weaving Riesz bases", J. Fourier Anal. Appl., vol. 31, Paper No. 4, 20, 2025.
[2] E. Candés, J. Romberg and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information", IEEE Trans. Inform. Theory, vol. 52, pp. 489–509, 2006.
[3] A. Caragea and D. G. Lee, "A note on exponential Riesz bases", Sampl. Theory Signal Process. Data Anal., vol. 20, Paper No. 13, 14, 2022.
[4] A. Caragea and D. G. Lee, "On the principal minors of Fourier matrices", arXiv preprint, https://arxiv.org/abs/2409.09793.
[5] P. E. Frenkel, "Simple proof of Chebotarëv's theorem on roots of unity", arXiv preprint, https://arxiv.org/abs/math/0312398.
[6] G. Kozma and S. Nitzan, "Combining Riesz bases", Invent. Math., vol. 199, pp. 267–285, 2015
[7] M. Loukaki, "Chebotarev's theorem for roots of unity of square free order", arXiv preprint, https://arxiv.org/abs/2412.08600.
[8] R. Meshulam, "An uncertainty inequality for finite abelian groups", European J. Combin., vol. 27, pp. 63–67, 2006.
[9] G. Pfander, S. Revay and D. Walnut, "Exponential bases for partitions of intervals", Appl. Comput. Harmon. Anal., vol. 68, Paper No. 101607, 22, 2024.
[10] T. Tao, "An uncertainty principle for cyclic groups of prime order," Math. Res. Lett., vol. 12, pp. 121–127, 2005.
[11] G. Zhang, "On the Chebotarëv theorem over finite fields", Finite Fields Appl., vol. 56, pp. 97–108, 2019