Temporally-Interpolated Funnel Synthesis for Nonlinear Systems

Taylor P. Reynolds, Danylo Malyuta, Mehran Mesbahi, and Behçet Açıkmeşe
Department of Aeronautics and Astronautics, University of Washington, Seattle, Washington 98105

***Abstract*—We propose a novel temporal interpolation scheme to enable Lyapunov-based convex synthesis of controlled invariant sets, called funnels, around nominal trajectories for a class of nonlinear systems. The approach scales well to high dimensional systems, and aims to maximize funnel volume.**

I. Introduction

This paper uses Lyapunov stability theory to design invariant funnels that enable efficient trajectory generation for a class of nonlinear systems. A byproduct of funnel synthesis is an exponentially stabilizing time-varying control law that respects state and input constraints. The goal is to design funnels that are as large as possible, because this implies that trajectories can be generated for a large set of initial states.

We use quadratic Lyapunov functions to design synthesis procedures because they often lead to (convex) linear matrix inequalities (LMIs) [5]. Similar methods for designing both controllers and observers have appeared in the literature [1, 2, 3, 4, 18]. Our approach is to use quadratic Lyapunov functions defined by time-varying matrices, which lead to differential linear matrix inequalities (DLMIs) [12, 15, 16]. Our contribution is to propose a special temporal parametrization for which these DLMIs reduce to a finite number of LMIs. The conversion is intimately related to the convex hull of quadratics [9, 10], but the key difference is that our parametrization is temporal, rather than spatial. As a result, the solution approach differs from either state-based partitioning [8, 11, 14] and sum-of-squares programming [13].

Notation. The argument of time is omitted wherever possible for conciseness. A function $f \in \mathcal{C}^k$ is k-times continuously differentiable. A positive (semi)definite matrix is denoted by $M(\succeq) \succ 0$. We denote an ellipsoid by $\mathcal{E}_M := \{x \in \mathbb{R}^n : x^\top M^{-1} x \le 1\}$ for $M \succ 0$, and a unit simplex by $\Delta_n := \{x \in \mathbb{R}^n_+ : \mathbf{1}^\top x = 1\}$. Horizontal concatenation is denoted by $\mathrm{hcat}\{\cdot\}$.

II. Theory

Consider controlling a dynamical system $\dot{x} = f(x,u)$ about a feasible reference trajectory $\bar{\Gamma} := \{\bar{x}(t), \bar{u}(t)\}_{t=0}^{t_f}$. Let $x \in \mathbb{R}^n$ be the state and $u \in \mathbb{R}^m$ be the control, and restrict $f \in \mathcal{C}^1$. With $\eta := x - \bar{x}$ and $\xi := u - \bar{u}$, the deviation dynamics are:

$$\dot{\eta} = A\eta + B\xi + g(x,u) - g(\bar{x},\bar{u}), \tag{1}$$

where $A = \nabla_x f(\bar{x},\bar{u})$, $B = \nabla_u f(\bar{x},\bar{u})$, and $g \in \mathcal{C}^1$ is a remainder. The generalized mean value theorem [6] guarantees the existence of $n_p \ge 1$ matrices $E_i \in \mathbb{R}^{n\times p_i}$, $\Theta_i \in \mathbb{R}^{p_i \times q_i}$, $C_{q,i} \in \mathbb{R}^{q_i \times n}$, and $D_{q,i} \in \mathbb{R}^{q_i \times m}$ such that:

$$g(x,u) - g(\bar{x},\bar{u}) = \textstyle\sum_{i=1}^{n_p} E_i \Theta_i (C_{q,i}\eta + D_{q,i}\xi). \tag{2}$$

The i-th summand in (2) describes the i-th nonlinearity, with $\{C_{q,i}, D_{q,i}, E_i\}$ acting as input and output *channel selectors*. To facilitate a convex control synthesis procedure, relax Θ_i from a singleton to any matrix satisfying $\|\Theta_i\|_2 \le \gamma_i$. The parameter $\gamma_i > 0$ is a local Lipschitz constant for $g \in \mathcal{C}^1$ around $\bar{\Gamma}$. Combining (1) with (2) yields a block-diagonal norm-bounded linear differential inclusion (LDI) [5]:

$$\begin{aligned}
&\dot{\eta} = A\eta + B\xi + \textstyle\sum_{i=1}^{n_p} E_i p_i, && && \text{(3a)}\\
&q_i = C_{q,i}\eta + D_{q,i}\xi, && i = 1,\dots,n_p, && \text{(3b)}\\
&p_i = \Theta_i q_i,\ \ \|\Theta_i\|_2 \le \gamma_i, && i = 1,\dots,n_p. && \text{(3c)}
\end{aligned}$$

Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ describe the feasible states and controls. The goal is to synthesize a control law $\xi = K(t)\eta$ that maintains $x \in \mathcal{X}$, $u \in \mathcal{U}$ and maximizes the size of the invariant set $\mathcal{E}_{Q(t)}$ centered around $\bar{\Gamma}$. This can be achieved by showing that $V = \eta^\top Q(t)^{-1}\eta$ is a Lyapunov function. Sufficient conditions are established by the following theorem.

Theorem 1. *Assume that* $\exists\ \bar{Q}(t) \succ 0$ *and* $R(t) \succ 0$ *such that* $\bar{x} + \mathcal{E}_{\bar{Q}} \in \mathcal{X}$ *and* $\bar{u} + \mathcal{E}_R \in \mathcal{U}$. *The control gain* $K(t) = Y(t)Q(t)^{-1}$ *quadratically stabilizes* (3) *with decay rate* $\alpha/2$ *if there exist* $Q(t) \succ 0$, $Y(t) \in \mathbb{R}^{m\times n}$ *and* $\lambda(t) \in \mathbb{R}^{n_p}_+$ *such that*

$$\max_{Q,Y,\lambda}\ \log\det Q(0), \tag{4a}$$

$$\text{s.t. } \kappa I \preceq Q \preceq \bar{Q},\ \lambda \ge 0, \tag{4b}$$

$$\begin{bmatrix} F - \dot{Q} & N_1 & N_2 \\ \star & D_1 & 0 \\ \star & \star & D_2 \end{bmatrix} \preceq 0,\quad \begin{bmatrix} Q & Y^\top \\ Y & R \end{bmatrix} \succeq 0. \tag{4c}$$

where $\kappa > 0$ *is a small constant, and*

$$F = QA^\top + AQ + B^\top Y^\top + YB + \alpha Q, \tag{5a}$$

$$N_1 = \mathrm{hcat}\{\lambda_i E_i\}_{i=1}^{n_p}, \tag{5b}$$

$$N_2 = \mathrm{hcat}\{\gamma_i (QC_{q,i}^\top + Y^\top D_{q,i}^\top)\}_{i=1}^{n_p}, \tag{5c}$$

$$D_1 = \mathrm{diag}\{-\lambda_i I_{p_i}\}_{i=1}^{n_p},\ D_2 = \mathrm{diag}\{-\lambda_i I_{q_i}\}_{i=1}^{n_p}. \tag{5d}$$

Proof: The core step is to write $\dot{V} \le -\alpha V$ subject to (3c), viewed as $\|p_i\|_2^2 \le \gamma_i^2 \|q_i\|_2^2$, via the $\mathcal{S}$-procedure [5]. ■

Problem (4) is challenging to solve due to the DLMI (4c). To make the synthesis tractable, we derive a finite parameterization by assuming:

$$\square(t) = \textstyle\sum_{i=1}^{n_T} \sigma_i(t) \square_i, \tag{6}$$

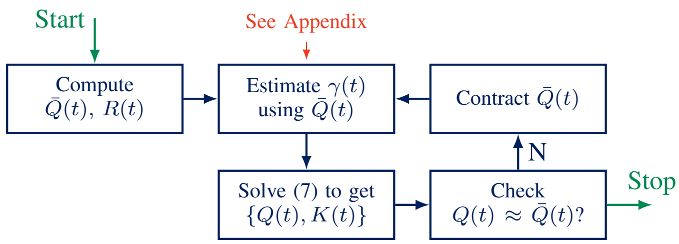

Fig. 1: Block diagram of the funnel synthesis procedure.

where placeholder $\square$ stands for $\{Q, Y, A, B, \bar{Q}, R\}$ and $\sigma \in \Delta_{n_T}$ interpolates within a convex hull of matrices. Let $t_i \in [0, t_f]$ denote the i-th time node. We assume a zeroth-order hold structure $\gamma_i(t) = \gamma_{ij}$ $\forall t \in [t_j, t_{j+1}]$. The following theorem establishes an SDP that ensures Theorem 1 holds.

Theorem 2. *Theorem 1 holds given the structure* (6) *if the following SDP admits a solution:*

$$\max_{Q,Y,\lambda} \quad \log\det Q(0), \tag{7a}$$

$$\text{s.t. } \lambda \geq 0,\ \kappa I \prec Q_i \preceq \bar{Q},\ i = 1, \ldots, n_T \tag{7b}$$

$$\begin{bmatrix} F_{ii} - \dot{Q} & N_1 & N_{2,i} \\ \star & D_1 & 0 \\ \star & \star & D_2 \end{bmatrix} \preceq 0,\ i = 1, \ldots, n_T, \tag{7c}$$

$$\begin{bmatrix} H_{ij} - 2\dot{Q} & 2N_1 & L_{ij} \\ \star & 2D_1 & 0 \\ \star & \star & 2D_2 \end{bmatrix} \preceq 0, \quad \begin{matrix} i = 1, \ldots, n_T - 1, \\ j = i+1, \ldots, n_T, \end{matrix} \tag{7d}$$

$$\begin{bmatrix} Q_i & Y_i^\top \\ Y_i & R_i \end{bmatrix} \succeq 0,\ i = 1, \ldots, n_T, \tag{7e}$$

where:

$$F_{ij} = Q_i A_j^\top + A_j Q_i + Y_i^\top B_j^\top + B_j Y_i + \alpha Q_i, \tag{8a}$$

$$N_{2,i} = \text{hcat}\{\gamma_i (Q_i C_{q,j}^\top + Y_i^\top D_{q,j}^\top)\}_{j=1}^{n_p}, \tag{8b}$$

$$H_{ij} = F_{ij} + F_{ji},\ L_{ij} = N_{2,i} + N_{2,j}. \tag{8c}$$

Proof: To ensure $M(t)N(t) \preceq 0$ *for matrices* M, N *given by* (6), *it is sufficient that* $M_i N_i \preceq 0$, $i = 1, \ldots, n_T$, *and* $M_i N_j + M_j N_i \preceq 0$, $i = 1, \ldots, n_T - 1$, $j = i+1, \ldots, n_T$. ■

For certain classes of $\sigma_i(t)$, (7c) and (7d) become LMIs. In addition, γ depends on K through (2) because $u = \bar{u} + K\eta$. Fig. 1 therefore provides a fixed-point procedure to compute a pair Q, K that induce a piecewise constant γ not larger than that for which they were designed.

III. Numerical Example

This section demonstrates funnel synthesis via Theorem 2 with an example[1]. Consider a planar quadrotor model:

$$\dot{r}_\mathcal{I} = v_\mathcal{I},\ \dot{v}_\mathcal{I} = m^{-1} F(\theta) u_\mathcal{B} + g e_2,\ \dot{\theta} = \omega,\ \dot{\omega} = {}^{\ell^\top u_\mathcal{B}}/_{J}, \tag{9}$$

where,

$$F(\theta) = \begin{bmatrix} -s\theta & -s\theta & -s\theta \\ c\theta & c\theta & c\theta \end{bmatrix}, \quad \ell = \begin{bmatrix} 0 & -l & l \end{bmatrix}. \tag{10}$$

[1]Our code is available at https://github.com/tpreynolds/RSS_2020.

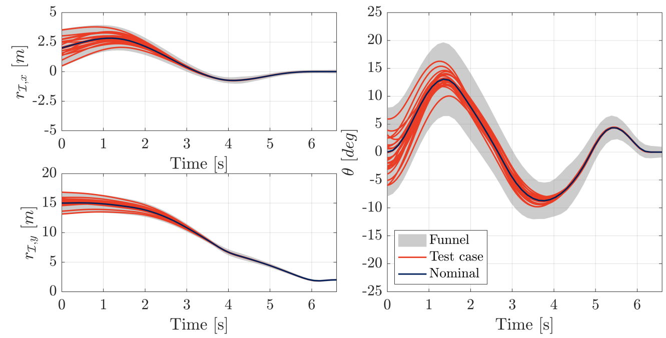

Fig. 2: The synthesized funnel (grey) with $2n$ test cases started from the furthest point along each principle semi-axis (red).

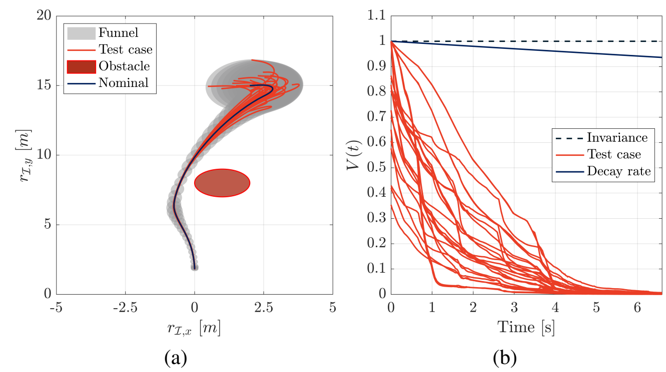

Fig. 3: (a) Synthesized funnel in position alongside the obstacle. (b) The value of the Lyapunov function $V(t)$ along each test case, along with the theoretical bound dictated by α.

Parameter m is mass, J is inertia, and l is arm length. The input $u_\mathcal{B} \in [\underline{u}, \overline{u}]$ and the state has similar box constraints. We allow for a single ellipsoidal obstacle defined by:

$$\| H\,(x - x_c)\,\|_2 \geq 1. \tag{11}$$

A nominal trajectory is obtained by sequential convex programming [17] and minimizes the integral of $\|u(t)\|_1$. A funnel is synthesized via Theorem 2 with $n_T = 2$. Geometrically, the funnel in this case is a convex combination of two ellipsoids during each time segment, the projection of which is shown in Figs. 2 and 3a. To verify invariance, several test trajectories are simulated using the actual dynamics (9). The first $2n$ initial conditions were selected as the furthest points from the center of $\mathcal{E}_{Q(0)}$ along the principal semi-axes, intended to represent a challenging set of initial conditions. Subsequent initial conditions were sampled uniformly from $\mathcal{E}_{Q(0)}$. Fig. 3b shows the value of $V = \eta^\top Q^{-1} \eta$ along the closed-loop trajectories in order to show that invariance was achieved for all test cases, as expected.

IV. Conclusion

Experience has shown Theorem 2 to scale to systems with $n \geq 12$ states and $m \geq 10$ inputs. Future research will aim to generalize the class of nonlinear systems used for funnel synthesis, and to enlarge the computable funnel volume.

Appendix

In this appendix, we briefly outline two methods that can be used to estimate the γ_i in (3) if it is not known a priori. For toy problems, it is often possible to compute γ_i analytically, perhaps as an upper bound on the Lipschitz constant of some nonlinear or uncertain terms. However, for more complex systems, this is not tractable in general, and estimation procedures become necessary. For simplicity, we assume in this Appendix that $n_p = 1$, and fix the temporal interval to be $t \in [t_1, t_2]$.

Sampling-Based Approach

The simplest method to estimate γ is to use a spatiotemporal sampling-based approach. Let $\mathcal{T} \in \mathbb{R}^N$ be a set of N uniformly sampled points in the interval $[t_1, t_2]$. For each sampled time $t_s \in \mathcal{T}$, $s = 1, \ldots, N$, we sample a state uniformly from the funnel $\mathcal{E}_{Q(t_s)}$, and label this sampled point η^s. After closing the loop with a controller $K(t)$, the dynamics (3a) can be recast as

$$\dot{\eta}^s = (A_{cl} + E\Theta C_{cl})\,\eta^s, \tag{12}$$

and the matrix Θ can be computed as the solution of the following optimization problem:

$$\begin{aligned} \vartheta^*(\eta^s) = \arg\min_{\Theta} \quad & \|\Theta\|_2 && (13a)\\ \text{s.t.} \quad & \dot{\eta}^s = (A_{cl} + E\Theta C_{cl})\,\eta^s. && (13b) \end{aligned}$$

The value of γ is then estimated by

$$\gamma = \max_{s=1,\ldots,N} \|\vartheta^*(\eta^s)\|_2. \tag{14}$$

Note that the estimation accuracy (i.e., the number of samples N required to achieve a prescribed accuracy on $|\gamma - \gamma_{\text{true}}|$) increases exponentially in the dimension of η. However, for simple problems (and even some complex ones), sampling with a reasonable $\mathcal{O}(100)$ number of points has been empirically observed to be sufficiently accurate. In general, this process may *underestimate* the true value γ_{true}.

Nonlinear Programming Approach

The idea behind solving (13) at each sample point is to find the *smallest* matrix (in the chosen norm) that is consistent with the model, in the sense that (12) is satisfied (this is equivalent to satisfying (2)). Minimizing the matrix's norm yields the least conservative bound for each sample point, and in turn (14) selects the largest such bound. Sampling an infinite number of points η^s for each time $t \in [t_1, t_2]$ would by construction yield the *least conservative bound* over that interval that is consistent with (3c).

This intuitive max-min description of γ can be formalized in the following nonlinear program. Let $\operatorname{vec} M$ denote the vectorization operator that maps a matrix $M \in \mathbb{R}^{n\times m}$ to a vector $\operatorname{vec} M \in \mathbb{R}^{mn}$. For a given $t \in [t_1, t_2]$, by solving

$$\begin{aligned} \delta^* = \max_{\eta} \quad & \frac{1}{2}\,\|\operatorname{vec}\vartheta^*(\eta)\|_2^2 && (15a)\\ \text{s.t.} \quad & \eta^\top Q(t)^{-1}\eta \le 1, && (15b) \end{aligned}$$

and performing a line search over t, we can compute the value of γ using

$$\gamma = \sqrt{2\delta^*}. \tag{16}$$

To render the NLP (15) solvable in practice, note that the cost function in (13) can be altered so that the solution of

$$\begin{aligned} \operatorname{vec}\vartheta^*(\eta) = \arg\min_{\operatorname{vec}\Theta} \quad & \|\operatorname{vec}\Theta\|_2 && (17a)\\ \text{s.t.} \quad & y(\eta) = M(\eta)\operatorname{vec}\Theta && (17b) \end{aligned}$$

upper bounds the cost (13a), for $y(\eta) = \dot{\eta} - A_{cl}\eta$ and the unique matrix $M(\eta)$ such that the equality $E\Theta C_{cl}\eta = M(\eta)\operatorname{vec}\Theta$ holds. This notation exposes $\vartheta^*(\eta)$ as the solution of a minimum norm least squares problem, whose solution is

$$\operatorname{vec}\vartheta^*(\eta) = M(\eta)^\dagger y(\eta). \tag{18}$$

Equation (18) facilitates computation of the gradient of the cost function (15a) [7].

Since no a priori knowledge of the system dynamics has been assumed to formulate (15), it is likely the most general approach to computing γ, and simultaneously the most practically challenging. For simple systems, we have observed the NLP formulation to work quite well; and even for more challenging real-world systems, it remains a viable strategy. The NLP (15) should be solved using a number of initial guesses to reduce the likelihood of falling into a local maximum.

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