Phase retrieval for solutions of the Schrödinger equations

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***Abstract*—In this note we present several questions about the phase retrieval problem for the Schrödinger equation. Some partial answers are given as well as some of the heuristics behind these questions.**

***Index Terms*—Phase retrieval, Schrödinger equation**

I. INTRODUCTION

The phase retrieval problem consists in reconstructing a function from its modulus or the modulus of some transform of it (frame coefficient, Fourier transform,...) and some structural information on the function (*e.g.* to be compactly supported). Such a problem occurs in many scientific fields: microscopy, holography, crystallography, neutron radiography, optical coherence tomography, optical design, radar signal processing and quantum mechanics to name a few.

As for quantum mechanics, the most popular phase retrieval problem in the area is Pauli's problem which asks whether a function $f \in L^2(\mathbb{R}^d)$ is uniquely determined (up to a global phase factor) by its modulus $|f|$ and the modulus of its Fourier transform $|\widehat{f}|$. In other words we are asking whether $f, g \in L^2(\mathbb{R}^d)$ such that $|f| = |g|$ and $|\widehat{f}| = |\widehat{g}|$ implies $f = cg$ with $c \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ (a global phase factor). The answer to this question is well known to be negative and there are now many counter-examples (*see e.g.* [AJ] and references therein).

It was further conjectured by Wright that there exists a third unitary operator $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $|f|, |\widehat{f}|, |Uf|$ uniquely determine f. As far as we know, this question is open at the time of writing this note, but if U is only asked to be self adjoint, then there are examples for which the answer is positive (*see* [JR] for more on the problem and the explanation on why the assumption of U being unitary is crucial).

Our aim here is in some sense more modest as we are looking for a one parameter family of unitary operators (actually a semi-group) such that $|U_\alpha f|$ uniquely determines f up to a global phase factor. As far as we know, the only family known so far that answers this question positively is given by the fractional Fourier transform [Ja]. It turns out that, thanks to minor renormalization computations, the result can be reformulated in terms of solutions of Schrödinger equations:

Proposition I.1 (Jaming [Ja]). *Let $u_0, v_0 \in L^2(\mathbb{R}^d)$ and let u, v be the solution of the* free Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\Delta u(t,x) \\ u(0,x) = u_0 \end{cases}, \quad \begin{cases} i\partial_t v(t,x) = -\Delta v(t,x) \\ v(0,x) = v_0 \end{cases}.$$

If $|u(t,x)| = |v(t,x)|$ for every $t \in \mathbb{R}$ and every $x \in \mathbb{R}^d$ then there exists $c \in \mathbb{R}$ such that $u_0 = cv_0$.

This proposition is very simple and most of the content of [Ja] deals with the following questions linked to Wright's conjecture: can one restrict the set of times t needed to obtain the same answer, at least if one restrict u_0 and/or v_0 to be in some set of functions with some additional structure. For instance, it is shown that if u_0, v_0 belong to the spectral subsets of the Laplacian (that is to the Paley-Wiener spaces) then an explicit discrete set of time suffices.

Our aim here is to start a different line of investigation. The previous proposition is restricted to the free Schrödinger equation on $\mathbb{R}^d$. It is natural to try to extend this result to a more general setting in which the Schrödinger equation has a natural physical meaning:

Let Ω be any structure on which the Laplace operator has a natural analogue. For instance

– The natural Euclidean metric on $\mathbb{R}^d$ could be deformed which would lead to $-\Delta$ to be replaced by a more general elliptic operator $-\mathrm{div}(A\nabla)$.

– Instead of $\Omega = \mathbb{R}^d$, one could consider the Schrödinger equation on an open set (say with smooth boundary) and add some boundary conditions.

– More generally Ω could be a Riemannian manifold and $-\Delta$ the Laplace Beltrami operator.

– In the opposite direction, Ω could be a graph (finite or not) and $-\Delta$ be the combinatorial Laplacian.

– The intermediate situation of quantum graphs (in short, graphs in which the edges are intervals with boundary conditions on the vertices) is also of interest.

Once this is set, a further question comes immediately, even in the simplest case of $\mathbb{R}^d$: what is the influence of the potential $W : \Omega \to \mathbb{R}^+$ (or $\Omega \to \mathbb{R}$). In short, we are asking the following:

Question I.2. *Let Ω be any of the above structures and $-\Delta$ be the (positive) Laplacian associated to it. Let $W : \Omega \to \mathbb{R}^+$*

be a potential. Let u_0, v_0 be two functions on Ω and let u, v be the solutions of the same Schrödinger equation

$$\begin{cases} i\partial_t u = -\Delta u + Wu \\ \quad u(0,x) = u_0 \end{cases} \quad \begin{cases} i\partial_t v = -\Delta v + Wv \\ \quad v(0,x) = v_0 \end{cases}$$

but with initial data u_0 and v_0 respectively. Assume that $|u(t,x)| = |v(t,x)|$ for every $t \in \mathbb{R}$ and every $x \in \Omega$. Does there exists $c \in \mathbb{R}$ such that $u_0 = cv_0$.

One may eventually ask the same question when u_0, v_0 belong only to a spectral subset of $-\Delta + W$ (or any other natural restriction). In particular, can the set of times be reduced?

Of course, one needs to impose some restrictions on W for the question to even make sense, but we will not elaborate in this direction in this note. Our main message in this note is that the question should be investigated in further detail and that

there are many more questions than answers so far!

Of course, we have only asked the question of uniqueness, the other natural questions on phase retrieval should also be asked: stability with respect to noise, possibility of sampling, reconstruction algorithms,...

The remaining of the paper is organised as follows: in the next section, we give some information on the Schrödinger equation on $\mathbb{R}^d$ that will lead to a conjecture that we think reasonable. The third section, we make some observation about our problem on finite graphs that will lead to some partial results on our question in this setting.

II. The Schrödinger equation on $\mathbb{R}^d$

In this section, we make some observations on the Schrödinger equation on $\mathbb{R}^d$ and the related phase retrieval problem. We here stay on an *heuristic level* and do not aim for full mathematical rigour as this would require a much longer exposition. One may refer to [LP] or to the survey [FOPT] for further detail.

We consider the *scaled* Schrödinger equation on $\mathbb{R} \times \mathbb{R}^d$

$$i\varepsilon\partial_t u^\varepsilon = -\frac{\varepsilon^2}{2}\Delta u^\varepsilon + W(x)u^\varepsilon$$

with initial data $u^\varepsilon(t=0,x) = u_0^\varepsilon(x)$. Here ε stands for the scaled Plank constant, $u^\varepsilon = u^\varepsilon(t,x)$ is the wave function and W is a potential. From the point of view of physics, the interesting quantity is not u^ε but the *position density* $|u^\varepsilon|^2$.

Next, we introduce the *Wigner transform* of a function $f \in L^2(\mathbb{R}^d)$ by

$$\mathcal{W}^\varepsilon[f](x,\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f\left(x - \varepsilon\frac{\eta}{2}\right) \overline{f\left(x + \varepsilon\frac{\eta}{2}\right)} e^{i\langle \eta,\xi\rangle}\,\mathrm{d}\eta.$$

Recall that

$$\mathcal{W}^\varepsilon[f](x,\xi) = \mathcal{W}^\varepsilon[g](x,\xi) \quad \Leftrightarrow \quad f = cg,\ c \in \mathbb{T}.$$

Now, in the free case, the scaling is not needed and we take $\varepsilon = 1$. If u solves the Schrödinger equation, $i\partial_t u = -\frac{1}{2}\Delta u$ then $\omega(t,x,\xi) := \mathcal{W}^1[u(t,\cdot)](x,\xi)$ solves the *transport equation*

$$\partial_t\omega + \langle \xi, \nabla_x \omega\rangle = 0.$$

It follows that $\omega(t,x,\xi) = \omega(0, x-\xi t, \xi)$. On the other hand, the well-known marginal properties of the Wigner transform read

$$\begin{aligned} |u(t,x)|^2 &= \int_{\mathbb{R}^d} \omega(t,x,\xi)\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} \omega(0, x-\xi t, \xi)\,\mathrm{d}\xi. \end{aligned}$$

This is nothing but the X-ray transform of $\omega(0,\cdot)$ through the line $(x,0) + \mathbb{R}(-t,1)$. Inversion properties of the X-ray transform then show that $|u(t,x)|$ determines the Wigner transform of u_0 and thus u_0 up to a global phase factor. This gives a new (but not totally rigorous at this stage) proof of Proposition I.1. The original proof was slightly different. It consisted in showing that $|u(t,x)|$ determines the restriction of the ambiguity function of u to certain lines. As the ambiguity function is the $\mathbb{R}^{2d}$ Fourier transform of the Wigner function, the link between the 2 proofs is the well-known Fourier Slice Theorem for the X-ray transform.

In the presence of a potential $W \neq 0$, the situation is more complicated and we have no answer to our problem so far. Nevertheless we have an heuristic that leads us to believe that the answer should be positive in many cases. To describe it, we will now need the scaled Planck constant ε. We then consider $\omega^\varepsilon(t,x,\xi) := \mathcal{W}^\varepsilon[u^\varepsilon(t,\cdot)](x,\xi)$ which still satisfies a PDE known as the *Wigner equation*. The formal limit when $\varepsilon \to 0$ of this equation leads to the following *Vlasov* equation

$$\partial_t\omega^0 + \left\langle \xi, \nabla_x\omega^0\right\rangle - \left\langle \nabla_x W(x), \nabla_\xi\omega^0\right\rangle = 0$$

satisfied by the limit ω^0 of ω^ε. The meaning of this limit needs to be made precise (and shown to exist) and is called the *Wigner measure*. It is constant along Hamiltonian trajectories $\big(x(t,y,\eta), \xi(t,y,\eta)\big)$ satisfying

$$\begin{cases} \partial_t x(t,y,\eta) = \xi(t,y,\eta) & x(0,y,\eta) = y \\ \partial_t\xi(t,y,\eta) = -\nabla_x W\big(x(t,y,\eta)\big) & \xi(0,y,\eta) = \eta \end{cases}.$$

For instance, in the case of the harmonic oscillator, $W(x) = \frac{x^2}{2}$, ($d=1$) we obtain that ω^0 is constant along circles and the circular Radon transform should play the same role as the X-ray transform. It is thus natural to conjecture the following:

Conjecture II.1. *Let $u_0, v_0 \in L^2(\mathbb{R}^d)$ and let u, v be the solution of the* free Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\Delta u(t,x) + |x|^2 u(t,x) \\ \qquad u(0,x) = u_0 \end{cases}$$

and

$$\begin{cases} i\partial_t v(t,x) = -\Delta v(t,x) + |x|^2 v(t,x) \\ \qquad v(0,x) = v_0 \end{cases}.$$

If $|u(t,x)| = |v(t,x)|$ for every $t \in \mathbb{R}$ and every $x \in \mathbb{R}^d$ then there exists $c \in \mathbb{R}$ such that $u_0 = cv_0$.

A similar proof as for [Ja, Theorem 5.5] shows that the answer is positive when u_0, v_0 belong to the spectral set of the harmonic oscillator, that is, when they are linear combinations of Hermite functions. It is natural to speculate that the same is true when the potential W is confining, or at least when it satisfies a lower bound of the form $W(x) \geq c|x|^\alpha$,$c, \alpha > 0$.

III. Schrödinger equations on finite graphs

In this section, we consider $\Gamma = (\mathcal{V}, \mathcal{E})$ to be a finite non-oriented graph. Without loss of generality, we assume that $\mathcal{V} = \{1, \ldots, n\}$, we also assume that $n \geq 3$. For $x \in \mathcal{V}$, we write $y \sim x$ to say that $y \in V$ is a neighbor of x *i.e.* $(x, y) \in \mathcal{E}$ (thus also $(y, x) \in \mathcal{E}$) and $d(x)$ the number of neighbors of x (the degree). For a function on $\mathcal{V}$, the *Laplace* operator Δ is defined by

$$\Delta u(x) = \left(\sum_{y\sim x} u(y)\right) - d(x)u(x).$$

Fix a function $W : \mathcal{V} \to \mathbb{R}$ (note that V is *real valued*), we say that a function $u : \mathbb{R} \times \mathcal{V} \to \mathbb{C}$ satisfies the *Schrödinger equation* on Γ with potential V if

$$\begin{cases} i\partial_t u(t, x) = -\Delta u(t, x) + W(x)u(t, x) & t \in \mathbb{R},\ x \in \mathcal{V}, \\ u(0, x) = u_0(x) & x \in \mathcal{V} \end{cases}. \tag{III.1}$$

The aim of this section is to start investigation of Question I.2 in this setting. This question is still largely unexplored and we will here focus on some simple observation

Our first observation deals with the connectedness assumption. Assume that Γ has several connected components, $\Gamma = \bigcup_{j=1}^m \Gamma_j$, $\Gamma_j = (\mathcal{V}_j, \mathcal{E}_j)$. Take a function u_0 on $\mathcal{V}$ and write u_0^j for its restriction to $\mathcal{V}_j$ and let u_j be the corresponding solution of Schrödinger equation on Γ_j. Write $u = (u_j)_{j=1,\ldots,m}$ for the function on $\mathcal{V}$ obtained by gluing back the u_j's. Then u is a solution of the Schrödinger equation on Γ with initial condition u_0. In particular, we may attribute an arbitrary phase $c_j \in \mathbb{T}$ to each u_0^j leading to $u_c = (c_j u_j)_{j=1,\ldots,m}$, a solution of the Schrödinger equation on Γ that has same modulus as u though u_c is not a multiple of u. From a practical point of view, this is harmless, nevertheless, the connectedness restriction is necessary to obtain uniqueness up to a global phase factor.

Next, note that $u \to -\Delta u + V u$ is a *real symmetric* operator on $\mathbb{R}^{\mathcal{V}} = \mathbb{R}^n$ *i.e.* its matrix is real symmetric. Therefore, there is an orthonormal basis of (real) eigenvectors $(\Phi_j)_{j=1,\ldots,n}$ with corresponding eigenvalues $\lambda_j \in \mathbb{R}$. To further fix ideas, we will assume that $V \geq 0$, so that the operator is even positive. One can then order the eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. The fact that $\lambda_1 = 0$ and is a simple eigenvalue is a basic fact on spectral graph theory and comes from our connectivity assumption. The corresponding eigenvector is $\phi_1 = n^{-1/2}(1, \ldots, 1)^t$. Therefore, if we decompose u_0 in this basis

$$u_0(x) = \sum_{j=1}^n a_j \phi_j(x) \quad a_j = \langle u_0, \phi_j\rangle,$$

we obtain the solution $u(t, x)$ at any time via

$$u(t, x) = \sum_{j=1}^n a_j e^{i\lambda_j t} \phi_j(x).$$

Our second observation is that multiple eigenvalues are a potential source of non uniqueness.

Before elaborating on this a bit further, it should also be noted that this situation is rare. Indeed, Tao and Vu [TV] have shown that if the graph is drawn randomly (e.g. when considering Erdös-Renyi random graphs $G(n, p)$) and the potential is chosen randomly as well (say n values drawn uniformly at random between 0 and 1), then the eigenvalues are simple with high probability (at least when n is large enough). Note that if $p \geq 2\dfrac{\log n}{n}$ (say) then $G(n, p)$ is also connected with high probability.

Let us now see how we are lead to a classical phase retrieval problem. Assume that one of the eigenvalues is not simple. Call it λ and let m be its multiplicity and E_λ be the corresponding eigenspace. Then, if $u_0 \in E_\lambda$, $u(t, x) = e^{i\lambda t} u_0(x)$ which has same modulus as u_0. In this case, we are considering a finite dimension phase retrieval problem. If $(e_j)_{j=1,\ldots,n}$ is the canonical basis of $\mathbb{R}^n$, we are asking whether $|\langle u_0, e_j\rangle|$ determines u_0 up to a global phase factor when $u_0 \in E_\lambda$.

Lemma III.1. *Let E be a finite dimensional subspace of $\mathbb{C}^m$. The following are equivalent:*

i) there exist f, g in E such that there coordinates satisfy $|f_j| = |g_j|$ for every j, though f is not a multiple of g,
ii) there exist f, g in E that are orthogonal and such that there coordinates satisfy $|f_j| = |g_j|$ for every j, though f is not a multiple of g,

This lemma tells us that we will have non-uniqueness for $u_0 \in E_m$ when there are already two orthogonal eigenfunction for the same eigenvalue that have same modulus (entrywise). It is easy to find two such eigenvectors on the complete graph K_n when $n \geq 3$.

This lemma seems to be known for some time, at least a more evolved version of the lemma can be found in a recent preprint by Freeman *et al* [FOPT]. For sake of completeness, here is a proof:

Proof. One way is obvious, for the second one, assume that $f \neq g$ are such that $|f_j| = |g_j|$ and $\langle f, g\rangle \neq 0$. Consider the quantity $\min_{|c|=1} \|f - cg\|^2$. Up to replacing g by a unimodular constant times g, we may assume that this minimum is attaigned for $c = 1$, that is $\min_{|c|=1} \|f - cg\|^2 = \|f - g\|^2$. In other words, when $|c| = 1$,

$$\|f\|^2 + \|g\|^2 - 2\Re\bar{c}\langle f, g\rangle \geq \|f\|^2 + \|g\|^2 - 2\Re\langle f, g\rangle.$$

Now $\Re\bar{c}\langle f, g\rangle$ is maximised if and only if c is the phase of $\langle f, g\rangle$ so that $\langle f, g\rangle$ is real, positive.

Next consider $f_\lambda = f - \lambda\frac{f+g}{2}$ and $g_\lambda = g - \lambda\frac{f+g}{2}$ so that $f_\lambda, g_\lambda \in E$. Note that $\langle f_0, g_0\rangle > 0$ while $\langle f_1, g_1\rangle =$

$-\left\|\frac{f-g}{2}\right\|^2 < 0$. Therefore, there exists λ such that $\langle f_\lambda, g_\lambda\rangle = 0$.

Further if $z, \zeta \in \mathbb{C}$ are such that $|z| = |\zeta|$ then , for $0 \leq \lambda \leq 1$,

$$\begin{aligned}\left|z - \lambda\frac{z+\zeta}{2}\right|^2 &= \left|\left(1-\frac{\lambda}{2}\right)z + \frac{\lambda}{2}\zeta\right|^2 \\ &= \left(1-\frac{\lambda}{2}\right)^2|z|^2 + \frac{\lambda^2}{4}|\zeta|^2 \\ &\quad + \left(1-\frac{\lambda}{2}\right)\lambda\Re(z\bar{\zeta}).\end{aligned}$$

Now as z and ζ have same modulus, we can exchange them and unwind the computation to obtain that this is $\left|\zeta - \lambda\frac{z+\zeta}{2}\right|^2$. This shows that the coordinates of f_λ and of g_λ have same modulus. □

The final observation so far is that a stronger situation allows for uniqueness. Indeed, looking at the m-th coordinate of $u(t,x)$, it is of the form $\sum_{j=1}^n a_j\phi_{j,m}e^{i\lambda_j t}$. The square modulus of this quantity is thus

$$\sum_{j,k=1}^n a_j\overline{a_k}\phi_{j,m}\phi_{k,m}e^{i(\lambda_j-\lambda_k)t}$$

and this quantity is known for all t.

Definition III.2. We will say that the multiset $\{\lambda_j,\ j = 1,\ldots,n\}$ is *totally dissociated* if none of the λ_j's is repeated and if $\lambda_j - \lambda_k \neq \lambda_{j'} - \lambda_{k'}$ if $(j,k) \neq (j',k')$.

Note that this is a generic condition for a set in $\mathbb{R}^n$. We will assume that the eigenvalues of $-\Delta + V$ are totally dissociated. Then, from the linear independence of $\{e^{i\lambda t}\}_{\lambda\in\mathbb{R}}$, we obtain that,

$$a_j\overline{a_k}\phi_{j,m}\phi_{k,m}, \quad j,k,m \in \{1,\ldots,n\}$$

is uniquely determined by $|u(t,x)|$.

We will need a second property, this time of the eigenvectors:

Definition III.3. Let $\Gamma = (\mathcal{V},\mathcal{E})$ be a finite non-oriented connected graph and $V : \mathcal{V} \to \mathbb{R}$ be a potential. Let $\phi_1,\ldots,\phi_n$ be a corresponding orthonormal basis of eigenfunctions. Define the graph $\Gamma_V = (\mathcal{V}_V, \mathcal{E}_V)$ with $\mathcal{V}_V = \{1,\ldots,n\}$ and $(j,k) \in \mathcal{E}_V$ if and only if there exists $m = m(x,y) \in \{1,\ldots,m\}$ such that $\phi_{j,m}\phi_{k,m} \neq 0$.

We say that V satisfies *property (S)* if Γ_V is *complete*.

This situation is generic since the eigenvectors depend continuously on V, any small change in V would thus imply that each ϕ_j has full support.

We can now prove our main result in this section:

Theorem III.4. *Let $\Gamma = (\mathcal{V},\mathcal{E})$ be a non-oriented finite connected graph and $V : \mathcal{R} \to \mathbb{R}$ be a potential. Assume that the eigenvalues of $-\Delta + V$ are totally dissocated and that V satisfies property (S). Then the modulus $|u(x,t)|$ of the solution of the Schrödinger equation* (III.1) *uniquely determines u_0 up to a global phase factor.*

Proof. Let ϕ_j be the set let

$$u_0 = \sum_{j=1}^n a_j\phi_j \quad v_0 = \sum_{j=1}^n b_j\phi_j$$

be such that $|u(t,x)| = |v(t,x)|$ for all $t \in \mathbb{R}$ and $x \in \mathcal{V}$. As said above, from the total dissociativity of the spectrum, for all j,k,m,

$$a_j\overline{a_k}\phi_{j,m}\phi_{k,m} = b_j\overline{b_k}\phi_{j,m}\phi_{k,m}. \tag{III.2}$$

Taking $j = k$ in (III.2), we obtain $|a_j|^2\phi_{j,m}^2 = |b_j|^2\phi_{j,m}^2$ for all m and, as at least for one m we have that $\phi_{j,m} \neq 0$, we obtain $|a_j|^2 = |b_j|^2$.

Now, let ℓ be the first j for which $a_j \neq 0$ so that $a_j = b_j = 0$ for $j < \ell$ and $|a_\ell|^2 = |b_\ell|^2 \neq 0$. Replacing v by cv for some $c \in \mathbb{T}$, that is replacing $(b_j)_j$ by (cb_j), we may assume that $a_\ell = b_\ell \neq 0$. Then, taking $k = \ell$ and $j > \ell$ in (III.2) we obtain $a_j\phi_{j,m}\phi_{\ell,m} = b_j\phi_{j,m}\phi_{\ell,m}$. Our hypothesis is that there is an m such that $\phi_{j,m}\phi_{\ell,m} \neq 0$ so that this implies that $a_j = b_j$ for all j and finally that $u_0 = v_0$. □

Note that if the hypothesis is not met, the result is false: assume that one of the vertices is not a neighbor to all of the others. Say 1 is a neighbor of $2,\ldots,m$ but not of $m+1,\ldots,n$. Take $a_1 = b_1 = 1$, $a_j = b_j = 0$ for $j = 2,\ldots,m$ and $a_j = -b_j = 1$ for $j = m+1,\ldots,n$. Then

$$a_j\overline{a_k}\phi_{j,m}\phi_{k,m} = \begin{cases}\phi_{j,m}\phi_{k,m} & \text{when } j=k=1 \text{ or } j,k \geq m+1\\ 0 & \text{otherwise}\end{cases}$$

and is therefore $= b_j\overline{b_k}\phi_{j,m}\phi_{k,m}$. Thus, if $u_0 = \phi_1 + \phi_{m+1} + \cdots + \phi_n$ and $v_0 = \phi_1 - \phi_{m+1} - \cdots + \phi_n$ then the corresponding solutions have same modulus at all time.

In this case, one may nevertheless obtain results, but no longer for every u_0. An inspection of the above proof and of the counterexample show that the problems come from a_j's that are 0. It is actually enough to have an $a_j \neq 0$ so that in the graph Γ_V, j is a neighbor of every other vertex. When Γ is connected and $V = 0$, $\phi_1 = n^{-1/2}(1,\ldots,1)^t$, is an eigenvector (corresponding to the 0 eigenvalue) so that all edges are connected to 1 in Γ_0. In particular, if $u_0 = (u_{0,j})_{j=1,\ldots,n}$ is such that

$$a_1 = \langle u_0, \phi_1\rangle = n^{-1/2}\sum_{j=1}^n u_{0,j} \neq 0$$

then $|u(x,t)|$ uniquely determines u_0.

One may perturb this result to show that if V is small enough, there exists $\varepsilon_V > 0$ such that, if $\left|\sum_{j=1}^n u_{0,j}\right| > \varepsilon_V$ then $|u(x,t)|$ still uniquely determines u_0.

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