# BAYESIAN-DRIVEN LEARNING OF A NEW WEIGHTED TENSOR NORM FOR TENSOR RECOVERY

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## ABSTRACT

This study addresses the performance limitations of t-SVD-based tensor recovery caused by non-smooth changes and imbalanced low-rankness in tensor data. We introduce a novel bilevel tensor completion model, integrating the learning of a data-dependent weighted tensor norm within the tensor completion framework as an upper-level problem. We treat the optimization of the bilevel problem as a black-box problem, employing Bayesian Optimization (BO) for efficient learning of the proposed tensor norm. Numerical experiments demonstrated the superior performance of our proposed method compared to state-of-the-art methods in tensor completion. The code of our method is available at https://github.com/jzheng20/TR-BO.git.

## 1 INTRODUCTION

Currently, the tensor recovery can classified into three main categories: CP decomposition-based methods (Hitchcock, 1927; 1928), Tucker decomposition-based methods (Tucker, 1963; Liu et al., 2013; Xie et al., 2017), and tensor product (t-product)-based methods (Lu et al., 2019b; 2018; Qin et al., 2022; Zheng et al., 2022; Wang et al., 2023; Zhang et al., 2022; Zhou et al., 2017). Due to the effectiveness of t-product-based methods in image and video processing, this work mainly focus on the developing of the current t-product-based methods.

Different with the first two defining ways for tensor rank, the t-product-based methods are looking for slice-wise tensor decomposition of the resulting tensor obtained by performing a fixed transform along the certain dimension of given tensor data. For example, Kilmer & Martin (2011) given Discrete Fourier Transform (DFT)-based t-product defined on the face product of the transformed tensors by DFT. After determining the t-product, the tensor tubal rank (or tensor average rank) (Lu et al., 2019a) for characterizing the internal correlation of tensor data can be naturally given by counting the number of non-zero singular tubal (or values) obtained by tensor Singular Value Decomposition (t-SVD). To study the low-rankness across different dimensions of higher order tensor data, Zheng et al. (2020) have proposed a new tensor norm by using the Weighted Sum of Tensor Nuclear Norm of all mode-$k_1k_2$ unfolding tensors (WSTNN). However, $\binom{h}{2}$ different unfolding tensors are considered in WSTNN for $h$-order tensor data and it leads to a difficult setting for the weight parameters, with the increasing $h$, especially when tensor data has imbalanced low-rankness across its different dimensions. Besides, since fixed transform are used in these methods, these tensor product-based methods suffer from significant performance degradation when dealing with tensor data that exhibits fast-changing or non-continuous variations as shown in the Fig. 1 of the appendix. The contribution of this work is to give a new tensor recovery framework based on a novel weighted tensor norm from bayesian-driven learning to solve the above issues, and thus can recover the low-rank component more effectively.

## 2 METHODOLOGY

In this section, we propose a novel weighted tensor norm formulized as

$$\|\boldsymbol{\mathcal{X}}\|_{*,\mathcal{U}}^{\boldsymbol{w}} = \sum_{1\le i<j\le h} w_{i,j}\|\boldsymbol{\mathcal{X}} \times_{k_1} \hat{\boldsymbol{U}}_{k_1} \cdots \times_{k_s} \hat{\boldsymbol{U}}_{k_s} \times_{k_{s+1}} \boldsymbol{U}_{i,j,k_{s+1}} \cdots \boldsymbol{U}_{i,j,k_h}\|_*^{(i,j)}, \tag{1}$$

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Table 1: Comparing the PSNR results by all methods on *Berkeley Segmentation Dataset* (*BSD*) (Martin et al., 2001) at different sampling rates $c$. 'Ours' and 'Ours-BO' represents the proposed norm with the equal weights and weights learned from BO, respectively.

| Sampling Rate $c$ | TNN-DCT | SNN | KBR | WSTNN | HTNN-DCT | Ours | Ours-BO |
|---|---|---|---|---|---|---|---|
| 0.3 | 23.25 | 21.86 | 25.45 | 25.75 | 25.21 | 27.87 | **28.33** |
| 0.5 | 27.25 | 25.50 | 31.57 | 31.07 | 30.72 | 33.35 | **33.82** |
| 0.7 | 32.04 | 29.84 | 38.81 | 37.11 | 38.22 | 40.47 | **40.81** |

which aims to study the low-rank characteristics of an h-order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times I_2 \cdots \times I_h}$ across its various dimensions. Here, $\|\boldsymbol{\mathcal{A}}\|_*^{(k_1,k_2)} = \sum_{1 \leq i_{k_n} \leq I_{k_n}(n=3,4,\cdots,h)} \|[\boldsymbol{\mathcal{A}}]_{i_1,\cdots,i_{k_1-1},:,i_{k_1+1},\cdots,i_{k_2-1},:,i_{k_2+1},\cdots,i_{k_h}}\|_*$, and $\times_n$ is mode-n product. We have proposed

$$[\boldsymbol{\mathcal{X}}^*, \boldsymbol{U}_{i,j,k_n}^*] = \underset{\boldsymbol{\mathcal{X}}, \boldsymbol{U}_{i,j,k_n}^T \boldsymbol{U}_{i,j,k_n} = \boldsymbol{I}}{\arg\min} \|\boldsymbol{\mathcal{X}}\|_{*,\mathcal{U}}^{\boldsymbol{w}} \qquad s.t.\ \mathbf{P}_\Omega(\boldsymbol{\mathcal{X}}) = \mathbf{P}_\Omega(\boldsymbol{\mathcal{M}}) \tag{2}$$

for a better tensor recovery and the following bilevel tensor completion model for the learning of the proposed weighted norm:

$$\begin{aligned}[\boldsymbol{w}^{\mathrm{bo}}, \boldsymbol{U}_{i,j,k_n}^{\mathrm{bo}}] = & \underset{\boldsymbol{w}, \boldsymbol{U}_{i,j,k_n}^{\mathrm{bo}} = \boldsymbol{U}_{i,j,k_n}^*}{\arg\min} \|\mathbf{P}_\Phi(\boldsymbol{\mathcal{Z}}^*) - \mathbf{P}_\Phi(\boldsymbol{\mathcal{M}})\|_F \\ & s.t.\ [\boldsymbol{\mathcal{Z}}^*, \boldsymbol{U}_{i,j,k_n}^*] = \underset{\boldsymbol{\mathcal{Z}}, \boldsymbol{U}_{i,j,k_n}^T \boldsymbol{U}_{i,j,k_n} = \boldsymbol{I}}{\arg\min} \|\boldsymbol{\mathcal{Z}}\|_{*,\mathcal{U}}^{\boldsymbol{w}} \qquad s.t.\ \mathbf{P}_{\Phi^c \cap \Omega}(\boldsymbol{\mathcal{Z}}) = \mathbf{P}_{\Phi^c \cap \Omega}(\boldsymbol{\mathcal{M}}),\end{aligned} \tag{3}$$

where $\Omega$ denotes the set of locations of the observed elements in the tensor $\boldsymbol{\mathcal{M}}$, and $\Phi \subset \Omega$. Once we get the learned weighted norm based on $\boldsymbol{U}_{i,j,k_n}^{\mathrm{bo}}(n = s+1, \cdots, h)$ and $\boldsymbol{w}^{\mathrm{bo}}$, we can obtain a better estimation $\boldsymbol{\mathcal{X}}^*$ for $\boldsymbol{\mathcal{M}}$ by minimizing the proposed weighted norm $\|\boldsymbol{\mathcal{X}}\|_{*,\mathcal{U}}^{\boldsymbol{w}^{\mathrm{bo}}}$ under the constraint of $\mathbf{P}_\Omega(\boldsymbol{\mathcal{X}}) = \mathbf{P}_\Omega(\boldsymbol{\mathcal{M}})$. [1] From convergence guarantee for the solving $\boldsymbol{\mathcal{Z}}^*$ by using APMM (Zheng et al., 2023) that is given in the appendix, we can provides an approximate representation for $\boldsymbol{\mathcal{Z}}^*$ as $f_{\boldsymbol{w}}^{(N)} \circ \cdots \circ f_{\boldsymbol{w}}^{(2)} \circ f_{\boldsymbol{w}}^{(1)}(\mathbf{P}_{\Phi^c \cap \Omega}(\boldsymbol{\mathcal{M}}))$, where $f_{\boldsymbol{w}}^{(t)}(\cdot)$ denotes the $t$-th iteration of the optimization for solving $\boldsymbol{\mathcal{Z}}^{(t)}$ and it depend on the choice of $\boldsymbol{w}$. Consequently, we turn to minimize the approximation of the upper-level objective function, $F_{\boldsymbol{w}} = \|\mathbf{P}_\Phi(f_{\boldsymbol{w}}^{(N)} \circ \cdots \circ f_{\boldsymbol{w}}^{(2)} \circ f_{\boldsymbol{w}}^{(1)}(\mathbf{P}_{\Phi^c \cap \Omega}(\boldsymbol{\mathcal{M}}))) - \mathbf{P}_\Phi(\boldsymbol{\mathcal{M}})\|_F$, using BO. We assume that $F_{\boldsymbol{w}}$ follows a Gaussian Process $\mathcal{GP}(m(\boldsymbol{w}), k(\boldsymbol{w}, \boldsymbol{w}'))$, and employ the knowledge gradient (Garnett, 2023) as our acquisition function.

## 3 Experiments

In this section, we compared the proposed methods with five state-of-the-art methods, including TNN-DCT (Lu et al., 2019b), SNN (Liu et al., 2013), KBR (Xie et al., 2017), WSTNN (Zheng et al., 2020), and HTNN-DCT (Qin et al., 2022), on image sequence inpainting to demonstrate the effectiveness of the proposed methods in tensor completion. The Peak Signal-To-Noise Ratio (PSNR) results for all methods are presented in the Table 1[2]. From the table, we can see that our methods (Ours and Ours-BO) have achieved the best performance for all cases. Particularly, the results achieved by Ours-BO exhibit an improvement of approximately 0.5 dB compared to those attained by Ours. Furthermore, when compared to other methods, our approaches demonstrate a superiority of at least 2 dB across all cases.

## 4 Conclusion

In this work, we have proposed a new data-dependent weighted tensor norm learned by BO for handling the case when tensor data is imbalanced low-rankness and non-continuous changing. The experiments demonstrates the superior performance of the proposed method. It worth noting that, beyond tensor completion, the proposed framework can be applied to other tensor recovery model, offering numerous possibilities for future research directions.

[1] In practice, we sample the slices in $\mathbf{P}_{\Phi^c \cap \Omega}(\boldsymbol{\mathcal{M}})$ to construct a smaller tensor for a more efficient learning of $\boldsymbol{w}$, and we have detailed the algorithm for that case in the appendix.

[2] Following Lu et al. (2019a), we randomly selected 50 color images.

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URM STATEMENT

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(a) Imbalanced low-rankness across different dimensions (b) Image sequence obtained by concatenating different scenes (c) Rapidly changing content between frames (d) Disordered image samples for classification task

Figure 1: Imbalanced low-rankness and fast-changing in tensor data

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**Algorithm 1:** BO for the learning of weighted tensor norm

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**Input:** $\mathbf{P}_\Omega(\boldsymbol{\mathcal{M}})$, $\Phi \subset \Omega$, initial dataset $\mathbb{D}$, $\hat{\boldsymbol{U}}_{k_n}$ for $1 \leq n \leq s$.
**Output:** $\boldsymbol{w}^{\text{bo}}$, $\{\boldsymbol{U}^*_{i,j,k_{n_0}}\}_{n_0=s+1}^{h}$ for $1 \leq i < j \leq h$, and $\boldsymbol{\mathcal{X}}^*$.
1. Sampling the slices in $\mathbf{P}_\Phi(\boldsymbol{\mathcal{M}})$ to obtain $\mathbf{P}_\Phi(\boldsymbol{\mathcal{M}}_{\text{small}})$
2.**Repeat**
3. $\boldsymbol{w} \leftarrow \text{POLICY}(\mathbb{D})$
4. $[\boldsymbol{\mathcal{Z}}^*_{\text{small}}, \boldsymbol{U}^*_{i,j,k_n}] = \arg\min_{\boldsymbol{\mathcal{Z}}_{\text{small}}, \boldsymbol{U}^T_{i,j,k_n}\boldsymbol{U}_{i,j,k_n}=\boldsymbol{I}} \|\boldsymbol{\mathcal{Z}}_{\text{samll}}\|^{\boldsymbol{w}}_{*,\mathcal{U}},\ s.t.\ \mathbf{P}_{\Phi^c\cap\Omega}(\boldsymbol{\mathcal{M}}_{\text{small}}) = \mathbf{P}_{\Phi^c\cap\Omega}(\boldsymbol{\mathcal{Z}}_{\text{samll}})$
5. $F_{\boldsymbol{w}} \leftarrow \text{MSE}(\mathbf{P}_\Phi(\boldsymbol{\mathcal{Z}}^*_{\text{small}}), \mathbf{P}_\Phi(\boldsymbol{\mathcal{M}}_{\text{small}}))$, $\mathbb{D} \leftarrow \mathbb{D} \cup \{(\boldsymbol{w}, F_{\text{mse}})\}$
7.**Until** termination reached
8. Obtain $\boldsymbol{\mathcal{X}}^*$ and $\{\boldsymbol{U}^*_{i,j,k_{n_0}}\}_{n_0=s+1}^{h}$ for $1 \leq i < j \leq h$ by solving
$\arg\min_{\boldsymbol{\mathcal{X}}, \boldsymbol{U}_{i,j,k_n}} \|\boldsymbol{\mathcal{X}}\|^{\boldsymbol{w}^{\text{ob}}}_{*,\mathcal{U}} \quad s.t.\ \mathbf{P}_\Omega(\boldsymbol{\mathcal{M}}) = \mathbf{P}_\Omega(\boldsymbol{\mathcal{X}}),\ \boldsymbol{U}^T_{i,j,k_n}\boldsymbol{U}_{i,j,k_n} = \boldsymbol{I}(n = s+1, \cdots, h,\ 1 \leq i < j \leq h).$

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# A Appendix

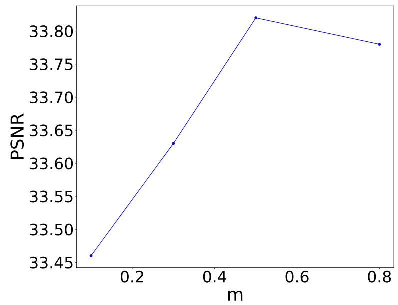
## A.1 The Algorithm for learning the weighted tensor norm by BO (see Algorithm 1 )

## A.2 Parameters analysis

To explore the influence of the choice of $\Phi$, we study the relation between $m = |\Phi^c \cap \Omega|/|\Omega|$ and the recovery performance of the proposed method (Ours-BO). As shown in the Fig. A.2, the performance of Ours-BO have achieved the best when $m = 0.5$.

## A.3 Some properties of the proposed norm

For the convenience of discussion, we define $\|\boldsymbol{\mathcal{A}}\|^{(i,j)}_{*,\mathcal{U}_{i,j}} = \|\mathcal{U}_{i,j}(\boldsymbol{\mathcal{A}})\|^{(i,j)}_* = \|\boldsymbol{\mathcal{A}} \times_{k_1} \hat{\boldsymbol{U}}_{k_1} \cdots \times_{k_s} \hat{\boldsymbol{U}}_{k_s} \times_{k_{s+1}} \boldsymbol{U}_{i,j,k_{s+1}} \cdots \times_{k_h} \boldsymbol{U}_{i,j,k_h}\|^{(i,j)}_*$, and we can get the following results by the convexity of $\|\cdot\|^{(i,j)}_{*,\mathcal{U}_{i,j}}$ and the linearity of the subgradient.




**Property 1.** *The proposed norm defined as* $\|\mathcal{A}\|_{*,\mathcal{U}}^{\boldsymbol{w}} = \sum_{1\le i<j\le h} w_{i,j}\|\mathcal{A}\|_{*,\mathcal{U}_{i,j}}^{(i,j)}$ *for tensor* $\mathcal{A} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_h}$ *is convex if* $w_{i,j} \ge 0$ *for any* $1 \le i < j \le h$.

**Definition 1.** *(tensor product under mode* $(k_1, k_2)$ *for given* $\mathcal{U}$*) For an h-order tensor* $\mathcal{A} \in \mathbb{R}^{I_1\times I_2\times\cdots I_{k_2-1}\times L\times I_{k_2+1}\times\cdots\times I_h}$ *and* $\mathcal{B} \in \mathbb{R}^{I_1\times I_2\cdots\times I_{k_1-1}\times L\times I_{k_1+1}\cdots\times I_h}$, *the tensor product of* $\mathcal{A}$ *and* $\mathcal{B}$ *under mode* $(k_1, k_2)$ *is defined as* $\mathcal{A} *_{k_1,k_2} \mathcal{B} = \mathcal{U}^{-1}(\mathcal{U}(\mathcal{A}) \odot_{\mathrm{slice}_{k_1,k_2}} \mathcal{U}(\mathcal{B}))$, *where* $[\bar{\mathcal{A}} \odot_{\mathrm{slice}_{k_1,k_2}} \bar{\mathcal{B}}]_{i_1,\cdots,i_{k_1-1},:,i_{k_1+1},\cdots,i_{k_2-1},:,i_{k_2+1},\cdots,i_{k_h}} = [\bar{\mathcal{A}}]_{i_1,\cdots,i_{k_1-1},:,i_{k_1+1},\cdots,i_{k_2-1},:,i_{k_2+1},\cdots,i_{k_h}}[\bar{\mathcal{B}}]_{i_1,\cdots,i_{k_1-1},:,i_{k_1+1},\cdots,i_{k_2-1},:,i_{k_2+1},\cdots,i_{k_h}}$.

**Lemma 1.** *For tensor* $\mathcal{A} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_h}$, *if its skinny t-SVD under mode* $(i,j)$ *is* $\mathcal{A} = \mathcal{U}_{i,j} *_{i,j} \mathcal{S}_{i,j} *_{i,j} \mathcal{V}_{i,j}^T$, *then the subgradient of* $\|\mathcal{A}\|_{*,\mathcal{U}}^{\boldsymbol{w}} = \sum_{1\le i<j\le h} w_{i,j}\|\mathcal{A}\|_{*,\mathcal{U}_{i,j}}^{(i,j)}$ *can be given as* $\partial\|\mathcal{A}\|_{*,\mathcal{U}}^{\boldsymbol{w}} = \{\sum_{1\le i<j\le h} w_{i,j}(\mathcal{U}_{i,j} *_{i,j} \mathcal{V}_{i,j}^T + \mathcal{W}_{i,j}) | \mathcal{U}_{i,j}^T *_{i,j} \mathcal{W}_{i,j} = \mathbf{0}, \mathcal{W}_{i,j} *_{i,j} \mathcal{V}_{i,j} = \mathbf{0}, \max_{1\le i<j\le h}\|\mathcal{W}_{i,j}\|_{2,\mathcal{U}_{i,j}}^{(i,j)} \le 1\}$, *where* $\|\mathcal{W}_{i,j}\|_{2,\mathcal{U}_{i,j}}^{(i,j)}$ *is defined as* $\|\mathcal{W}_{i,j}\|_{2,\mathcal{U}_{i,j}}^{(k_1,k_2)} = \max_{i_1,i_2,\cdots,i_{k_1-1},i_{k_1+1},\cdots,i_{k_2-1},i_{k_2+1},\cdots,i_h}\|[\mathcal{U}_{i,j}(\mathcal{W}_{i,j})]_{i_1,\cdots,i_{k_1-1},:,i_{k_1+1},\cdots,i_{k_2-1},:,i_{k_2+1},\cdots,i_{k_h}}\|_2$.

### A.4 Optimization for equation 4 and the corresponding convergence analysis

In this subsection, we are going to solve equation 4 by using APMM Zheng et al. (2023).

$$\begin{aligned}\mathop{\arg\min}_{\mathcal{X},\boldsymbol{U}_{i,j,k_n}} \|\mathcal{X}\|_{*,\mathcal{U}}^{\boldsymbol{w}} \qquad & s.t.\ \mathbf{P}_\Omega(\mathcal{M}) = \mathbf{P}_\Omega(\mathcal{X}),\\ & \boldsymbol{U}_{i,j,k_n}^T\boldsymbol{U}_{i,j,k_n} = \boldsymbol{I}(n = s+1,\cdots,h,\ 1\le i<j\le h). \end{aligned} \tag{4}$$

For easily solving of equation 4, we introduce auxiliary variables $\mathcal{E} \in \mathbb{E} = \{\mathcal{E}|\mathbf{P}_\Omega(\mathcal{E}) = \mathbf{0}\}$ and $\mathcal{Z}_{i,j} = \mathcal{X} \times_{k_{s+1}} \boldsymbol{U}_{i,j,k_{s+1}} \cdots \times_{k_h} \boldsymbol{U}_{i,j,k_h}$ for $1 \le i < j \le h$, and thus obtain the following problem:

$$\begin{aligned}\mathop{\arg\min}_{\mathcal{X},\boldsymbol{U}_{i,j,k_n}^T\boldsymbol{U}_{i,j,k_n}=\boldsymbol{I},\mathcal{E}\in\mathbb{E}} & \sum_{1\le i<j\le h} w_{i,j}\|\mathcal{Z}_{i,j}\|_*^{(i,j)}\\ & s.t.\ \mathbf{P}_\Omega(\mathcal{M}) = \mathcal{X} + \mathcal{E}\\ & \mathcal{X} = \mathcal{Z}_{i,j} \times_{k_1} \hat{\boldsymbol{U}}_{k_1}^T \cdots \times_{k_s} \hat{\boldsymbol{U}}_{k_s}^T \times_{k_{s+1}} \boldsymbol{U}_{i,j,k_{s+1}}^T \cdots \times_{k_h} \boldsymbol{U}_{i,j,k_h}^T\\ & (n = s+1,\cdots,h,\ 1\le i<j\le h),\end{aligned} \tag{5}$$

where the Lagrangian function of equation 5 is given as follows

$$\begin{aligned}&\mathcal{L}(\mathcal{X},\mathcal{E},\boldsymbol{U}_{i,j,k},\mathcal{Z}_{i,j},\boldsymbol{\Lambda}_{i,j},\boldsymbol{\Lambda},\mu,\mu_{ij}) = \sum_{1\le i\le j\le h} w_{i,j}\|\mathcal{Z}_{i,j}\|_*^{(i,j)} + \frac{\mu}{2}\|\mathbf{P}_\Omega(\mathcal{M}) - \mathcal{X} - \mathcal{E}\|_F^2\\ &\langle\mathbf{P}_\Omega(\mathcal{M}) - \mathcal{X} - \mathcal{E},\boldsymbol{\Lambda}\rangle + \sum_{1\le i<j\le h}\Big(\frac{\mu_{ij}}{2}\|\mathcal{Z}_{i,j} \times_{k_1} \hat{\boldsymbol{U}}_{k_1}^T \cdots \times_{k_s} \hat{\boldsymbol{U}}_{k_s}^T \times_{k_{s+1}} \boldsymbol{U}_{i,j,k_{s+1}}^T \cdots \times_{k_h} \boldsymbol{U}_{i,j,k_h}^T - \mathcal{X}\|_F^2\\ &+ \langle\mathcal{Z}_{i,j} \times_{k_1} \hat{\boldsymbol{U}}_{k_1}^T \cdots \times_{k_s} \hat{\boldsymbol{U}}_{k_s}^T \times_{k_{s+1}} \boldsymbol{U}_{i,j,k_{s+1}}^T \cdots \times_{k_h} \boldsymbol{U}_{i,j,k_h}^T - \mathcal{X},\boldsymbol{\Lambda}_{ij}\rangle\Big).\end{aligned} \tag{6}$$

According to the framework of APMM, the above optimization problem equation 5 can be iteratively solved by minimizing the Lagrangian function as presented in the Algorithm 2. The convergence of the algorithm is guaranteed by the Theorem 1.

**Algorithm 2:** APMM-based Iterative Solver to equation 5

**Input:** $\mathbf{P}_\Omega(\boldsymbol{\mathcal{M}})$, $\{\boldsymbol{U}_{i,j,k_n}^{(0)}\}_{n=s+1}^h$, $\{\hat{\boldsymbol{U}}_{k_n}\}_{n=1}^s$, $\boldsymbol{\mathcal{E}}^{(0)}$, $\boldsymbol{\mathcal{Y}}^{(0)}$, $t=0$, $\rho_\mu, \rho_{\mu_{ij}}, \rho_\xi > 1$, $\bar{\mu}$, $\bar{\xi}$, $\mu^{(0)}$, and $\xi^{(0)}$.

**Output:** $\boldsymbol{\mathcal{E}}^{(t+1)}$ and $\boldsymbol{\mathcal{X}}^{(t+1)}$

1. **While not converge do**
2. Calculate $\boldsymbol{\mathcal{Z}}_{i,j}^{(t+1)} (1 \le i < j \le h)$ by
 $\boldsymbol{\mathcal{Z}}_{i,j}^{(t+1)} = \arg\min_{\boldsymbol{\mathcal{Z}}_{i,j}} \mathcal{L}(\boldsymbol{\mathcal{X}}^{(t)}, \boldsymbol{\mathcal{E}}^{(t)}, \boldsymbol{U}_{i,j,k}^{(t)}, \boldsymbol{\mathcal{Z}}_{i,j}, \boldsymbol{\Lambda}_{i,j}^{(t)}, \boldsymbol{\Lambda}^{(t)}, \mu^{(t)}, \mu_{ij}^{(t)}) + \frac{\xi^{(t)}}{2}\|\boldsymbol{\mathcal{Z}}_{i,j}^{(t)} - \boldsymbol{\mathcal{Z}}_{i,j}\|_F^2$;
3. Update $\boldsymbol{\mathcal{X}}^{(t+1)}$ by
 $\boldsymbol{\mathcal{X}}^{(t+1)} = \arg\min_{\boldsymbol{\mathcal{X}}} \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{E}}^{(t)}, \boldsymbol{U}_{i,j,k}^{(t)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t+1)}, \boldsymbol{\Lambda}_{i,j}^{(t)}, \boldsymbol{\Lambda}^{(t)}, \mu^{(t)}, \mu_{ij}^{(t)}) + \frac{\xi^{(t)}}{2}\|\boldsymbol{\mathcal{X}}^{(t)} - \boldsymbol{\mathcal{X}}\|_F^2$;
4. Calculate $\boldsymbol{\mathcal{E}}^{(t+1)}$ by
 $\boldsymbol{\mathcal{E}}^{(t+1)} = \arg\min_{\boldsymbol{\mathcal{E}} \in \mathbb{E}} \mathcal{L}(\boldsymbol{\mathcal{X}}^{(t+1)}, \boldsymbol{\mathcal{E}}, \boldsymbol{U}_{i,j,k}^{(t)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t+1)}, \boldsymbol{\Lambda}_{i,j}^{(t)}, \boldsymbol{\Lambda}^{(t)}, \mu^{(t)}, \mu_{ij}^{(t)}) + \frac{\xi^{(t)}}{2}\|\boldsymbol{\mathcal{E}}^{(t)} - \boldsymbol{\mathcal{E}}\|_F^2$;
5. Calculate $\boldsymbol{U}_{i,j,k_{n_0}}^{(t+1)}$ $(s+1 \le n \le h, 1 \le i < j \le h)$ by
 $\boldsymbol{U}_{i,j,k_{n_0}}^{(t+1)} = \arg\min_{\boldsymbol{U}_{i,j,k_{n_0}}^T \boldsymbol{U}_{i,j,k_{n_0}} = \boldsymbol{I}} \mathcal{L}(\boldsymbol{\mathcal{X}}^{(t+1)}, \boldsymbol{\mathcal{E}}^{(t+1)}, \{\boldsymbol{U}_{i,j,k_n}\}_{n=s=1}^{n_0}, \boldsymbol{U}_{i,j,k_{n_0}}$
 $, \{\boldsymbol{U}_{i,j,k_n}\}_{n=n_0+1}^h, \boldsymbol{\mathcal{Z}}_{i,j}^{(t+1)}, \boldsymbol{\Lambda}_{i,j}^{(t)}, \boldsymbol{\Lambda}^{(t)}, \mu^{(t)}, \mu_{ij}^{(t)}) + \frac{\xi^{(t)}}{2}\|\boldsymbol{U}_{k_{n_0}}^{(t)} - \boldsymbol{U}_{k_{n_0}}\|_F^2$
6. Calculate $\boldsymbol{\Lambda}_{i,j}^{(t+1)} (1 \le i < j \le h)$ by
 $\boldsymbol{\Lambda}_{i,j}^{(t+1)} = \boldsymbol{\Lambda}_{i,j}^{(t)} + \mu_{ij}^{(t)}(\boldsymbol{\mathcal{Z}}_{i,j} \times_{k_1} \hat{\boldsymbol{U}}_{k_1}^T \cdots \times_{k_s} \hat{\boldsymbol{U}}_{k_s}^T \times_{k_{s+1}} \boldsymbol{U}_{i,j,k_{s+1}}^{(t+1)T} \cdots \times_{k_h} \boldsymbol{U}_{i,j,k_h}^{(t+1)T} - \boldsymbol{\mathcal{X}}^{(t+1)})$
7. Calculate $\boldsymbol{\Lambda}^{(t+1)}$ by $\boldsymbol{\Lambda}^{(t+1)} = \boldsymbol{\Lambda}^{(t)} + \mu^{(t)}(\mathbf{P}_\Omega(\boldsymbol{\mathcal{M}}) - \boldsymbol{\mathcal{X}}^{(t+1)} - \boldsymbol{\mathcal{E}}^{(t+1)})$
8. Update $\mu^{(t+1)}$, $\mu_{ij}^{(t+1)}$ and $\xi^{(t+1)}$ by $\mu^{(t+1)} = \min(\rho_\mu \mu^{(t)}, \bar{\mu})$,
 $\mu_{k_1 k_2}^{(t+1)} = \min(\rho_{k_1 k_2} \mu_{k_1 k_2}^{(t)}, \bar{\mu})$ and $\xi^{(t+1)} = \min(\rho_\xi \xi^{(t)}, \bar{\xi})$, respectively;
9. Check the convergence condition: $\|\boldsymbol{\mathcal{Z}}^{(t+1)} - \boldsymbol{\mathcal{Z}}^{(t)}\|_\infty < \varepsilon$, $\|\boldsymbol{\mathcal{X}}^{(t+1)} - \boldsymbol{\mathcal{X}}^{(t)}\|_\infty < \varepsilon$,
 $\|\boldsymbol{U}_{i,j,k_n}^{(t+1)} - \boldsymbol{U}_{i,j,k_n}^{(t)}\|_\infty < \varepsilon$ for $n = s+1, s+2, \cdots, h$ and $1 \le i < j \le h$;
10. $t = t+1$.
11. **end while**

**Theorem 1.** *For the sequence* $\{[\boldsymbol{\mathcal{X}}^{(t)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t)}, \{\boldsymbol{U}_{i,j,k_n}^{(t)}\}_{n=s+1}^h, \boldsymbol{\mathcal{E}}^{(t)}, \boldsymbol{\Lambda}^{(t)}, \boldsymbol{\Lambda}_{i,j}^{(t)}, \mu^{(t)}, \mu_{ij}^{(t)}, \xi^{(t)}]\}$ *generated by the proposed algorithm 2, we have the following properties if* $\{\boldsymbol{\Lambda}_{i,j}^{(t)}\}$ *and* $\{\boldsymbol{\Lambda}^{(t)}\}$ *are bounded,* $\sum_{t=1}^\infty (\mu^{(t)})^{-2}\mu^{(t+1)} < +\infty$, $\sum_{t=1}^\infty (\mu_{ij}^{(t)})^{-2}\mu_{ij}^{(t+1)} < +\infty$ *and* $\lim\limits_{n \longrightarrow \infty} \mu^{(n)} \sum\limits_{t=n}^{\infty} (\xi^{(t)})^{-1/2} = \lim\limits_{n \longrightarrow \infty} \mu_{ij}^{(n)} \sum\limits_{t=n}^{\infty} (\xi^{(t)})^{-1/2} = 0$.

*(i)* $\lim\limits_{t \longrightarrow \infty} \mathbf{P}_\Omega(\boldsymbol{\mathcal{M}}) - \boldsymbol{\mathcal{X}}^{(t)} - \boldsymbol{\mathcal{E}}^{(t)} = \mathbf{0}$ *and* $\lim\limits_{t \longrightarrow \infty} \boldsymbol{\mathcal{Z}}_{i,j}^{(t)} - \boldsymbol{\mathcal{X}}^{(t)} \times_{k_1} \hat{\boldsymbol{U}}_{k_1} \cdots \times_{k_s} \hat{\boldsymbol{U}}_{k_s} \times_{k_{s+1}} \boldsymbol{U}_{i,j,k_{s+1}}^{(t)} \cdots \times_{k_h} \boldsymbol{U}_{i,j,k_h}^{(t)} = \mathbf{0}$

*(ii)* $\{[\boldsymbol{\mathcal{X}}^{(t)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t)}, \{\boldsymbol{U}_{i,j,k_n}^{(t)}\}_{n=s+1}^h\}, \boldsymbol{\mathcal{E}}^{(t)}]\}$ *is bounded.*

*(iii)* $\sum_{t=1}^\infty \xi^{(t)} \|[\boldsymbol{\mathcal{X}}^{(t)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t)}, \{\boldsymbol{U}_{i,j,k_n}^{(t)}\}_{n=s+1}^h\}, \boldsymbol{\mathcal{E}}^{(t)}] - [\boldsymbol{\mathcal{X}}^{(t+1)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t+1)}, \{\boldsymbol{U}_{i,j,k_n}^{(t+1)}\}_{n=s+1}^h\}, \boldsymbol{\mathcal{E}}^{(t+1)}]\|_F^2$ *is convergent. Thus, we have* $\|[\boldsymbol{\mathcal{X}}^{(t)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t)}, \{\boldsymbol{U}_{i,j,k_n}^{(t)}\}_{n=s+1}^h\}, \boldsymbol{\mathcal{E}}^{(t)}] - [\boldsymbol{\mathcal{X}}^{(t+1)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t+1)}, \{\boldsymbol{U}_{i,j,k_n}^{(t+1)}\}_{n=s+1}^h\}, \boldsymbol{\mathcal{E}}^{(t+1)}]\|_F^2 \le \mathcal{O}(\frac{1}{\xi^{(t)}})$.

*(iv)* $\lim\limits_{t \longrightarrow \infty} \|\boldsymbol{\Lambda}_{i,j}^{(t+1)} - \boldsymbol{\Lambda}_{i,j}^{(t)}\|_F = 0$ *and* $\lim\limits_{t \longrightarrow \infty} \|\boldsymbol{\Lambda}^{(t+1)} - \boldsymbol{\Lambda}^{(t)}\|_F = 0$.

*(v) Let* $[\boldsymbol{\mathcal{X}}^*, \boldsymbol{\mathcal{Z}}_{i,j}^*, \{\boldsymbol{U}_{i,j,k_n}^*\}_{n=s+1}^h, \boldsymbol{\mathcal{E}}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Lambda}_{i,j}^*]$ *be any limit point of* $\{[\boldsymbol{\mathcal{X}}^{(t)}, \boldsymbol{\mathcal{Z}}_{i,j}^{(t)}, \{\boldsymbol{U}_{i,j,k_n}^{(t)}\}_{n=s+1}^h, \boldsymbol{\mathcal{E}}^{(t)}, \boldsymbol{\Lambda}^{(t)}, \boldsymbol{\Lambda}_{i,j}^{(t)}]\}$. *Then,* $[\boldsymbol{\mathcal{X}}^*, \boldsymbol{\mathcal{Z}}_{i,j}^*, \{\boldsymbol{U}_{i,j,k_n}^*\}_{n=s+1}^h, \boldsymbol{\mathcal{E}}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Lambda}_{i,j}^*]$ *is a KKT point to equation 5.*