UNIFYING RENORMALIZATION WITH MARKOV CATEGORIES

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ABSTRACT

This paper explores a novel approach for modeling renormalization processes using Markov categories, a formalism rooted in category theory. By leveraging the abstraction provided by Markov categories, we aim to provide a coherent framework that bridges stochastic processes with renormalization theory, potentially enhancing the interpretability and application of these crucial transformations. Our study elucidates theoretical insights, outlines computational benefits, and suggests interdisciplinary applications, especially in conjunction with machine learning methodologies. Key comparisons with existing models highlight the advantages in terms of flexibility and abstraction.

Background. In the realm of modern physics, renormalization is an indispensable tool that addresses some of the most complex challenges faced in theoretical and practical applications (Pes18). It provides a means of managing some infinities that arise in quantum field theories, allowing physicists to make accurate predictions across varying scales (Wei95). Despite its pivotal role, traditional approaches to renormalization, while successful in many aspects, often fall short of offering a coherent framework that integrates seamlessly with both computational and theoretical ideas (BSH80; Pes18).

Renormalization has been a cornerstone technique for achieving scale consistency and extracting meaningful physical predictions in diverse fields such as quantum electrodynamics and statistical mechanics (Zin02; Car96). These methods have historically guided the translation of vague theoretical constructs into quantifiable hypotheses and results (Fey61). However, the pursuit of a unified approach remains a significant hurdle, primarily due to the fragmented nature of existing methods, which are effective, but lack cohesive integration (Car96).

Conceptual framework. In this work we venture into innovative territory by positing the use of Markov categories (CJ19; Fri20), in particular categories of stochastic matrices and Markov kernels, as a formal structural framework for renormalization. Using the rich structures provided by these categories, we can formalize renormalization processes in a structural way. The objective is not only to systematically capture the stochastic nature inherent in these physical processes, but also to elucidate the underlying conceptual structures, which could in turn enhance computational efficiency and provide a more unified theoretical basis for future applications.

In a Markov category, objects can be interpreted as "spaces of outcomes" (e.g., all possible microstates of a physical system), and morphisms are stochastic kernels (i.e. conditional probability distributions). In renormalization:

- The **fine-grained space** X is the set of all microstates (e.g., spin configurations at high resolution).
- The **coarse-grained space** Y is the set of fewer, lower-resolution effective states (e.g., block spins or averaged magnetization).
- A **coarse-graining map** $\varphi \colon X \to Y$ is a *Markov kernel*: for each microstate $x \in X$, it assigns a probability distribution $\varphi(\cdot \mid x)$ over Y.

As renormalization typically proceeds by iterating such block transformations, we regard

$$\varphi_1 \colon X \longrightarrow Y, \quad \varphi_2 \colon Y \longrightarrow Z, \quad \ldots$$

as a sequence of Markov morphisms. Composition in the Markov category then matches the repeated application of these coarse-graining steps, i.e. $\varphi_2 \circ \varphi_1$ represents further reduction in degrees of freedom. One might think of this iterative process as a discrete version of the renormalization group flow.

Usually, the coarse-graining step involves a form of "averaging" over the quotiented degrees of freedom. Sometimes this is a deterministic procedure (a function), sometimes, such as in the next example, the procedure is probabilistic. A Markov kernel is therefore a way of modeling all these possible ways.

Example: Ising spin chain. Consider a finite one-dimensional (1D) Ising chain of N spins with periodic boundary conditions (Kad66). Each spin σ_i can take values in $\{+1, -1\}$. Thus the microstate space is

$$X \;=\; \{\pm 1\}^N.$$

We assume a Hamiltonian of the typical form

$$H(\sigma) \;=\; -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1},$$

where $J > 0$ and $\sigma_{N+1} \equiv \sigma_1$ (periodic boundary). A simple real-space renormalization step groups spins in blocks of size b (for instance, $b = 2$) and replaces each block with an *effective spin* that depends stochastically on the spins in that block. Denote by

$$Y \;=\; \{\pm 1\}^{N/b}$$

the coarse-grained configuration space, where each "block spin" $\tilde{\sigma}_k \in \{\pm 1\}$ represents block k of original spins $(\sigma_{(k-1)b+1}, \ldots, \sigma_{kb})$.

A common (though still somewhat simplistic) choice for the block-spin rule might be the **majority rule**:

$$\tilde{\sigma}_k \;=\; \begin{cases} +1, & \text{with probability proportional to } e^{-\beta \tilde{H}(\sigma)}, \\ -1, & \text{with probability proportional to } e^{-\beta \tilde{H}(\sigma)}, \end{cases}$$

where $\tilde{H}(\sigma)$ is an effective Hamiltonian constructed from the block's original spins. More explicitly, for each block k one might define

$$P(\tilde{\sigma}_k = +1 \,|\, \sigma_{(k-1)b+1}, \ldots, \sigma_{kb}) \;=\; \frac{e^{-\beta \Delta E_+}}{e^{-\beta \Delta E_+} + e^{-\beta \Delta E_-}} \quad \text{and similarly for } \tilde{\sigma}_k = -1,$$

with $\Delta E_\pm$ representing the energy cost of assigning $+1$ or -1 to the block spins, given the original microscopic configuration.

Taken together, these conditional probabilities for each block define a single Markov kernel

$$\varphi : X \;\longrightarrow\; Y, \qquad \sigma \;\mapsto\; P\big(\tilde{\sigma} \,|\, \sigma\big),$$

where $\tilde{\sigma}$ represents the entire coarse-grained configuration. In the *category of finite sets and stochastic matrices*, $\varphi(\tilde{\sigma}|\sigma)$ is simply the product of each block's transition probability.

The power of the Markov category viewpoint emerges when we *iterate* block-spin transformations. For a second-level coarse-graining,

$$\psi : Y \;\longrightarrow\; Z,$$

one obtains a *composed* Markov morphism

$$\psi \circ \varphi : \; X \;\longrightarrow\; Z,$$

capturing the effect of applying two successive coarse-graining steps (first from X to Y, then from Y to Z). This repeated application of Markov morphisms mimics the iterative RG flow in a rigorous categorical sense: each new morphism lumps fewer degrees of freedom at each step, maintaining track of how spin configurations evolve probabilistically under repeated blocking. In diagrams:

$$X \;-\; \boxed{\varphi_1} \xrightarrow{\;X^{(1)}\;} \boxed{\varphi_2} \xrightarrow{\;X^{(2)}\;} \cdots$$

Note that the Markov kernels can be obtained canonically from (measurable) taking values in probability distributions, by postcomposing with the **sampling** kernel (see for example (FGPR23)), meaning that at each step, the following diagram commutes,

$$\begin{tikzcd} X^{(k)} \arrow[r, "\text{function}"] \arrow[dr, "\text{kernel}"'] & P(X^{(k+1)}) \arrow[d, "\text{sample}"] \\ & X^{(k+1)} \end{tikzcd}$$

where $P(\dots)$ denotes forming the space of probability measures. Therefore, the Markov kernel language is equivalent to to the usual way, but it is more concise and structured.

Conclusion. This paper presents a novel perspective on renormalization by recasting its core processes in the language of Markov categories. By viewing block-spin transformations and other coarse-graining steps as morphisms in a category of stochastic maps, we obtain a unified framework for handling the iterative, multi-scale aspects of renormalization. This categorical approach offers high-level clarity and organizes the stochastic transformations that can otherwise be obscured in traditional settings. The underlying string-diagram calculus helps track marginalizations, conditionals, and effective parameters in a more transparent way, aligning renormalization steps with broader principles of probabilistic programming and Bayesian inference.

Beyond providing a disciplined bookkeeping tool, the Markov-category perspective paves the way for cross-disciplinary applications, particularly in machine learning, where multi-scale modeling and hierarchical abstractions share conceptual links with the renormalization group. Although established challenges, such as strong coupling or sign problems, remain, casting renormalization in this compositional framework opens new routes for theoretical and computational innovations. We hope that this work fosters further exploration of the categorical underpinnings of physical theory, stimulating a deeper synergy between statistical physics, probability theory, and machine learning.

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